Applications of hypersingular equation to free-surface seepage problems

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Outlines

- Problem statement
- Literature
- Dual boundary integral equations
- Flowchart of iteration
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Problem statement

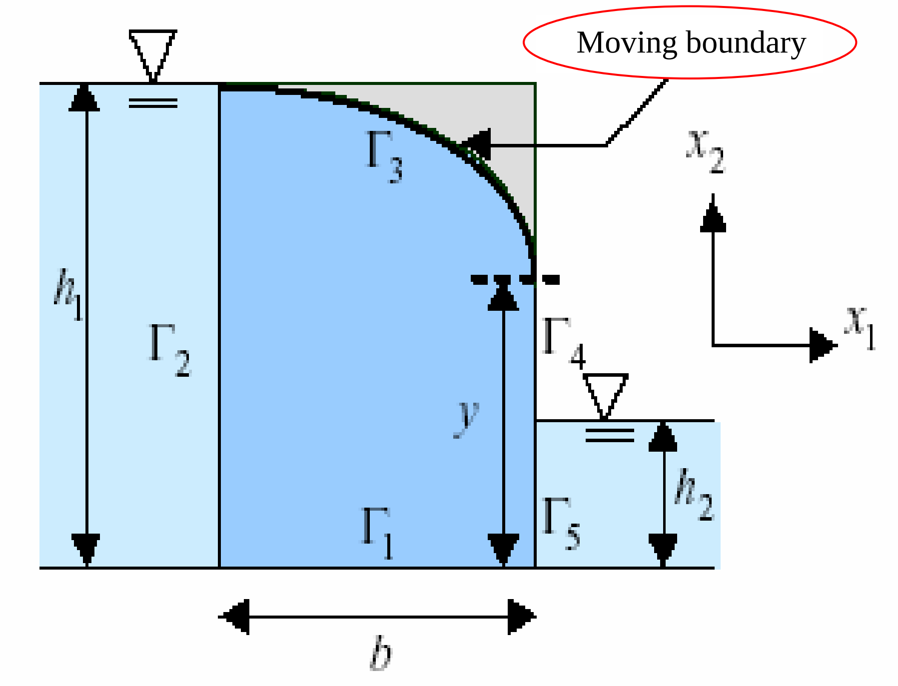
- G.E. $: \nabla^2 \phi = 0$
- B.C. $: \phi = h_1 \text{ on } \Gamma_2$

$$\phi = h_2 \text{ on } \Gamma_5$$

$$\frac{\partial \phi}{\partial n} = 0 \quad \text{on} \quad \Gamma_1$$

$$\phi = y(\underset{\sim}{x}) \quad \text{on} \quad \Gamma_4$$

$$\frac{\partial \phi}{\partial n} = 0, \ \phi = y(\underset{\sim}{x}) \quad \text{on} \quad \Gamma_3$$



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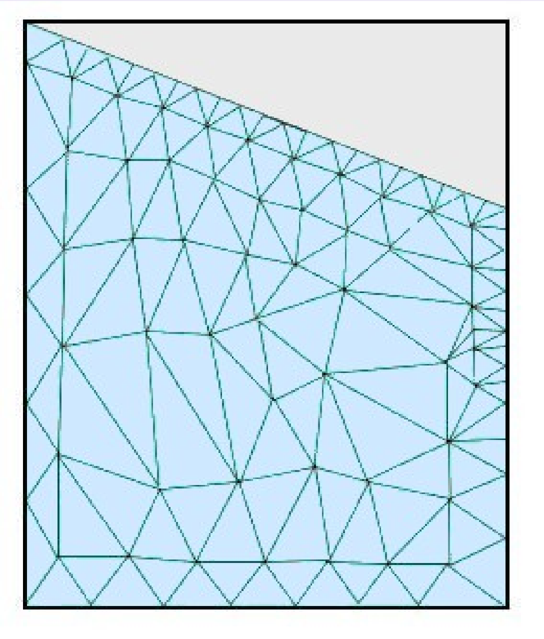
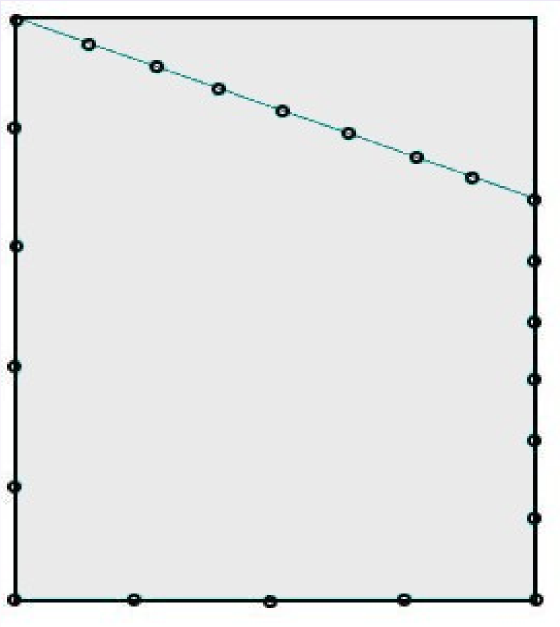
Literature

- Polubarinova-Kochina developed an analytical solution of free surface for the rectangular dam, 1962.
- Aitchison used the FDM to determine the free surface, 1972.
- Liggett and Liu used the BIEM to analyze the free surface, 1983.
- Westbrook used FEM to determine the free surface, 1985.
- Cabral and Wrobel uesd B-Spline boundary elements to determine the free surface, 1991.

FEM mesh & BEM mesh



Finite element mesh

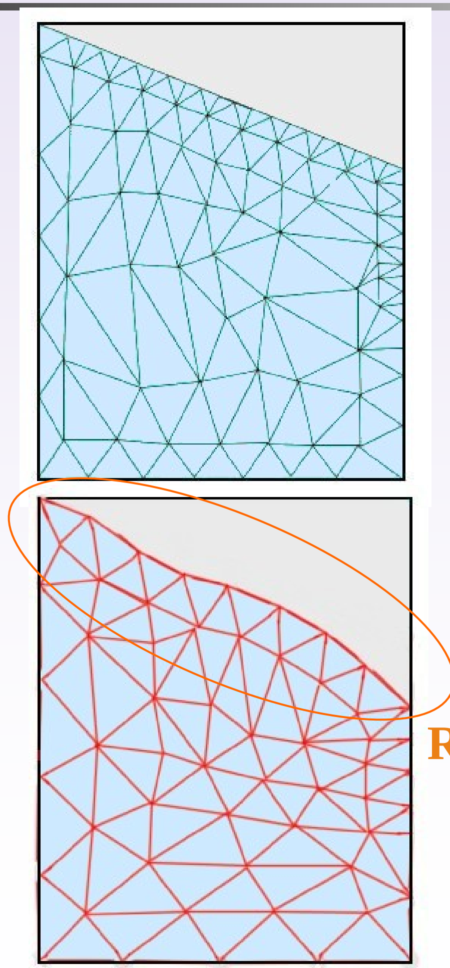


Boundary element mesh

FEM

Initial guess

After iteration

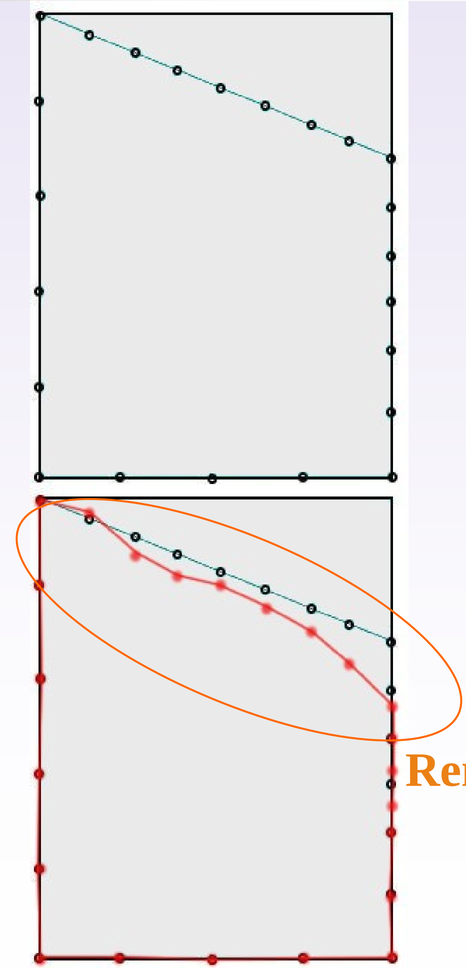


Remesh area

BEM

Initial guess

After iteration



Remesh line

- B-Spline BEM was used to approach the free surface by increasing the order of elements.
- In this paper, we utilized the higher order kernels to approach free surface instead of increasing the order of elements.

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Dual boundary integral equation

- Dual boundary integral equations are derived from the Green identity：

Singular equation

$$2\pi\phi(x) = \int_B T(s,x)\phi(s)\,dB(s) - \int_B U(s,x)\frac{\partial\phi(s)}{\partial n_s}\,dB(s)\ ,\ \ x \in D,$$

Hypersingular equation

$$2\pi\frac{\partial\phi(x)}{\partial n_x} = \int_B M(s,x)\phi(s)\,dB(s) - \int_B L(s,x)\frac{\partial\phi(s)}{\partial n_s}\,dB(s)\ ,x \in D,$$

where $U(s,x) = \ln(r)$, $T(s,x) = \frac{\partial U(s,x)}{\partial n_s}$, $L(s,x) = \frac{\partial U(s,x)}{\partial n_x}$, $M(s,x) = \frac{\partial^2 U(s,x)}{\partial n_s \partial n_x}$

r denotes the distance between source point s and field point x .

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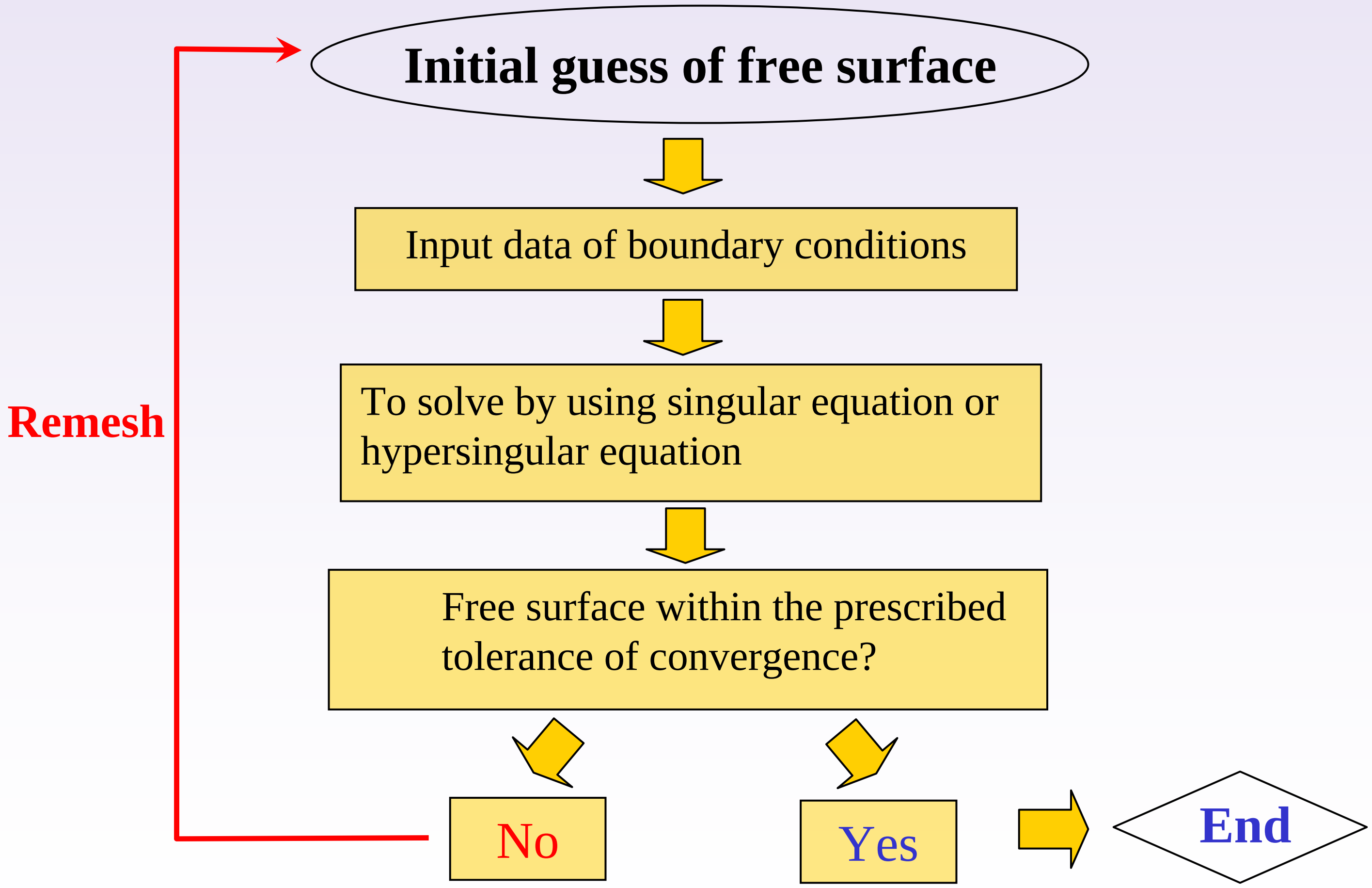
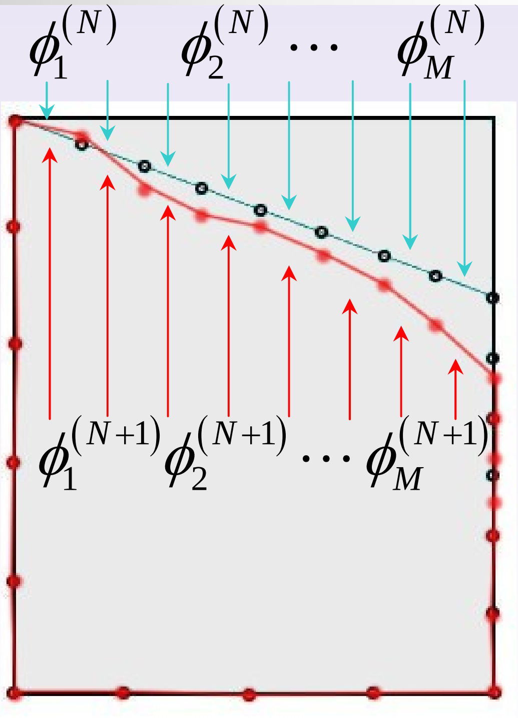
Flowchart of iteration



- Tolerance

$$\varepsilon = \frac{\sqrt{\sum_{i=1}^{M}\left(\phi_i^{(N+1)} - \phi_i^{(N)}\right)^2}}{\sqrt{\sum_{i=1}^{M}\left(\phi_i^{(N)}\right)^2}} < 10^{-4}$$



where the symbol M is the number of elements on the free surface

$\phi_i^{(N+1)}$ is the location of free surface for the ($N+1$)th number of iteration.

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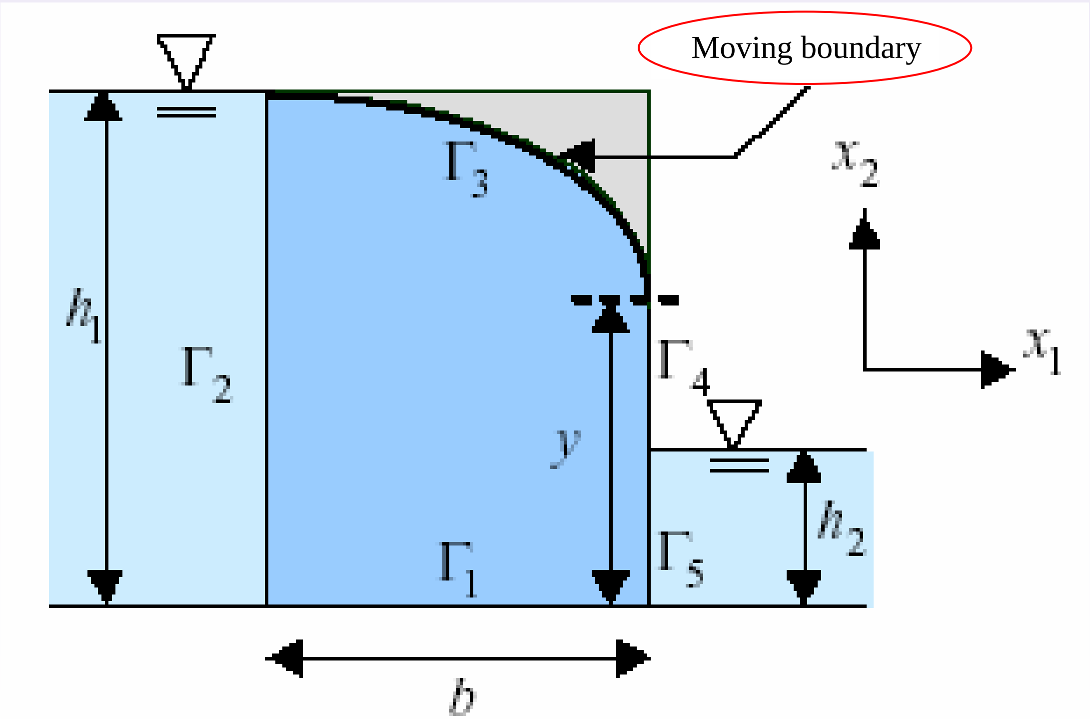
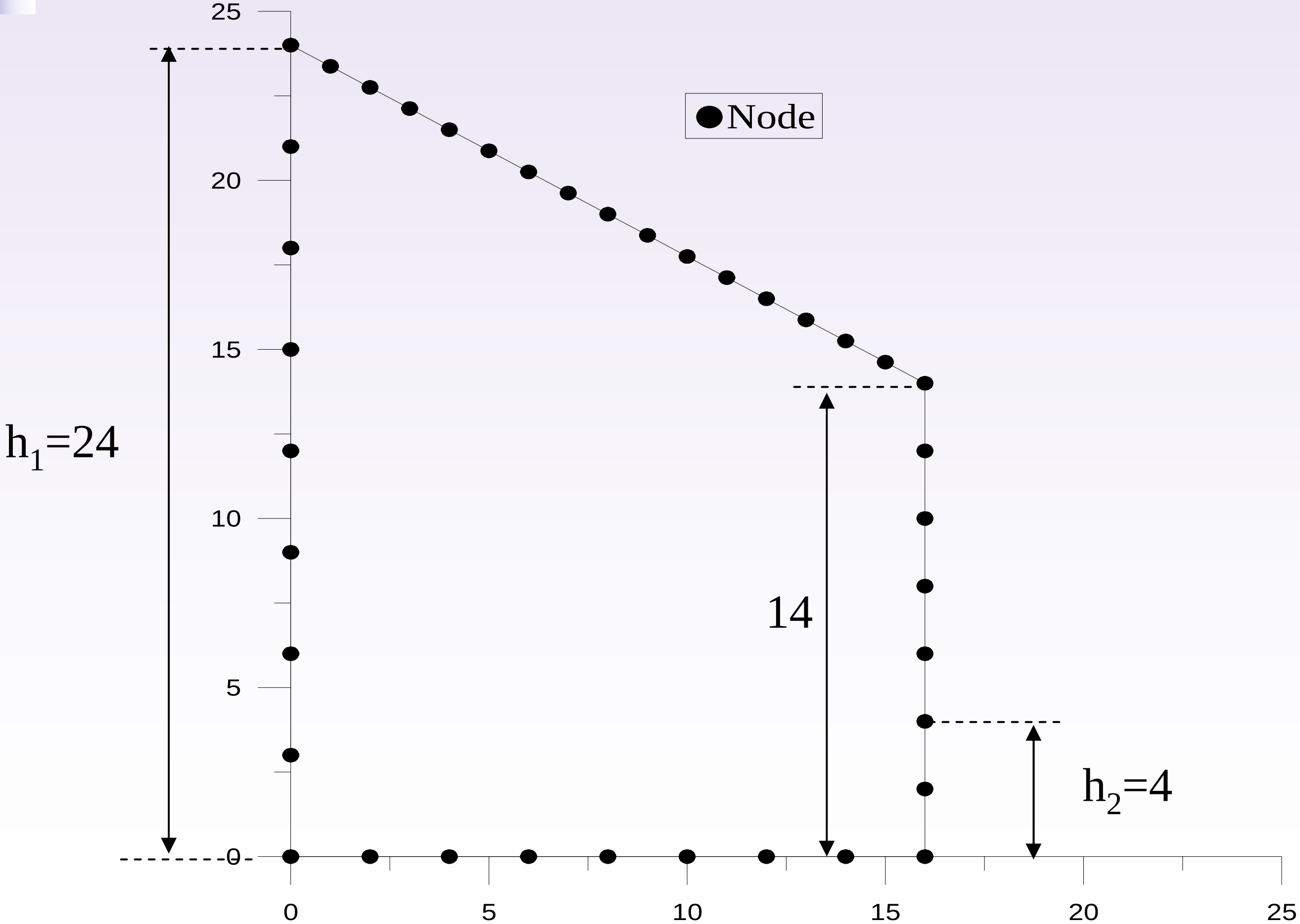
Numerical examples (Case 1)

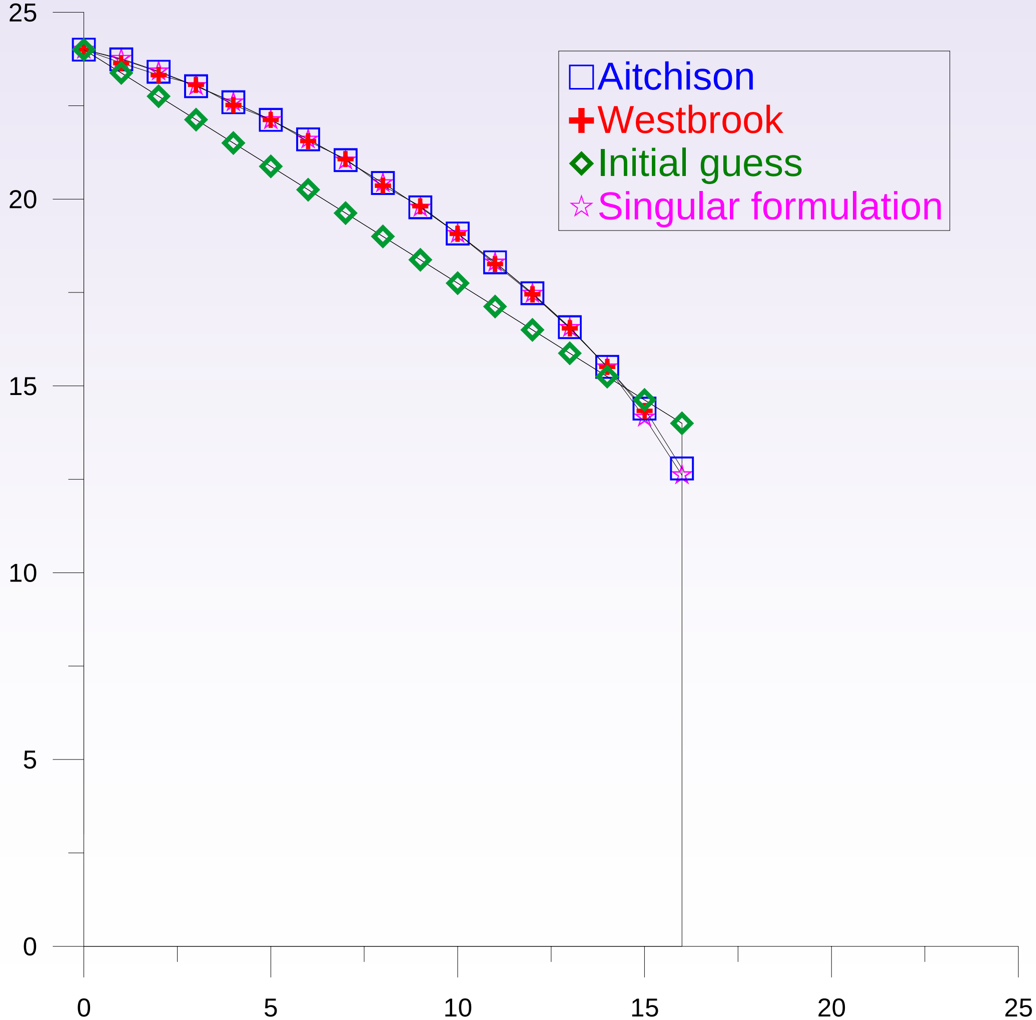
- Case 1： $h_1 = 24, h_2 = 4, b = 16$



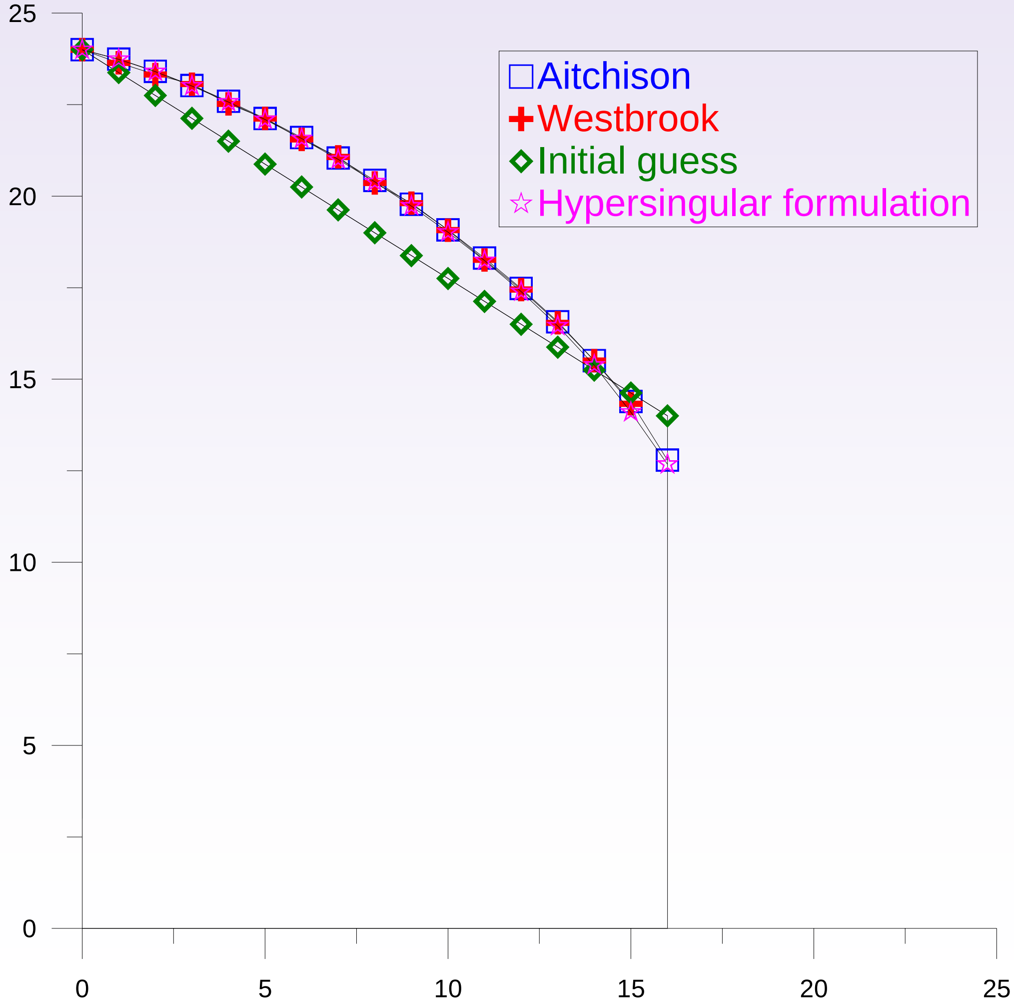
Boundary element mesh of case 1



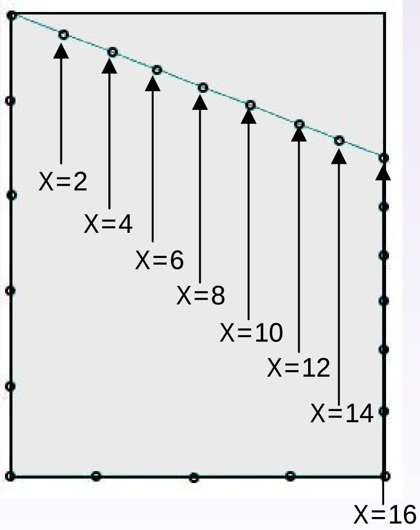
Free surface (Singular equation)



Free surface (Hypersingular equation)



Free surface obtained by different methods



x	2	4	6	8	10	12	14	16
Aitchison	23.41	22.59	21.60	20.43	19.08	17.48	15.54	12.79
Westbrook	23.32	22.52	21.55	20.36	19.07	17.45	15.51	—
Present (Singular equation)	23.42	22.59	21.60	20.43	19.07	17.47	15.50	12.61
Present (Hypersingular equation)	23.40	22.52	21.57	20.39	19.02	17.39	15.39	12.68

Further investigation of the separation point

Final position of separation point using different methods

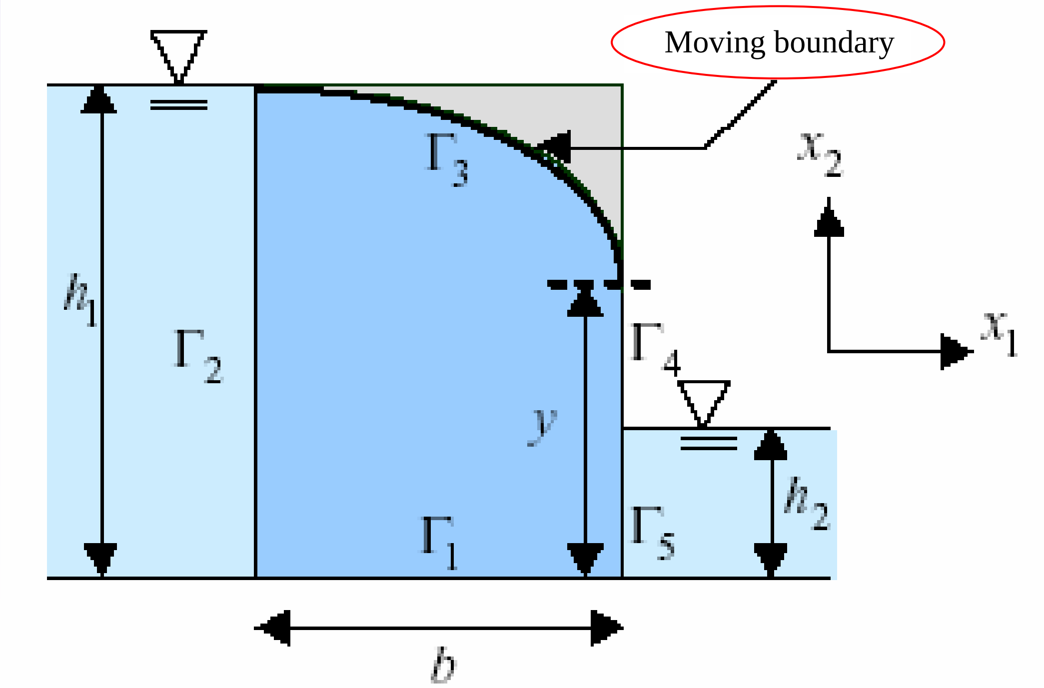
References	Height
Polubarinova-Kochina (1962)	12.95
Cryer (1976)	12.7132
Ozis (1981)	**12.7070**
Westbrook (1985), FEM	NA
Bruch (1988), BEM, Linear element	12.98
Cabral and Wrobel (1991), BEM, B-spline	12.74
Present, (2004), BEM, constant element, Singular equation	12.61
Present, (2004), BEM, constant element, Hypersingular equation	**12.68**

Number of iterations using different methods

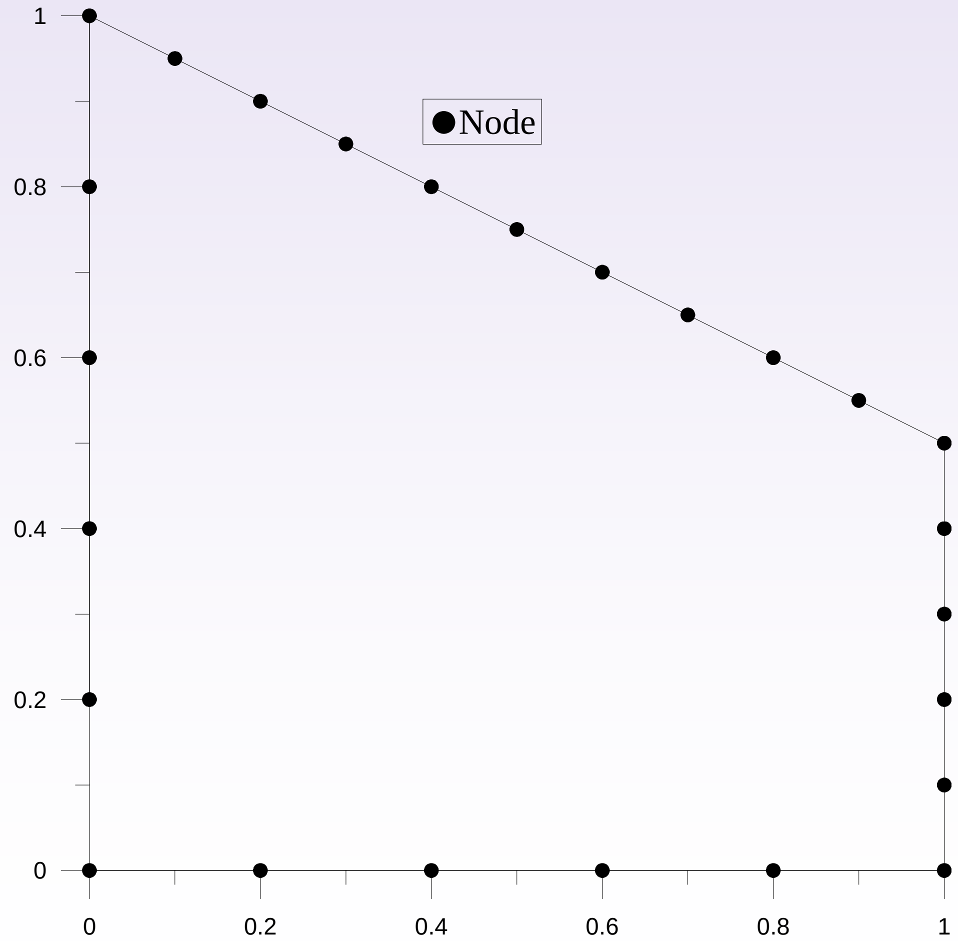
Method	Mesh	Number of iterations
FEM	17╳25	49
Singular equation	39	14
Hypersingular equation	39	13 (better)

Numerical examples (Case 2)

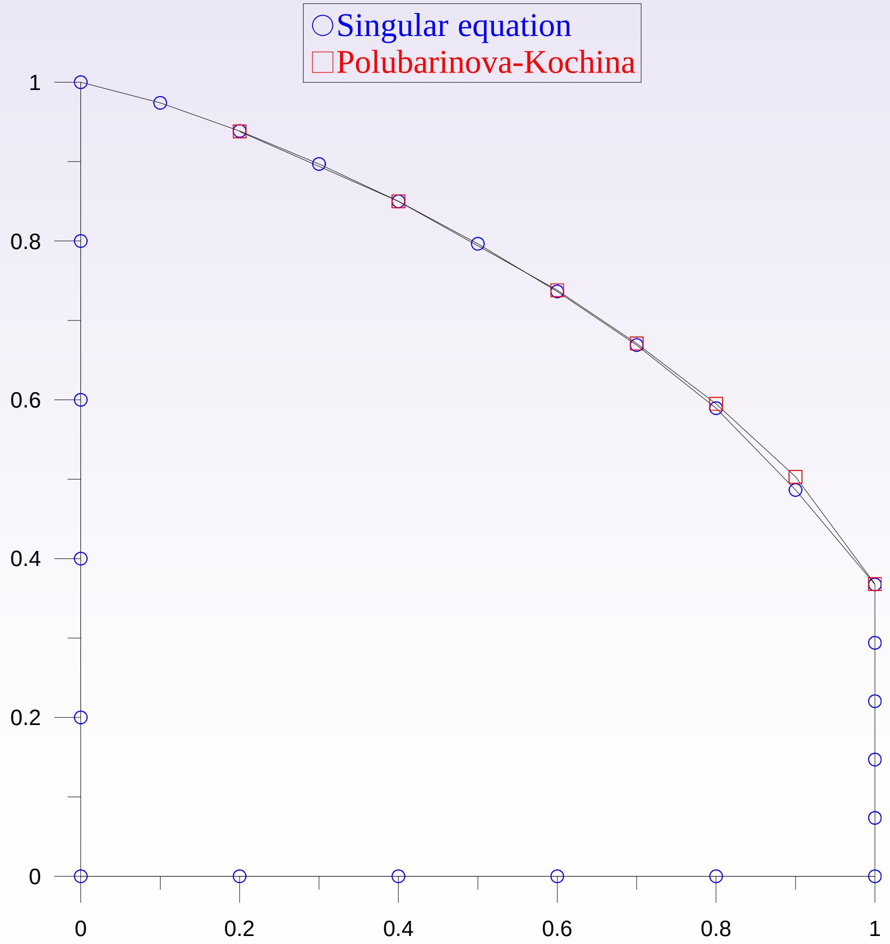
- Case 2：$h_1 = 1, h_2 = 0, b = 1$



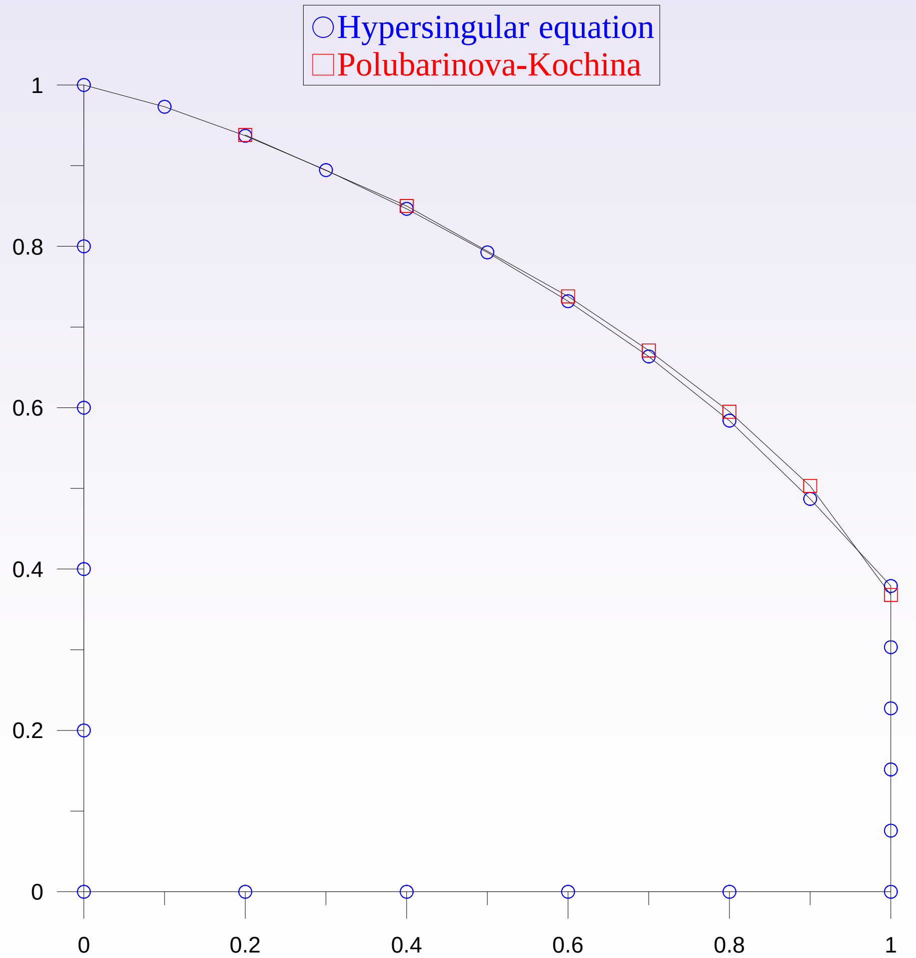
Boundary element mesh of case 2



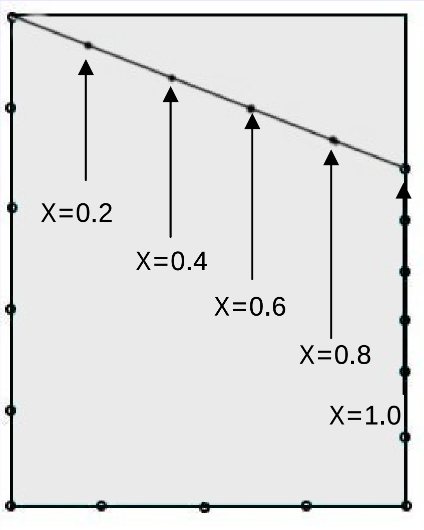
Free surface (Singular equation)



Free surface (Hypersingular equation)



Free surface obtained by different methods

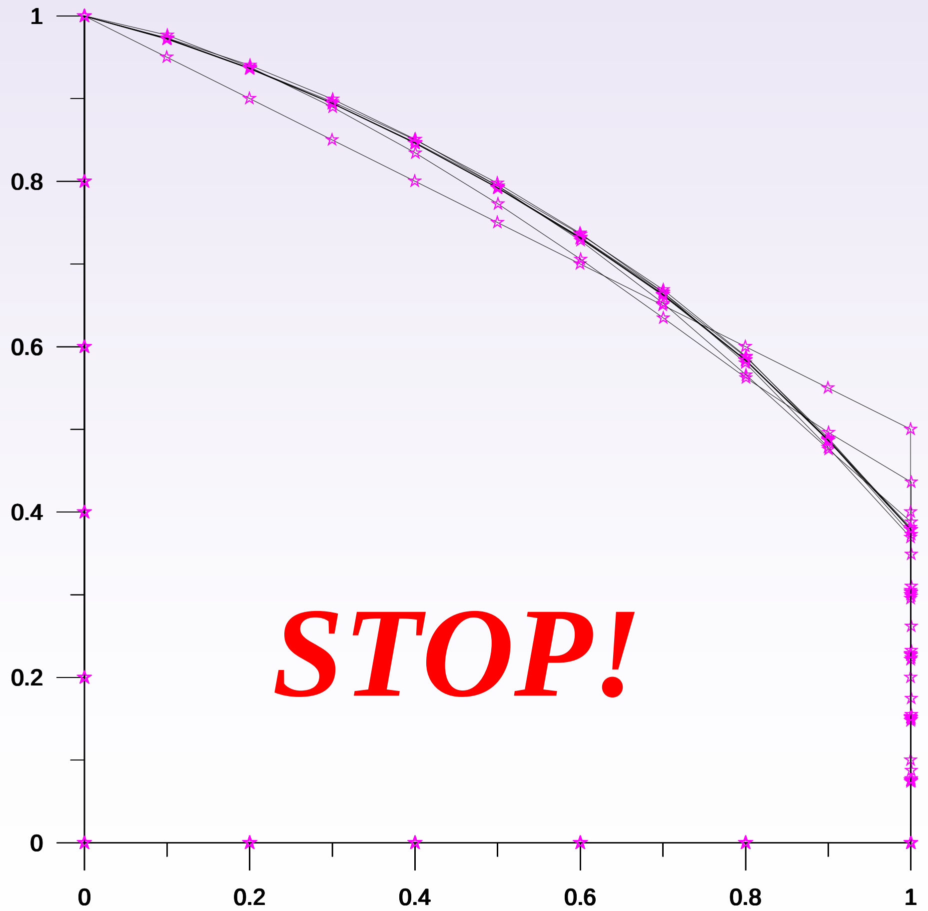


x	0.2	0.4	0.6	0.8	1.0
Polubarinova-Kochina	0.938	0.850	0.738	0.595	0.368
Singular equation	0.939	0.850	0.737	0.590	0.368
Hypersingular equation	0.937	0.847	0.732	0.584	0.379

Iteration number by using the singular equation and hypersingular equation

Method	Mesh	Number of iteration
Present (Singular equation)	25	12
Present (Hypersingular equation)	25	9 (better)

Procedure of iteration (Case2) (Hypersingular equation)



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Conclusions

- Free-surface seepage problems were solved by using the hypersingular equation successfully and compared with the analytical solution, FEM and conventional BEM.
- It is found that the convergence rate and the mesh generation of the present method are superior to the other methods, we can save much time in the process of remesh.
- Two examples were demonstrated to check the accuracy and efficiency for the present method.

THE END

Thank you for your attention

You can get more information in our website

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