Null-field integral equation approach for boundary value problems with circular boundaries

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Summary

In this paper, a systematic approach is proposed to deal with boundary value problems containing circular boundaries. Null-field integral equations are employed to solve the problem. The mathematical tools, degenerate kernels and Fourier series, are utilized. The kernel function is expanded to degenerate form and the boundary density is expressed in terms of Fourier series. By moving the null-field point to the boundary, the singularity novelly disappears. By matching the boundary condition, a linear algebraic system is obtained. After obtaining the unknown Fourier coefficients, the solution can be obtained by using the integral representation. This systematic approach can be applied to the Laplace, Helmhotz and biharmonic problems. Finally, several examples are demonstrated to check the validity of present formulation.

Introduction

Boundary value problems with circular boundaries have numerous applications in engineering. Analytical approach using bi-polar coordinate [1] was developed for two-holes problems. For a problem with several holes, many numerical methods, e.g. FEM and BEM, were resorted to solve. To developing a systematic approach is not trivial. In this paper, we develop a null-field integral equation approach for boundary value problems (BVPs) with circular boundaries. The key idea is the expansion of kernel functions and boundary densities in the null-field integral equations. Applications to the Laplace, Helmholtz and biharmonic problems are addressed. Not only interior problems but also exterior cases are examined. Several examples were demonstrated to see the validity of the new formulation.

Null-field integral equation approach for boundary value problems

Suppose there are N randomly distributed circular cavities bounded in the domain D and enclosed with the boundary, B_k ($k = 0, 1, 2, \cdots, N$) as shown in Figure 1. We define

$$B = \bigcup_{k=0}^{N} B_k \, . \qquad (1)$$

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In mathematical physics, boundary value problems can be modelled by the governing equation,

$$\mathcal{L}\, u(\mathrm{x}) = 0, \ \mathrm{x} \in D, \tag{2}$$

where $\mathcal{L}$ may be the Laplace, Helmholtz or biharmonic operator, $u(\mathrm{x})$ is the potential function and D is the domain of interest. The integral equation for the domain point can be derived from the third Green's identity or Rayleigh Green identity, we have

$$2\pi u(\mathrm{x}) = \int_B T(\mathrm{s},\mathrm{x})u(\mathrm{s})dB(\mathrm{s}) - \int_B U(\mathrm{s},\mathrm{x})t(\mathrm{s})dB(\mathrm{s}), \ \mathrm{x} \in D, \tag{3}$$

$$2\pi \frac{\partial u(\mathrm{x})}{\partial \mathbf{n}_{\mathrm{x}}} = \int_B M(\mathrm{s},\mathrm{x})u(\mathrm{s})dB(\mathrm{s}) - \int_B L(\mathrm{s},\mathrm{x})t(\mathrm{s})dB(\mathrm{s}), \ \mathrm{x} \in D, \tag{4}$$

where s and x are the source and field points, respectively, B is the boundary, $\mathbf{n}_{\mathrm{x}}$ denotes the outward normal vector at field point x and the kernel function $U(\mathrm{s},\mathrm{x})$, is the fundamental solution, and the other kernel functions, $T(\mathrm{s},\mathrm{x})$, $L(\mathrm{s},\mathrm{x})$ and $M(\mathrm{s},\mathrm{x})$, are defined in the dual BIEM[2]. It is noted that more potentials are needed in Eqs. (3) and (4) for biharmonic case.

By moving the field point to the boundary, the Eqs. (3) and (4) reduce to

$$\pi u(\mathrm{x}) = C.P.V.\int_B T(\mathrm{s},\mathrm{x})u(\mathrm{s})dB(\mathrm{s}) - R.P.V\int_B U(\mathrm{s},\mathrm{x})t(\mathrm{s})dB(\mathrm{s}), \ \mathrm{x} \in B, \tag{5}$$

$$\pi \frac{\partial u(\mathrm{x})}{\partial \mathbf{n}_{\mathrm{x}}} = H.P.V.\int_B M(\mathrm{s},\mathrm{x})u(\mathrm{s})dB(\mathrm{s}) - C.P.V\int_B L(\mathrm{s},\mathrm{x})t(\mathrm{s})dB(\mathrm{s}), \ \mathrm{x} \in B, \tag{6}$$

where $C.P.V$, $R.P.V$ and $H.P.V$ denote the Cauchy principal value, Riemann principal value and Hadamard principal value, respectively. Once the field point x locates outside the domain, the null-field integral equation in Eqs. (5) and (6) yield

$$0 = \int_B T(\mathrm{s},\mathrm{x})u(\mathrm{s})dB(\mathrm{s}) - \int_B U(\mathrm{s},\mathrm{x})t(\mathrm{s})dB(\mathrm{s}), \ \mathrm{x} \in D^c, \tag{7}$$

$$0 = \int_B M(\mathrm{s},\mathrm{x})u(\mathrm{s})dB(\mathrm{s}) - \int_B L(\mathrm{s},\mathrm{x})t(\mathrm{s})dB(\mathrm{s}), \ \mathrm{x} \in D^c, \tag{8}$$

where D^c is the complementary domain.

Expansions of fundamental solution and boundary density

Based on the separable property, the kernel function $U(\mathrm{s},\mathrm{x})$ can be expanded into separable form by dividing the source point and field point in the polar coordinate:

$$U(s,x) = \begin{cases} U^i(s,x) = A(s)B(x), & |s| \geq |x| \\ U^e(s,x) = A(x)B(s), & |x| \geq |s| \end{cases}, \tag{9}$$

where the $A(x)$ and $B(x)$ can be found for the Laplace [3], Helmholtz [4] and biharmonic [5] operators and the superscripts "i" and "e" denote the interior ($|s| > |x|$) and exterior ($|x| > |s|$) cases, respectively. For the degenerate forms of T, L, M kernels, it can be derived according to their definitions.

We apply the Fourier series expansions to approximate the potential u and its normal derivative on the boundary

$$u(s_k) = a_0^k + \sum_{n=1}^{\infty} (a_n^k \cos n\theta_k + b_n^k \sin n\theta_k), \; s_k \in B_k, \; k = 1, 2, \cdots, N, \tag{10}$$

$$t(s_k) = p_0^k + \sum_{n=1}^{\infty} (p_n^k \cos n\theta_k + q_n^k \sin n\theta_k), \; s_k \in B_k, \; k = 1, 2, \cdots, N, \tag{11}$$

where a_n^k, b_n^k, p_n^k and q_n^k ($n = 0, 1, 2, \cdots$) are the Fourier coefficients and θ_k is the polar angle measured related to the x-direction.

After collocating points in the null-field integral equation of Eq. (7), the boundary integrals through all the circular contours are required. The observer system is adaptively to locate the origin at the center of circle in the boundary integrals. Adaptive observer system is chosen to fully employ the property of degenerate kernels. Figures 1 shows the boundary integration for the circular boundaries in the adaptive observer system. It is worthy noted that the origin of the observer system is located on the center of the corresponding circle under integration to entirely utilize the geometry of circular boundary for the expansion of degenerate kernels and boundary densities.

By collocating the null-field point x_k on the *kth* circular boundary for Eq. (7) in Figure 1, we have

$$0 = \sum_{k=0}^{N} \int_{B_k} T(s_k, x_j) u_k(s) dB_k(s) - \sum_{k=0}^{N} \int_{B_k} U(s_k, x_j) t_k(s) dB_k(s), \; x \in D^c, \tag{12}$$

where N is the number of circles including the outer boundary and the inner circular holes. Therefore, a linear algebraic system is obtained

$$[\mathbf{U}]\{\mathbf{t}\} = [\mathbf{T}]\{\mathbf{u}\}, \tag{13}$$

where $[\mathbf{U}]$ and $[\mathbf{T}]$ are the influence matrices with a dimension of $N+1(2M+1)$ by $N+1(2M+1)$, $\{\mathbf{u}\}$ and $\{\mathbf{t}\}$ denote the column vectors of Fourier coefficients with a dimension of $N+1(2M+1)$ by 1 in which M indicates the truncated terms of Fourier series. After the boundary unknowns are solved by using Eq. (13), the interior potential can be easily obtained according to Eq. (3).

Illustrative examples

Case 1: Infinite medium with two circular holes under anti-plane shear (Laplace problem)

A hole centered at the origin of radius a_1 and the other hole of radius $a_2 = 2a_1$ centered on x axis at $a_1 + a_2 + d$ are shown in Fig. 2. In order to be compared with the Honein's data [6] obtained by using the Möbius transformation, the stress along the boundary of radius a_1 is shown in Fig. 2 and good agreement is made.

Case 2: Eigensolution for an eccentric membrane (interior acoustics)

An eccentric case with radii r_1 and r_2 ($r_1 = 0.5, r_2 = 2.0$) is considered as shown in Fig. 3. The boundary condition is subject to the Dirichlet type. The result matches well with those of FEM and BEM [7] as shown in Fig. 4.

Case 3: Five scatters of cylinders (exterior acoustics).

Plane wave scattering by five soft circular cylinders is solved by the present method. The real-part solution agrees well with that of multiple DtN method [7].

Case 4: Eccentric case (Stokes' problem)

An eccentric case of Stokes' flow problem is considered. The inner cylinder is rotating with a constant angular velocity and the outer one is stationary. The stream function is shown in Fig. 5. and matches well with that of BEM [8].

Conclusions

A semi-analytical approach was proposed for solving BVPs with circular boundaries. Although the BIE for the boundary point was employed, we need not to face the problems of CPV and HPV after introducing the degenerate kernels. In order to verity the formulation, applications to Laplace, Helmholtz and biharmonic problems were done. Extension to other shapes, e.g. ellipse, is straightforward once the degenerate kernel is available.

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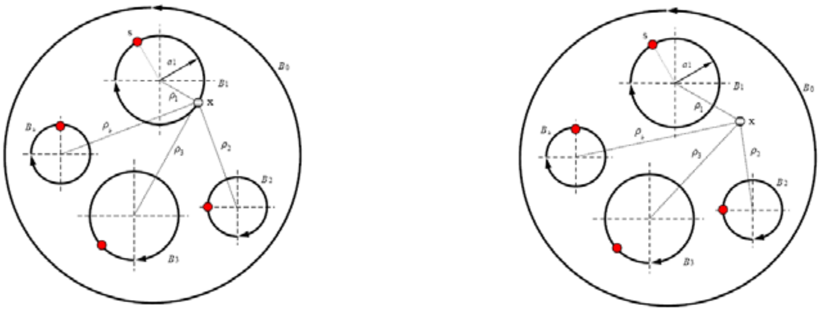

Fig. 1. Sketch of null-field and domain points in conjunction with the adaptive observer system (left: boundary point, right: null-field point)

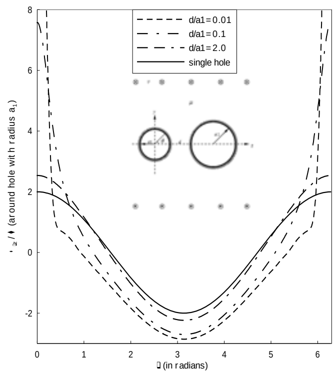

Fig. 2. Stress around the small hole

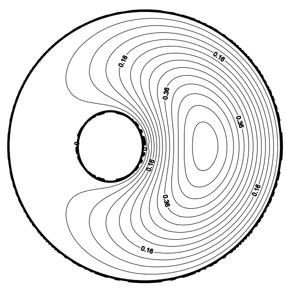

Fig. 3. The first mode of eccentric membrane

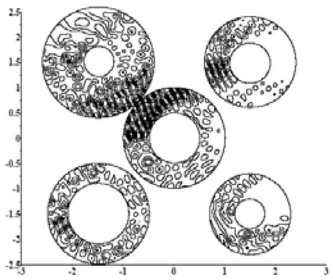

Fig. 4. Contour of the real-part solution

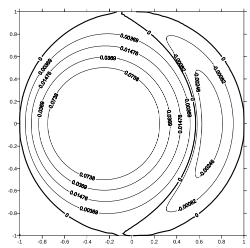

Fig. 5. Stream function of Stokes' problem