Derivation of the Green's function for Laplace and Helmholtz problems with circular boundaries by using the null-field integral equation approach

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Outlines

- **Motivation and literature review**
- **Derivation of the Green's function**
  - Expansions of fundamental solution and boundary density
  - Adaptive observer system
  - Vector decomposition technique
  - Linear algebraic equation
  - Take free body
  - Image technique for solving half-plane problems
- **Numerical examples**
  - Green's function for Laplace problems
  - Green's function for Helmholtz problems
- **Conclusions**

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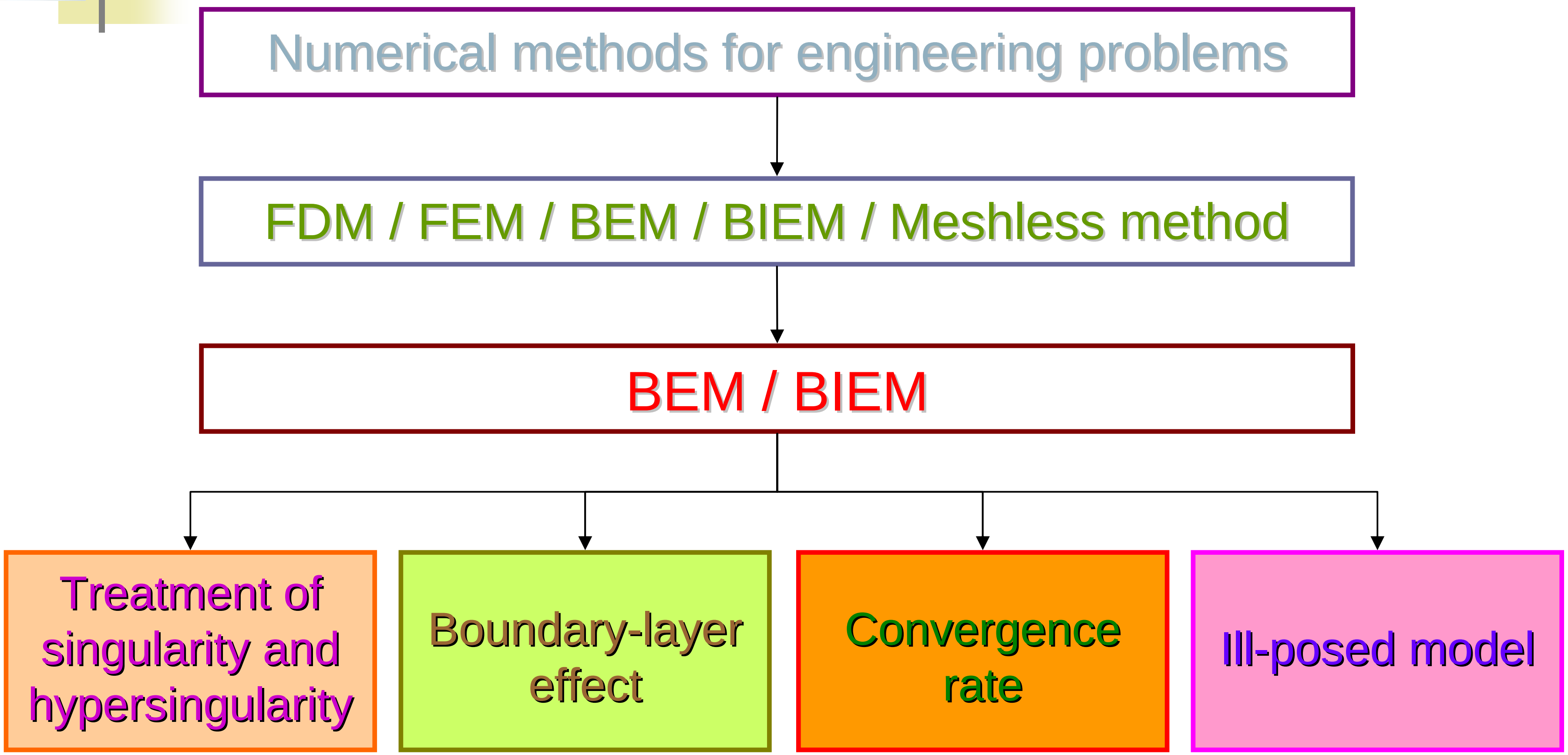
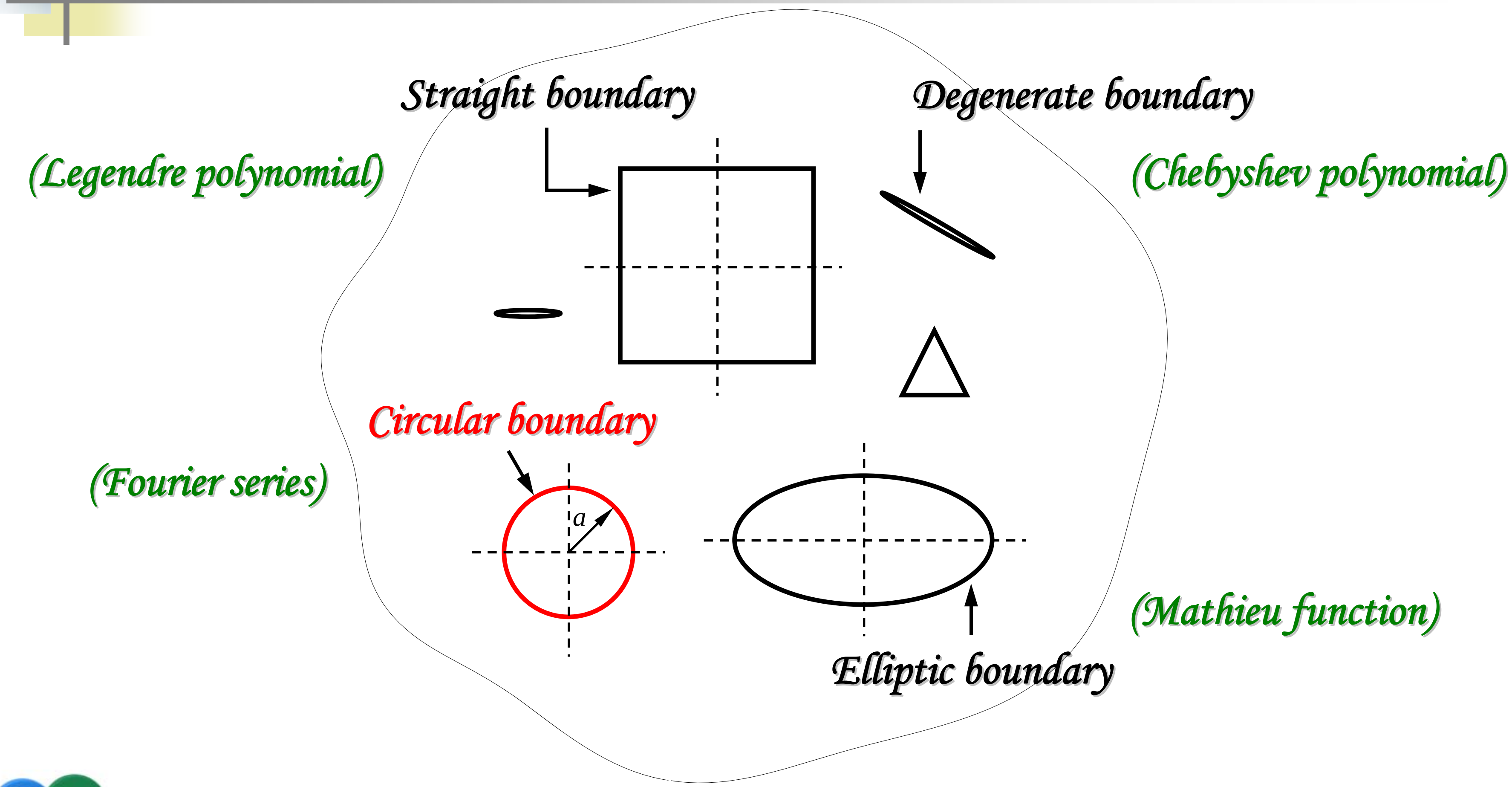
Motivation



Numerical methods for engineering problems
FDM / FEM / BEM / BIEM / Meshless method
BEM / BIEM
Treatment of singularity and hypersingularity
Boundary-layer effect
Convergence rate
Ill-posed model

Engineering problem with arbitrary geometries



Literature review

Derivation of the Green's function

Successive iteration method	Modified potential method	Trefftz bases
Boley, 1956, "A method for the construction of Green's functions,", Quarterly of Applied Mathematics	*Melnikov, 2001, "Modified potential as a tool foor computing Green's functions in continuum mechanics", Computer Modeling in Engineering Science*	*Wang and Sudak, 2007, "Antiplane time-harmonic Green's functions for a circular inhomogeneity with an imperfect interface", Mechanics Research Communications*

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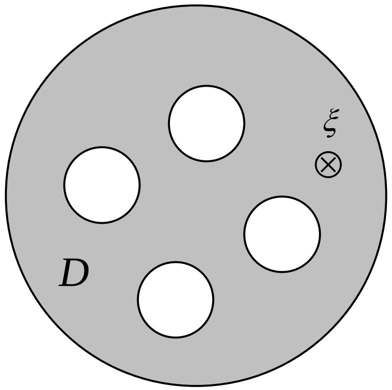
Null-field integral approach to construct the Green's function

Original Problem



Governing equation: $\nabla^2 G(x,\xi) = \delta(x-\xi),\ x \in D$

Boundary condition: Subjected to given B. C.

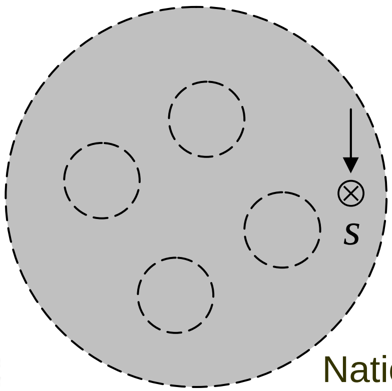
Green's third identity

$$\iint_D [u(x)\nabla^2 v(x) - v(x)\nabla^2 u(x)]dD(x) = \int_B [(u(x)\frac{\partial v(x)}{\partial n} - v(x)\frac{\partial u(x)}{\partial n}]dB(x)$$

BIE for Green's function

$$2\pi G(x,\xi) = \int_B \frac{\partial U(s,x)}{\partial n_s} G(s,\xi) dB(s) - \int_B U(s,x)\frac{\partial G(s,\xi)}{\partial n_s} dB(s) + U(\xi,x)$$

Auxiliary system



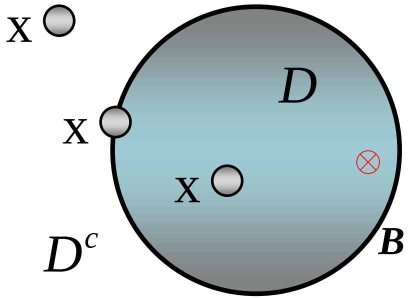
Governing equation: $\nabla^2 U(x,s) = 2\pi\delta(x-s)$

$v(x) = U(s,x)$ Fundamental solution

$u(x) = G(x,\xi)$

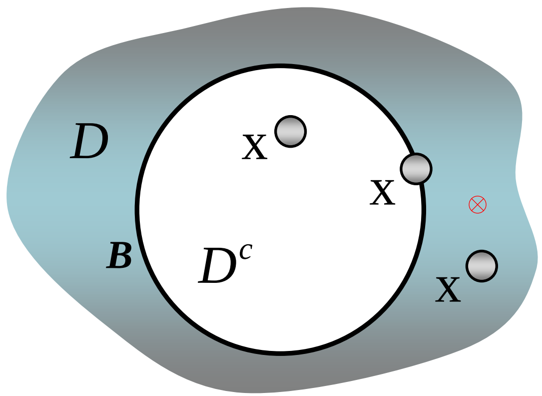
Boundary integral equation and null-field integral equation

Interior case



$$T(s,x)=\frac{\partial U(s,x)}{\partial n_s}$$

Exterior case



$$2\pi G(x,\xi)=\int_B T(s,x)G(s,\xi)dB(s)-\int_B U(s,x)\frac{\partial G(s,\xi)}{\partial n_s}dB(s)+U(\xi,x),\quad x\in D\cup B$$

Degenerate (separate) form

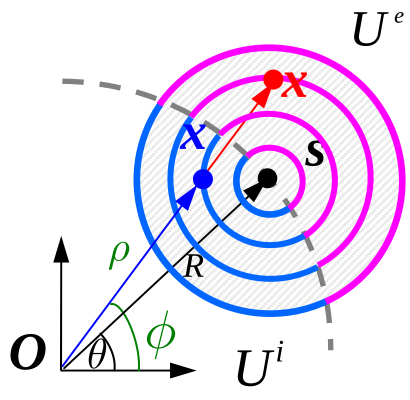
$$\pi G(x,\xi)=C.P.V.\int_B T(s,x)G(s,\xi)dB(s)-R.P.V.\int_B U(s,x)\frac{\partial G(s,\xi)}{\partial n_s}dB(s)+U(\xi,x),\quad x\in B$$

$$0=\int_B T(s,x)G(s,\xi)dB(s)-\int_B U(s,x)\frac{\partial G(s,\xi)}{\partial n_s}dB(s)+U(\xi,x),\quad x\in D^c\cup B$$

Expansions of fundamental solution (2D)

Laplace problem-- $U(\mathrm{s},\mathrm{x}) = \ln|\mathrm{x}-\mathrm{s}| = \ln r$

$$U(\mathrm{s},\mathrm{x}) = \begin{cases} U^i(R,\theta;\rho,\phi) = \ln R - \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{\rho}{R}\right)^m \cos m(\theta-\phi), & R \ge \rho \\ U^e(R,\theta;\rho,\phi) = \ln \rho - \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{R}{\rho}\right)^m \cos m(\theta-\phi), & \rho > R \end{cases}$$

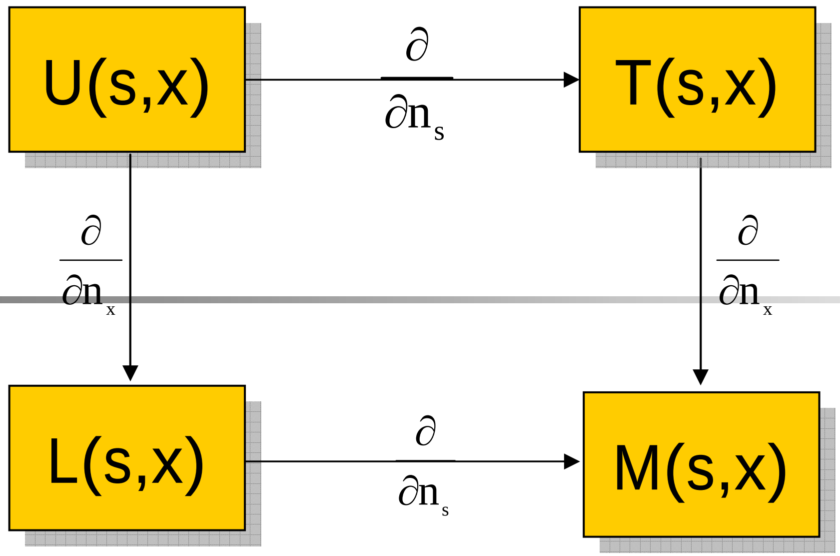


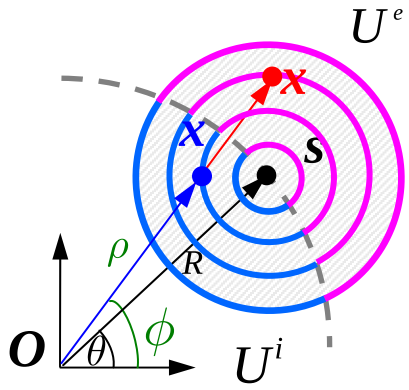
Helmholtz problem-- $U(s,x) = -i\pi H_0^{(1)}(kr)/2$

$$U(\mathrm{s},\mathrm{x}) = \begin{cases} U^i(R,\theta;\rho,\phi) = \frac{-\pi i}{2} \sum_{m=0}^{\infty} \varepsilon_m J_m(k\rho) H_m^{(1)}(kR) \cos(m(\theta-\phi)), & R \ge \rho \\ U^e(R,\theta;\rho,\phi) = \frac{-\pi i}{2} \sum_{m=0}^{\infty} \varepsilon_m H_m^{(1)}(k\rho) J_m(kR) \cos(m(\theta-\phi)), & \rho > R \end{cases}$$

Neumann factor

$$\varepsilon_M = \begin{cases} 1, & m = 0 \\ 2, & m = 1, 2, \cdots \end{cases}$$





Laplace problem--

$$T(s,x)=\begin{cases} T^{i}(R,\theta;\rho,\phi)=\dfrac{1}{R}+\displaystyle\sum_{m=1}^{\infty}\left(\dfrac{\rho^{m}}{R^{m+1}}\right)\cos m(\theta-\phi), & R>\rho \\ T^{e}(R,\theta;\rho,\phi)=-\displaystyle\sum_{m=1}^{\infty}\left(\dfrac{R^{m-1}}{\rho^{m}}\right)\cos m(\theta-\phi), & \rho>R \end{cases}$$

Helmholtz problem--

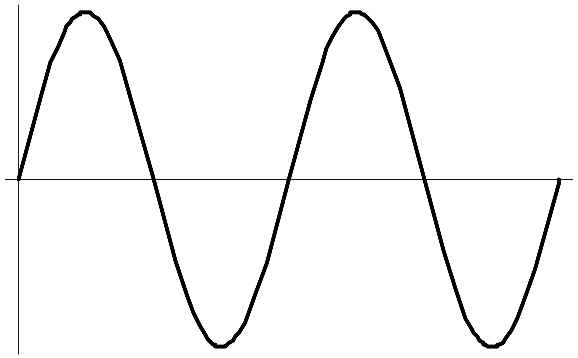
$$T(s,x)=\begin{cases} T^{i}(R,\theta;\rho,\phi)=\dfrac{-\pi ki}{2}\displaystyle\sum_{m=0}^{\infty}\varepsilon_m J_m(k\rho)H_m^{'(1)}(kR)\cos(m(\theta-\phi)), R>\rho \\ T^{e}(R,\theta;\rho,\phi)=\dfrac{-\pi ki}{2}\displaystyle\sum_{m=0}^{\infty}\varepsilon_m J_m'(kR)H_m^{(1)}(k\rho)\cos(m(\theta-\phi)), \rho>R \end{cases}$$

Boundary density discretization

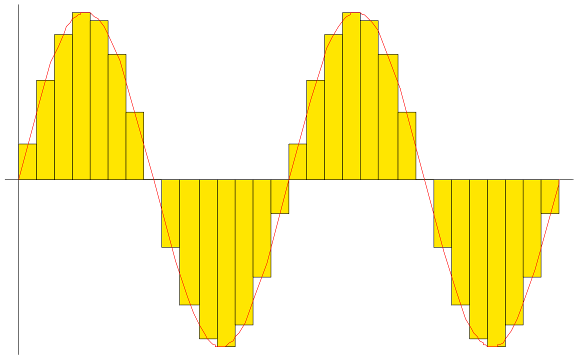
Fourier series expansions - boundary density

$$u(s) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta), \quad s \in B$$

$$t(s) = p_0 + \sum_{n=1}^{\infty} (p_n \cos n\theta + q_n \sin n\theta), \quad s \in B$$

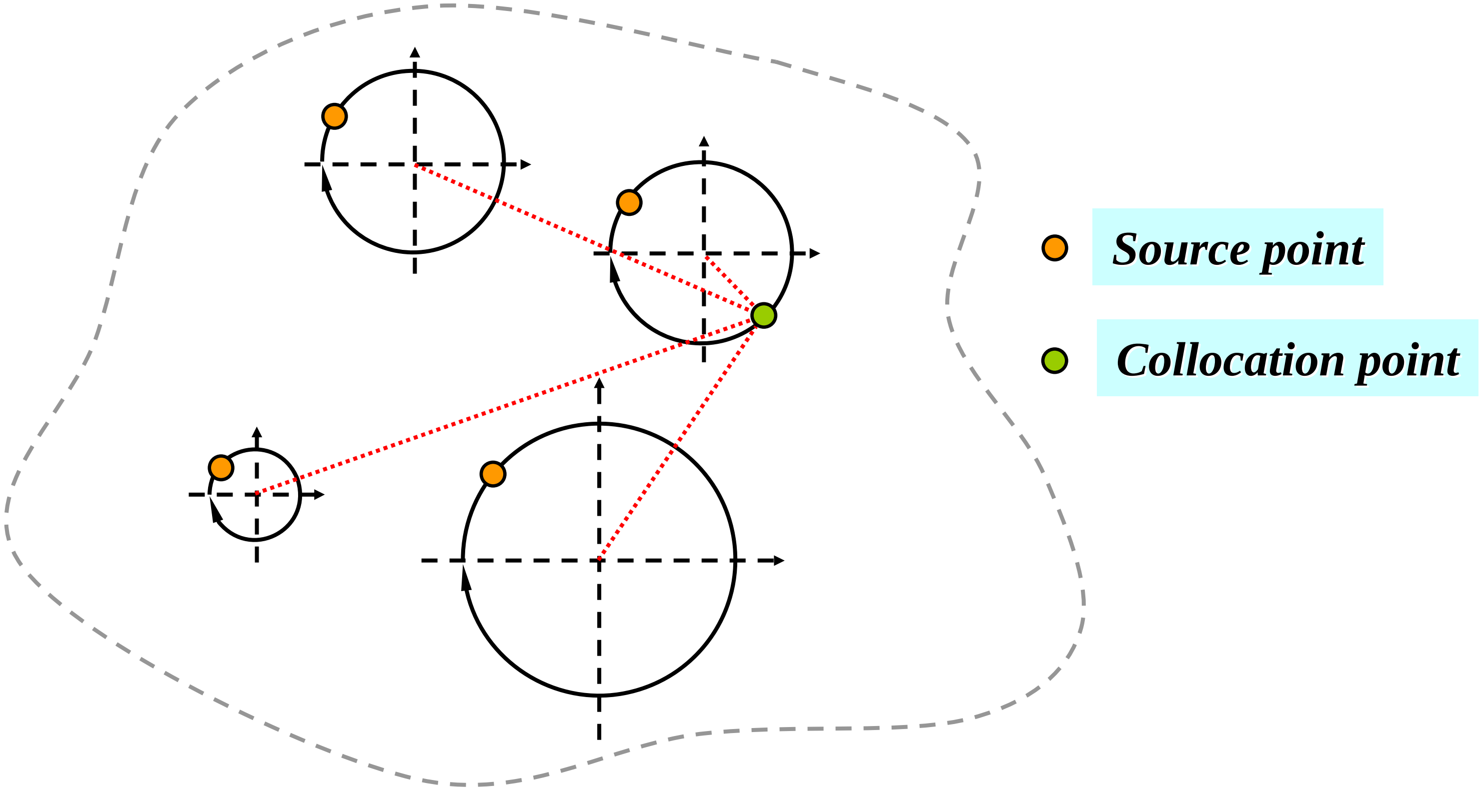


Fourier series



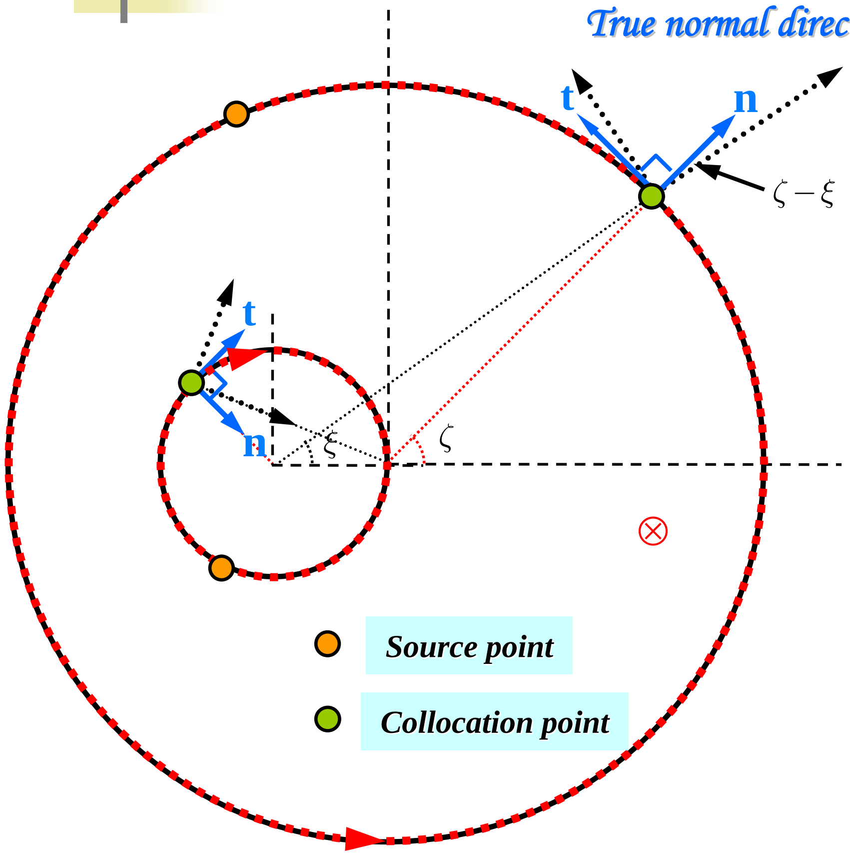
Ex. constant element

Adaptive observer system



National Taiwan Ocean University
Department of Harbor and River Engineering

Vector decomposition technique for potential gradient



$$2\pi \frac{\partial G(x,\xi)}{\partial \mathbf{n}} = \int_B M_\rho(s,x) G(s) dB(s) - \int_B L_\rho(s,x) \frac{\partial G(s)}{\partial \mathbf{n}} dB(s) + L_\rho(\xi, x)$$

$$2\pi \frac{\partial G(x,\xi)}{\partial \mathbf{t}} = \int_B M_\phi(s,x) G(s) dB(s) - \int_B L_\phi(s,x) \frac{\partial G(s)}{\partial \mathbf{n}} dB(s) + L_\phi(\xi, x)$$

Non-concentric case:

$$L_\rho(s,x) = \frac{\partial U(s,x)}{\partial \rho} \boxed{\cos(\zeta-\xi)} + \frac{1}{\rho}\frac{\partial U(s,x)}{\partial \phi} \boxed{\cos(\frac{\pi}{2}-\zeta+\xi)}$$

$$M_\rho(s,x) = \frac{\partial T(s,x)}{\partial \rho} \boxed{\cos(\zeta-\xi)} + \frac{1}{\rho}\frac{\partial T(s,x)}{\partial \phi} \boxed{\cos(\frac{\pi}{2}-\zeta+\xi)}$$

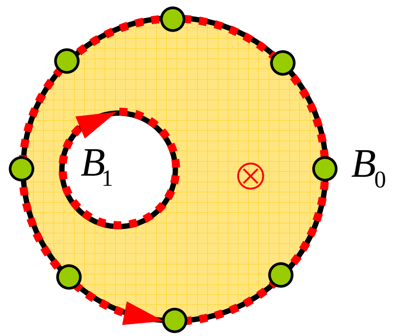
Concentric case (special case): $\zeta = \xi$

$$L_\rho(s,x) = \frac{\partial U(s,x)}{\partial \rho} \qquad M_\rho(s,x) = \frac{\partial T(s,x)}{\partial \rho}$$

Linear algebraic equation

$$0 = \int_B T(\mathrm{s},\mathrm{x})G(\mathrm{s},\xi)dB(\mathrm{s}) - \int_B U(\mathrm{s},\mathrm{x})\frac{\partial G(\mathrm{s},\xi)}{\partial n_{\mathrm{s}}}dB(\mathrm{s}) + U(\xi,\mathrm{x})$$

$$\Rightarrow [\mathbf{U}]\{\mathbf{t}\} = [\mathbf{T}]\{\mathbf{u}\} + \{\mathrm{b}\}$$



$$[\mathbf{U}] = \begin{bmatrix} \mathbf{U}_{00} & \mathbf{U}_{01} & \cdots & \mathbf{U}_{0N} \\ \mathbf{U}_{10} & \mathbf{U}_{11} & \cdots & \mathbf{U}_{1N} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{U}_{N0} & \mathbf{U}_{N1} & \cdots & \mathbf{U}_{NN} \end{bmatrix} \quad \{\mathbf{t}\} = \begin{Bmatrix} \mathbf{t}_0 \\ \mathbf{t}_1 \\ \mathbf{t}_2 \\ \vdots \\ \mathbf{t}_N \end{Bmatrix} \quad \{\mathbf{b}\} = \begin{Bmatrix} \mathbf{b}_0 \\ \mathbf{b}_1 \\ \mathbf{b}_2 \\ \vdots \\ \mathbf{b}_N \end{Bmatrix}$$

Take free body

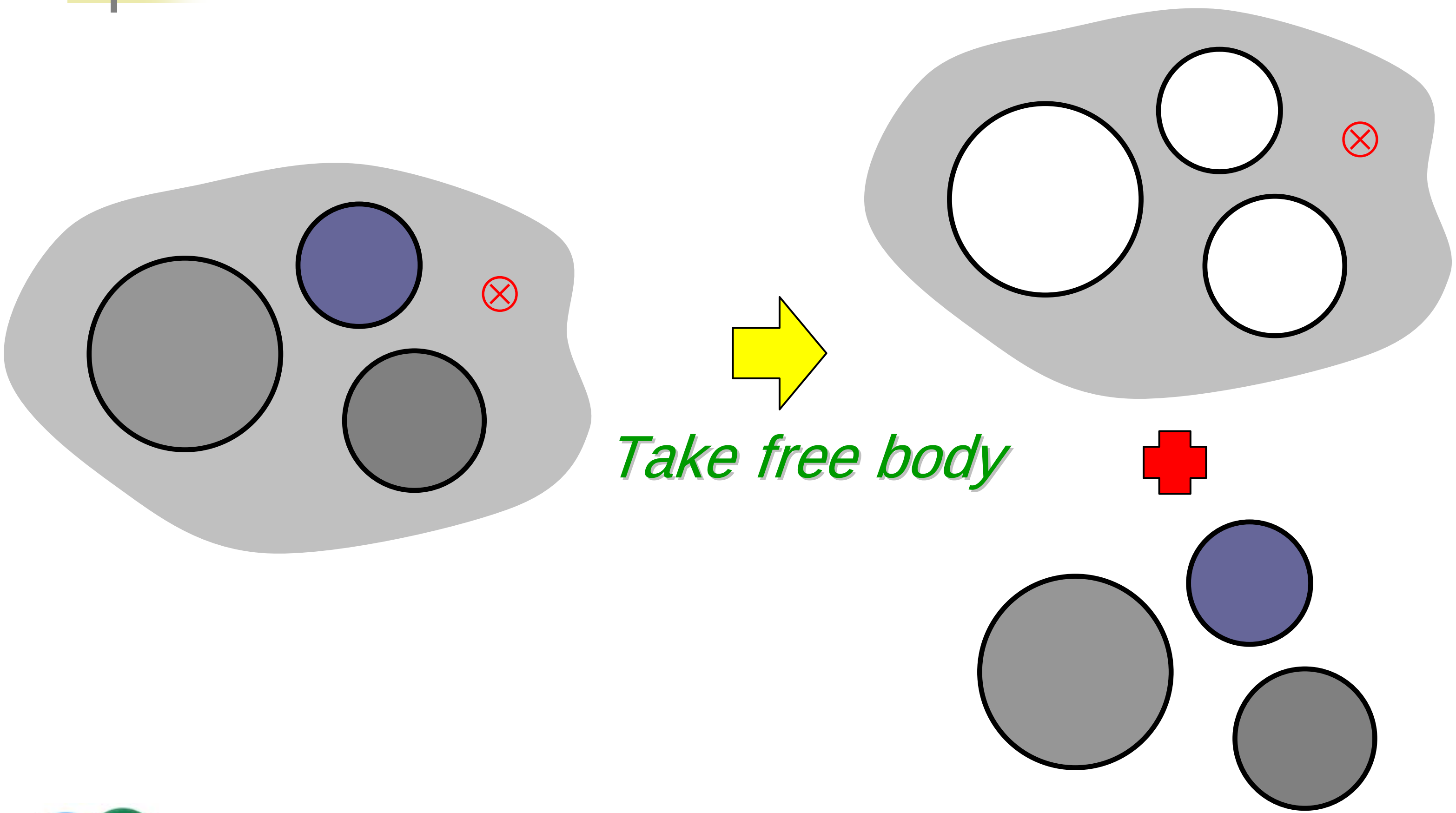
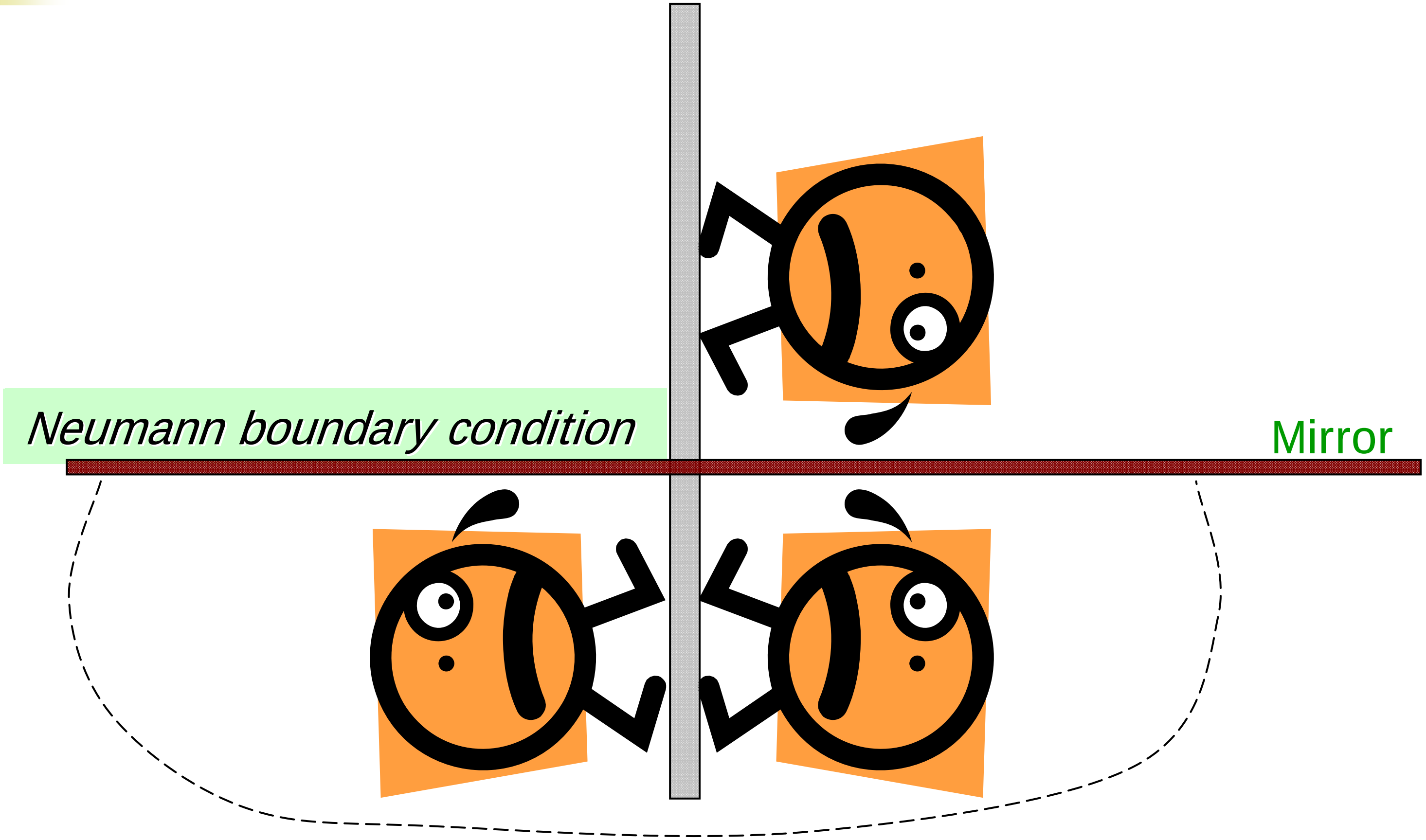
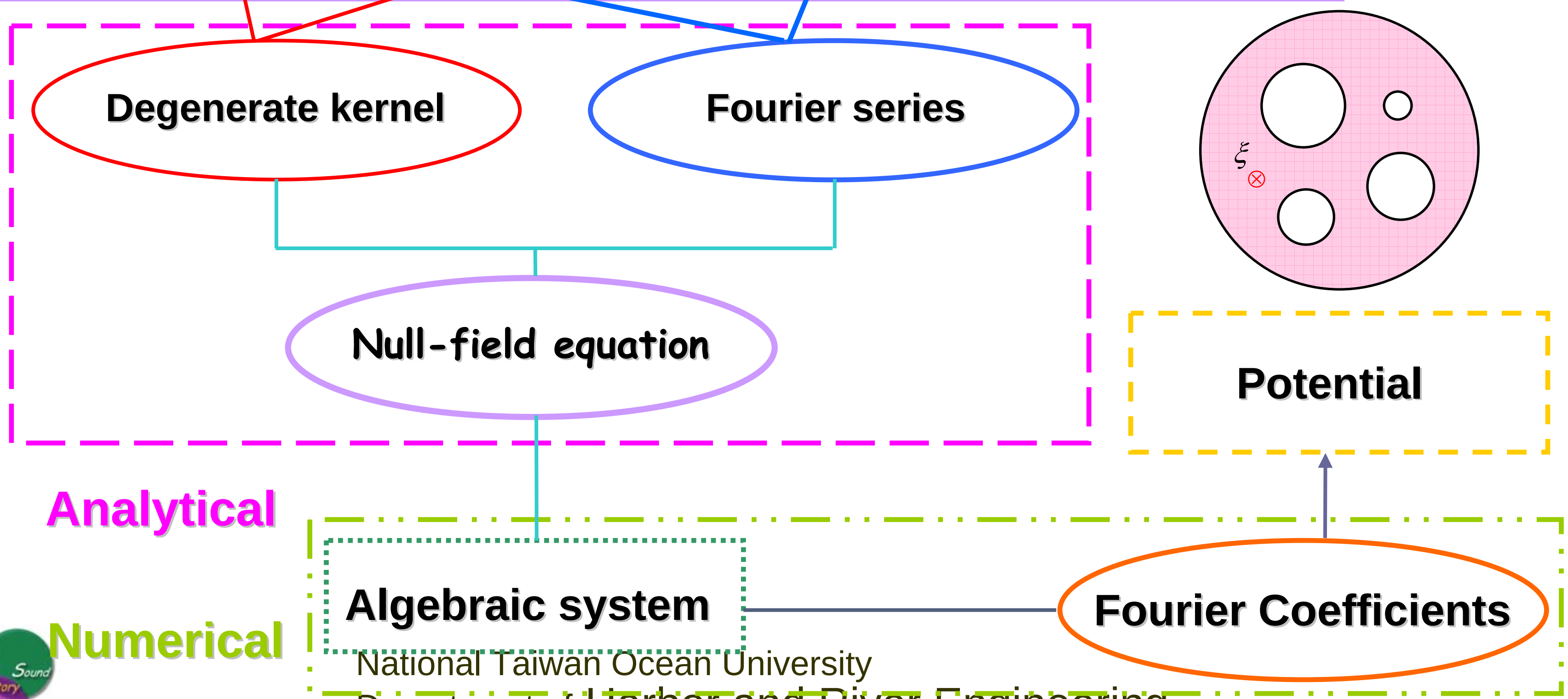


Image technique for solving half-plane problems



Flowchart of present method

$$0 = \int_B [T(s,x)G(s,\xi) - U(s,x)\frac{\partial G(s,\xi)}{\partial n_s}]dB(s) + U(\xi,x)$$



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  - Green's function for Helmholtz problems
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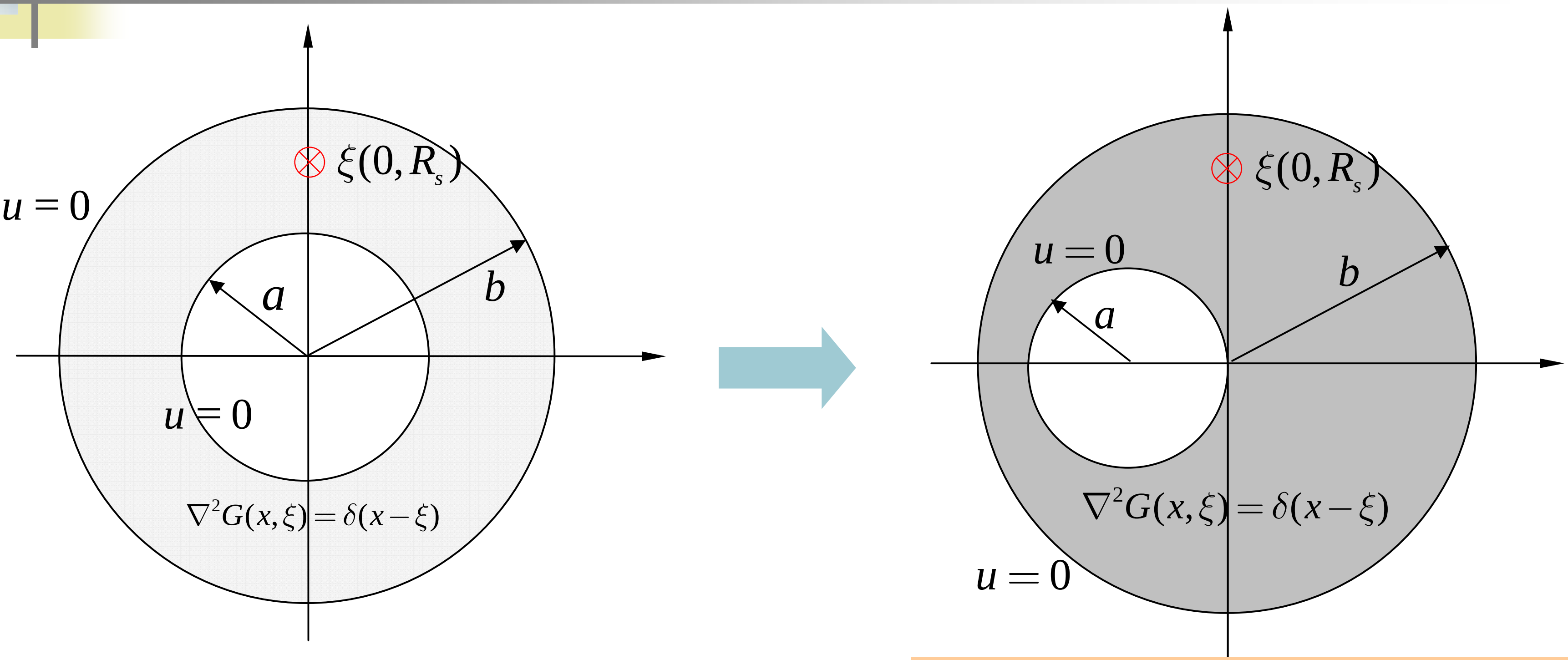
Numerical examples

- Laplace problems
  - Eccentric ring
  - A half-plane with an aperture
    (1) Dirichlet boundary condition
    (2) Robin boundary condition
  - A half-plane problem with a circular hole and a half-circular inclusion
- Helmholtz problems
  - An infinite matrix containing a circular inclusion with a concentrated force in the matrix or inclusion
  - Special cases and parameter study
  - An infinite matrix containing two circular inclusions with a concentrated force in the matrix

Present study for Laplace equation



Analytical Green's function → Semi-Analytical Green's function

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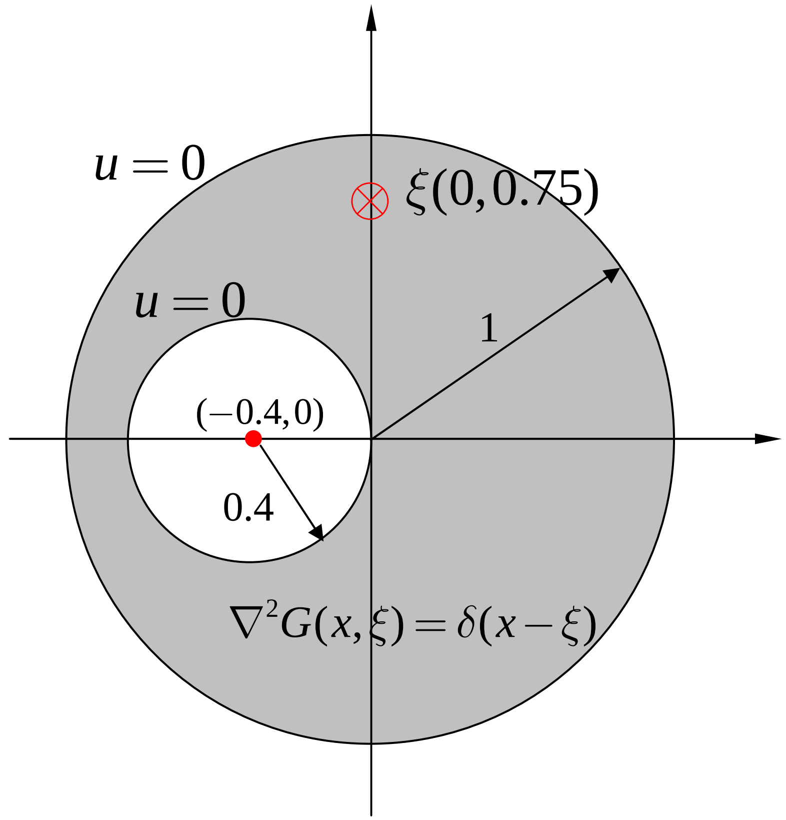
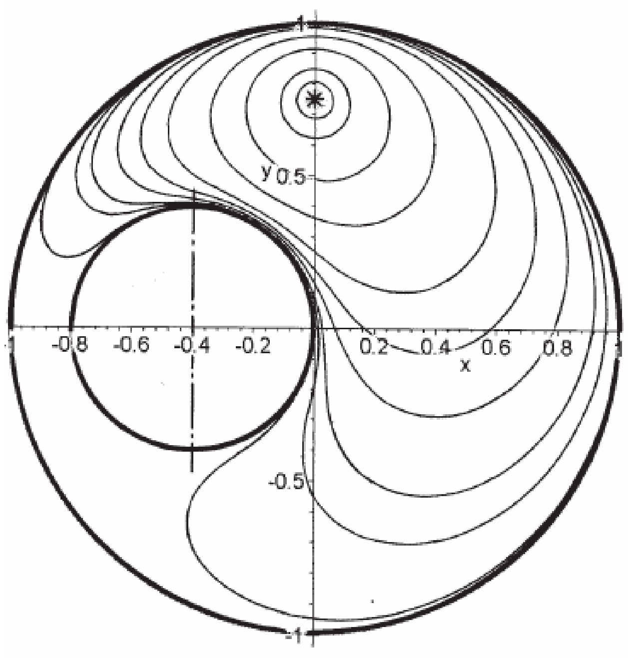
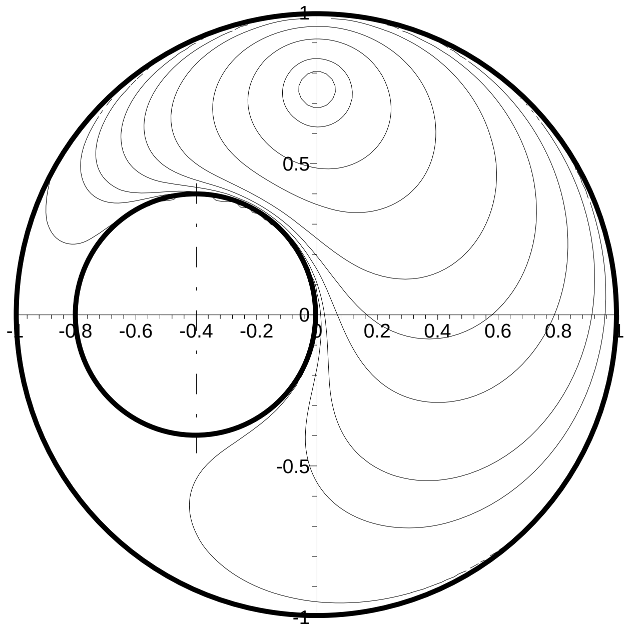
Eccentric ring



Eccentric ring



Potential contour using the Melnikov's method



Potential contour using the present method (M=50)

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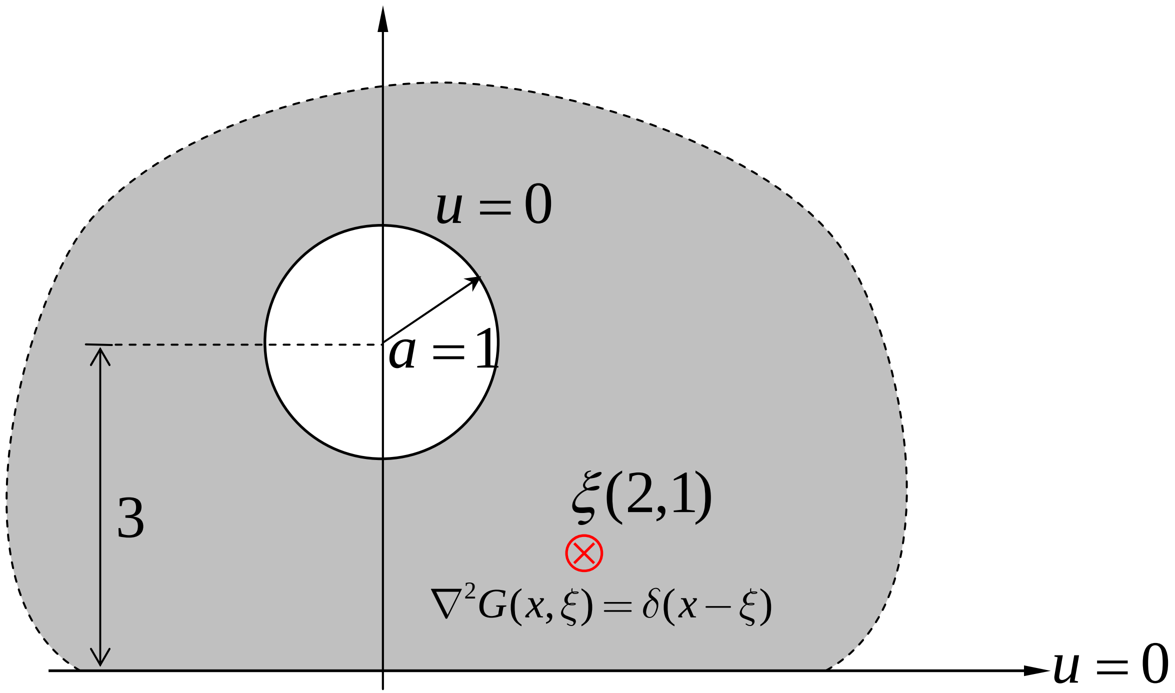
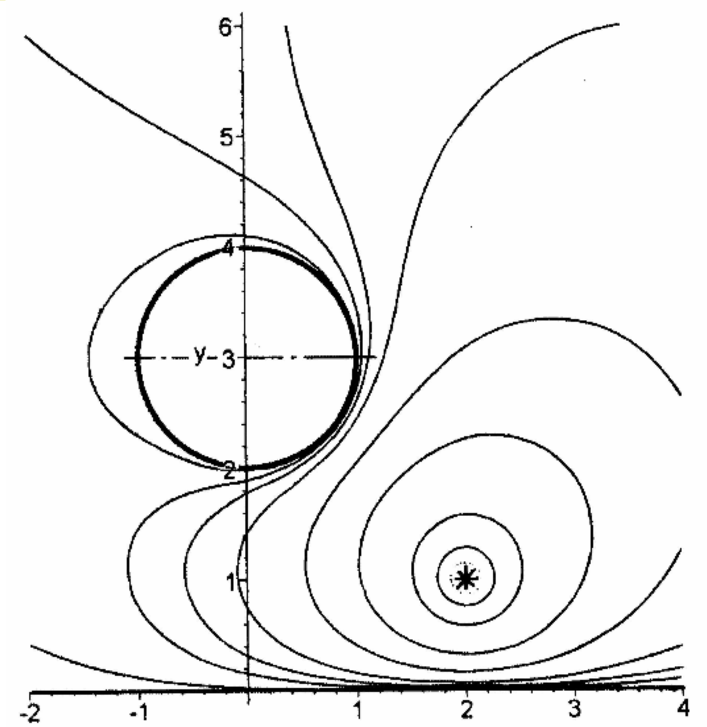
A half plane with an aperture subjected to Dirichlet boundary condition



Result of a half-plane problem with an aperture subjected to Dirichlet boundary condition



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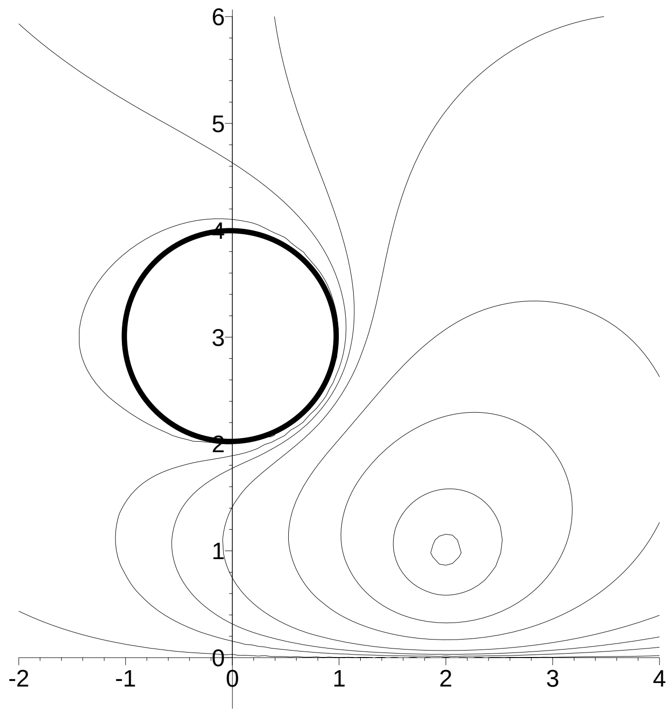
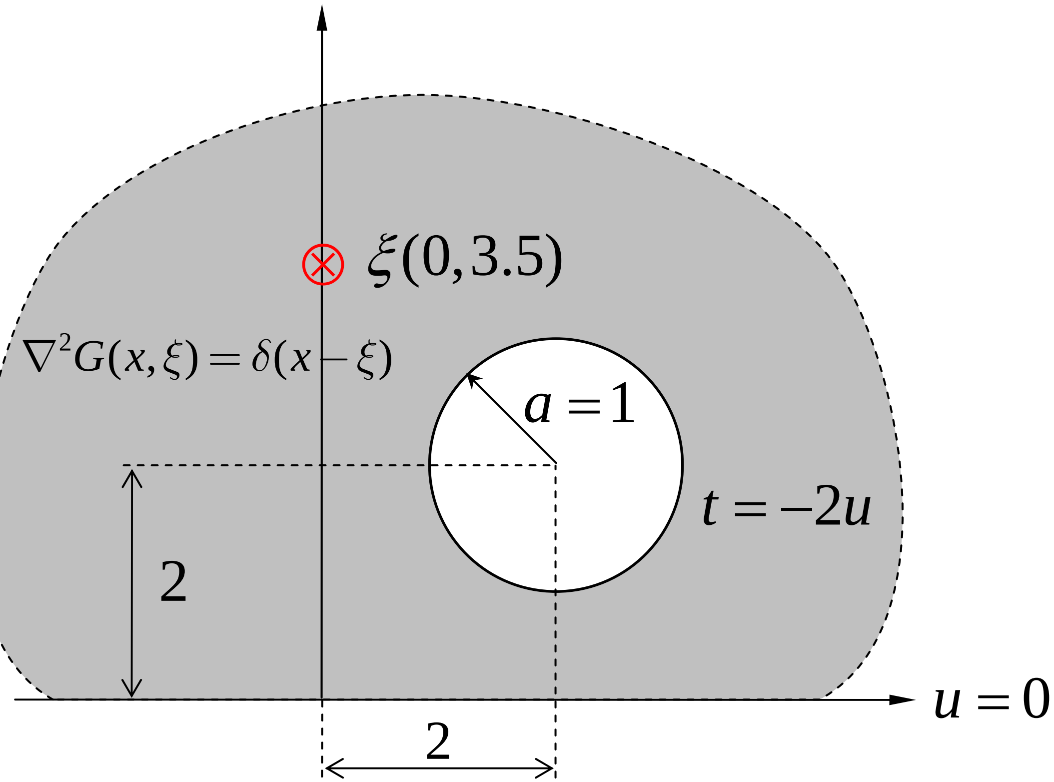
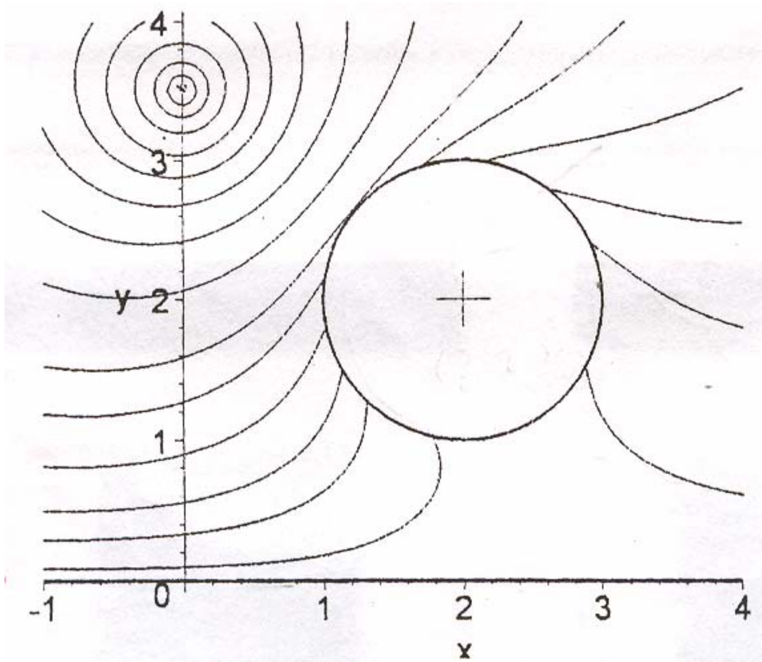


Potential contour using the present method (M=50)

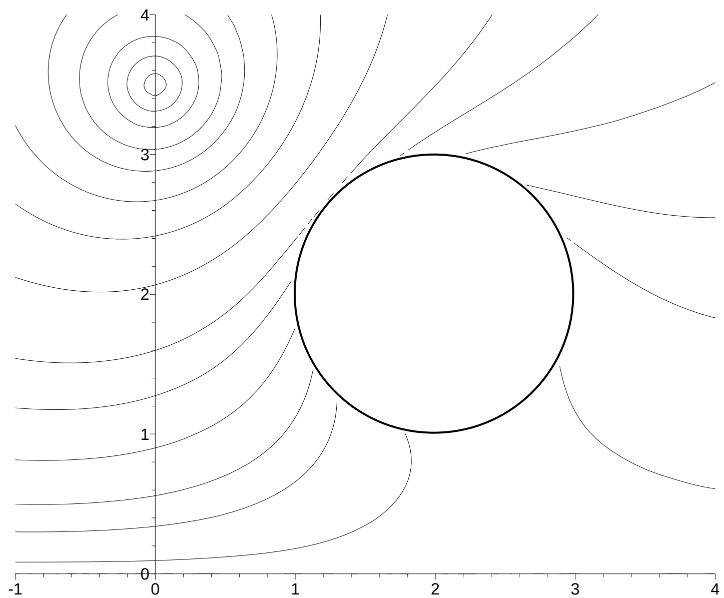
A half plane with an aperture subjected to Robin boundary condition



Result of a half-plane problem with an aperture subjected to Robin boundary condition



Potential contour using the Melnikov's method



Potential contour using the present method (M=50)

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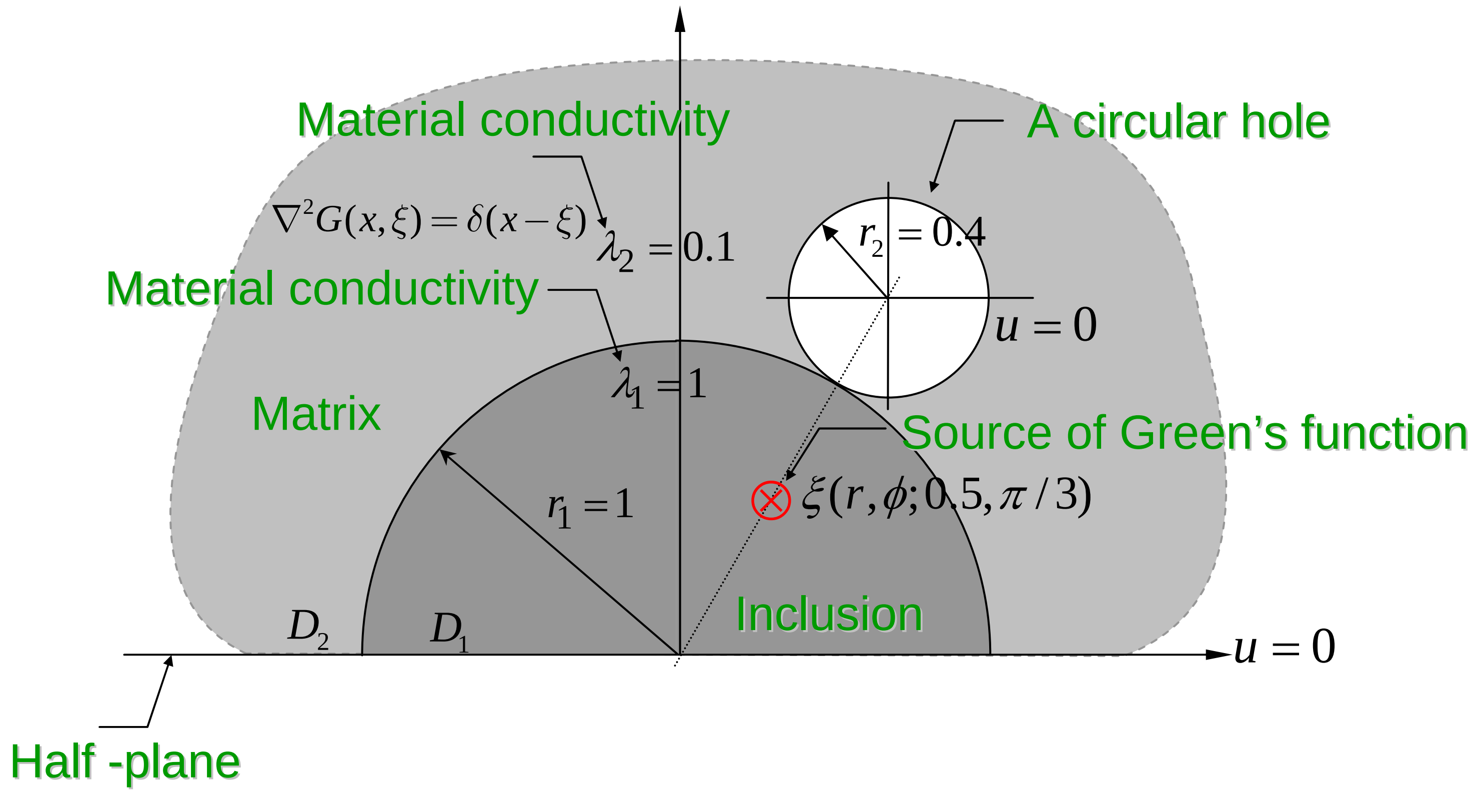
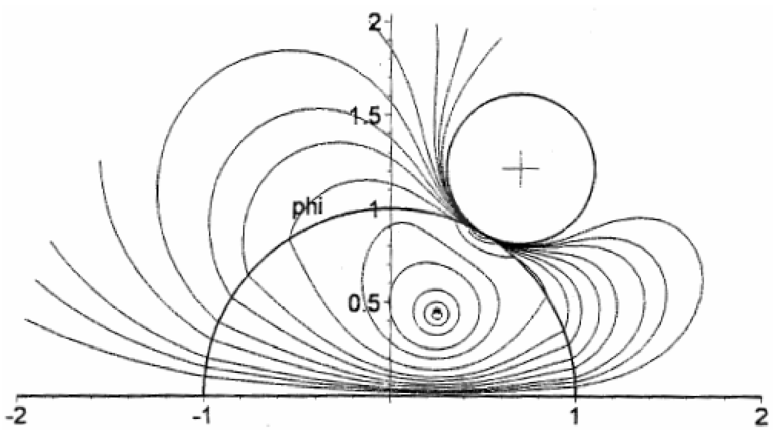
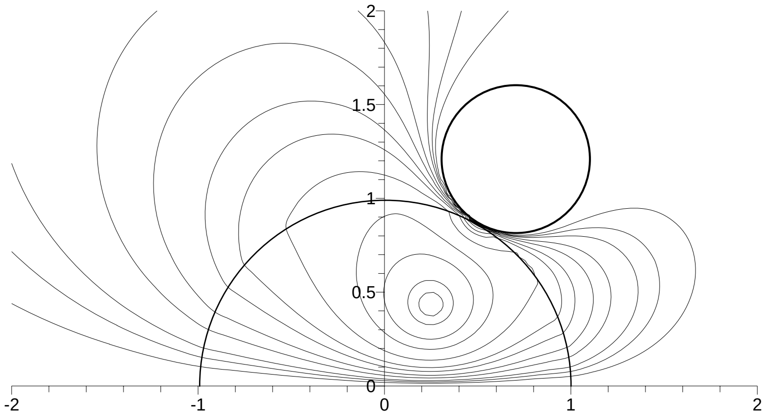
A half-plane problem with a circular hole and a half-circular inclusion



Result of a half-plane problem with a circular hole and a half-circular inclusion



Contour plot by using the Melikov's approach (2006)



Contour plot by using the null-field integral equation approach

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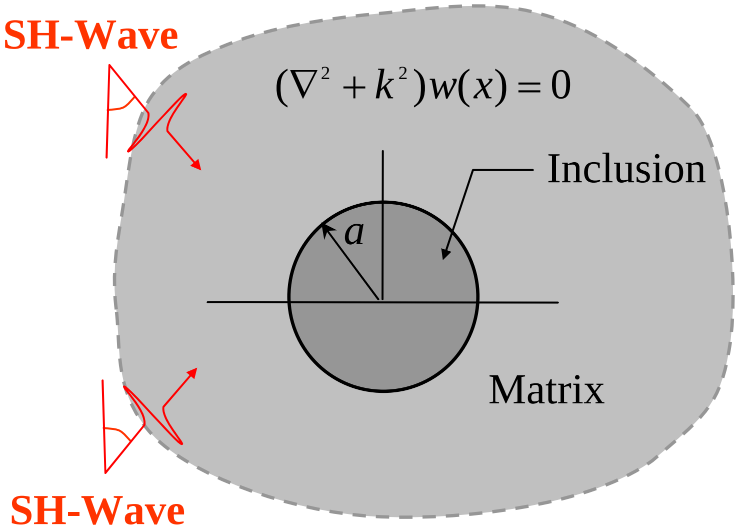
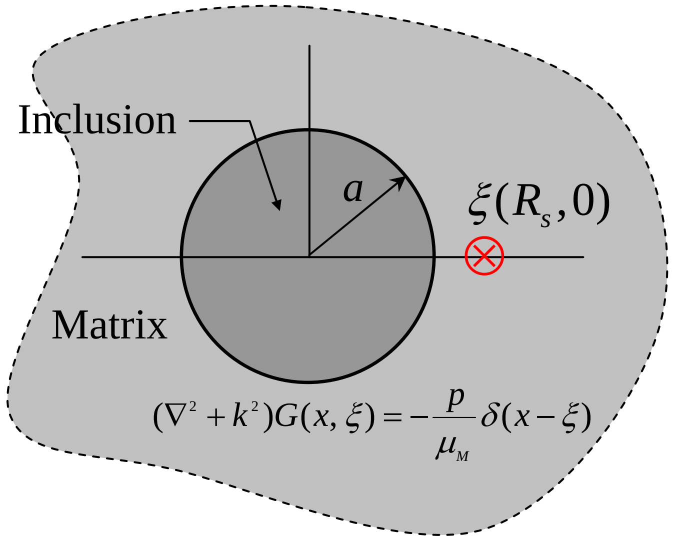
Present study for Helmholtz equation

Perfect interface boundary → Imperfect interface boundary





SH-wave problem (Chen P. Y.)

Green's function problem (Ke J. N.)

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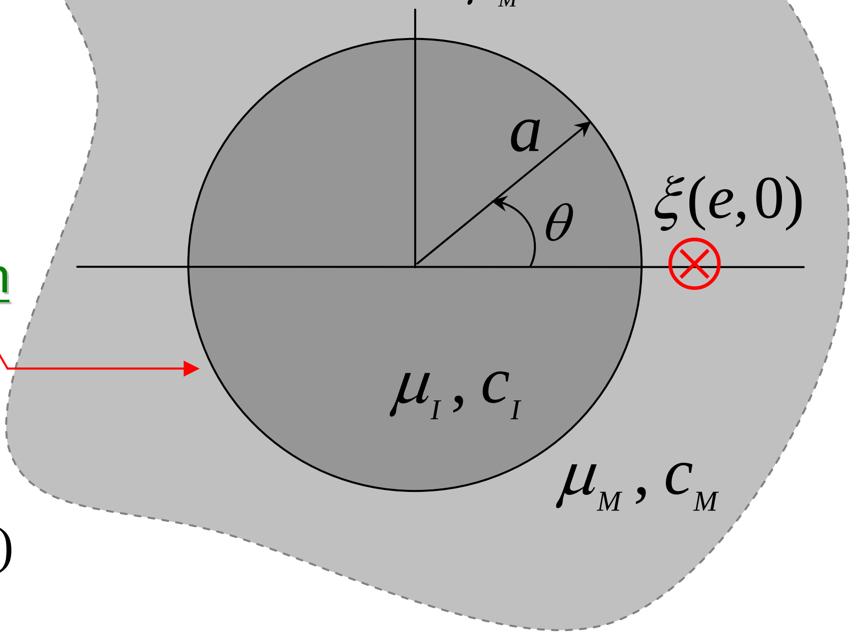
An infinite matrix containing a circular inclusion with a concentrated force at ξ in the matrix

$$(\nabla^2 + k^2)G(x,\xi) = -\frac{p}{\mu_M}\delta(x-\xi)$$



Interface condition

$$t^M = -\frac{\mu_I}{\mu_M}t^I$$

$$t^I = \frac{\beta}{\mu_I}(u^M - u^I)$$

$$e = 1.1a$$

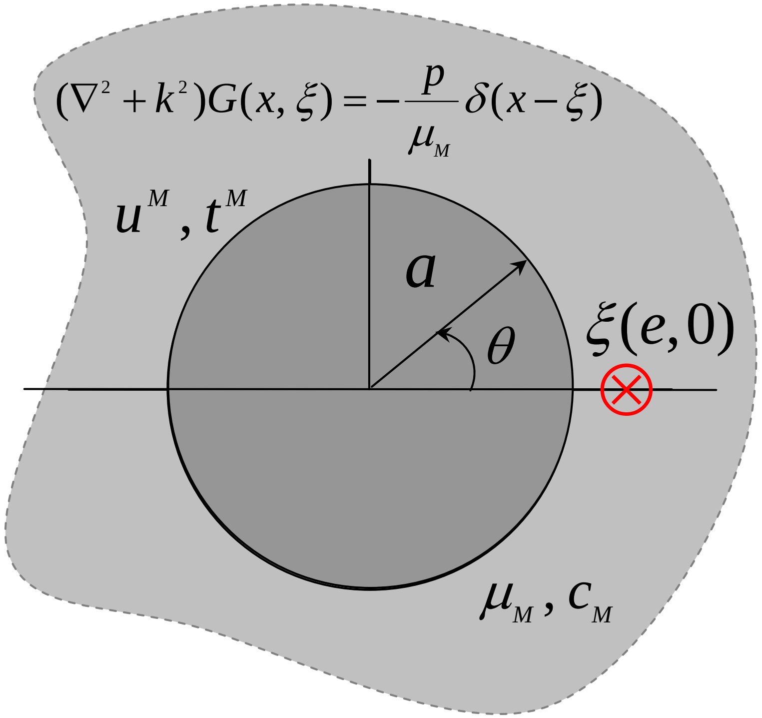
$$\mu_I = 4\mu_M \qquad c_I = 2c_M$$

μ is the shear modulus

c is the wave speed

β is the imperfect interface parameter

Take free body



u^I, t^I

a

θ

μ_I, c_I

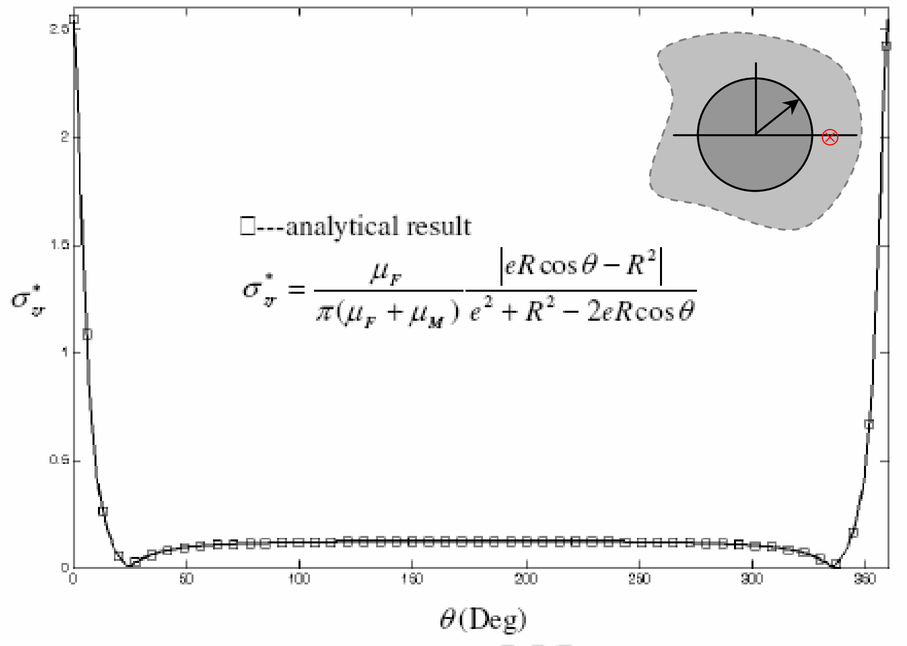
$$t^M = -\frac{\mu_I}{\mu_M} t^I$$

$$t^I = \frac{\beta}{\mu_I}(u^M - u^I)$$

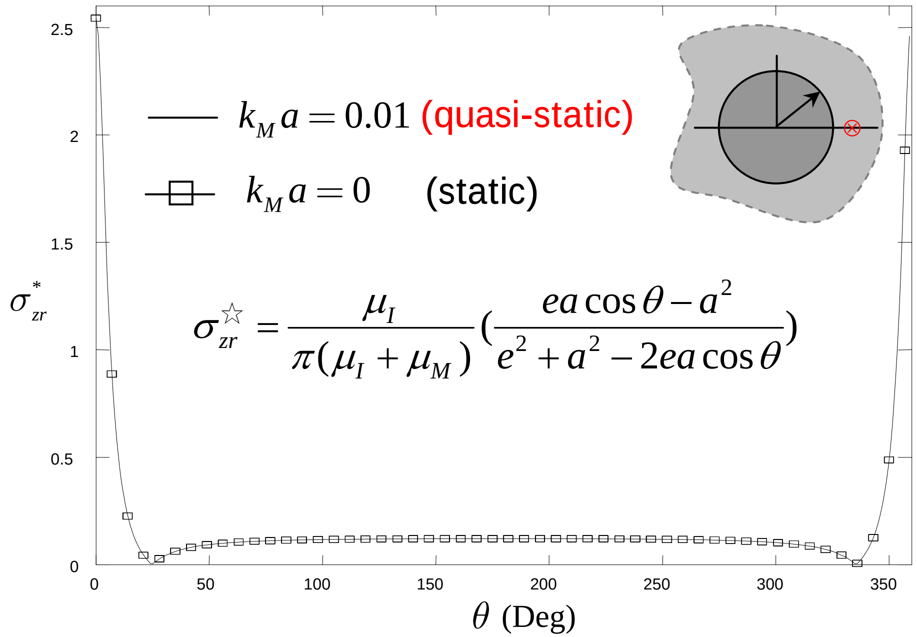
Distribution of σ^*_{zr} for the quasi-static $(k_M a = 0.01)$ solution along the circular boundary

$$\sigma^*_{zr} = a\left|\sigma^I_{zr}\right| / p = a\left|\sigma^M_{zr}\right| / p$$

$$\sigma^{\star}_{zr} = a\sigma^I_{zr} / p = a\sigma^M_{zr} / p$$



Wang and Sudak's solution

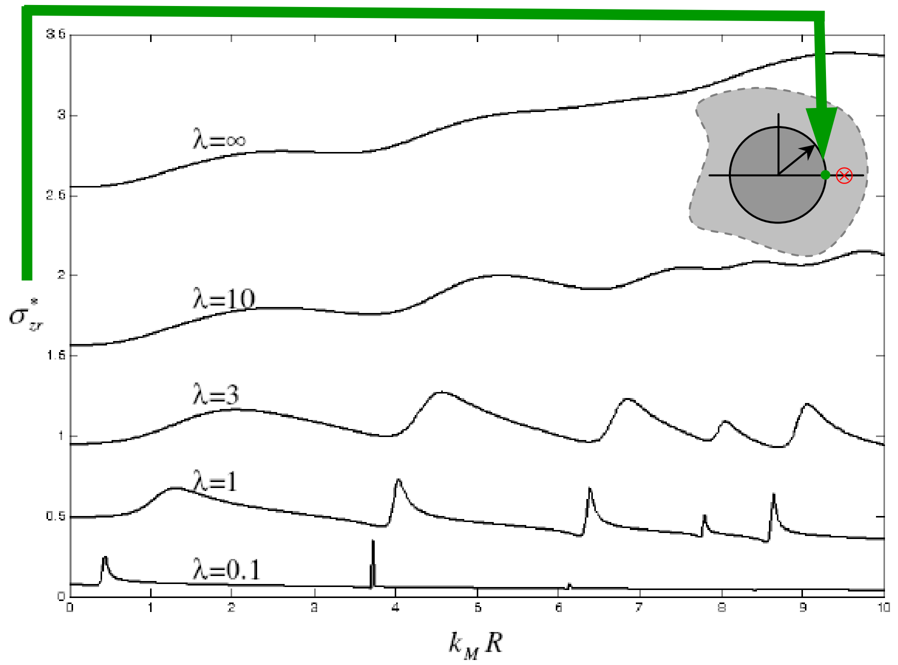


The present solution

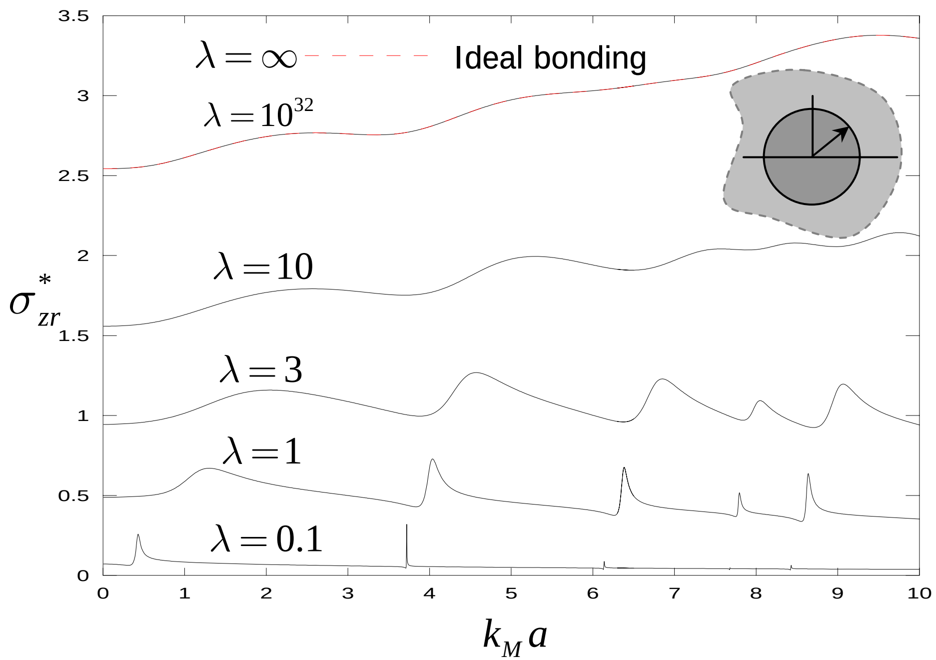
Parameter study of $\lambda = a\beta / \mu_M$ for the stress response

$$\sigma^*_{zr} = a\left|\sigma^I_{zr}\right| / p = a\left|\sigma^M_{zr}\right| / p$$

Bonding behavior



Wang and Sudak's solution

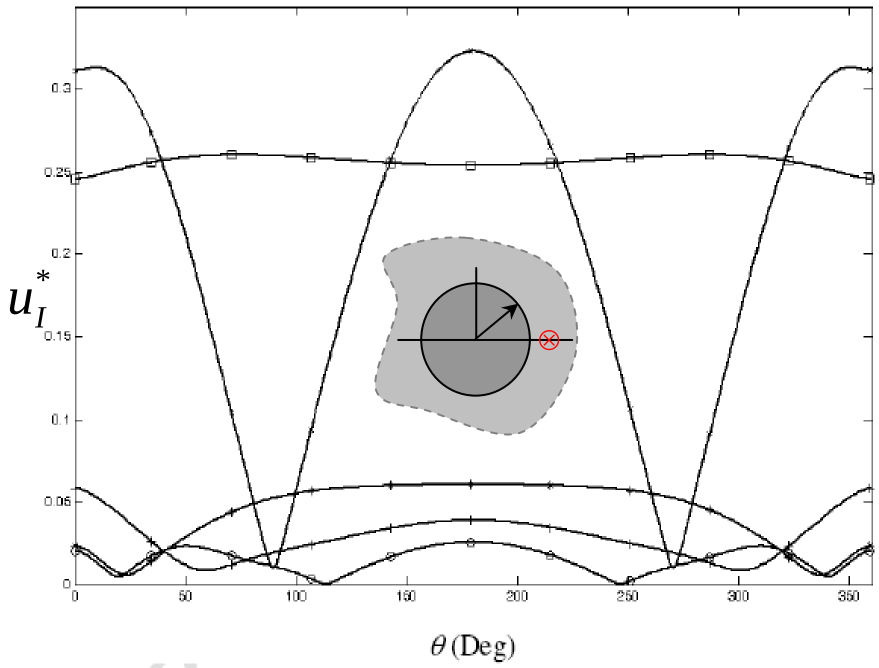


The present solution

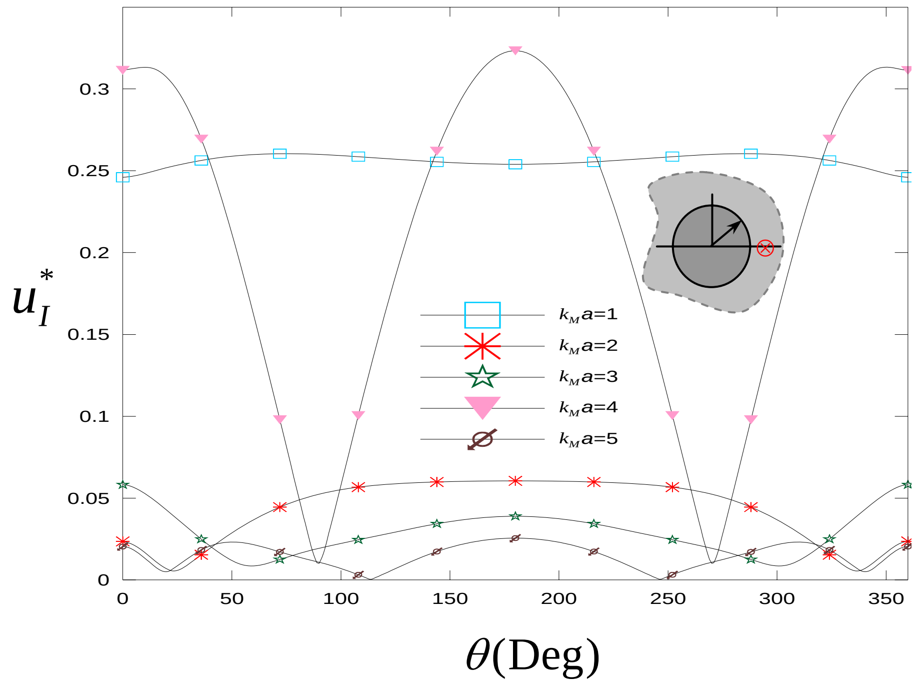
The distribution of displacement u_I^* along the circular boundary for the case $(k_M a = 1,2,3,4,5)$

$$u_I^* = \mu_M \left| u_I \right| / p$$

Dynamic effect



Wang and Sudak's solution



The present solution

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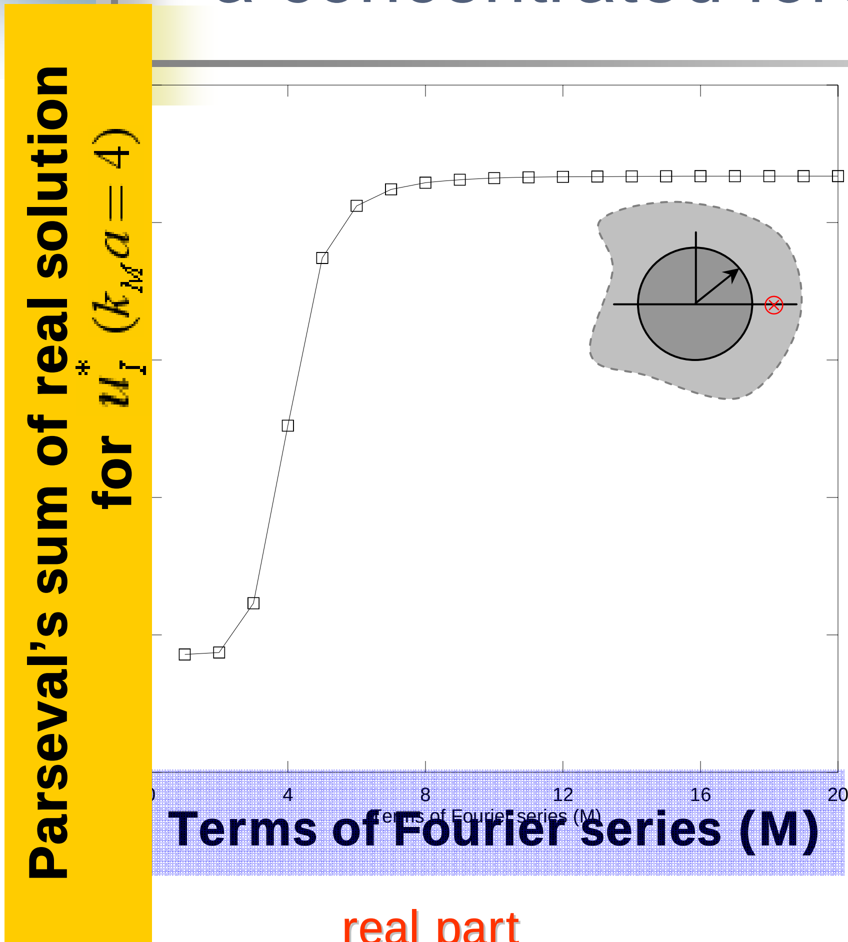
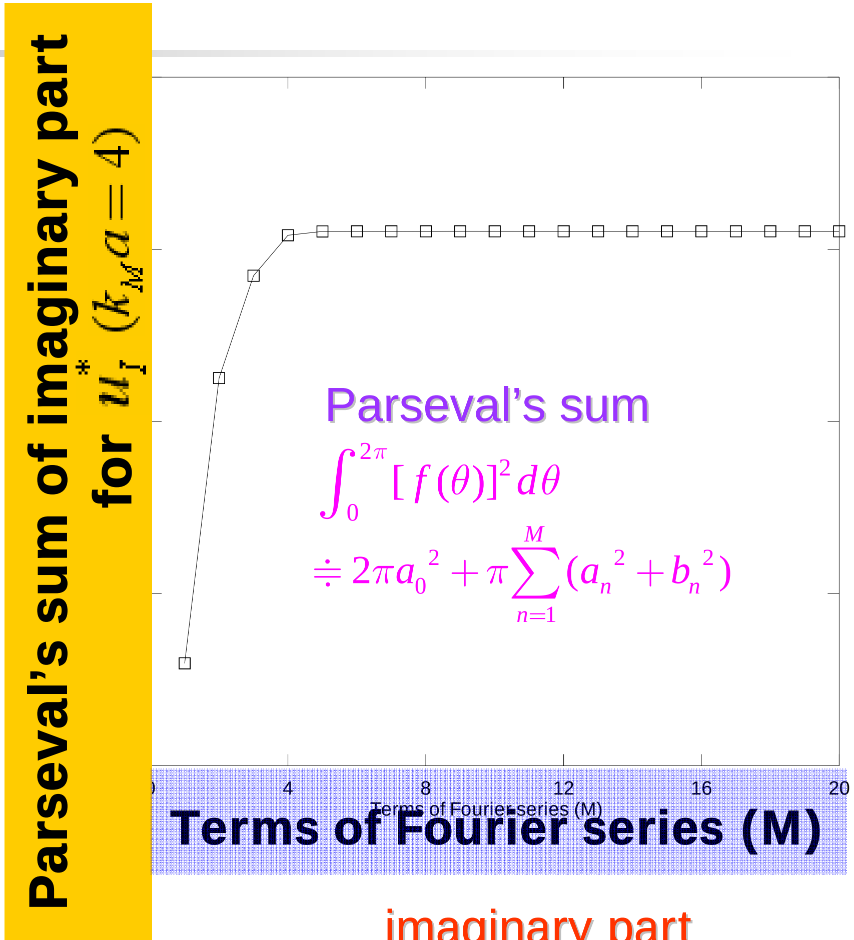
Test of convergence for the Fourier series with a concentrated force in the inclusion

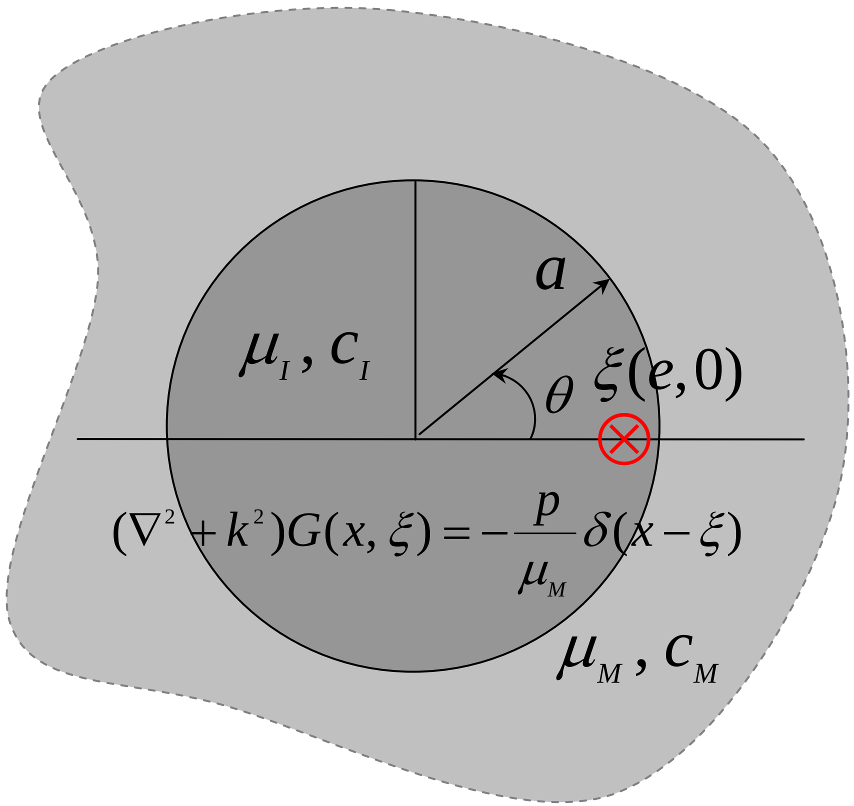


real part



imaginary part

An infinite matrix containing a circular inclusion with a concentrated force at ξ in the inclusion



$$t^M = -\frac{\mu_I}{\mu_M} t^I$$

$$t^I = \frac{\beta}{\mu_I}(u^M - u^I)$$

$$e = 0.9a$$

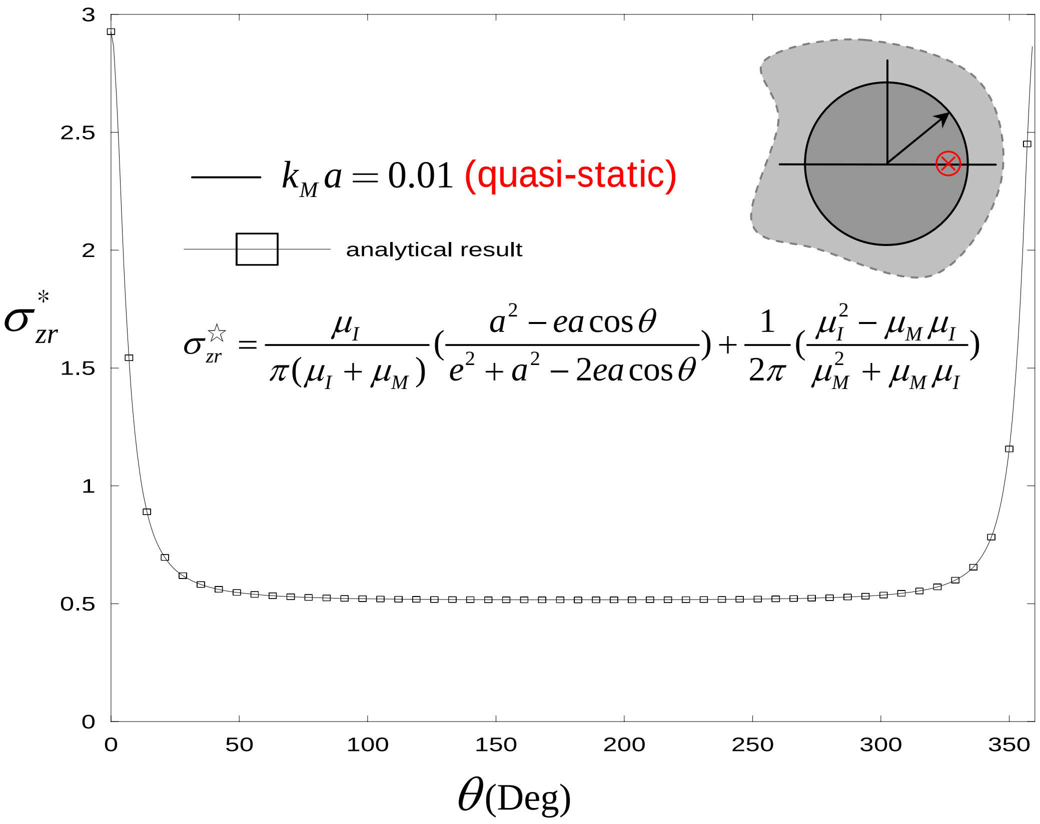
$$\mu_I = 4\mu_M, \quad c_I = 2c_M$$

μ is the shear modulus

c is the wave speed

β is the imperfect interface parameter

Distribution of σ_{zr}^{*} for the quasi-static $(k_M a = 0.01)$ solution along the circular boundary $(e = 0.9a)$

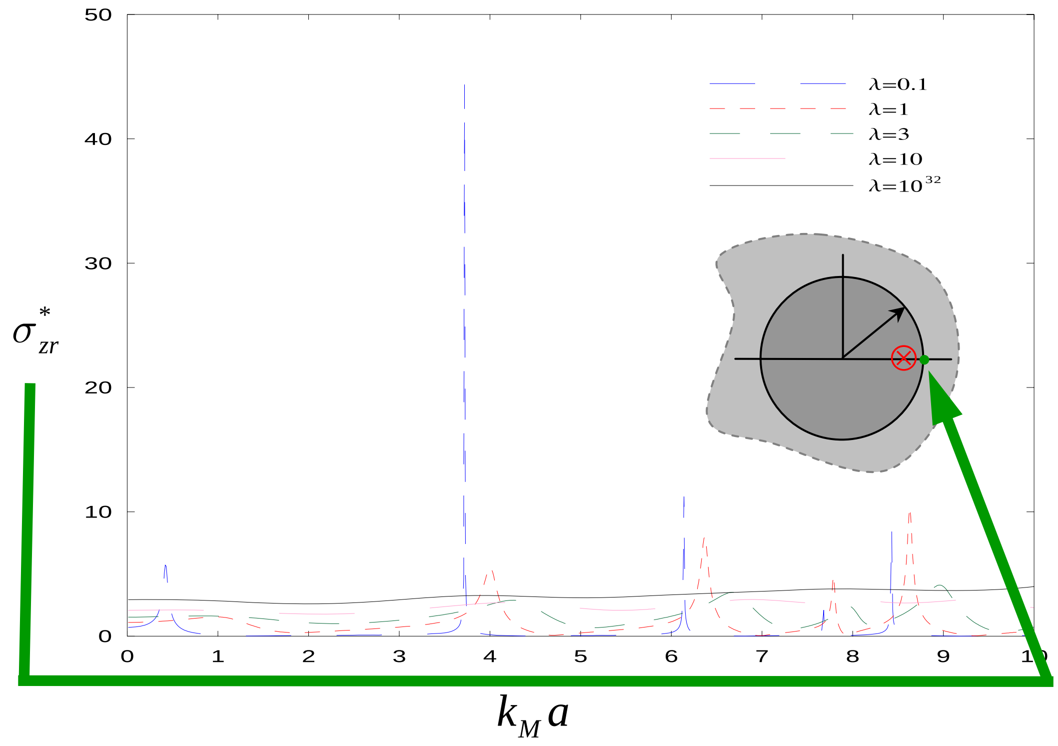


$$\sigma_{zr}^{*} = a\left|\sigma_{zr}^{I}\right| / p = a\left|\sigma_{zr}^{M}\right| / p$$

$$\sigma_{zr}^{\star} = a\sigma_{zr}^{I} / p = a\sigma_{zr}^{M} / p$$

Parameter study of $\lambda = a\beta / \mu_M$ for the stress response $(e = 0.9a)$

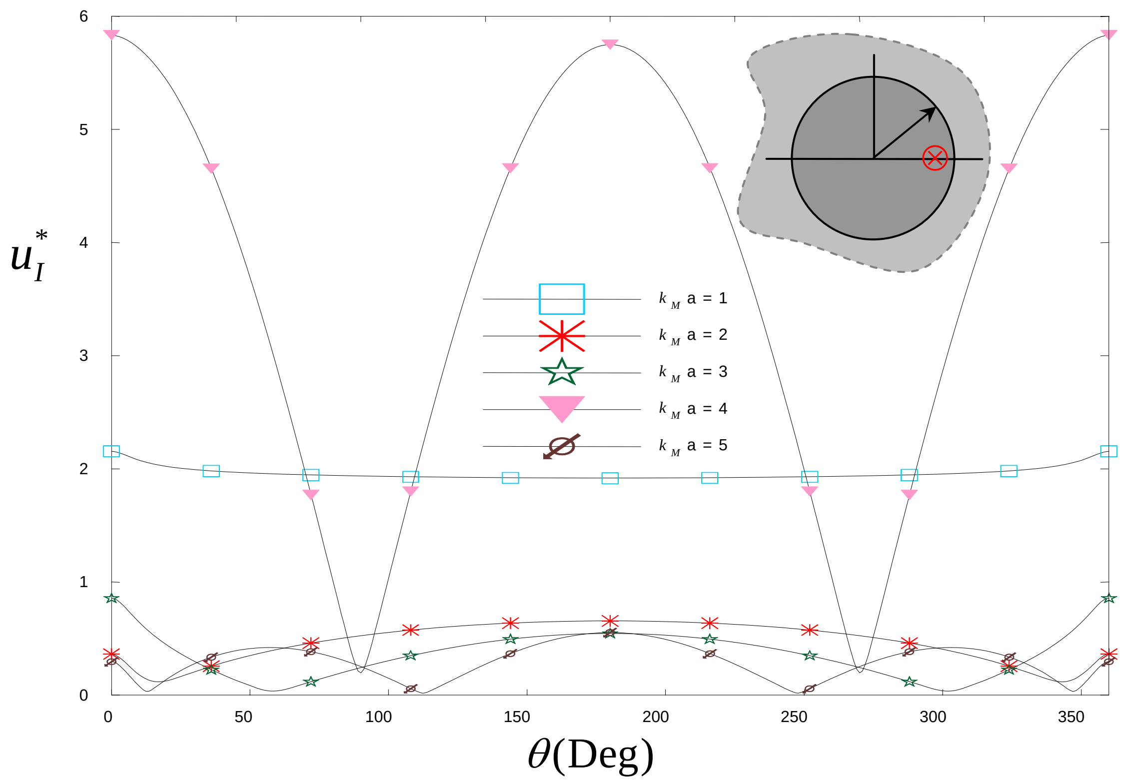
$$\sigma^*_{zr} = a\left|\sigma^I_{zr}\right| / p = a\left|\sigma^M_{zr}\right| / p$$



Bonding behavior

The distribution of displacement u_I^* along the circular boundary for the case of $\lambda = 1 \ (e = 0.9a)$

$$u_I^* = \mu_M \left| u_I \right| / p$$

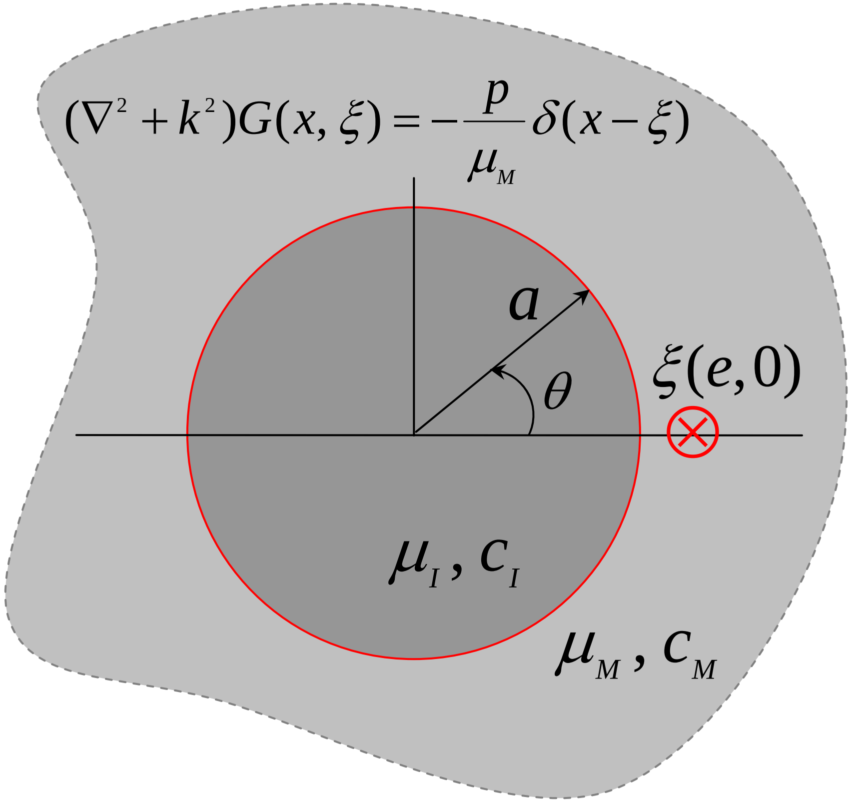


Dynamic effect

Numerical examples

- Laplace problems
  - Eccentric ring
  - A half-plane with an aperture
    (1) Dirichlet boundary condition
    (2) Robin boundary condition
  - A half-plane problem with a circular hole and a half-ircular inclusion
- Helmholtz problems
  - An infinite matrix containing a circular inclusion with a concentrated force in the matrix or inclusion
  - Special cases and parameter study
  - An infinite matrix containing two circular inclusions with a concentrated force in the matrix

Special case of an ideally bonded case $(\beta = \infty)$



$$t^M = -\frac{\mu_I}{\mu_M} t^I$$

$$t^I = \frac{\beta}{\mu_I}(u^M - u^I)$$

Imperfect bonding

$(\beta \to \infty)$

Ideal bonding

$$t^M = -\frac{\mu_I}{\mu_M} t^I$$

$$u^M = u^I$$

$\mu_I = 4\mu_M$

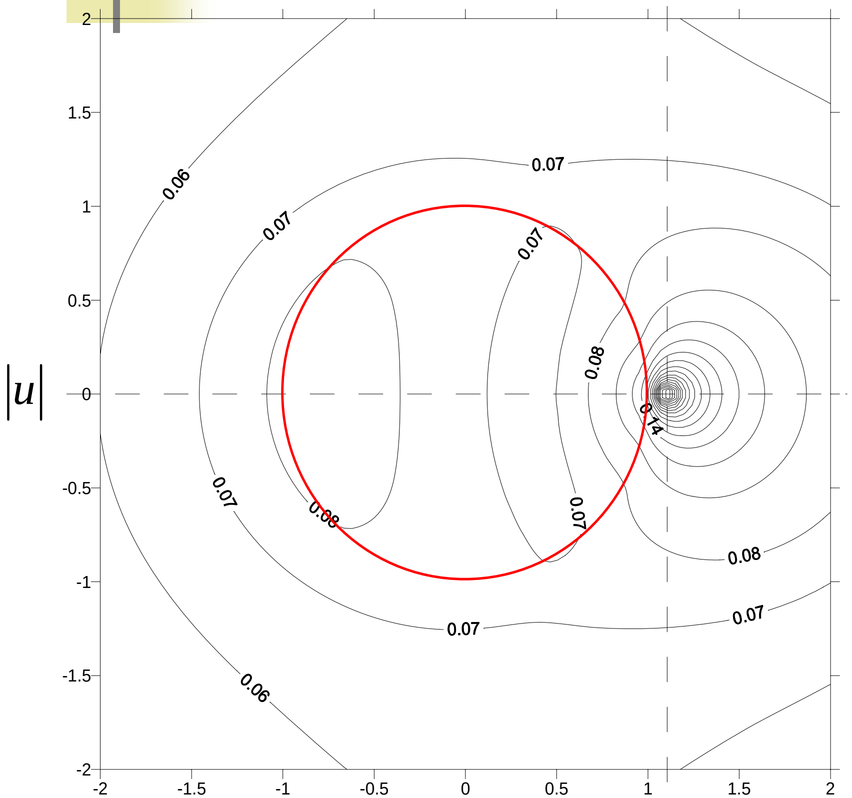
$c_I = 2c_M$

μ is the shear modulus

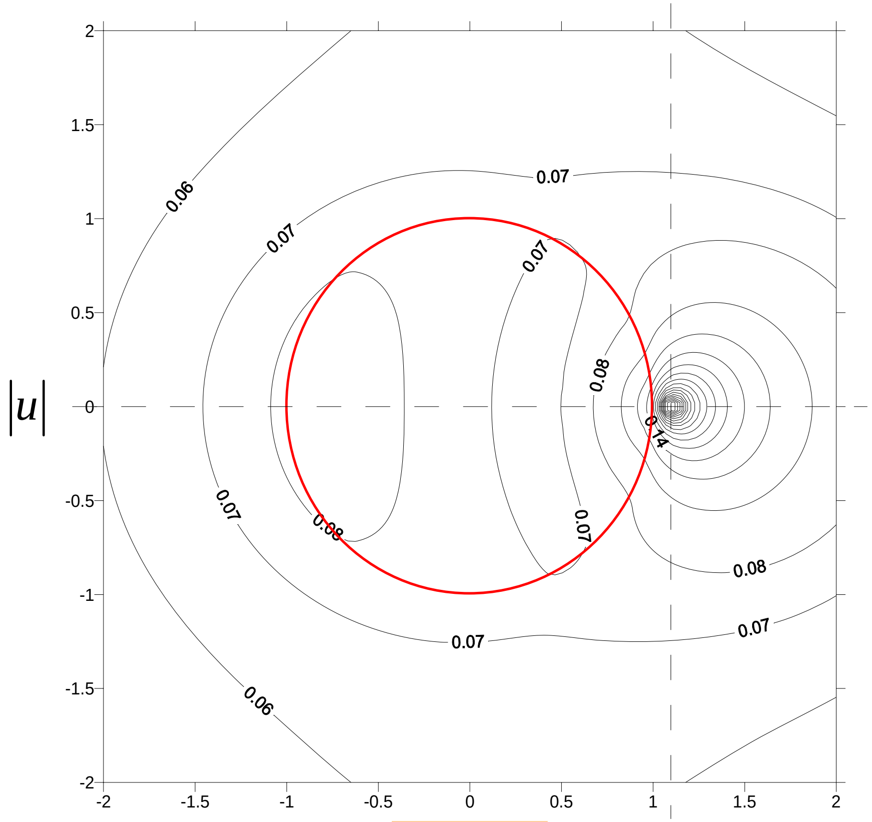
c is the wave speed

β is the imperfect interface parameter

The absolute amplitude of displacement by the present method



$\beta = \infty$ (ideal bonding interface)



$\beta = 10^{32}$

Special case of cavity $(\beta = 0)$

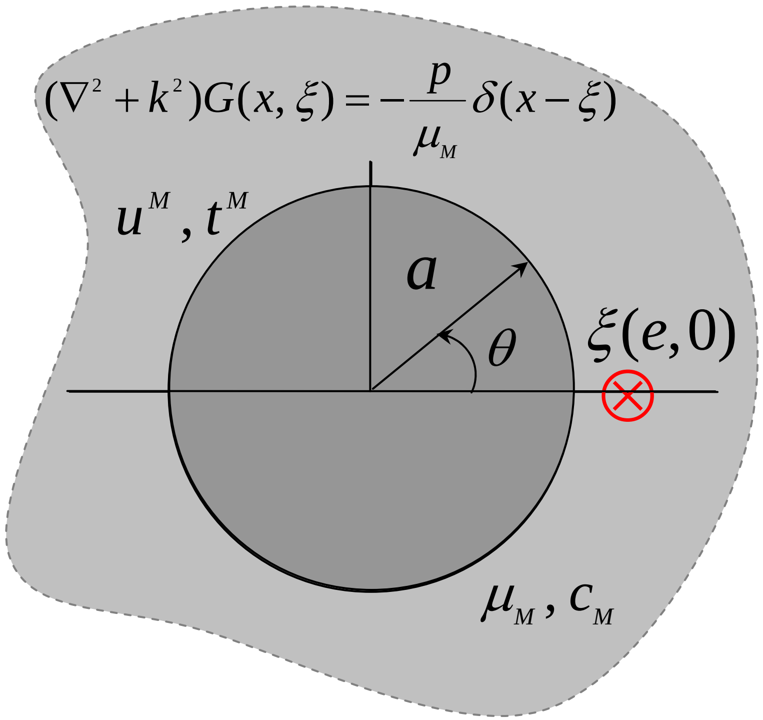
$\mu_I = 4\mu_M$

$c_I = 2c_M$

μ is shear modulus

c is wave speed

β is the imperfect interface parameter



$$t^M = -\frac{\mu_I}{\mu_M} t^I$$

$$t^I = \frac{\beta}{\mu_I}(u^M - u^I)$$

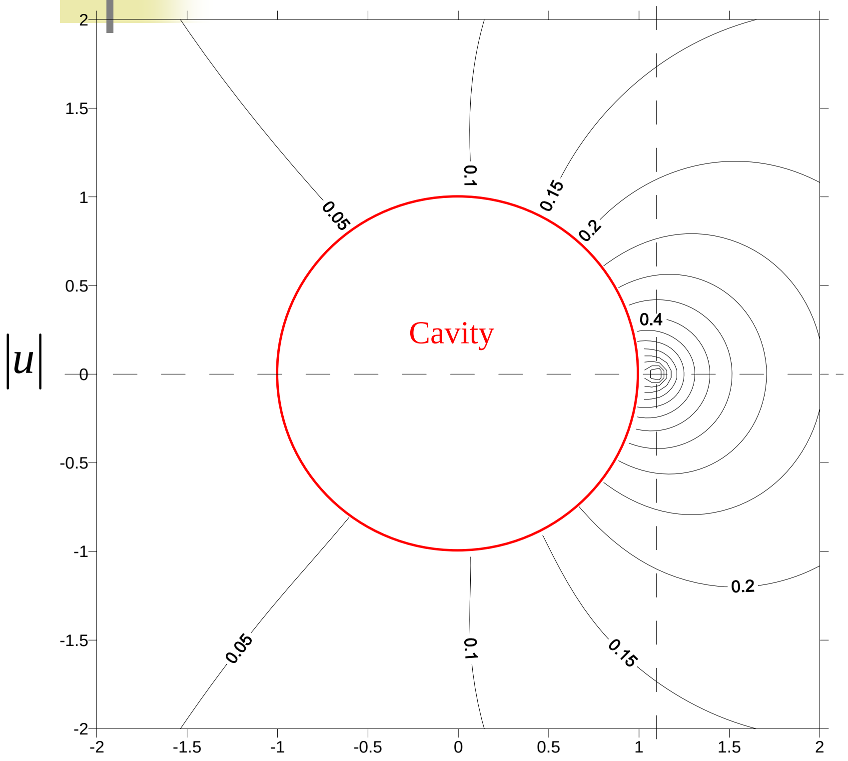
Imperfect bonding

Cavity

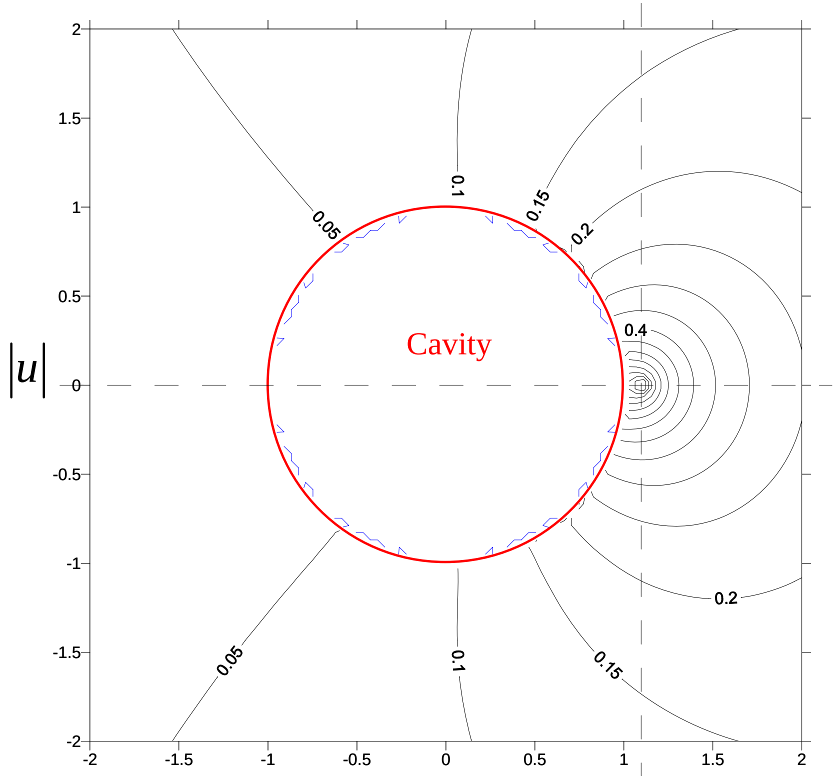
$$t^M = 0$$

$$u^M = ?$$

The absolute amplitude of displacement by the present method

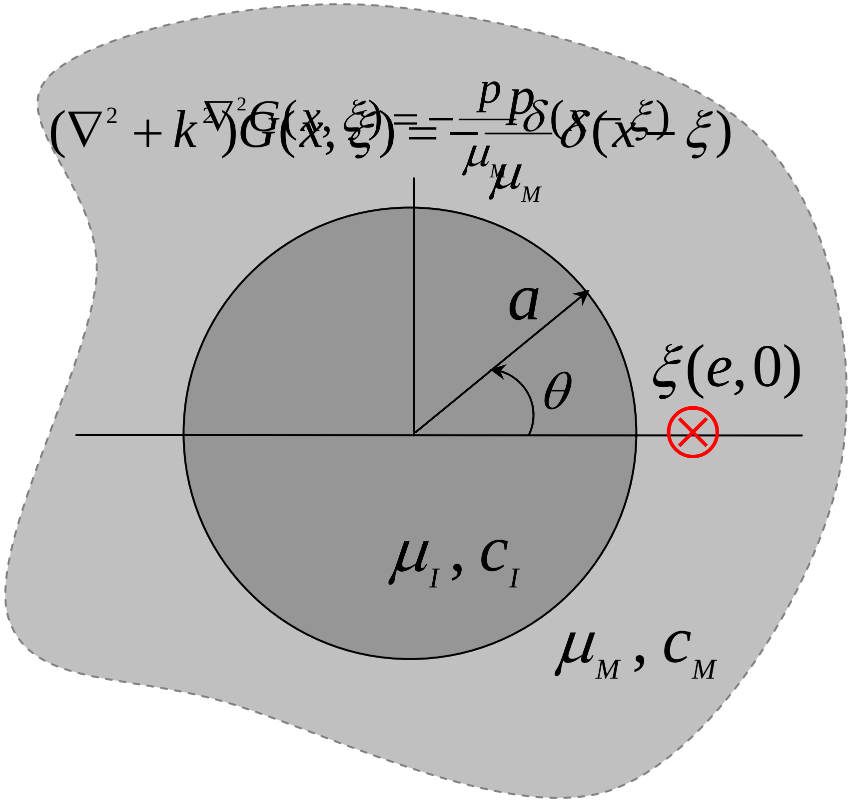


$\beta = 0$ (cavity)



$\beta = 10^{-32}$

Parameter study $(k=0)$ for ideal bonding



$$t^M = -\frac{\mu_I}{\mu_M} t^I$$

$$u^M = u^I$$

Fundamental solution

$$U(s,x) = -i\pi H_0^{(1)}(kr)/2$$

$k=0$

$$U(s,x) = \ln|x-s| = \ln r$$

$\mu_I = 4\mu_M$

μ is the shear modulus

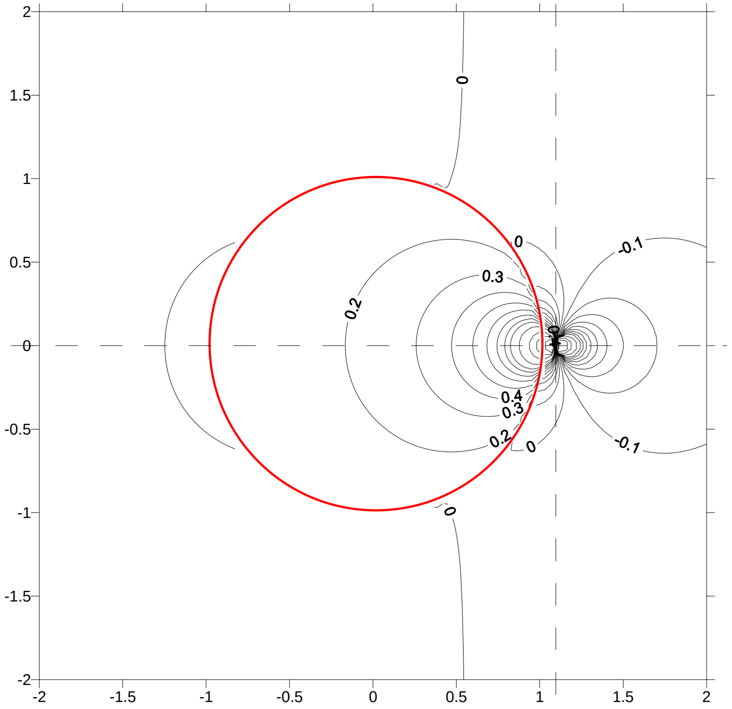
β is the imperfect interface parameter

Stress contours of σ_{zx} and σ_{zy} for the static solutions (a concentrated force in the matrix)

$$\sigma_{zx} = \sigma_{zr}\cos\phi - \sigma_{z\theta}\sin\phi$$

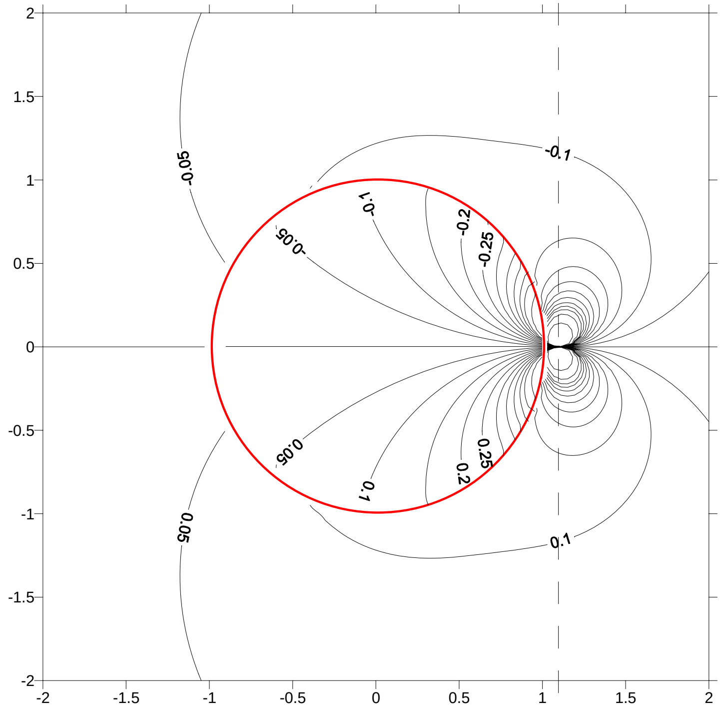
$$\sigma_{zy} = \sigma_{zr}\sin\phi + \sigma_{z\theta}\cos\phi$$

σ_{zx}



$k = 0,\ ideal\ \ bonding$

σ_{zy}



$k = 0,\ ideal\ \ bonding$

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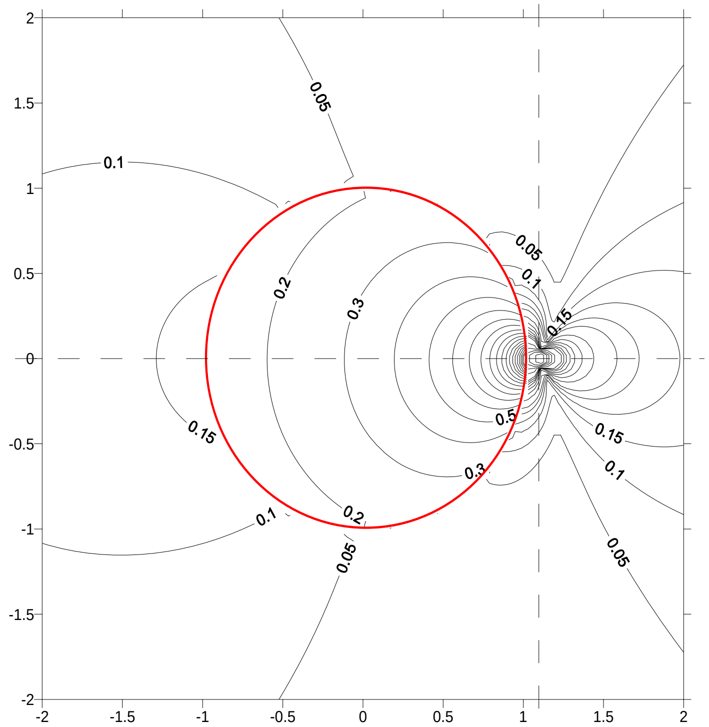
Stress contours of σ_{zx} and σ_{zy} for the dynamic solutions (a concentrated force in the matrix)

$$\sigma_{zx} = \sigma_{zr}\cos\phi - \sigma_{z\theta}\sin\phi$$

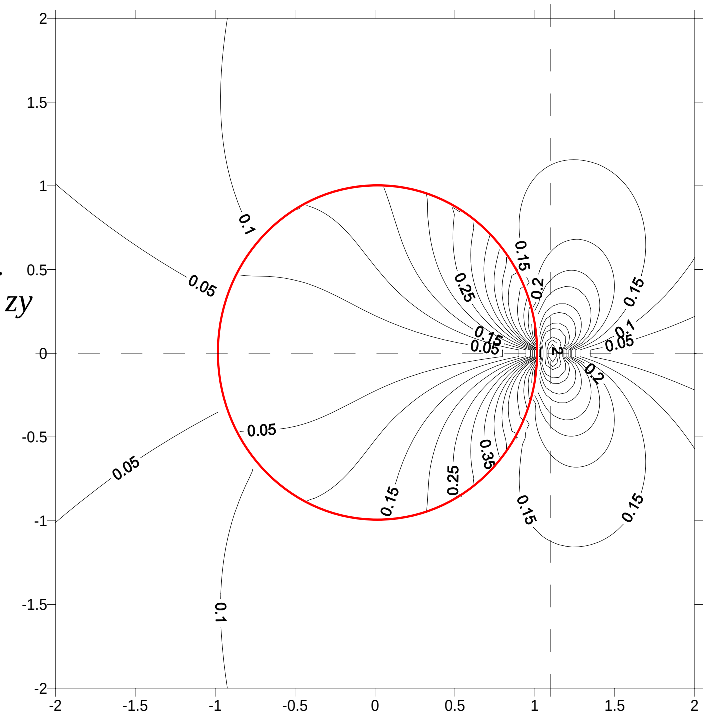
σ_{zx}



$k_I = 1,\ k_M = 2,\ \beta = 10^{32}$ (*ideal bonding*)

$$\sigma_{zy} = \sigma_{zr}\sin\phi + \sigma_{z\theta}\cos\phi$$

σ_{zy}

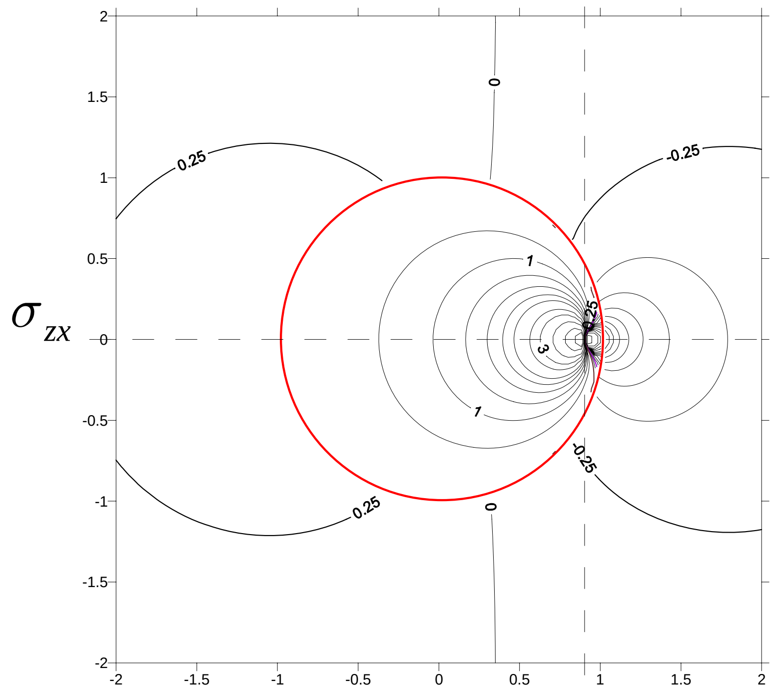


$k_I = 1,\ k_M = 2,\ \beta = 10^{32}$ (*ideal bonding*)

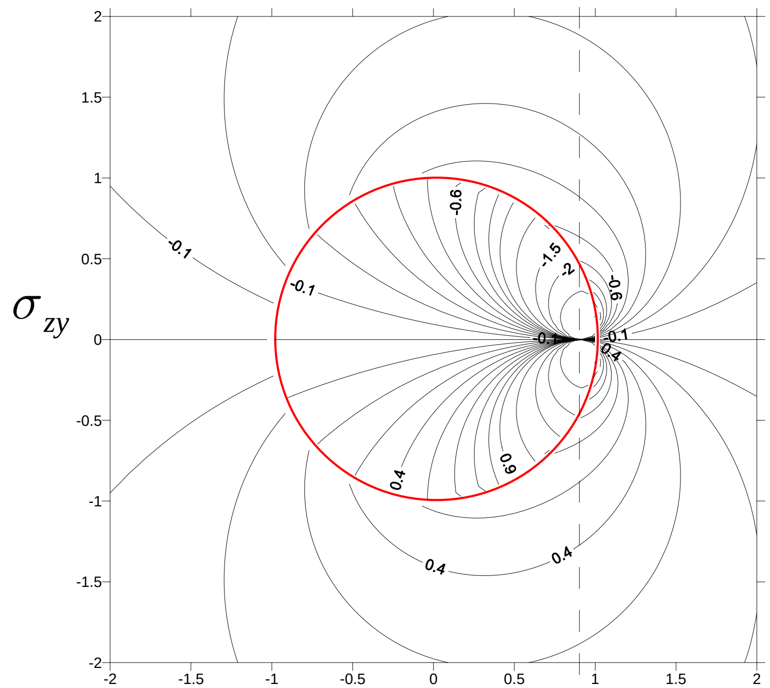
Stress contours of σ_{zx} and σ_{zy} for the static solutions (a concentrated force in the inclusion)

$$\sigma_{zx} = \sigma_{zr}\cos\phi - \sigma_{z\theta}\sin\phi$$

$$\sigma_{zy} = \sigma_{zr}\sin\phi + \sigma_{z\theta}\cos\phi$$



$k = 0, ideal\ \ bonding$



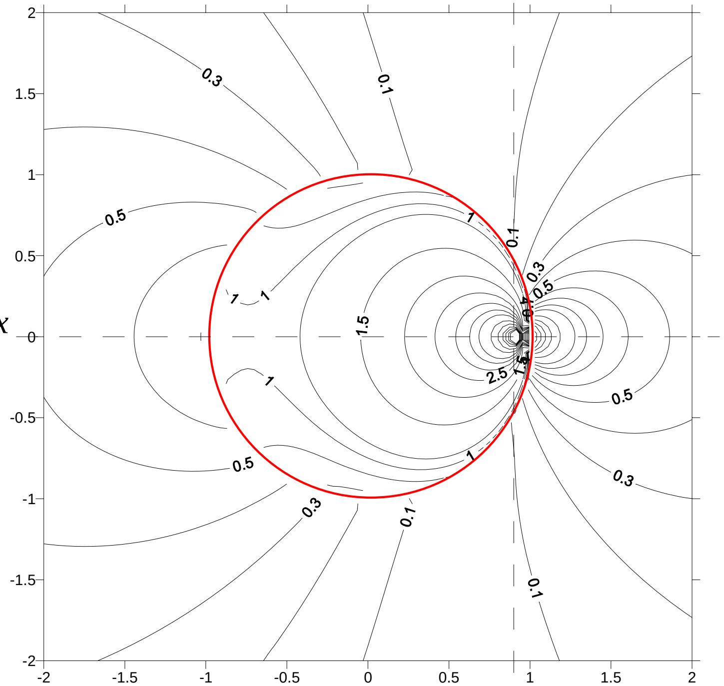
$k = 0, ideal\ \ bonding$

Stress contours of σ_{zx} and σ_{zy} for the dynamic solutions (a concentrated force in the inclusion)

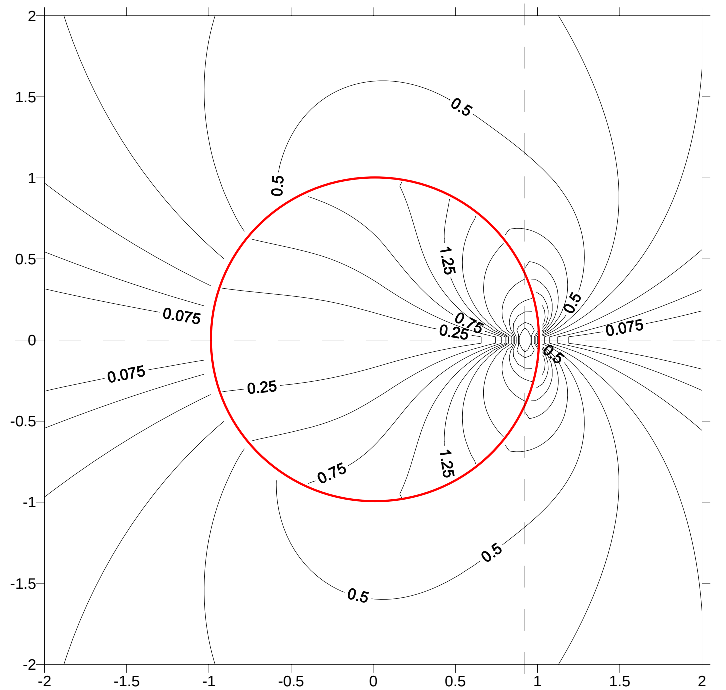
$$\sigma_{zx} = \sigma_{zr}\cos\phi - \sigma_{z\theta}\sin\phi$$

$$\sigma_{zy} = \sigma_{zr}\sin\phi + \sigma_{z\theta}\cos\phi$$

σ_{zx}



σ_{zy}



$k_I = 1, k_M = 2, \beta = 10^{32}$ $(ideal\ \ bonding)$

$k_I = 1, k_M = 2, \beta = 10^{32}$ $(ideal\ \ bonding)$

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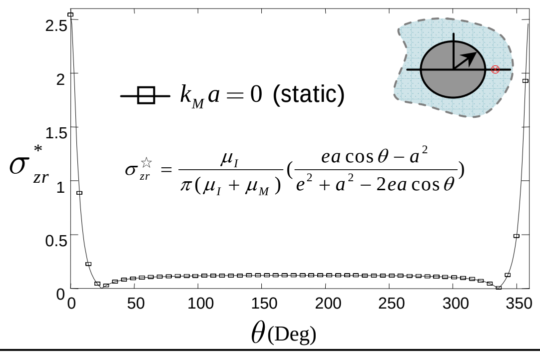
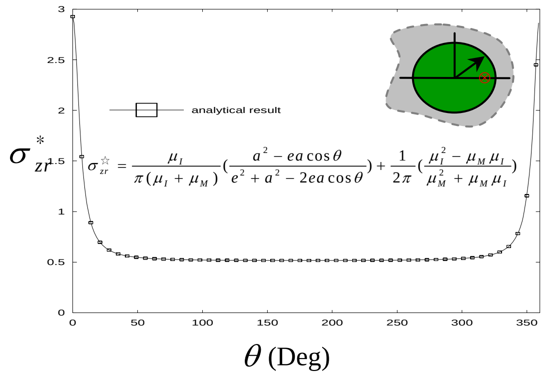
Series-form & closed-form solutions for the static case (ideally bonded interface)

$\sigma_{zr}^{\star} = a\sigma_{zr}^{I} / p = a\sigma_{zr}^{M} / p$

	Concentrated force in the matrix	Concentrated force in the inclusion
Stress distribution along the interface		
Closed-form solution	$\sigma_{zr}^{\star} = \frac{\mu_I}{\pi(\mu_I+\mu_M)}(\frac{ea\cos\theta - a^2}{e^2+a^2-2ea\cos\theta})$	$\sigma_{zr}^{\star} = \frac{\mu_I}{\pi(\mu_I+\mu_M)}(\frac{a^2 - ea\cos\theta}{e^2+a^2-2ea\cos\theta}) + \frac{1}{2\pi}(\frac{\mu_I^2-\mu_M\mu_I}{\mu_M^2+\mu_M\mu_I})$
Seires-form solution	$\sigma_{zr}^{\star} = \frac{\mu_I}{\pi(\mu_I+\mu_M)}\sum_{m=1}^{\infty}(\frac{a}{e})^m \cos m\theta$	$\sigma_{zr}^{\star} = \frac{\mu_I}{2\pi\mu_M} + \frac{\mu_I}{\pi(\mu_I+\mu_M)}\sum_{m=1}^{\infty}(\frac{e}{a})^m \cos m\theta$

Fourier series
(Poisson integral formula)

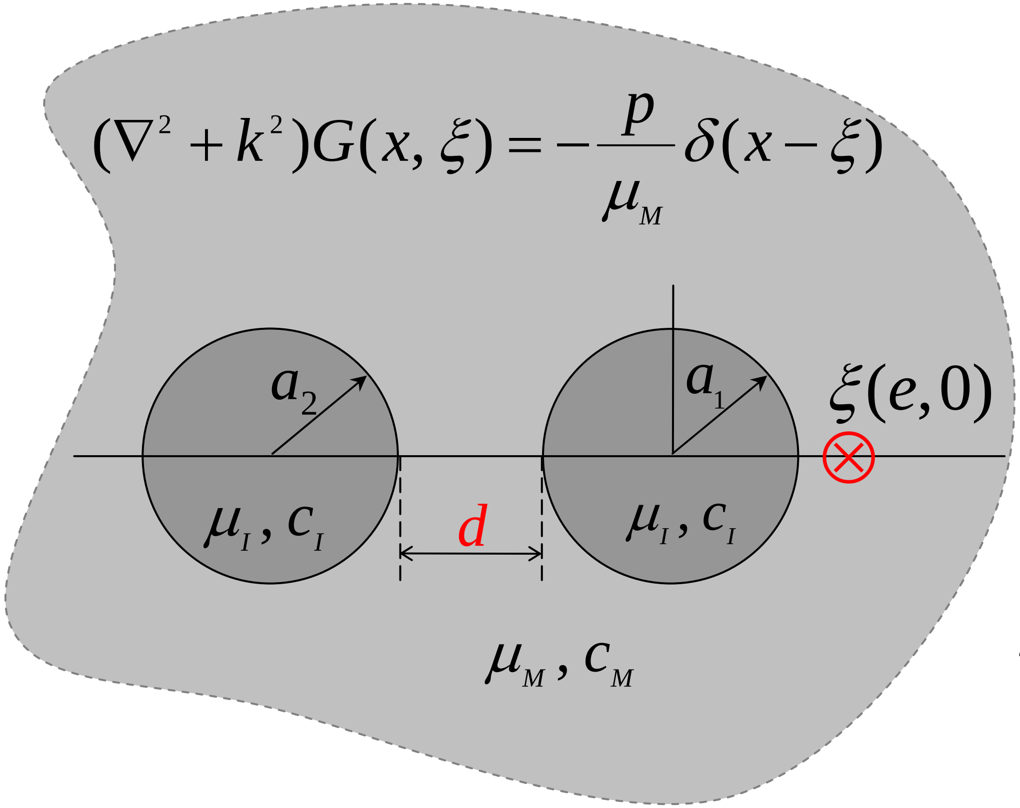
(easy)

(not easy)

Numerical examples

- Laplace problems
  - Eccentric ring
  - A half-plane with an aperture
    (1) Dirichlet boundary condition
    (2) Robin boundary condition
  - A half-plane problem with a circular hole and a half-ircular inclusion
- Helmholtz problems
  - An infinite matrix containing a circular inclusion with a concentrated force in the matrix or inclusion
  - Special cases and parameter study
  - An infinite matrix containing two circular inclusions with a concentrated force in the matrix

An infinite matrix containing two circular inclusions with a concentrated force at ξ in the matrix



$$t^M = -\frac{\mu_I}{\mu_M} t^I$$

$$t^I = \frac{\beta}{\mu_I}(u^M - u^I)$$

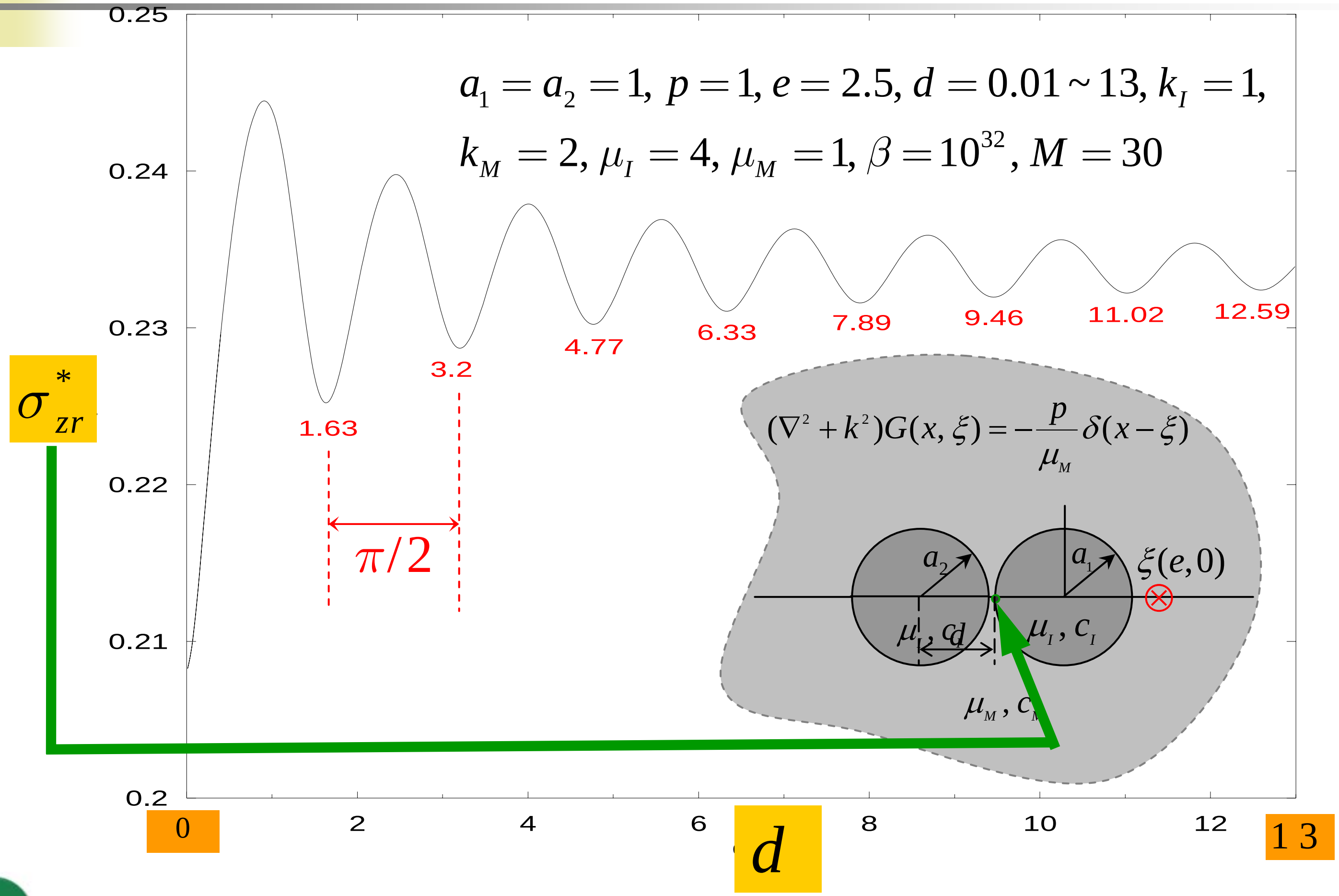
$\mu_I = 4\mu_M \quad c_I = 2c_M$

μ is the shear modulus

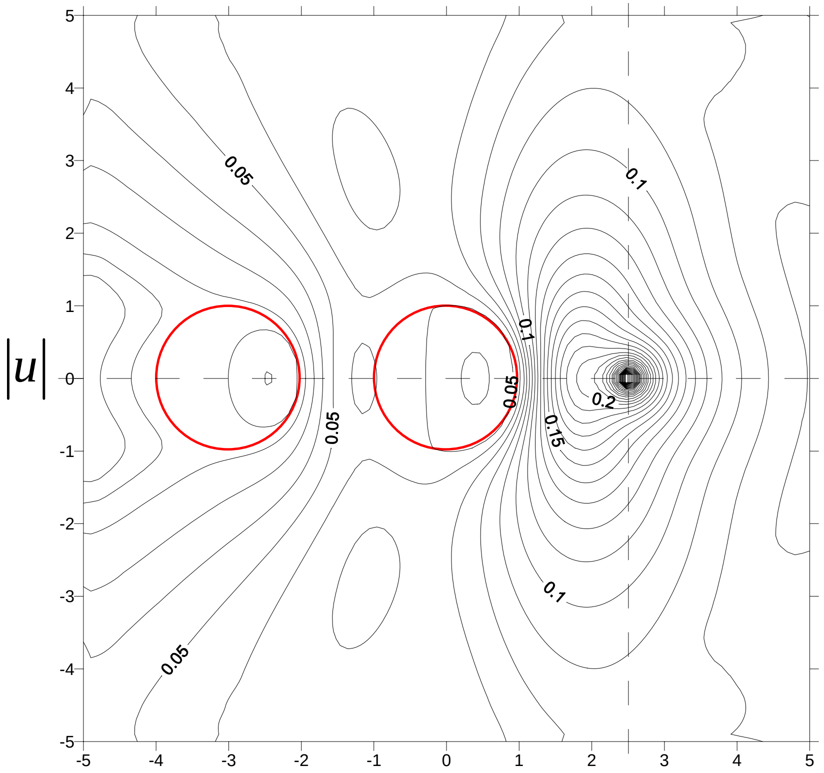
c is the wave speed

β is the imperfect interface parameter

Distribution of σ^*_{zr} of the matrix at the position of d various (a_1, π)



The contour of the displacement for an infinite matrix containing two inclusions with a concentrated force at ξ in the matrix for ideal bonding



Potential contour using the present method (M=30)

Outlines

- **Motivation and literature review**
- **Derivation of the Green's function**
  - Expansions of fundamental solution and boundary density
  - Adaptive observer system
  - Vector decomposition technique
  - Linear algebraic equation
  - Take free body
  - Image technique for solving half-plane problems
- **Numerical examples**
  - Green's function for Laplace problems
  - Green's function for Helmholtz problems
- **Conclusions**

Conclusions

- After introducing the degenerate kernel, the BIE is nothing more than the linear algebra.

- We derived the analytic Green's function for one inclusion problem by using the null-field integral equation. Also, the present approach can be utilized to construct semi-analytic Green's functions for several circular inclusions.

Conclusions

- Several examples, Laplace and Helmholtz problems were demonstrated to check the validity of the present formulation and the results match well with available solutions in the literature.
- A general-purpose program for deriving the Green's function of Laplace or Helmholtz problems with arbitrary number of circular apertures and/or inclusions of arbitrary radii and various positions involving Dirichlet or Neumann or mixed boundary condition was developed.

Further studies

- The imperfect circular interface is homogeneous → nonhomogeneous.

  $\beta \rightarrow \beta(\theta)$

- According to our successful experiences for half-plane problems, it is straightforward to quarter-plane problems.

The end

Thanks for your attentions.

You can get more information on our website.

http://msvlab.hre.ntou.edu.tw