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Application of hypersingular equations to free-surface seepage problems

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Abstract

In this paper, the Laplace problem with a free surface is solved by using the hypersingular equation. In the conventional boundary element method, only singular integral equation was used. The boundary of free surface can be determined by trial and error after initial guess and iterations. By introducing the hypersingular equation, the convergence rate of free surface can be accelerated. It is found that the result by using a higher-order singularity approach for the kernel is more accurate than introducing a higher-order element for the boundary density. Finally, numerical examples were demonstrated to show the validity of the present method.

Keywords: singular equation, hypersingular equation, dual BEM, free surface

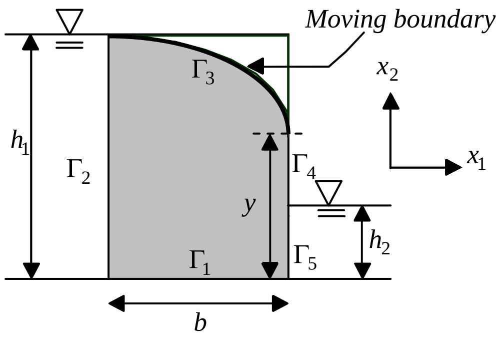
Introduction

The analysis of seepage problems is strongly influenced by porous media, hydraulic gradient, and pore pressure. In order to study these problems, accurately defining the position of free surface is very important and necessary. In this decade, many researchers utilized boundary element method (BEM) to determine the free surface but only the conventional BEM of singular equation was used. Since the dual boundary integral equation method has been proposed for the Laplace equation by Chen and Hong (Chen and Hong, 1992), the hypersingular equation can be considered as an alternative to solve the free-surface seepage problems. Based on the theory of dual boundary integral equations, a BEPO2D program was developed by Chen and Hong (Chen and Hong, 1992) to solve the Laplace equation. In this paper, the dual BEM was used to determine the location of free surface by iteration.

In the past decades, several methods were used to determine the location of free surface. For example, Aitchison (Aitchison, 1972), Liggett and Liu (Liggett and Liu, 1983), and Westbrook (Westbrook, 1985) etc. used FDM, BIEM, and FEM, to solve the position of the free surface, respectively. Regarding to these methods, free surface can be determined by using all of these methods but with different rates of convergence. Comparing with these methods, domain-based approach spends much time on mesh generation. In particular, it needs to remesh in the domain for each iteration. Therefore, BEM was proposed to analyze the problem with easier mesh generation. Higher-order element of B-spline (Cabral and Wrobel, 1991) as well as linear element (Bruch, 1988) was employed to study the problem. In this study, hypersingular formulation in conjunction with constant element scheme was considered and free surface was initially guessed before iteration. The main purpose of this paper is to employ the hypersingular equation for determining the location of free surface.

Problem statement and hypersingular formulation

The steady state flow through the homogeneous dam is considered. The problem is to find the potential ϕ which satisfies the Laplace equation $\nabla^2\phi = 0$. Referring to Fig. 1, the boundary conditions are



$$\frac{\partial\phi}{\partial n} = 0 \text{ on } \Gamma_1, \ \phi = h_1 \text{ on } \Gamma_2,$$

$$\frac{\partial\phi}{\partial n} = 0, \ \phi = y(\underset{\sim}{x}) \text{ on } \Gamma_3,$$

$$\phi = y(\underset{\sim}{x}) \text{ on } \Gamma_4, \ \phi = h_2 \text{ on } \Gamma_5,$$

Fig. 1 Flow through a rectangular dam

where $y(\underset{\sim}{x})$ is the free surface curve to be determined. We solve the problem by using the hypersingular formulation which can be written as

$$\pi\frac{\partial\phi(x)}{\partial n_x} = H.P.V.\int_B M(s,x)\phi(s)dB(s) - C.P.V.\int_B L(s,x)\frac{\partial\phi(s)}{\partial n_s}dB(s), \ x \in B,$$

where $L(s,x) = \frac{\partial U(s,x)}{\partial n_x}$, $M(s,x) = \frac{\partial^2 U(s,x)}{\partial n_s \partial n_x}$, $U(s,x) = \ln(r)$, and r denotes the distance between source point s and collocation point x, n_s is the unit outer normal at point s on the boundary, and n_x is the unit outer normal at point x on the boundary. $C.P.V.$ and $H.P.V.$ are the Cauchy principal value, and Hadamard principal value, respectively.

Numerical examples

Two cases of the free surface of the homogeneous rectangular dam are considered in Fig. 1. (Case 1：$h_1 = 24$, $h_2 = 4$, and $b = 16$. Case 2： $b = 1$, $h_1 = 1$, and $h_2 = 0$.) By using the BEPO2D program, the iteration for the location of free surface stops by checking the criterion of convergence

$$\varepsilon = \sqrt{\sum_{i=1}^{M}\left(\phi_i^{(N+1)} - \phi_i^{(N)}\right)^2} \Big/ \sqrt{\sum_{i=1}^{M}\left(\phi_i^{(N)}\right)^2} < 10^{-4},$$

where the symbol M is the number of elements on the free surface, $\phi_i^{(N+1)}$ is the location of free surface for the $(N+1)$th number of iteration, and the allowable tolerance used in this paper is 10^{-4}. Table 1 shows the results of free surface of case 1 solved by using different methods. The number of iterations and elements of case 1 by using different methods are shown in Table 2. Although Polubarinova-Kochina developed an analytical solution for the rectangular dam, it was adjusted by Cryer later (Cryer, 1976). Ozis used Cryer's formulation and improved the integrals by using Gaussian quadrature with 64 integration points (Ozis, 1981). Then, Bruch used linear boundary elements and an iterative technique to determine the separation point (Bruch, 1988). Now, singular and hypersingular equations by using constant elements are both employed to solve the problem. The separation point of case 1 by using different methods is shown in Table 3 which indicates the accuracy of the hypersingular formulation. Table 4 and Table 5 also present the results of case 2. In Table 2 and Table 5, the number of iterations and elements by using the hypersingular equation is fewer than that by using singular equation and FEM.

Table 1 Free surface obtained by using different methods (Case 1)

x	1	2	3	4	5	6	7	8
Aitchison (1972)	23.74	23.41	22.12	21.60	21.04	20.43	22.12	21.60
Westbrook (1985)	23.64	23.32	22.12	21.55	21.07	20.36	22.12	21.55
Present (Singular equation)	23.76	23.42	22.12	21.60	21.04	20.43	22.12	21.60
Present (Hypersingular equation)	23.74	23.40	22.09	21.57	21.00	20.39	22.09	21.57
x	9	10	11	12	13	14	15	16
Aitchison (1972)	19.78	19.08	18.31	17.48	16.57	15.54	14.39	12.79
Westbrook (1985)	19.81	19.07	18.26	17.45	16.54	15.51	14.33	—
Present (Singular equation)	19.78	19.07	18.30	17.47	16.56	15.50	14.15	12.61
Present (Hypersingular equation)	19.73	19.02	18.24	17.39	16.45	15.39	14.09	12.68

Table 2 Number of iterations by using different methods (Case 1)

Method	Mesh	Number of iterations
FEM (Westbrook, 1985)	17X25	49
Singular equation	39	14
Hypersingular equation	39	13

Table 3 Final position of separation point using different methods (Case 1)

Reference	Height
Polubarinova-Kochina (1962)	12.95
Cryer (1976)	12.7132
Ozis (1981)	12.7070
Westbrook (1985), FEM	NA
Bruch (1988), BEM, Linear element	12.98

Cabral and Wrobel (1991), BEM, B-spline	12.74
Present (2004), BEM, constant element, Singular equation	12.61
Present (2004), BEM, constant element, Hypersingular equation	12.68

Table 4 Free surface obtained by using different methods (Case 2)

x	0.2	0.4	0.6	0.8	1.0
Polubarinova-Kochina (1962)	0.938	0.850	0.738	0.595	0.368
Singular equation	0.939	0.850	0.737	0.590	0.368
Hypersingular equation	0.937	0.847	0.732	0.584	0.379

Table 5 Iteration number by using the present methods (Case 2)

Method	Mesh	Number of iterations
Present (Singular equation)	25	12
Present (Hypersingular equation)	25	9

Conclusions

Free-surface seepage problems were solved by using the hypersingular equation and the results were compared with other solutions. It is found that the convergence rate of the present method as well as its accuracy is superior to the other methods. It is suggested that increasing the order of kernel singularity can obtain more accurate results than increasing the order of boundary element. Two examples were demonstrated to check the accuracy and efficiency of the present method.

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