

## **Free vibration analysis of multiply-connected plates using the method of fundamental solutions**

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### **Abstract**

In this paper, the method of fundamental solutions (MFS) for solving the eigenfrequencies of multiply-connected plates is proposed. The coefficients of influence matrices are easily determined when the fundamental solution is known. True and spurious eigensolutions appear at the same time. It is found that the spurious eigensolution using the MFS depends on the location of the inner boundary where the fictitious sources are distributed. To verify this finding, mathematical analysis for the appearance of spurious eigenequations using degenerate kernels and circulants is done by demonstrating an annular plate with a discrete model. In order to obtain the true eigensolution, the Burton & Miller method is utilized to filter out the spurious eigensolutions. One example is demonstrated analytically and numerically to see the validity of the present method.

**Keywords:** Method of fundamental solutions; Biharmonic equation; Circulant; Degenerate kernel; SVD updating document; Burton & Miller method

### **Introduction**

The method of fundamental solutions (MFS) is a numerical approach as well as finite difference method (FDM), finite element method (FEM) and boundary element method (BEM). This method was attributed to Kupradze in 1964. The MFS was applied to many problems (Fairweather and Karageorghis, 1998; Chen *et al.*, 1998; Karageorghis, 2001), and can be regarded as one kind of meshless method. It has several advantages over boundary element method, *e.g.*, no boundary integrals, no singularity and mesh-free model. Although MFS has been applied to solve many engineering problems, most of them are for cases of simply-connected domains. Chen *et al.* have tried to solve the eigenproblem of multiply-connected membrane and found that spurious eigenvalues also appear (Chen *et al.*, 2004) as well as BEM (Chen *et al.*, 2003). We may wonder whether spurious solutions also occur for the plate case rather than membrane.

In this paper, the MFS for solving the eigenfrequencies of annular plate is proposed. The occurring mechanism of the spurious eigensolution of an annular plate is studied analytically. The degenerate kernels and circulants are employed to determine the spurious eigensolution. In order to filter out the spurious eigenvalues, singular value decomposition updating technique and Burton & Miller method are utilized. An annular case is demonstrated analytically to see the validity of the present method.

### Analytical derivation of free vibration for annular plate using the method of fundamental solutions

The governing equation for an annular plate vibration in Fig.1 is the biharmonic equation as follows:

$$\nabla^4 u(x) = \lambda^4 u(x), \quad x \in \Omega, \quad (1)$$

where  $\nabla^4$  is the biharmonic operator,  $u$  is the lateral displacement,  $\lambda^4 = \omega^2 \rho_0 h / D$ ,  $\lambda$  is the frequency parameter,  $\omega$  is the angular frequency,  $\rho_0$  is the surface density,  $D$  is the flexural rigidity expressed as  $D = Eh^3 / 12(1 - \nu^2)$  in terms of Young's modulus  $E$ , the Poisson ratio  $\nu$  and the plate thickness  $h$ ,  $\Omega$  is the domain of the thin plate. The fundamental solution is chosen as

$$U(s, x) = \frac{1}{8\lambda^2} [Y_0(\lambda r) - iJ_0(\lambda r) + \frac{2}{\pi} (K_0(\lambda r) - iI_0(\lambda r))], \quad (2)$$

where  $r \equiv |s - x|$ ,  $i^2 = -1$ ,  $J_0(\lambda r)$  and  $Y_0(\lambda r)$  are the first kind and second kind zeroth-order Bessel functions, respectively,  $I_0(\lambda r)$  and  $K_0(\lambda r)$  are the first and second kind zeroth-order modified Bessel functions, respectively. Based on the MFS, we can represent the displacement field of plate vibration by

$$u(x_i) = \sum_{j=1}^{2N} P(s_j, x_i) \phi_j + \sum_{j=1}^{2N} Q(s_j, x_i) \varphi_j, \quad (3)$$

where  $2N$  is the number of fictitious source nodes.  $\phi_j$  and  $\varphi_j$  are the known densities with respect to  $P$  and  $Q$ . The two kernels ( $P$  and  $Q$ ) are obtained from either the two of the kernel  $U(s, x)$  and the other three kernels,  $\Theta(s, x)$ ,  $M(s, x)$  and  $V(s, x)$  (Chen *et al.*, 2004). The slope ( $\theta$ ), normal moment ( $m$ ) and effective shear force ( $v$ ), are also obtained as reference (Chen *et al.*, 2004). In order to derive the exact eigensolution, degenerate kernel and circulant are considered for an annular plate. The field and source points are distributed as shown in Fig.2. Here, we consider the clamped case ( $u=0$  and  $\theta=0$ ) by using  $U$  and  $\Theta$  kernels. We distributed  $2N$  field points on the real boundary, and the same  $2N$  sources are distributed on the fictitious boundary. By matching the boundary condition, we obtain

$$[SM^{cc}] \begin{Bmatrix} \phi 1 \\ \phi 2 \\ \varphi 1 \\ \varphi 2 \end{Bmatrix} = \begin{bmatrix} U11 & U12 & \Theta11 & \Theta12 \\ U21 & U22 & \Theta21 & \Theta22 \\ U11_\theta & U12_\theta & \Theta11_\theta & \Theta12_\theta \\ U21_\theta & U22_\theta & \Theta21_\theta & \Theta22_\theta \end{bmatrix}_{8N \times 8N} \begin{Bmatrix} \phi 1 \\ \phi 2 \\ \varphi 1 \\ \varphi 2 \end{Bmatrix} = \{0\}, \quad (4)$$

where  $\{\phi 1\}$ ,  $\{\phi 2\}$ ,  $\{\varphi 1\}$  and  $\{\varphi 2\}$  are the generalized coefficients for  $B_I$  and  $B_2$  with a dimension of  $2N \times 1$ , the matrices  $[Uij]$ ,  $[\Thetaij]$ ,  $[Uij_\theta]$  and  $[\Thetaij_\theta]$  mean the influence matrices of  $U$ ,  $\Theta$ ,  $U_\theta$  and  $\Theta_\theta$  kernels which are obtained by collocating the field and source points on  $B_i$  and  $B'_j$  with a dimension of  $2N \times 2N$ , respectively. For the existence of nontrivial solution, the determinant of the matrix versus the eigenvalue must be zero, *i.e.*,

$$\det[SM^{cc}] = \prod_{m=-(N-1)}^N \det([T_m^{cc}][S_m^{U\Theta}]) = 0, \quad (5)$$

where

$$[T_m^{cc}] = \begin{bmatrix} J_m(\lambda a) & Y_m(\lambda a) & I_m(\lambda a) & K_m(\lambda a) \\ J_m(\lambda b) & Y_m(\lambda b) & I_m(\lambda b) & K_m(\lambda b) \\ J'_m(\lambda a) & Y'_m(\lambda a) & I'_m(\lambda a) & K'_m(\lambda a) \\ J'_m(\lambda b) & Y'_m(\lambda b) & I'_m(\lambda b) & K'_m(\lambda b) \end{bmatrix} \quad (6)$$

and

$$[S_m^{U\Theta}] = \begin{bmatrix} -iJ_m(\lambda a') & Y_m(\lambda b') - iJ_m(\lambda b') & -iJ'_m(\lambda a') & Y'_m(\lambda b') - iJ'_m(\lambda b') \\ J_m(\lambda a') & 0 & J'_m(\lambda a') & 0 \\ -(-1)^m i \frac{2}{\pi} I_m(\lambda a') & \frac{2}{\pi} [k_m(\lambda b') - (-1)^m i I_m(\lambda b')] & -(-1)^m i \frac{2}{\pi} I'_m(\lambda a') & \frac{2}{\pi} [k'_m(\lambda b') - (-1)^m i I'_m(\lambda b')] \\ \frac{2}{\pi} I_m(\lambda a') & 0 & \frac{2}{\pi} I'_m(\lambda a') & 0 \end{bmatrix}. \quad (7)$$

It is noted that the matrix  $[T_m^{cc}]$  denotes the matrix of true eigenequation for the C-C case and the matrix  $[S_m^{U\Theta}]$  denotes the matrix of spurious eigenequation in the U- $\Theta$  formulation after comparing with the analytical solution for the annular plate (Leissa, 1969). The matrix in Eq.(7) can be further decomposed into

$$\det[S_m^{U\Theta}] = \begin{vmatrix} J_m(\lambda a') & J'_m(\lambda a') \\ I_m(\lambda a') & I'_m(\lambda a') \end{vmatrix} \begin{vmatrix} Y_m(\lambda b') - iJ_m(\lambda b') & Y'_m(\lambda b') - iJ'_m(\lambda b') \\ K_m(\lambda b') - i(-1)^m I_m(\lambda b') & K'_m(\lambda b') - i(-1)^m I'_m(\lambda b') \end{vmatrix} = 0. \quad (8)$$

Since the latter part of Eq.(8) is never zero, the spurious eigenequation depends on  $a'$ . It is noted that the spurious eigensolution happens to be true eigensolution of the clamped circular plate with a radius  $a'$ . Therefore, the positions of spurious eigenvalues for the annular problem depend on the location of inner fictitious boundary  $a'$  where the sources are distributed.

### A numerical example

An annular plate with the inner radius of 0.5 meter and the outer radius of 1 meter are considered, respectively. The source points are distributed at  $a' = 0.4$  meter and  $b' = 1.2$  meter. Forty-six nodes are uniformly distributed on the inner and outer fictitious boundaries. Fig.3 shows the determinant versus frequency parameter by using the U- $\Theta$  formulation. The drop location indicates the possible eigenvalues. Fig.4 shows the determinant versus frequency parameter by using the Burton & Miller method for the annular plate. It is found that the appearance of spurious eigenvalues is suppressed. After comparing the result with the analytical solution, good agreement is made.

### Conclusions

The mathematical analysis has shown that spurious eigenvalues occur by using degenerate kernels and circulants when the method of fundamental solutions is used to solve the eigenvalue of annular plates. The positions of spurious eigenvalues for the annular problem depend on the location of inner fictitious boundary where the sources are distributed. The spurious eigenvalues in the annular problem are found to be the true eigenvalues of the associated simply-connected problem bounded by the inner sources. We have employed the Burton & Miller method to filter out the spurious eigenvalues successfully.

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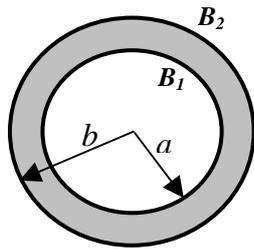
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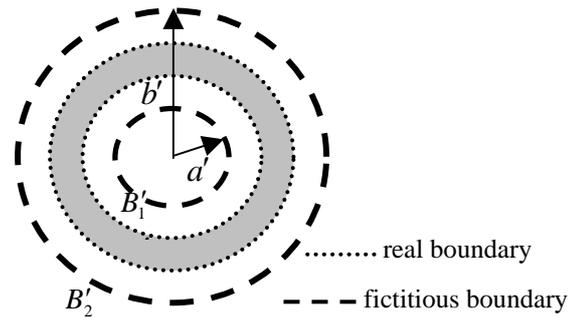
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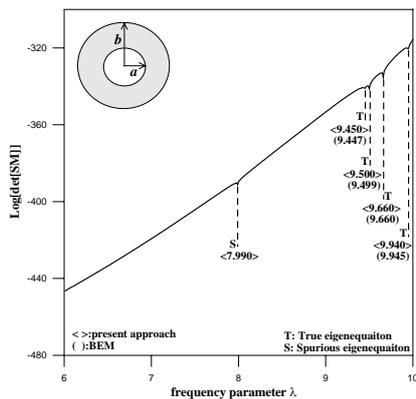
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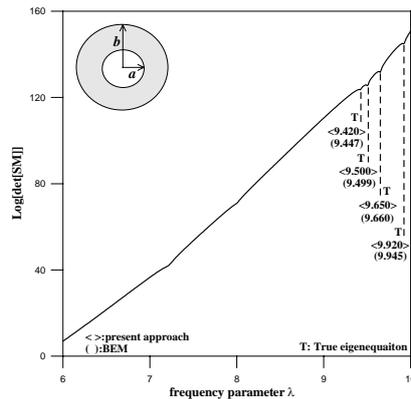
**Fig.1** An annular problem



**Fig.2** Figure sketch for source distribution



**Fig.3** The determinant versus frequency parameter by using the  $U-\Theta$  formulation



**Fig.4** The determinant versus frequency parameter by using the  $U-\Theta$  formulation in conjunction with Burton & Miller method