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A study of free terms for plate problems in the dual BEM

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Abstract

In this paper, we review the free terms of Lapalce and Navier equations for 2-D and 3-D problems and extend to biharmonic equation for plate problem. We derive the free terms of the dual BEM with a smooth boundary by means of the bump-contour technique surrounding the singularity. After using the limiting approach, the free terms and boundary terms for the sixteen improper integrals in the dual formulation for the plate problems are derived. The improper integrals due to the sixteen kernels with singularity or hypersingularity are interpreted as the finite part.

Key Words: dual boundary integral formulation, free term, boundary term, bump-contour technique, biharmonic equation, smooth boundary, hypersingularity

對偶邊界元素法中板之自由項之研究

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中文摘要

本文針對二維及三維的勢能及彈力問題的自由項做回顧並推廣至四階雙諧和函數的板問題。藉由繞過平滑邊界上奇異點的半圓弧路徑積分的技巧, 並以極限方法求得十六個核函數邊界 積分的自由項與跳躍項。最後, 此十六個核函數的主值問題所屬的奇異 性或超強奇異性均可以有限部分觀念予以詮釋。

關鍵字: 對偶邊界積分方程式、自由項、邊界項、繞半圓技巧、雙諧和方程、平滑邊界、超強奇異性

Introduction

Boundary integral equations (BIE) with strongly singular and hypersingular kernels are currently employed in many fields of applied mechanics, most of the mathematical issues have been clarified for the evaluation of the singular integrals. The treatment of singularities has always been a chief subject in the development of boundary element method (BEM).

Dual boundary integral equations (DBIEs) for crack problems were derived using the limiting and trace approaches proposed by Hong and Chen [1]. Also, the DBIEs for the Laplace equation with a degenerate boundary was developed by Chen and Hong [2]. The numerical implementation has been termed the dual boundary element method by Portela *et al.* [11]. How to determine free terms in a hypersingular equation accurately has received attention in the dual BEM by Guiggiani [7, 8, 9, 10]. Later, an additional free term in the hypersingular equation for the Laplace probelm was independently obtained by Guiggiani [10] and Chen and Hong [3]. Since the hypersingular integral equation can provide an additional constraint for the Dirichlet problems, the free terms must be examined. Many researchers, for example, Guiggiani has derived the free terms in the boundary integral formulation by employing the direct method for the Laplace equation, the Navier equation and the biharmonic equation. He also found an additional free term for the corner problem using the biharmonic equation instead of using the "dual" formulation. In 2000, Chen *et al.* [4, 6] have proposed the bump-contour technique and the limiting approach to determine the free terms of the two- or three-dimensional Laplace and Navier equations successfully. Also, the free terms of dual BIE for the 2-D Helmholtz equation were solved [5]. In this paper, we focus on the bending of thin plates where the BEM must face the improper integrals of hypersingular kernel or finte-part integrals. The unnamed higher-order singularities than hypersingularity occur in the dual formulation. We derive the free terms on a smooth boundary by means of the bump-contour technique surrounding the singularity. After using the bump-contour technique and limiting approach, the free terms and boundary terms for the sixteen improper integrals in the dual formulation are derived. The improper integrals due to the sixteen kernels with weak singularity, strong singularity, hypersingularity or unnamed singularity are interpreted as the finite parts.

Review of free terms of the dual integral formulation for 2-D and 3-D Laplace and Navier equations with a smooth boundary

According to the papers of Chen and his students [4, 5, 6] as well as his colleagues, they derived the free terms of the dual integral equations in conjunction with the bump-contour technique for the Laplce, Helmholtz and Navier probelms. By adopting the bump-contour technique, we have the free terms and boundary terms. We summarize the results for the two and three-dimensional problems in Table 1. It is found that the contributions from both the hypersingular integrals and the strongly singular integrals for the free terms of the BIE are, respectively, half and half for the 2-D case, one third and two thirds for 3-D problem. The dual boundary integral equations for the 2-D and 3-D Navier equations are also considered. Similarly, we summarize the results for the two and three-dimensional problems in Table 2. Comparing the results of the Laplace equation with those of the Navier equation, it is found that the free coefficients are the same to half for the smooth boundary.

Free terms of the DBIEs with a smooth boundary for the biharmonic problems

The dual integral equations for the domain point can be derived from the Rayleigh-Green identity as follows:

$$8\pi\, u(x) = \int_B \{-U(s,x)v(s) + \Theta(s,x)m(s) - M(s,x)\theta(s) + V(s,x)u(s)\} dB(s),\ x \in \Omega, \tag{1}$$

$$8\pi\, \theta(x) = \int_B \{-U_\theta(s,x)v(s) + \Theta_\theta(s,x)m(s) - M_\theta(s,x)\theta(s) + V_\theta(s,x)u(s)\} dB(s),\ x \in \Omega, \tag{2}$$

$$8\pi\, m(x) = \int_B \{-U_m(s,x)v(s) + \Theta_m(s,x)m(s) - M_m(s,x)\theta(s) + V_m(s,x)u(s)\} dB(s),\ x \in \Omega, \tag{3}$$

$$8\pi\, v(x) = \int_B \{-U_v(s,x)v(s) + \Theta_v(s,x)m(s) - M_v(s,x)\theta(s) + V_v(s,x)u(s)\} dB(s),\ x \in \Omega, \tag{4}$$

where B is the boundary, Ω is the domain of interest, u, θ, m and v mean the displacement, slope, normal moment and effective shear force, s and x are the source and field points, respectively.

For the biharmonic equation, we can obtain the fundamental solution as follows:

$$U(x,s) = r^2 ln(r), \tag{5}$$

where r is the distance between the field point and the source point written as $r \equiv |s-x|$. The other three kernels, $\Theta(s,x)$, $M(s,x)$ and $V(s,x)$, are defined as follows:

$$\Theta(s,x) \quad = \mathcal{K}_{\theta,s}(U(s,x)), \tag{6}$$

$$M(s,x) \quad = \mathcal{K}_{m,s}(U(s,x)), \tag{7}$$

$$V(s,x) \quad = \mathcal{K}_{v,s}(U(s,x)), \tag{8}$$

where $\mathcal{K}_{\theta,s}(\cdot)$, $\mathcal{K}_{m,s}(\cdot)$ and $\mathcal{K}_{v,s}(\cdot)$ mean the slope, moment and shear force operators with respect to s, respectively, which are defined as follows:

$$\mathcal{K}_{\theta,s}(\cdot) = \frac{\partial(\cdot)}{\partial n_s}, \tag{9}$$

$$\mathcal{K}_{m,s}(\cdot) = \nu\nabla_s^2(\cdot) + (1-\nu)\frac{\partial^2(\cdot)}{\partial n_s^2}, \tag{10}$$

$$\mathcal{K}_{v,s}(\cdot) = \frac{\partial \nabla_s^2(\cdot)}{\partial n_s} + (1-\nu)\frac{\partial}{\partial t_s}[(\frac{\partial^2(\cdot)}{\partial n_s \partial t_s})], \tag{11}$$

where n and t are the normal and tangential vectors, respectively. By moving the point to the smooth boundary, Eqs.(1)-(4) reduce to

$$4\pi u(x) = -F.P.\int_B U(s,x)v(s)\,dB(s) + F.P.\int_B \Theta(s,x)m(s)\,dB(s) - F.P.\int_B M(s,x)\theta(s)\,dB(s) + F.P.\int_B V(s,x)u(s)\,dB(s),\ x \in B, \tag{12}$$

$$4\pi\theta(x) = -F.P.\int_B U_\theta(s,x)v(s)\,dB(s) + F.P.\int_B \Theta_\theta(s,x)m(s)\,dB(s) - F.P.\int_B M_\theta(s,x)\theta(s)\,dB(s) + F.P.\int_B V_\theta(s,x)u(s)\,dB(s),\ x \in B, \tag{13}$$

$$4\pi m(x) = -F.P.\int_B U_m(s,x)v(s)\,dB(s) + F.P.\int_B \Theta_m(s,x)m(s)\,dB(s) - F.P.\int_B M_m(s,x)\theta(s)\,dB(s) + F.P.\int_B V_m(s,x)u(s)\,dB(s),\ x \in B, \tag{14}$$

$$4\pi v(x) = -F.P.\int_B U_v(s,x)v(s)\,dB(s) + F.P.\int_B \Theta_v(s,x)m(s)\,dB(s) - F.P.\int_B M_v(s,x)\theta(s)\,dB(s) + F.P.\int_B V_v(s,x)u(s)\,dB(s),\ x \in B, \tag{15}$$

where $F.P.$ denotes the finite part for all the improper integrals. The free terms for the sixteen singular integrals will be derived by using the bump-contour technique and the limiting process in the next section.

Taylor expansion of boundary density functions

Before deriving the free terms of the improper integral equations, the density functions (displacement, slope, moment and shear force) are needed to be expanded to series form for order analysis. Therefore, we expand the density functions by using the Taylor series in the BIE formulation as shown in Table 3. The density functions are the Taylor expansions at x and they should be substituted into the dual integral equations when deriving the free terms. The simplified forms of the density functions, $u(x)$, $\theta(x)$, $m(x)$ and $v(x)$, under the condition of $n_x = (0, 1)$ and $t_x = (-1, 0)$ are shown in Table 4 without loss of generality.

Explicit forms for the kernel functions and the order analysis for the asymptotic behavior

From the dual boundary integral equations in Eqs.(1)-(4), the contour integration path B can be separated as $B = B' + B^- + B_\epsilon + B^+$ including the domain Ω surrounding the singularity as shown in Fig.1. For convenience, it was assumed that B_ϵ is an arc of a semi-circle centered at the field point x with radius ϵ. The integration path B_ϵ denotes the contour integration around the singular point, and $B'+B^-+B^+$ is the definition of the integration region of the Cauchy principal value. By adopting the boundary integral formulations and the sixteen kernel functions, the notations generally employed in the Kirchhoff plate theory are briefly summarized. Without loss of generality, we have the following notations as shown in Fig.1: (1) The position of the field point: $x = (x_1, x_2) = (0, 0)$. (2) The position of the source point: $s = (s_1, s_2) = (\epsilon\cos\theta, \epsilon\sin\theta)$. (3) Distance: $r = |s - x|$. (4) Vector component: $y_i = x_i - s_i$, $i = 1, 2$. (5) Normal vector of the field point: $n(x) = (\overline{n}_1, \overline{n}_2) = (0, 1)$. (6) Normal vector of the source point along the arc: $n(s) = (n_1, n_2) = (\cos\theta, \sin\theta)$. (7) Tangential vector of the field point: $t(x) = (\bar{t}_1, \bar{t}_2) = (-1, 0)$. (8) Tangential vector of the source point: $t(s) = (t_1, t_2) = (-\sin\theta, \cos\theta)$. By employing the bump-contour technique and substituting the notations of (1)$\sim$(8) in Fig.1, we can derive the explicit forms of the sixteen kernels of the dual integral formulation. From the asymptotic analysis of the above-mentioned kernels, the order analysis for the asymptotic behavior in the kernels can be found in Table 5.

Potential due to the sixteen kernels for the bump integral

After defining the related symbols, sixteen kernel funtions and the density functions for the bump-contour technique, we substitute them into the boundary integral formulations in Eqs.(1)-(4) and derive the free terms. It is interesting to find that the order descends in a successive order for the sixteen integrals. Collecting all the previous results after using the bump-contour technique, the finite part can be defined as

$$F.P.\int_B U(s,x)v(s)dB(s) = C.P.V.\int_{B'+B^-+B^+} U(s,x)v(s)dB(s), \tag{16}$$

$$F.P.\int_B \Theta(s,x)m(s)dB(s) = C.P.V.\int_{B'+B^-+B^+} \Theta(s,x)m(s)dB(s), \tag{17}$$

$$F.P.\int_B M(s,x)\theta(s)dB(s) = C.P.V.\int_{B'+B^-+B^+} M(s,x)\theta(s)dB(s), \quad (18)$$

$$F.P.\int_B V(s,x)u(s)dB(s) = C.P.V.\int_{B'+B^-+B^+} V(s,x)u(s)dB(s), \quad (19)$$

$$F.P.\int_B U_\theta(s,x)v(s)dB(s) = C.P.V.\int_{B'+B^-+B^+} U_\theta(s,x)v(s)dB(s), \quad (20)$$

$$F.P.\int_B \Theta_\theta(s,x)m(s)dB(s) = C.P.V.\int_{B'+B^-+B^+} \Theta_\theta(s,x)m(s)dB(s), \quad (21)$$

$$F.P.\int_B M_\theta(s,x)\theta(s)dB(s) = C.P.V.\int_{B'+B^-+B^+} M_\theta(s,x)\theta(s)dB(s), \quad (22)$$

$$F.P.\int_B V_\theta(s,x)u(s)dB(s) = C.P.V.\int_{B'+B^-+B^+} V_\theta(s,x)u(s)dB(s) + \frac{4(3-\nu)}{\epsilon}u(x), \quad (23)$$

$$F.P.\int_B U_m(s,x)v(s)dB(s) = C.P.V.\int_{B'+B^-+B^+} U_m(s,x)v(s)dB(s), \quad (24)$$

$$F.P.\int_B \Theta_m(s,x)m(s)dB(s) = C.P.V.\int_{B'+B^-+B^+} \Theta_m(s,x)m(s)dB(s), \quad (25)$$

$$F.P.\int_B M_m(s,x)\theta(s)dB(s) = C.P.V.\int_{B'+B^-+B^+} M_m(s,x)\theta(s)dB(s) - 4(1-\nu)(1+\frac{5\nu}{3})\frac{\theta(x)}{\epsilon}, \quad (26)$$

$$F.P.\int_B V_m(s,x)u(s)dB(s) = C.P.V.\int_{B'+B^-+B^+} V_m(s,x)u(s)dB(s) + [\frac{8(\nu-1)(\nu+7)}{3\epsilon}]m(x) + [\frac{(1-\nu)}{3\epsilon^2}[3\pi(\nu-5)+16\nu]]\theta(x), \quad (27)$$

$$F.P.\int_B U_v(s,x)v(s)dB(s) = C.P.V.\int_{B'+B^-+B^+} U_v(s,x)v(s)dB(s), \quad (28)$$

$$F.P.\int_B \Theta_v(s,x)m(s)dB(s) = C.P.V.\int_{B'+B^-+B^+} \Theta_v(s,x)m(s)dB(s) + [\frac{4(-\nu^2+6\nu+7)}{3}]\frac{m(x)}{\epsilon}, \quad (29)$$

$$F.P.\int_B M_v(s,x)\theta(s)dB(s) = C.P.V.\int_B M_v(s,x)\theta(s)dB(s) + [\frac{8(\nu-1)(\nu+7)}{3\epsilon}]m(x) + [\frac{(1-\nu)}{3\epsilon^2}[3\pi(\nu-5)+16\nu]]\theta(x), \quad (30)$$

$$F.P.\int_B V_v(s,x)u(s)dB(s) = C.P.V.\int_B V_v(s,x)u(s)dB(s) + [\frac{4(\nu-1)(\nu-1)}{\epsilon}]m(x) + [\frac{1-\nu}{3\epsilon^2}[3\pi(\nu-5)+16(\nu-3)]]\theta(x) + [\frac{8(\nu-1)^2}{\epsilon^3}]u(x), \quad (31)$$

where $C.P.V.$ and $F.P.$ denote the Cauchy principal value and finite part, respectively. For the biharmonic problem, we have the kernels with higher singularity than the hypersingularity which is termed unnamed singularity by Guiggiani. Therefore, we denote them as the "finite part". The boundary terms of the kernel integration arise from the boundary integral equations naturally and can compensate the infinity of C.P.V.. Combining the sixteen improper integrals, we have the boundary integral equations with the free coefficient of 4π for a smooth boundary. The free terms may result from different kernels instead of only one Cauchy kernel. Finally, all the results of free terms and boundary terms are summarized in Table 6.

Conclusions

In this paper, the free terms of the DBIEs for the biharmonic problem were derived successfully. We adopted the bump-contour technique surrounding the singularity and expanded the density functions by using the Taylor series. After collecting the sixteen improper integrals for the smooth boundary, it is interesting to find that the sum of the free terms in the four boundary integral equations are 4π. Finally, the order analysis and the free terms of the sixteen kernels of the biharmonic equation are summarized in Table 6. The potentials of the sixteen kernels can be interpreted as finite part and Cauchy pricipal value.

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Table 1 Free terms of dual BIE for the 2-D and 3-D Laplace problems

2-D [1,4,6]	
$U(s,x)$	$T(s,x)$
0	$\pi\, u(x)$
$L(s,x)$	$M(s,x)$
$-\dfrac{\pi}{2}t(x)$	$\dfrac{\pi}{2}t(x)+\dfrac{2}{\varepsilon}u(x)$

3-D [1,4,6]	
$U(s,x)$	$T(s,x)$
0	$2\pi\, u(x)$
$L(s,x)$	$M(s,x)$
$-\dfrac{2\pi}{3}t(x)$	$\dfrac{4\pi}{3}t(x)-\dfrac{2\pi}{\varepsilon}u(x)$

Table 2 Free terms of dual BIE for the 2-D and 3-D elasticity problems

2-D [4,6]		
	$U_{ki}(s,x)$	$T_{ki}(s,x)$
$i=1,\ k=1$ $i=2,\ k=1$ $i=1,\ k=2$ $i=2,\ k=2$	No jump	$\frac{-u_1(x)}{2}$ 0 0 $\frac{-u_2(x)}{2}$
	$L_{ki}(s,x)$	$M_{ki}(s,x)$
$i=1,\ k=1$ $i=2,\ k=1$ $i=1,\ k=2$ $i=2,\ k=2$	$\frac{G(3-4\nu)}{16(1-\nu)}\left\{\frac{\partial u_1}{\partial s_2}+\frac{\partial u_2}{\partial s_1}\right\}\Big\vert_{s=x}$ $\frac{G(-1+4\nu)}{8(1-\nu)(1-2\nu)}\left\{(1-\nu)\frac{\partial u_1}{\partial s_1}+\nu\frac{\partial u_2}{\partial s_2}\right\}\Big\vert_{s=x}$ $\frac{G(3-4\nu)}{16(1-\nu)}\left\{\frac{\partial u_1}{\partial s_2}+\frac{\partial u_2}{\partial s_1}\right\}\Big\vert_{s=x}$ $\frac{G(5-4\nu)}{8(1-\nu)(1-2\nu)}\left\{(1-\nu)\frac{\partial u_2}{\partial s_2}+\nu\frac{\partial u_1}{\partial s_1}\right\}\Big\vert_{s=x}$	$\frac{-G}{8(1-\nu)}\frac{\partial u_1}{\partial s_2}\Big\vert_{s=x}$ $\frac{-G}{8(1-\nu)}\frac{\partial u_1}{\partial s_1}\Big\vert_{s=x}$ $\frac{-G}{8(1-\nu)}\frac{\partial u_2}{\partial s_1}\Big\vert_{s=x}$ $\frac{-3G}{8(1-\nu)}\frac{\partial u_2}{\partial s_2}\Big\vert_{s=x}$

3-D [4,6]		
	$U_{ki}(s,x)$	$T_{ki}(s,x)$
$i=1,\ k=1$ $i=2,\ k=1$ $i=3,\ k=1$ $i=1,\ k=2$ $i=2,\ k=2$ $i=3,\ k=2$ $i=1,\ k=3$ $i=2,\ k=3$ $i=3,\ k=3$	No jump	$\frac{-u_1(x)}{2}$ 0 0 0 $\frac{-u_2(x)}{2}$ 0 0 0 $\frac{-u_3(x)}{2}$
	$L_{ki}(s,x)$	$M_{ki}(s,x)$
$i=1,\ k=1$ $i=2,\ k=1$ $i=3,\ k=1$ $i=1,\ k=2$ $i=2,\ k=2$ $i=3,\ k=2$ $i=1,\ k=3$ $i=2,\ k=3$ $i=3,\ k=3$	$\frac{G(4-5\nu)}{30(1-\nu)}\left\{\frac{\partial u_1}{\partial s_3}+\frac{\partial u_3}{\partial s_1}\right\}\Big\vert_{s=x}$ 0 $\frac{G(-1+5\nu)}{15(1-\nu)(1-2\nu)}\left\{(1-\nu)\frac{\partial u_1}{\partial s_1}+\nu\left(\frac{\partial u_2}{\partial s_2}+\frac{\partial u_3}{\partial s_3}\right)\right\}\Big\vert_{s=x}$ 0 $\frac{G(4-5\nu)}{30(1-\nu)}\left\{\frac{\partial u_2}{\partial s_3}+\frac{\partial u_3}{\partial s_2}\right\}\Big\vert_{s=x}$ $\frac{G(-1+5\nu)}{15(1-\nu)(1-2\nu)}\left\{(1-\nu)\frac{\partial u_2}{\partial s_2}+\nu\left(\frac{\partial u_1}{\partial s_1}+\frac{\partial u_3}{\partial s_3}\right)\right\}\Big\vert_{s=x}$ $\frac{G(4-5\nu)}{30(1-\nu)}\left\{\frac{\partial u_1}{\partial s_3}+\frac{\partial u_3}{\partial s_1}\right\}\Big\vert_{s=x}$ $\frac{G(4-5\nu)}{30(1-\nu)}\left\{\frac{\partial u_2}{\partial s_3}+\frac{\partial u_3}{\partial s_2}\right\}\Big\vert_{s=x}$ $\frac{G(7-5\nu)}{15(1-\nu)(1-2\nu)}\left\{(1-\nu)\frac{\partial u_3}{\partial s_3}+\nu\left(\frac{\partial u_1}{\partial s_1}+\frac{\partial u_2}{\partial s_2}\right)\right\}\Big\vert_{s=x}$	$\frac{G(-7+5\nu)}{30(1-\nu)}\frac{\partial u_1}{\partial s_3}\Big\vert_{s=x}$ 0 $-\frac{G(1+5\nu)}{15(1-\nu)}\frac{\partial u_1}{\partial s_1}\Big\vert_{s=x}$ 0 $\frac{G(-7+5\nu)}{30(1-\nu)}\frac{\partial u_2}{\partial s_3}\Big\vert_{s=x}$ $-\frac{G(1+5\nu)}{15(1-\nu)}\frac{\partial u_2}{\partial s_2}\Big\vert_{s=x}$ $\frac{G(-7+5\nu)}{30(1-\nu)}\frac{\partial u_3}{\partial s_1}\Big\vert_{s=x}$ $\frac{G(-7+5\nu)}{30(1-\nu)}\frac{\partial u_3}{\partial s_2}\Big\vert_{s=x}$ $\frac{-8G}{15(1-\nu)}\frac{\partial u_3}{\partial s_3}\Big\vert_{s=x}$

Table 3 Taylor expansions for the density functions in the BIE formulation of plate problem

$u(s)=u(x)+[\frac{\partial u}{\partial s_1}\cos\theta+\frac{\partial u}{\partial s_2}\sin\theta]\varepsilon+\frac{1}{2!}[\frac{\partial^2 u}{\partial s_1^2}\cos^2\theta+\frac{\partial^2 u}{\partial s_2^2}\sin^2\theta+\frac{\partial^2 u}{\partial s_1\partial s_2}\cos\theta\sin\theta]\varepsilon^2+$ $\frac{1}{3!}[\frac{\partial^3 u}{\partial s_1^3}\cos^3\theta+\frac{\partial^3 u}{\partial s_1\partial s_2^2}\cos\theta\sin^2\theta+\frac{\partial^3 u}{\partial s_1^2\partial s_2}\cos^2\theta\sin\theta+\frac{\partial^3 u}{\partial s_2^3}\sin^3\theta]\varepsilon^3+O(\varepsilon^4)$	$\theta(s)=[\frac{\partial u}{\partial s_1}\cos\theta+\frac{\partial u}{\partial s_2}\sin\theta]+[\frac{\partial^2 u}{\partial s_1^2}\cos^2\theta+\frac{\partial^2 u}{\partial s_2^2}\sin^2\theta+\frac{\partial^2 u}{\partial s_1\partial s_2}\cos\theta\sin\theta]\varepsilon$ $+\frac{1}{2}[\frac{\partial^3 u}{\partial s_1^3}\cos^3\theta+\frac{\partial^3 u}{\partial s_1\partial s_2^2}\cos\theta\sin^2\theta+\frac{\partial^3 u}{\partial s_1^2\partial s_2}\cos^2\theta\sin\theta+\frac{\partial^3 u}{\partial s_2^3}\sin^3\theta]\varepsilon^2+O(\varepsilon^3)$
$m(s)=\frac{\partial^2 u}{\partial s_1^2}[\cos^2\theta+\nu\sin^2\theta]+\frac{\partial^2 u}{\partial s_2^2}[\sin^2\theta+\nu\cos^2\theta]+\{\frac{\partial^3 u}{\partial s_1^3}[\cos^3\theta+\nu\cos\theta\sin^2\theta]+$ $\frac{\partial^3 u}{\partial s_1\partial s_2^2}[\frac{\nu}{3}\cos^3\theta+(1-\frac{2}{3}\nu)\cos\theta\sin^2\theta]+\frac{\partial^3 u}{\partial s_1^2\partial s_2}[\frac{\nu}{3}\sin^3\theta+(1-\frac{2}{3}\nu)\cos^2\theta\sin\theta]+$ $\frac{\partial^3 u}{\partial s_2^3}\varepsilon^3[\cos^3\theta+\nu\cos^2\theta\sin\theta]\}\varepsilon+O(\varepsilon^2)$	$v(s)=(1-\nu)\{\frac{-1}{\varepsilon}\frac{\partial^2 u}{\partial s_1^2}[\cos^2\theta-\sin^2\theta]+\frac{1}{\varepsilon}\frac{\partial^2 u}{\partial s_2^2}[\cos^2\theta-\sin^2\theta]+\frac{-2}{\varepsilon}\frac{\partial^2 u}{\partial s_1\partial s_2}\cos\theta\sin\theta\}+$ $\frac{\partial^3 u}{\partial s_1^3}[\cos\theta+(1-\nu)(\cos^3\theta-2\cos\theta\sin^2\theta)]+\frac{1}{3}\frac{\partial^3 u}{\partial s_1\partial s_2^2}[\cos\theta+(1-\nu)(2\cos^3\theta-7\cos\theta\sin^2\theta)]$ $+\frac{1}{3}\frac{\partial^3 u}{\partial s_1^2\partial s_2}[\sin\theta+(1-\nu)(2\sin^3\theta-7\sin\theta\cos^2\theta)]+\frac{\partial^3 u}{\partial s_2^3}[\sin\theta+(1-\nu)(2\sin\theta\cos^2\theta-\sin^3\theta)]+O(\varepsilon)$

Table 4 Simplified forms of the density functions

Displacement	$u(x)$
Slope	$\theta(x)=\frac{\partial u(x)}{\partial n_x}=\frac{\partial u(x)}{\partial x_2}$
Moment	$m(x)=\nu\nabla^2 u(x)+(1-\nu)\frac{\partial^2 u(x)}{\partial n_x^2}=\nu\frac{\partial^2 u(x)}{\partial x_1^2}+\frac{\partial^2 u(x)}{\partial x_2^2}$
Shear force	$v(x)=\frac{\partial \nabla^2 u(x)}{\partial n_x}+(1-\nu)\frac{\partial}{\partial t_x}\left[\frac{\partial^2 u(x)}{\partial n_x \partial t_x}\right]=(2-\nu)\frac{\partial^3 u(x)}{\partial x_1^2 \partial x_2}+\frac{\partial^3 u(x)}{\partial x_2^3}$

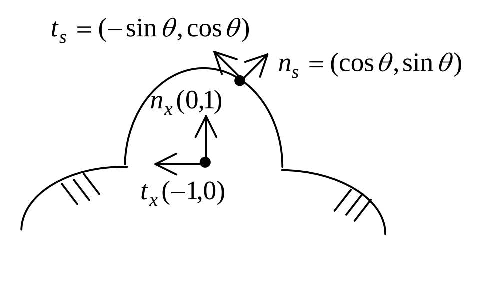

Table 5 Order analysis for the sixteen kernels of biharmonic problem

$U(s,x)$	$\Theta(s,x)$	$M(s,x)$	$V(s,x)$
$O(\varepsilon^2\ln\varepsilon)$	$O(\varepsilon\ln\varepsilon)$	$O(\ln\varepsilon)$	$O\left(\frac{1}{\varepsilon}\right)$
$U_\theta(s,x)$	$\Theta_\theta(s,x)$	$M_\theta(s,x)$	$V_\theta(s,x)$
$O(\varepsilon\ln\varepsilon)$	$O(\ln\varepsilon)$	$O\left(\frac{1}{\varepsilon}\right)$	$O\left(\frac{1}{\varepsilon^2}\right)$
$U_m(s,x)$	$\Theta_m(s,x)$	$M_m(s,x)$	$V_m(s,x)$
$O(\ln\varepsilon)$	$O\left(\frac{1}{\varepsilon}\right)$	$O\left(\frac{1}{\varepsilon^2}\right)$	$O\left(\frac{1}{\varepsilon^3}\right)$
$U_v(s,x)$	$\Theta_v(s,x)$	$M_v(s,x)$	$V_v(s,x)$
$O\left(\frac{1}{\varepsilon}\right)$	$O\left(\frac{1}{\varepsilon^2}\right)$	$O\left(\frac{1}{\varepsilon^3}\right)$	$O\left(\frac{1}{\varepsilon^4}\right)$

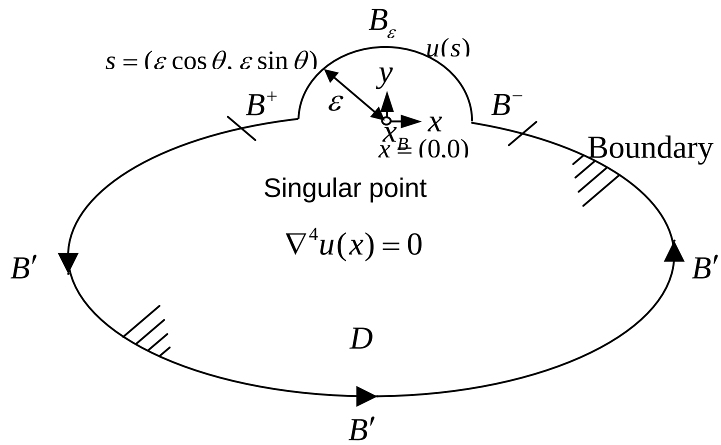

Fig. 1 The considered boundary integration path for the two-dimensional biharmonic problem.

Table 6 Free terms due to the bump integral for the biharmonic equation

$U(s,x)$	$\Theta(s,x)$	$M(s,x)$	$V(s,x)$
0	0	0	$[4\pi]u(x)$
$U_\theta(s,x)$	$\Theta_\theta(s,x)$	$M_\theta(s,x)$	$V_\theta(s,x)$
0	0	$[-\pi(1+\nu)]\theta(x)$	$[(3-\nu)\pi]\theta(x)+[4(3-\nu)]\frac{u(x)}{\varepsilon}$
$U_m(s,x)$	$\Theta_m(s,x)$	$M_m(s,x)$	$V_m(s,x)$
$\left[\frac{\pi}{2}(\nu-1)\right]m(x)$	$[\pi(1+\nu)]m(x)$	$[-\pi]m(x)+\left[-4(1-\nu)(1+\frac{5}{3}\nu)\right]\frac{\theta(x)}{\varepsilon}$	$\left[\frac{\pi}{2}(3-\nu)\right]m(x)+\left[\frac{8(1-\nu)(3-\nu)}{3}\right]\frac{\theta(x)}{\varepsilon}+\frac{0}{\varepsilon^2}u(x)$
$U_v(s,x)$	$\Theta_v(s,x)$	$M_v(s,x)$	$V_v(s,x)$
$\left[\frac{\pi}{2}(\nu+1)(\nu-3)\right]v(x)$	$[2\pi]v(x)+\left[\frac{4}{3}(-\nu^2+6\nu+7)\right]\frac{m(x)}{\varepsilon}$	$[-\pi(-1+\nu)(-2+\nu)]v(x)+\left[\frac{8(\nu-1)(\nu+7)}{3}\right]\frac{m(x)}{\varepsilon}+\left[\frac{(1-\nu)}{3}[3\pi(\nu-5)+16\nu]\right]\frac{\theta(x)}{\varepsilon^2}$	$\left[\frac{-\pi}{2}(-1+\nu)(-3+\nu)\right]v(x)+[4(\nu-1)(\nu-1)]\frac{m(x)}{\varepsilon}+\left[\frac{(1-\nu)}{3}[3\pi(\nu-5)+16(\nu-3)]\right]\frac{\theta(x)}{\varepsilon^2}+[8(-1+\nu)^2]\frac{u(x)}{\varepsilon^3}$