Free vibration analysis of multiply-connected plates using the method of fundamental solutions

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Outlines

1. Introduction
2. Methods of solution
3. Mathematical analysis
4. Treatment methods
5. Numerical example
6. Conclusions

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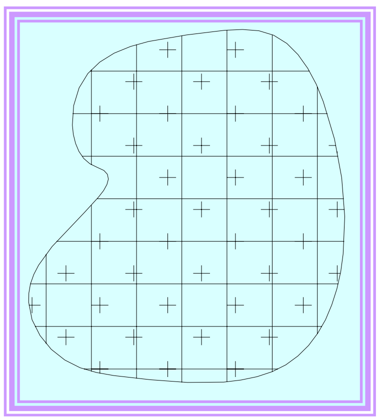
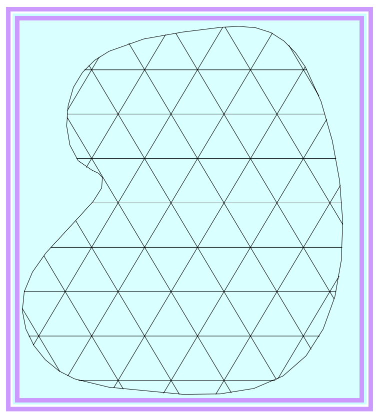
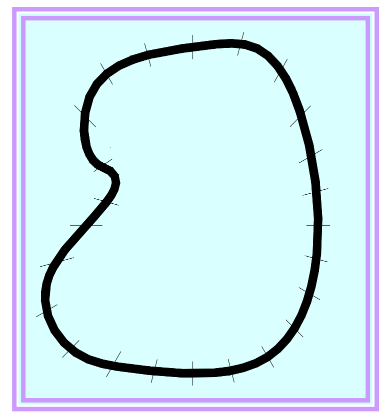
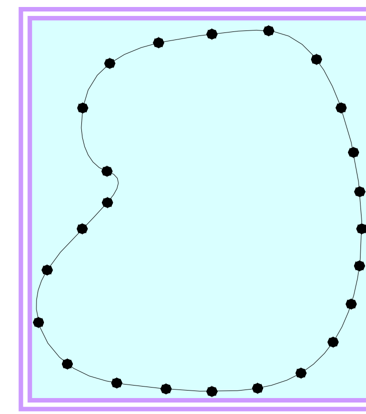
Numerical method

Numerical method

Finite Difference Method	Finite Element Method	Boundary Element Method	Meshless Method

Mesh X	Mesh X	Mesh X Singular integral X	

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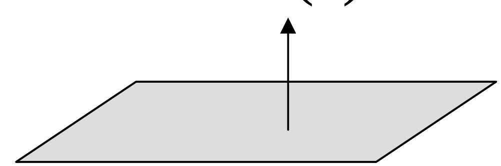
Vibration of plates

Governing Equation:



$$\nabla^4 u(x) = \lambda^4 u(x),\ x \in \Omega$$

$$\lambda^4 = \frac{\omega^2 \rho h}{D}$$

$$D = \frac{E h^3}{12(1-\nu)}$$

ω is the angle frequency

ρ is the surface density

h is the plates thickness

D is the flexural rigidity

E is the Young's modulus

ν is the Poisson ratio

rmonic operator

ral displacement

uency parameter,

nain of the thin plates

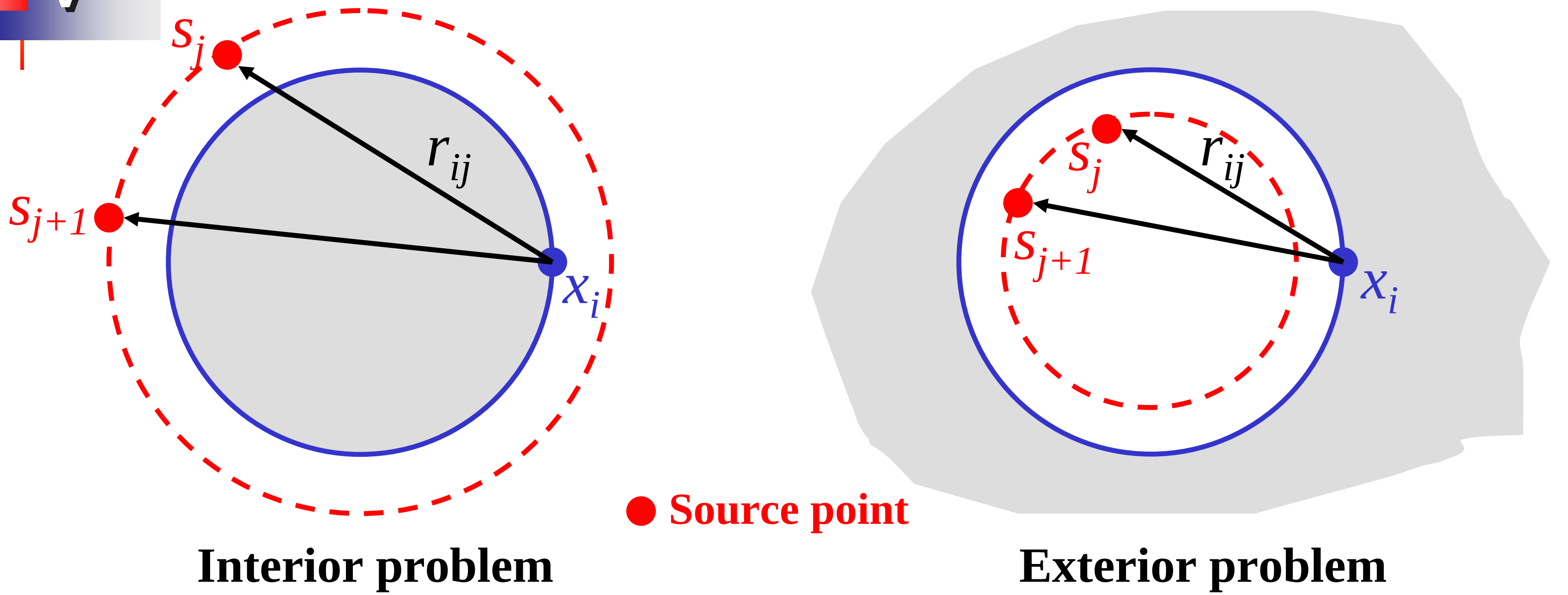
Fundamental solution

The fundamental solution satisfies

$$\nabla^4 U(s,x) - \lambda^4 U(s,x) = -\delta(x-s)$$

$$U(s,x) = \frac{1}{8\lambda^2}[Y_0(\lambda r) - iJ_0(\lambda r) + \frac{2}{\pi}(K_0(\lambda r) - iI_0(\lambda r))]$$

Field representation using MFS



Field representation

$$u(x_i) = \sum_j c_j \psi(s_j, x_i)$$

$$\psi(s_j, x_i) = \psi(r_{ij}), \quad r_{ij} \equiv |s_j - x_i|$$

The other three kernels

Displacement to slope operator

$$\Theta(s,x) = \aleph_{\theta}(U(s,x))$$

Displacement to normal moment operator

$$M(s,x) = \aleph_{m}(U(s,x))$$

Displacement to effective shear operator

$$V(s,x) = \aleph_{v}(U(s,x))$$

Explicit forms for operators

$$\aleph_{\theta}(\cdot) = \frac{\partial(\cdot)}{\partial n}$$

$$\aleph_{m}(\cdot) = \nu\nabla^{2}(\cdot) + (1-\nu)\frac{\partial^{2}(\cdot)}{\partial n^{2}}$$

$$\aleph_{v}(\cdot) = \frac{\partial\nabla^{2}(\cdot)}{\partial n} + (1-\nu)\frac{\partial}{\partial t}(\frac{\partial^{2}(\cdot)}{\partial n\partial t})$$

Displacement, Slope, Moment and Shear

Displacement

$$u(x) = \sum_{j=1}^{2N} P(s_j, x) p(s_j) + \sum_{j=1}^{2N} Q(s_j, x) q(s_j)$$

Slope

$$\theta(x) = \sum_{j=1}^{2N} P_\theta(s_j, x) p(s_j) + \sum_{j=1}^{2N} Q_\theta(s_j, x) q(s_j)$$

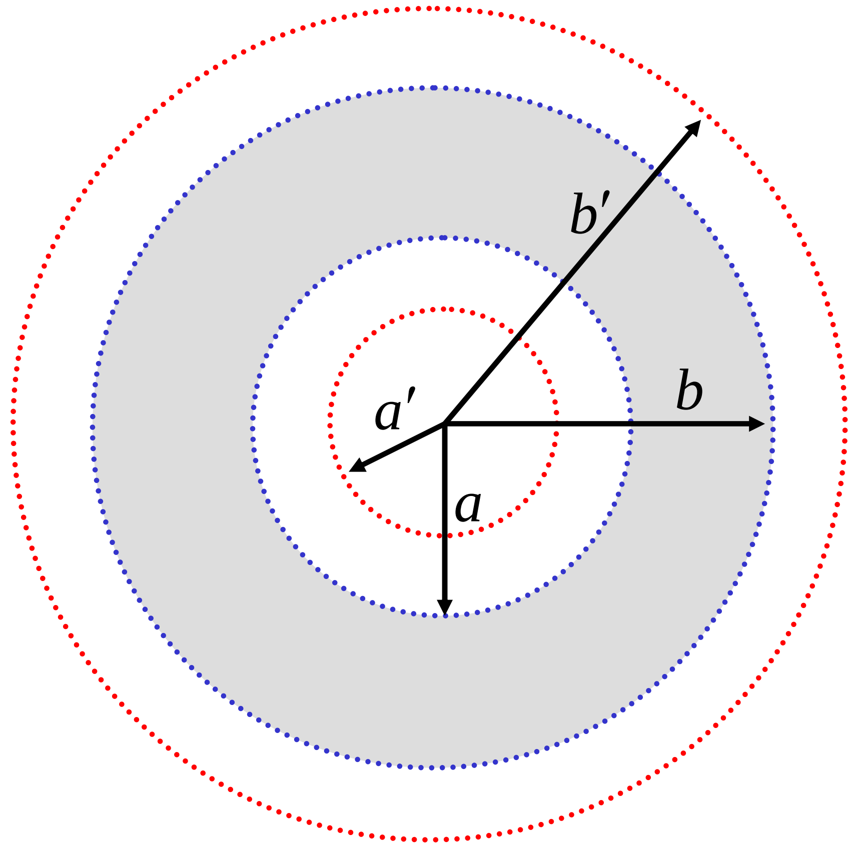
Normal moment

$$m(x) = \sum_{j=1}^{2N} P_m(s_j, x) p(s_j) + \sum_{j=1}^{2N} Q_m(s_j, x) q(s_j)$$

Effective shear

$$v(x) = \sum_{j=1}^{2N} P_v(s_j, x) p(s_j) + \sum_{j=1}^{2N} Q_v(s_j, x) q(s_j)$$

Problem statement



Collocation point distribution (real boundary)

Source point distribution (fictitious boundary)

Matching boundary condition

The clamped-clamped case is considered

$$\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} U11 & U12 \\ U21 & U22 \end{bmatrix} \begin{Bmatrix} \phi 1 \\ \phi 2 \end{Bmatrix} + \begin{bmatrix} \Theta 11 & \Theta 12 \\ \Theta 21 & \Theta 22 \end{bmatrix} \begin{Bmatrix} \varphi 1 \\ \varphi 2 \end{Bmatrix}$$

$$\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} U11_\theta & U12_\theta \\ U21_\theta & U22_\theta \end{bmatrix} \begin{Bmatrix} \phi 1 \\ \phi 2 \end{Bmatrix} + \begin{bmatrix} \Theta 11_\theta & \Theta 12_\theta \\ \Theta 21_\theta & \Theta 22_\theta \end{bmatrix} \begin{Bmatrix} \varphi 1 \\ \varphi 2 \end{Bmatrix}$$

Matrix assembling

$$\begin{bmatrix} U11 & U12 & \Theta 11 & \Theta 12 \\ U21 & U22 & \Theta 21 & \Theta 22 \\ U11_{\theta} & U12_{\theta} & \Theta 11_{\theta} & \Theta 12_{\theta} \\ U21_{\theta} & U22_{\theta} & \Theta 21_{\theta} & \Theta 22_{\theta} \end{bmatrix} \begin{Bmatrix} \phi 1 \\ \phi 2 \\ \varphi 1 \\ \varphi 2 \end{Bmatrix} = \{0\}$$

$\det[SM^{cc}]=0$

Nontrivial

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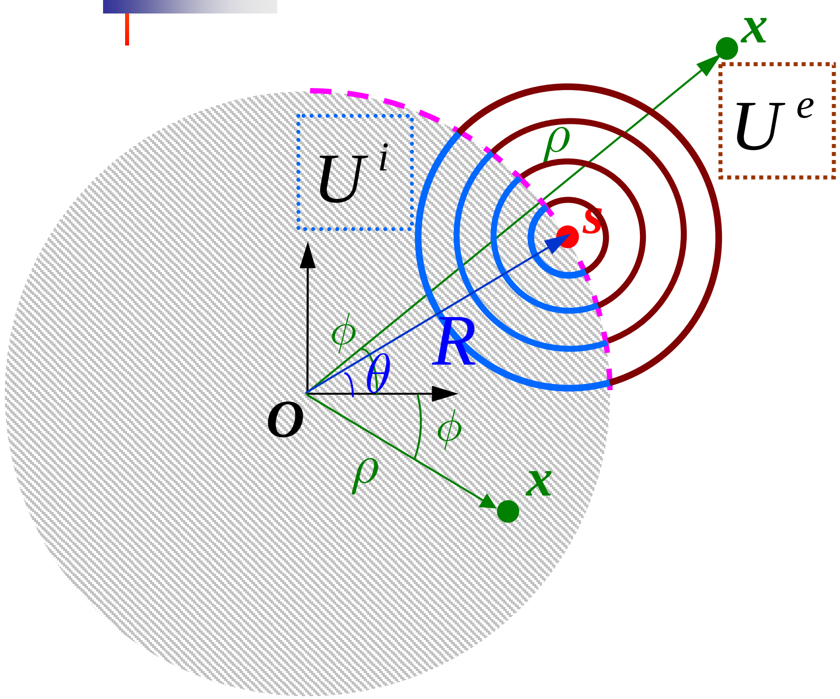
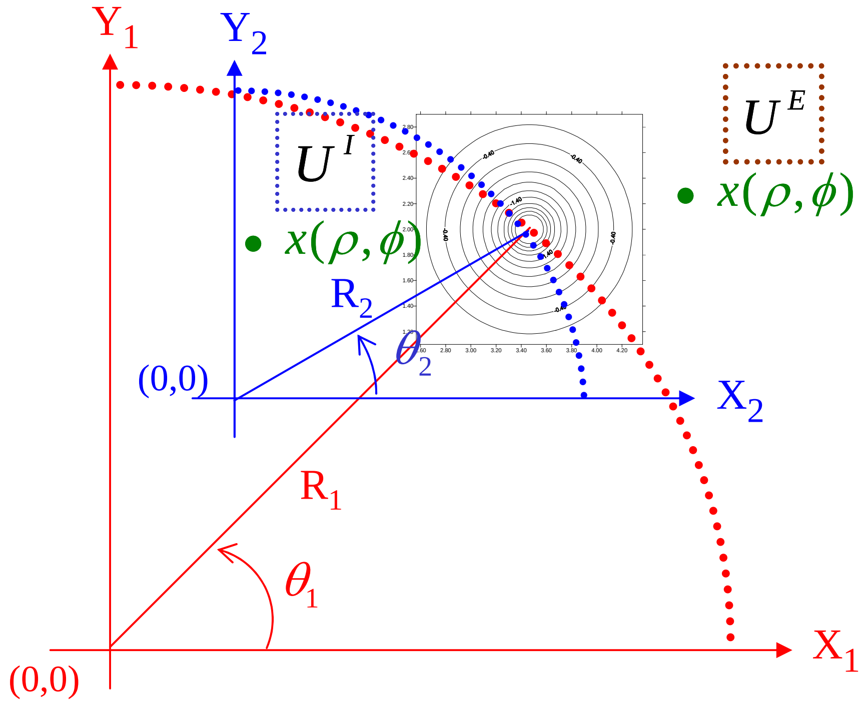
Degenerate kernels for circular case





$$U^I(R,\theta;\rho,\phi)=\sum_{m=-\infty}^{\infty}\frac{1}{8\lambda^2}\{J_m(\lambda\rho)[Y_m(\lambda R)-iJ_m(\lambda R)]+\frac{2}{\pi}(-1)^m I_m(\lambda\rho)[(-1)^m K_m(\lambda R)-iI_m(\lambda R)]\}\cos(m(\theta-\phi)),\quad R>\rho,$$

$$U^E(R,\theta;\rho,\phi)=\sum_{m=-\infty}^{\infty}\frac{1}{8\lambda^2}\{J_m(\lambda R)[Y_m(\lambda\rho)-iJ_m(\lambda\rho)]+\frac{2}{\pi}(-1)^m I_m(\lambda R)[(-1)^m K_m(\lambda\rho)-iI_m(\lambda\rho)]\}\cos(m(\theta-\phi)),\quad R<\rho,$$

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Circulants

Discritization into *2N* nodes on the circular boundary

$$[U11]=\begin{bmatrix} a_0 & a_1 & a_2 & \cdots & a_{2N-2} & a_{2N-1} \\ a_{2N-1} & a_0 & a_1 & \cdots & a_{2N-3} & a_{2N-2} \\ a_{2N-2} & a_{2N-1} & a_0 & \cdots & a_{2N-4} & a_{2N-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_2 & a_3 & a_4 & \cdots & a_0 & a_1 \\ a_1 & a_2 & a_3 & \cdots & a_{2N-1} & a_0 \end{bmatrix}$$

$$[U11]=a_0 I+a_1 C_{2N}+a_2 (C_{2N})^2+\cdots+a_{2N-1}(C_{2N})^{2N-1}$$

Circulants

$$C_{2N} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}_{2N \times 2N}$$

$$\alpha_\ell = e^{i\frac{2\pi\ell}{2N}} = \cos(\frac{2\pi\ell}{2N}) + i\sin(\frac{2\pi\ell}{2N})$$: eigenvalue of C_{2N}

$$\lambda_\ell^{[U11]} = a_0 + a_1\alpha_\ell + a_2\alpha_\ell^2 + \cdots + a_{2N-1}\alpha_\ell^{2N-1}$$: eigenvalue of $[U11]$

$$\ell = 0, \pm 1, \pm 2, \cdots, \pm(N-1), N$$

Eigenvalues

$$\lambda_m^{[U11]} = \frac{N}{4\lambda^2}\{J_m(\lambda a')[Y_m(\lambda a) - iJ_m(\lambda a)] + \frac{2}{\pi} I_m(\lambda a')[K_m(\lambda a) - (-1)^m iI_m(\lambda a)]\}$$

$$\lambda_m^{[U12]} = \frac{N}{4\lambda^2}\{J_m(\lambda a)[Y_m(\lambda b') - iJ_m(\lambda b')] + \frac{2}{\pi} I_m(\lambda a)[K_m(\lambda b') - (-1)^m iI_m(\lambda b')]\}$$

$$\lambda_m^{[U21]} = \frac{N}{4\lambda^2}\{J_m(\lambda a')[Y_m(\lambda b) - iJ_m(\lambda b)] + \frac{2}{\pi} I_m(\lambda a')[K_m(\lambda b) - (-1)^m iI_m(\lambda b)]\}$$

$$\lambda_m^{[U22]} = \frac{N}{4\lambda^2}\{J_m(\lambda b)[Y_m(\lambda b') - iJ_m(\lambda b')] + \frac{2}{\pi} I_m(\lambda b)[K_m(\lambda b') - (-1)^m iI_m(\lambda b')]\}$$

Similar transformation

By using the similar transformation

$$[U11] = \Phi \Sigma_{[U11]} \Phi^H$$

$$\Sigma_{[U11]} = diag(\lambda_0^{[U11]} \lambda_1^{[U11]} \lambda_{-1}^{[U11]} \cdots \lambda_{N-1}^{[U11]} \lambda_N^{[U11]})$$

$$\Phi = \frac{1}{\sqrt{2N}} \begin{bmatrix} 1 & (e^{2\pi i/2N})^0 & (e^{-2\pi i/2N})^0 & \cdots & (e^{-2(N-1)\pi i/2N})^0 & (e^{2N\pi i/2N})^0 \\ 1 & (e^{2\pi i/2N})^1 & (e^{-2\pi i/2N})^1 & \cdots & (e^{-2(N-1)\pi i/2N})^1 & (e^{2N\pi i/2N})^1 \\ 1 & (e^{2\pi i/2N})^2 & (e^{-2\pi i/2N})^2 & \cdots & (e^{-2(N-1)\pi i/2N})^2 & (e^{2N\pi i/2N})^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & (e^{2\pi i/2N})^{2N-2} & (e^{-2\pi i/2N})^{2N-2} & \cdots & (e^{-2(N-1)\pi i/2N})^{2N-2} & (e^{2N\pi i/2N})^{2N-2} \\ 1 & (e^{2\pi i/2N})^{2N-1} & (e^{-2\pi i/2N})^{2N-1} & \cdots & (e^{-2(N-1)\pi i/2N})^{2N-1} & (e^{2N\pi i/2N})^{2N-1} \end{bmatrix}$$

Similar transformation

$$[SM^{cc}]=\begin{bmatrix}\Phi & 0 & 0 & 0\\ 0 & \Phi & 0 & 0\\ 0 & 0 & \Phi & 0\\ 0 & 0 & 0 & \Phi\end{bmatrix}\begin{bmatrix}\Sigma_{[U11]} & \Sigma_{[U12]} & \Sigma_{[\Theta 11]} & \Sigma_{[\Theta 12]}\\ \Sigma_{[U21]} & \Sigma_{[U22]} & \Sigma_{[\Theta 21]} & \Sigma_{[\Theta 22]}\\ \Sigma_{[U11_\theta]} & \Sigma_{[U12_\theta]} & \Sigma_{[\Theta 11_\theta]} & \Sigma_{[\Theta 12_\theta]}\\ \Sigma_{[U21_\theta]} & \Sigma_{[U22_\theta]} & \Sigma_{[\Theta 21_\theta]} & \Sigma_{[\Theta 22_\theta]}\end{bmatrix}\begin{bmatrix}\Phi & 0 & 0 & 0\\ 0 & \Phi & 0 & 0\\ 0 & 0 & \Phi & 0\\ 0 & 0 & 0 & \Phi\end{bmatrix}^H$$

$$\det[SM^{cc}]=\det\begin{bmatrix}\Sigma_{[U11]} & \Sigma_{[U12]} & \Sigma_{[\Theta 11]} & \Sigma_{[\Theta 12]}\\ \Sigma_{[U21]} & \Sigma_{[U22]} & \Sigma_{[\Theta 21]} & \Sigma_{[\Theta 22]}\\ \Sigma_{[U11_\theta]} & \Sigma_{[U12_\theta]} & \Sigma_{[\Theta 11_\theta]} & \Sigma_{[\Theta 12_\theta]}\\ \Sigma_{[U21_\theta]} & \Sigma_{[U22_\theta]} & \Sigma_{[\Theta 21_\theta]} & \Sigma_{[\Theta 22_\theta]}\end{bmatrix}_{8N\times 8N}$$

True and spurious eigenequations

$$\det\left[SM^{cc}\right] = \prod_{m=-(N-1)}^{N} \det([T_m^{cc}][S_m^{U\Theta}])$$

$$[T_m^{cc}] = \begin{bmatrix} J_m(\lambda a) & Y_m(\lambda a) & I_m(\lambda a) & K_m(\lambda a) \\ J_m(\lambda b) & Y_m(\lambda b) & I_m(\lambda b) & K_m(\lambda b) \\ J'_m(\lambda a) & Y'_m(\lambda a) & I'_m(\lambda a) & K'_m(\lambda a) \\ J'_m(\lambda b) & Y'_m(\lambda b) & I'_m(\lambda b) & K'_m(\lambda b) \end{bmatrix}$$

True eigenequation

$$[S_m^{U\Theta}] = \begin{bmatrix} -iJ_m(\lambda a') & Y_m(\lambda b') - iJ_m(\lambda b') & -iJ'_m(\lambda a') & Y'_m(\lambda b') - iJ'_m(\lambda b') \\ J_m(\lambda a') & 0 & J'_m(\lambda a') & 0 \\ -(-1)^m i\frac{2}{\pi}I_m(\lambda a') & \frac{2}{\pi}[k_m(\lambda b') - (-1)^m iI_m(\lambda b')] & -(-1)^m i\frac{2}{\pi}I'_m(\lambda a') & \frac{2}{\pi}[k'_m(\lambda b') - (-1)^m iI'_m(\lambda b')] \\ \frac{2}{\pi}I_m(\lambda a') & 0 & \frac{2}{\pi}I'_m(\lambda a') & 0 \end{bmatrix}$$

Spurious eigenequation

Discussion of spurious eigenequation

$$\det[S_m^{U\Theta}] = \det[S_{a'}(a')]\det[S_{b'}(b')] = 0$$

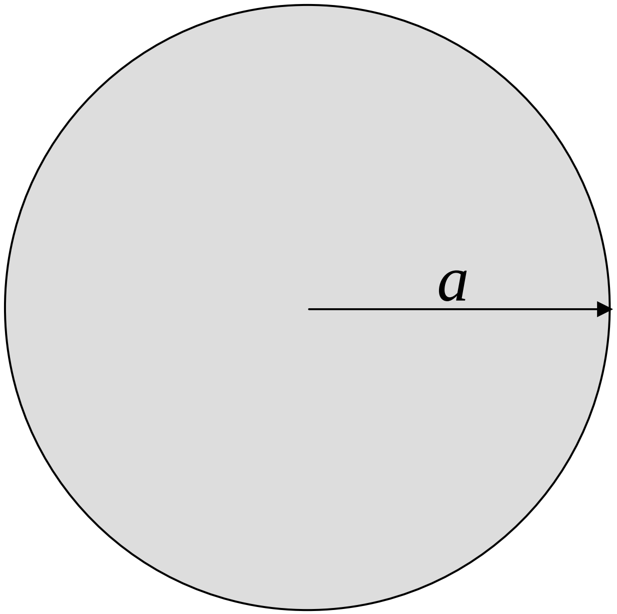
$$\det[S_{b'}(b')] = \frac{2}{\pi}\begin{vmatrix} Y_m(\lambda b') - iJ_m(\lambda b') & Y'_m(\lambda b') - iJ'_m(\lambda b') \\ K_m(\lambda b') - i(-1)^m I_m(\lambda b') & K'_m(\lambda b') - i(-1)^m I'_m(\lambda b') \end{vmatrix}$$

Never zero for any λ

$$\det[S_{a'}(a')] = \frac{2}{\pi}\begin{vmatrix} J_m(\lambda a') & J'_m(\lambda a') \\ I_m(\lambda a') & I'_m(\lambda a') \end{vmatrix} = 0$$

$$J_\ell(\lambda a')I_{\ell+1}(\lambda a') + I_\ell(\lambda a')J_{\ell+1}(\lambda a') = 0$$

True eigenequation of clamped plate



$$J_\ell(\lambda a)I_{\ell+1}(\lambda a)+I_\ell(\lambda a)J_{\ell+1}(\lambda a)=0$$

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SVD updating technique

$$[SM^{cc}]\begin{Bmatrix}\phi 1\\ \phi 2\\ \varphi 1\\ \varphi 2\end{Bmatrix}=\begin{bmatrix}U11 & U12 & \Theta 11 & \Theta 12\\ U21 & U22 & \Theta 21 & \Theta 22\\ U11_\theta & U12_\theta & \Theta 11_\theta & \Theta 12_\theta\\ U21_\theta & U22_\theta & \Theta 21_\theta & \Theta 22_\theta\end{bmatrix}\begin{Bmatrix}\phi 1\\ \phi 2\\ \varphi 1\\ \varphi 2\end{Bmatrix}=\{0\}$$

$$[SM_1^{cc}]\begin{Bmatrix}\phi' 1\\ \phi' 2\\ \varphi' 1\\ \varphi' 2\end{Bmatrix}=\begin{bmatrix}M11 & M12 & V11 & V12\\ M21 & M22 & V21 & V22\\ M11_\theta & M12_\theta & V11_\theta & V12_\theta\\ M21_\theta & M22_\theta & V21_\theta & V22_\theta\end{bmatrix}\begin{Bmatrix}\phi' 1\\ \phi' 2\\ \varphi' 1\\ \varphi' 2\end{Bmatrix}=\{0\}$$

SVD updating technique

$$[C]=\begin{bmatrix}(SM^{cc})^{H}\\(SM_{1}^{cc})^{H}\end{bmatrix}$$

By using SVD technique and the least-squares method, we have

$$\begin{vmatrix}J_m(\lambda a) & Y_m(\lambda a) & I_m(\lambda a) & K_m(\lambda a)\\J_m(\lambda b) & Y_m(\lambda b) & I_m(\lambda b) & K_m(\lambda b)\\J'_m(\lambda a) & Y'_m(\lambda a) & I'_m(\lambda a) & K'_m(\lambda a)\\J'_m(\lambda b) & Y'_m(\lambda b) & I'_m(\lambda b) & K'_m(\lambda b)\end{vmatrix}=0$$

Burton & Miller method

$$\left[[SM^{cc}]+i[SM_1^{cc}]\right]\begin{Bmatrix}\psi 1\\ \psi 2\end{Bmatrix}=\{0\}$$

True eigenequation is obtained

$$\begin{vmatrix} J_m(\lambda a) & Y_m(\lambda a) & I_m(\lambda a) & K_m(\lambda a)\\ J_m(\lambda b) & Y_m(\lambda b) & I_m(\lambda b) & K_m(\lambda b)\\ J'_m(\lambda a) & Y'_m(\lambda a) & I'_m(\lambda a) & K'_m(\lambda a)\\ J'_m(\lambda b) & Y'_m(\lambda b) & I'_m(\lambda b) & K'_m(\lambda b) \end{vmatrix}=0$$

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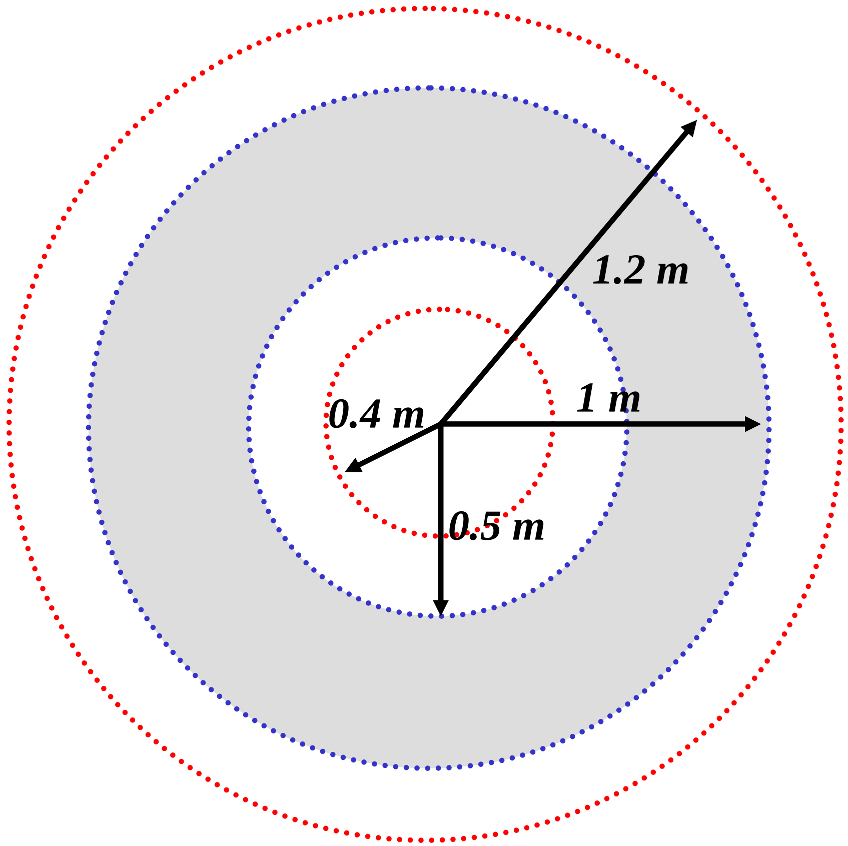
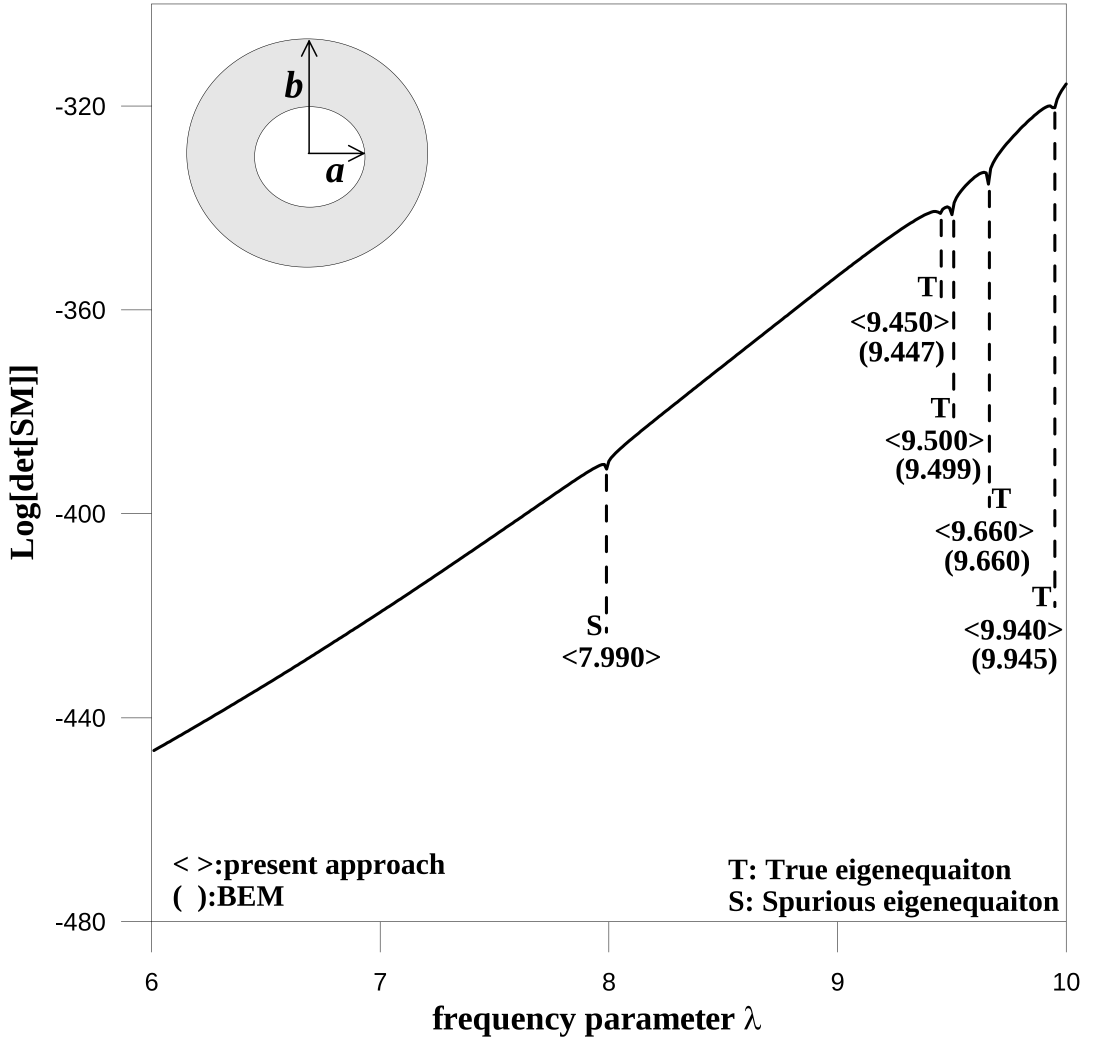
Numerical example



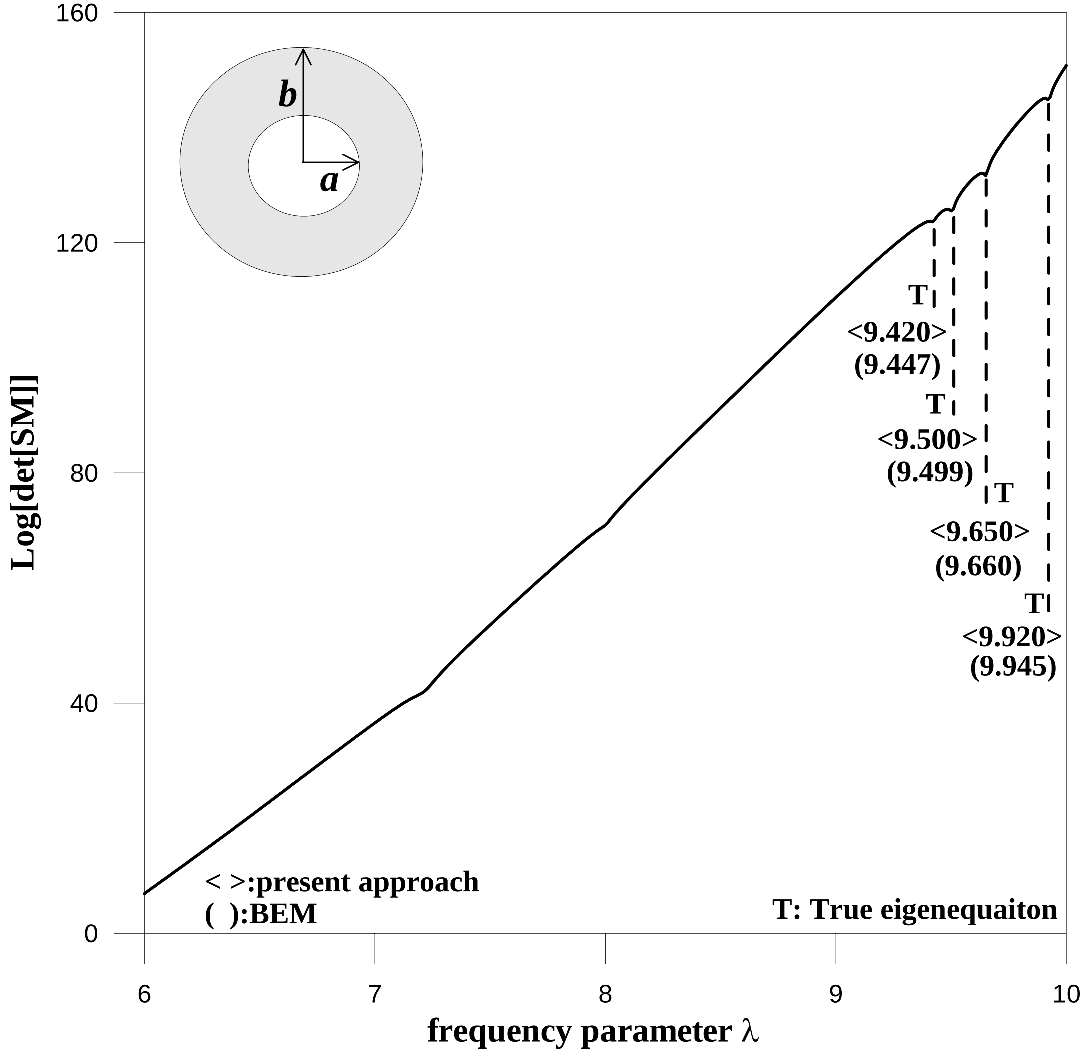
Collocation point distribution

Source point distribution

Numerical result (U- Θ formulation)



Burton & Miller method



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Conclusions

1. The mathematical analysis has shown that spurious eigenvalues occur by using degenerate kernels and circulants.
2. The positions of spurious eigenvalues for the annular problem depend on the location of inner fictitious boundary where the sources are distributed.
3. The spurious eigenvalues in the annular problem are found to be the true eigenvalues of the associated simply-connected problem bounded by the inner sources.
4. SVD updating technique and Burton & Miller method were used to filter out the spurious eigenvalues successfully.
5. For the membrane case, one paper of EABE is in press.

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The End

Thanks for your kind attention