Null-field integral equation approach for boundary value problems with circular boundaries

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ICCES2005, India
December, 3, 17:05-17:30, 2005
Room B
(ICCES2005-JTCHEN.ppt)

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Outlines

- Motivation and literature review
- Mathematical formulation
  - Expansions of fundamental solution and boundary density
  - Adaptive observer system
  - Vector decomposition technique
  - Linear algebraic equation
- Numerical examples
- Conclusions

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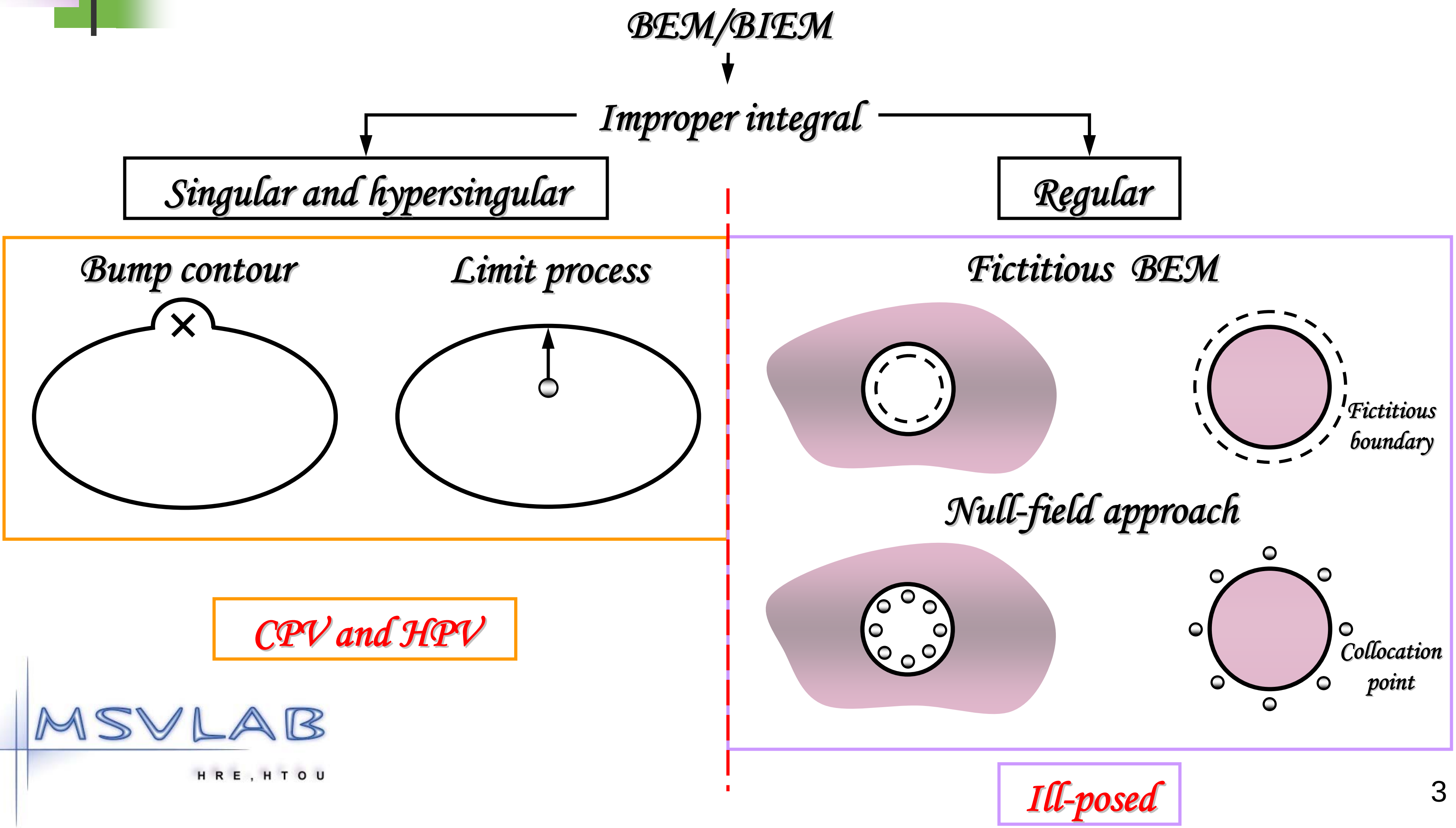
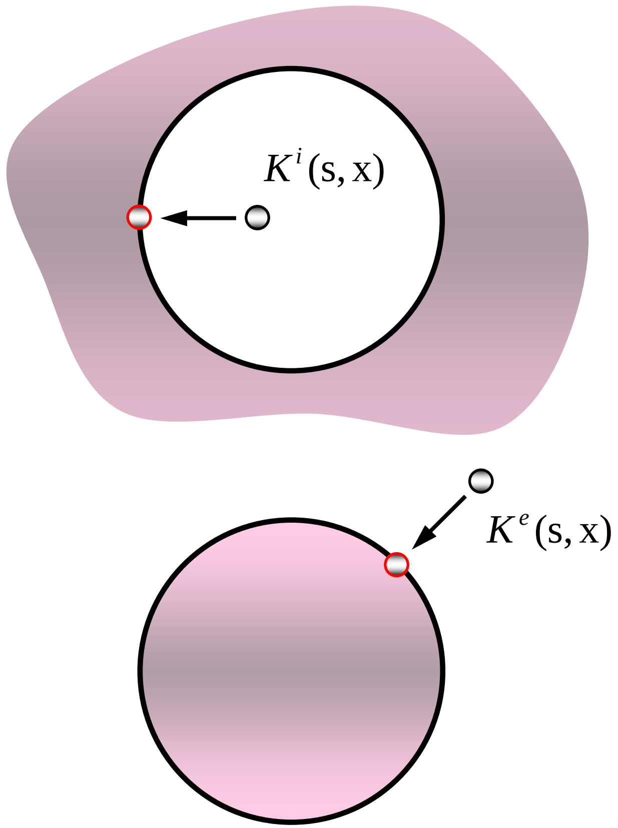
Motivation and literature review



Present approach



$a(x)b(s)$

$$\varphi(x) = \int_B K(s,x)\phi(s)\,dB(s)$$

$$\varphi(x) = \int_B a(x)b(s)\phi(s)\,dB(s)$$

$$O\left(\frac{1}{|x-s|}\right),\; O\left(\frac{1}{|x-s|^2}\right)$$

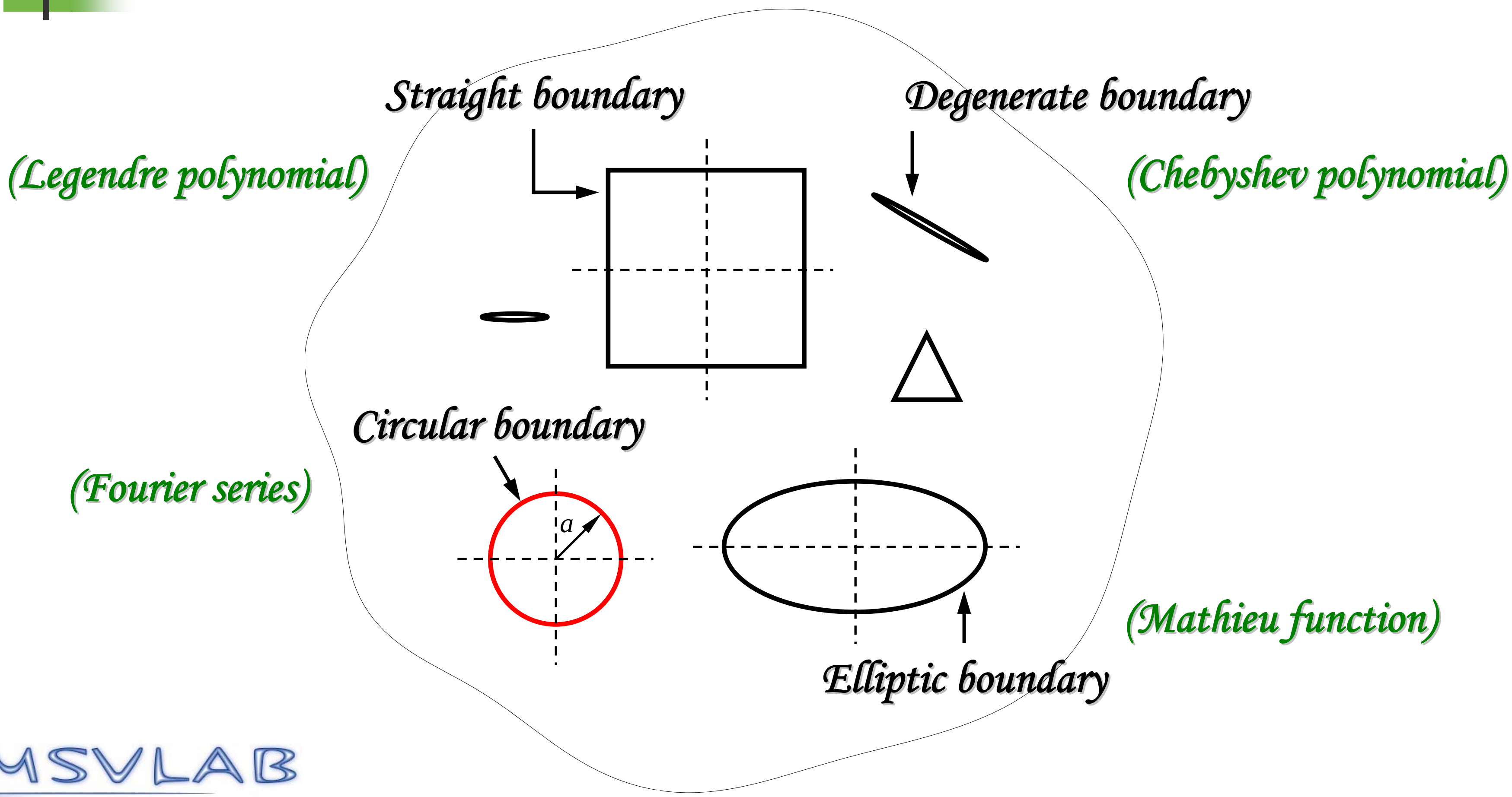
Degenerate kernel → *No principal value*

Fundamental solution → *CPV and HPV*

Advantages of degenerate kernel

1. *No principal value*
2. *Well-posed*
3. *No boundary-layer effect*
4. *Exponetial convergence*

Engineering problem with arbitrary geometries



Motivation and literature review

Analytical methods for solving Laplace problems with circular holes

Conformal mapping	Bipolar coordinate	Special solution
Chen and Weng, 2001, "Torsion of a circular compound bar with imperfect interface", ASME Journal of Applied Mechanics	*Lebedev, Skalskaya and Uyand, 1979, "Work problem in applied mathematics", Dover Publications*	*Honein, Honein and Hermann, 1992, "On two circular inclusions in harmonic problem", Quarterly of Applied Mathematics*

Limited to doubly connected domain

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Fourier series approximation

- *Ling (1943) - torsion of a circular tube*
- *Caulk et al. (1983) - steady heat conduction with circular holes*
- *Bird and Steele (1992) - harmonic and biharmonic problems with circular holes*
- *Mogilevskaya et al. (2002) - elasticity problems with circular boundaries*

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Contribution and goal

- *However, they didn't employ the null-field integral equation and degenerate kernels to fully capture the circular boundary, although they all employed Fourier series expansion.*
- *To develop a systematic approach for solving Laplace problems with multiple holes is our goal.*

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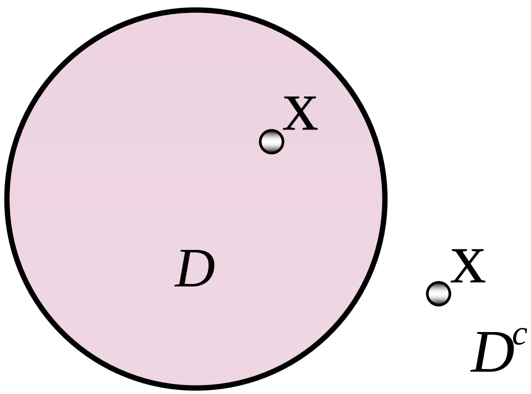
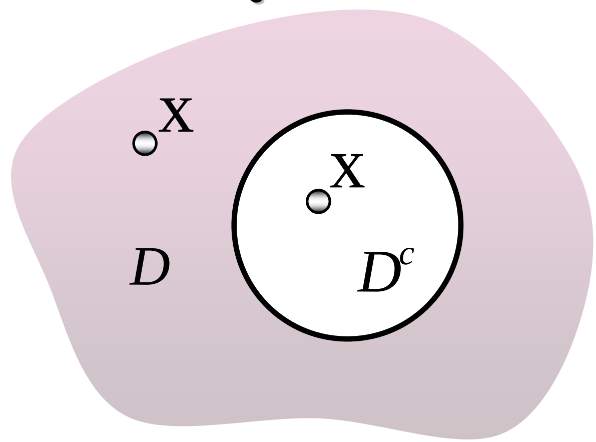
Boundary integral equation and null-field integral equation

Interior case



Exterior case



$$U(\mathrm{s},\mathrm{x}) = \ln|\mathrm{x}-\mathrm{s}| = \ln r$$

$$T(\mathrm{s},\mathrm{x}) = \frac{\partial U(\mathrm{s},\mathrm{x})}{\partial \mathbf{n}_{\mathrm{s}}}$$

$$t(\mathrm{s}) = \frac{\partial u(\mathrm{s})}{\partial \mathbf{n}_{\mathrm{s}}}$$

$$2\pi u(\mathrm{x}) = \int_B T(\mathrm{s},\mathrm{x})u(\mathrm{s})dB(\mathrm{s}) - \int_B U(\mathrm{s},\mathrm{x})t(\mathrm{s})dB(\mathrm{s}), \quad \mathrm{x} \in D$$

$$0 = \int_B T(\mathrm{s},\mathrm{x})u(\mathrm{s})dB(\mathrm{s}) - \int_B U(\mathrm{s},\mathrm{x})t(\mathrm{s})dB(\mathrm{s}), \quad \mathrm{x} \in D^c$$

Null-field integral equation

Outlines (Direct problem)

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- Numerical examples
- Degenerate scale
- Conclusions

Expansions of fundamental solution and boundary density

- *Degenerate kernel - fundamental solution*

$$U(\mathrm{s},\mathrm{x}) = \begin{cases} U^i(R,\theta;\rho,\phi) = \ln R - \sum_{m=1}^{\infty} \frac{1}{m}(\frac{\rho}{R})^m \cos m(\theta-\phi), & R \geq \rho \\ U^e(R,\theta;\rho,\phi) = \ln \rho - \sum_{m=1}^{\infty} \frac{1}{m}(\frac{R}{\rho})^m \cos m(\theta-\phi), & \rho > R \end{cases}$$

- *Fourier series expansions - boundary density*

$$u(\mathrm{s}) = a_0 + \sum_{n=1}^{M}(a_n \cos n\theta + b_n \sin n\theta), \ \ \mathrm{s} \in B$$

$$t(\mathrm{s}) = p_0 + \sum_{n=1}^{M}(p_n \cos n\theta + q_n \sin n\theta), \ \ \mathrm{s} \in B$$

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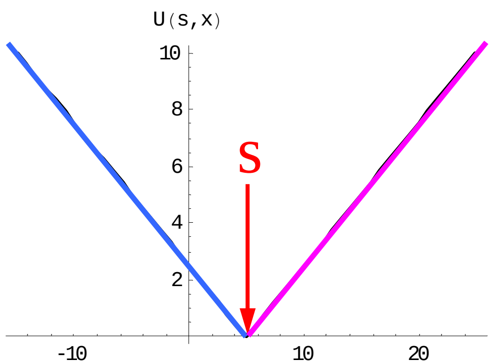
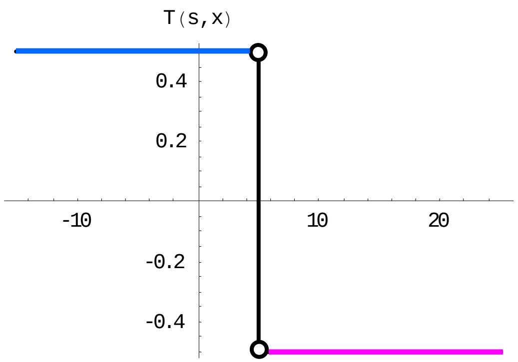
Separable form of fundamental solution (1D)

Separable property

$$U(s,x)=\begin{cases}\sum_{i=1}^{2} a_i(x)b_i(s), & s\ge x\\ \sum_{i=1}^{2} a_i(s)b_i(x), & x> s\end{cases}$$



continuous



discontinuous

$$U(s,x)=\frac{1}{2}r=\begin{cases}\frac{1}{2}(s-x), & s\ge x\\ \frac{1}{2}(x-s), & x> s\end{cases}$$

$$T(s,x)=\begin{cases}\frac{1}{2}, & s> x\\ \frac{-1}{2}, & x> s\end{cases}$$

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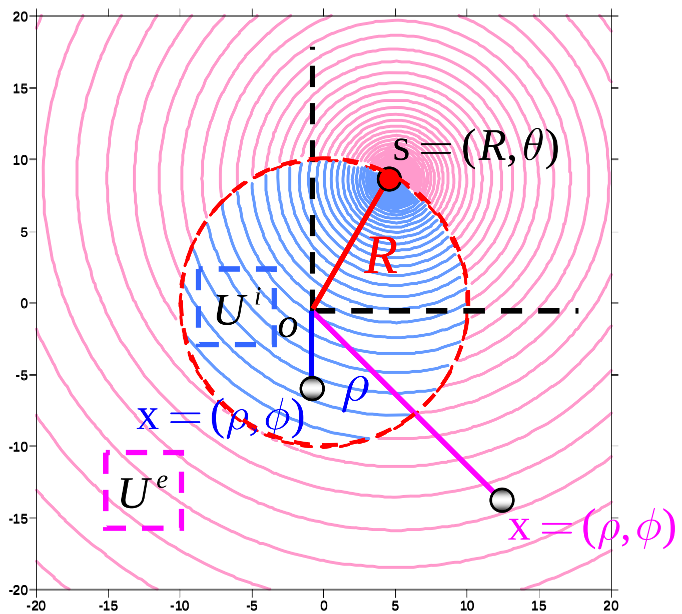
Separable form of fundamental solution (2D)

$$U(\mathrm{s},\mathrm{x}) = \begin{cases} U^i(R,\theta;\rho,\phi) = \ln R - \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{\rho}{R}\right)^m \cos m(\theta-\phi), & R \ge \rho \\ U^e(R,\theta;\rho,\phi) = \ln \rho - \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{R}{\rho}\right)^m \cos m(\theta-\phi), & \rho > R \end{cases}$$



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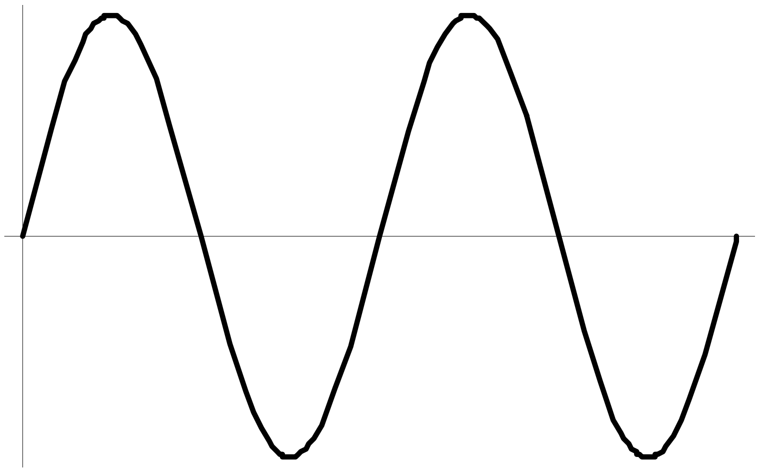
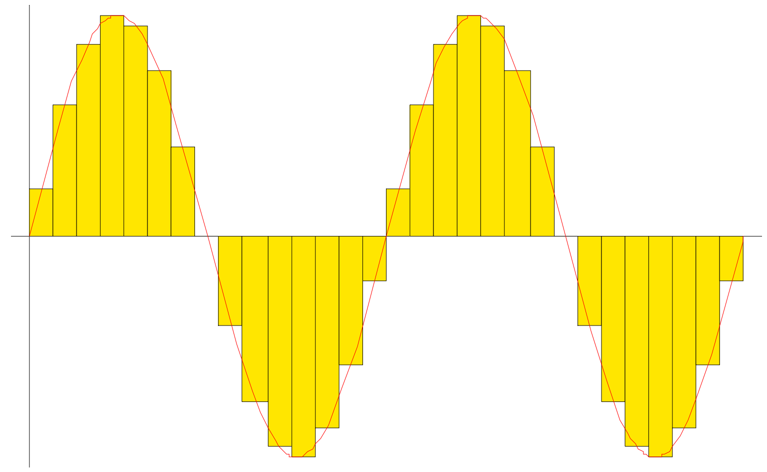
Boundary density discretization

Fourier series

Ex. constant element



Present method



Conventional BEM

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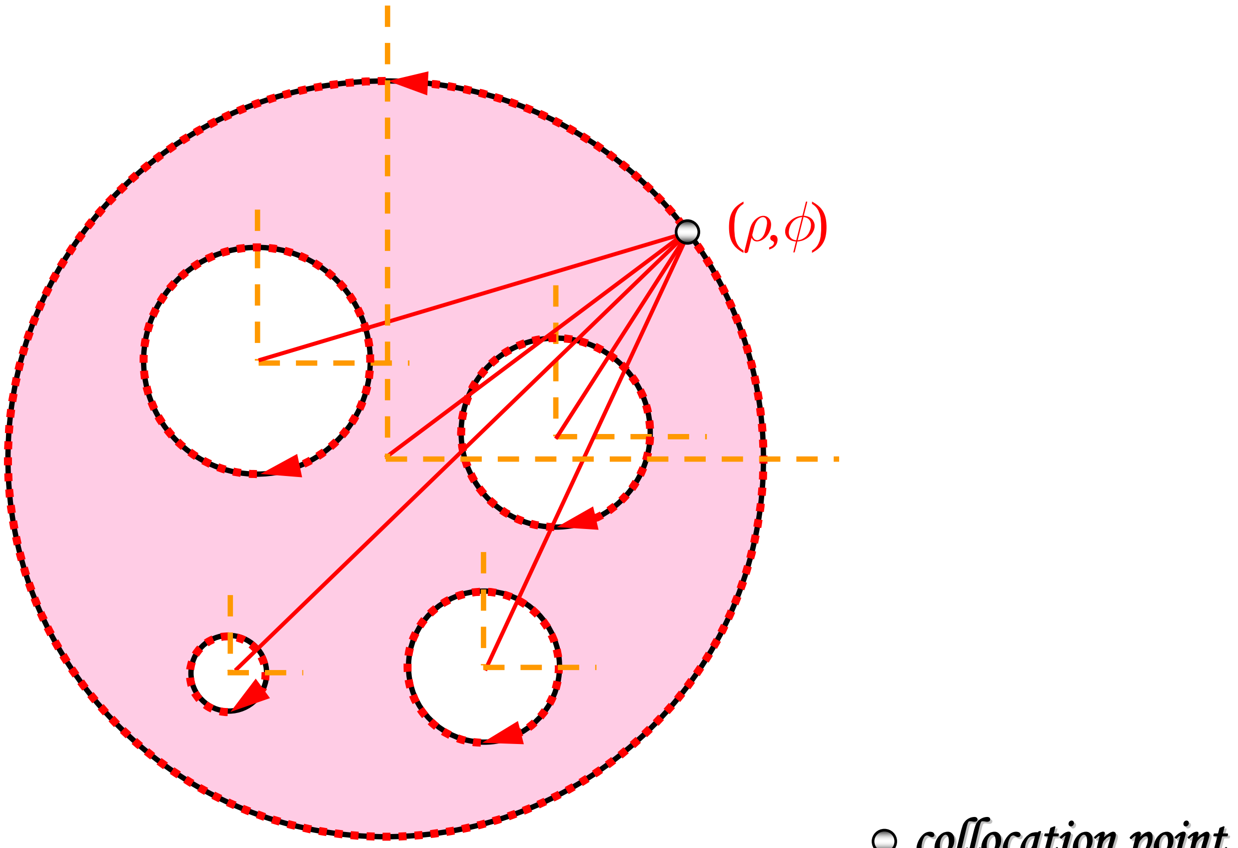
Adaptive observer system



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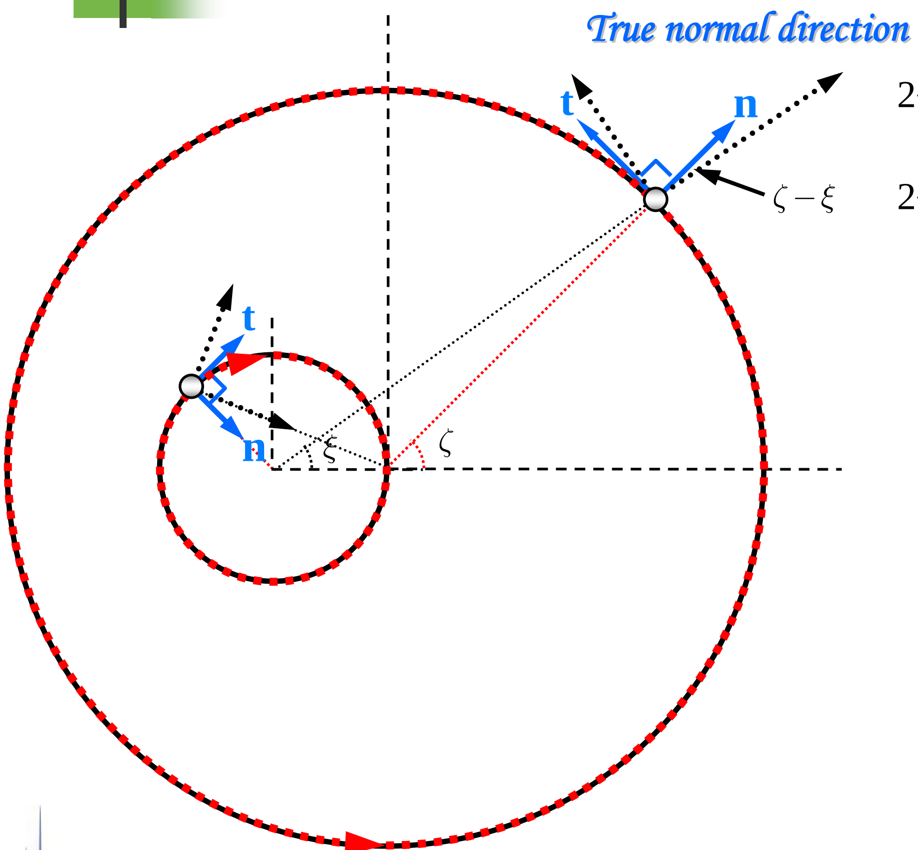
Vector decomposition technique for potential gradient



$$2\pi\frac{\partial u(\mathrm{x})}{\partial \mathbf{n}} = \int_B M_\rho(\mathrm{s},\mathrm{x})u(\mathrm{s})dB(\mathrm{s}) - \int_B L_\rho(\mathrm{s},\mathrm{x})t(\mathrm{s})dB(\mathrm{s}),\ \ \mathrm{x}\in D$$

$$2\pi\frac{\partial u(\mathrm{x})}{\partial \mathbf{t}} = \int_B M_\phi(\mathrm{s},\mathrm{x})u(\mathrm{s})dB(\mathrm{s}) - \int_B L_\phi(\mathrm{s},\mathrm{x})t(\mathrm{s})dB(\mathrm{s}),\ \ \mathrm{x}\in D$$

Non-concentric case:

$$L_\rho(\mathrm{s},\mathrm{x}) = \frac{\partial U(\mathrm{s},\mathrm{x})}{\partial \rho}\boxed{\cos(\zeta-\xi)} + \frac{1}{\rho}\frac{\partial U(\mathrm{s},\mathrm{x})}{\partial \phi}\boxed{\cos(\frac{\pi}{2}-\zeta+\xi)}$$

$$M_\rho(\mathrm{s},\mathrm{x}) = \frac{\partial T(\mathrm{s},\mathrm{x})}{\partial \rho}\boxed{\cos(\zeta-\xi)} + \frac{1}{\rho}\frac{\partial T(\mathrm{s},\mathrm{x})}{\partial \phi}\boxed{\cos(\frac{\pi}{2}-\zeta+\xi)}$$

Special case (concentric case): $\zeta = \xi$

$$L_\rho(\mathrm{s},\mathrm{x}) = \frac{\partial U(\mathrm{s},\mathrm{x})}{\partial \rho} \qquad M_\rho(\mathrm{s},\mathrm{x}) = \frac{\partial T(\mathrm{s},\mathrm{x})}{\partial \rho}$$

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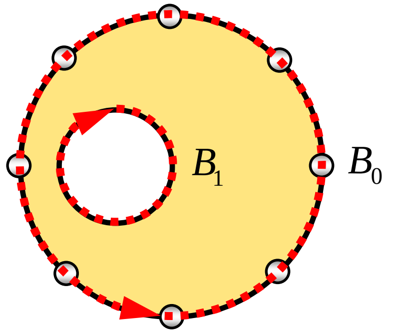
Linear algebraic equation

$$[\mathbf{U}]\{\mathbf{t}\}=[\mathbf{T}]\{\mathbf{u}\}$$



where

Index of collocation circle

$$[\mathbf{U}]=\begin{bmatrix} \mathbf{U}_{00} & \mathbf{U}_{01} & \cdots & \mathbf{U}_{0N} \\ \mathbf{U}_{10} & \mathbf{U}_{11} & \cdots & \mathbf{U}_{1N} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{U}_{N0} & \mathbf{U}_{N1} & \cdots & \mathbf{U}_{NN} \end{bmatrix} \qquad \{\mathbf{t}\}=\begin{Bmatrix} \mathbf{t}_0 \\ \mathbf{t}_1 \\ \mathbf{t}_2 \\ \vdots \\ \mathbf{t}_N \end{Bmatrix}$$

Index of routing circle

Column vector of Fourier coefficients (Nth routing circle)

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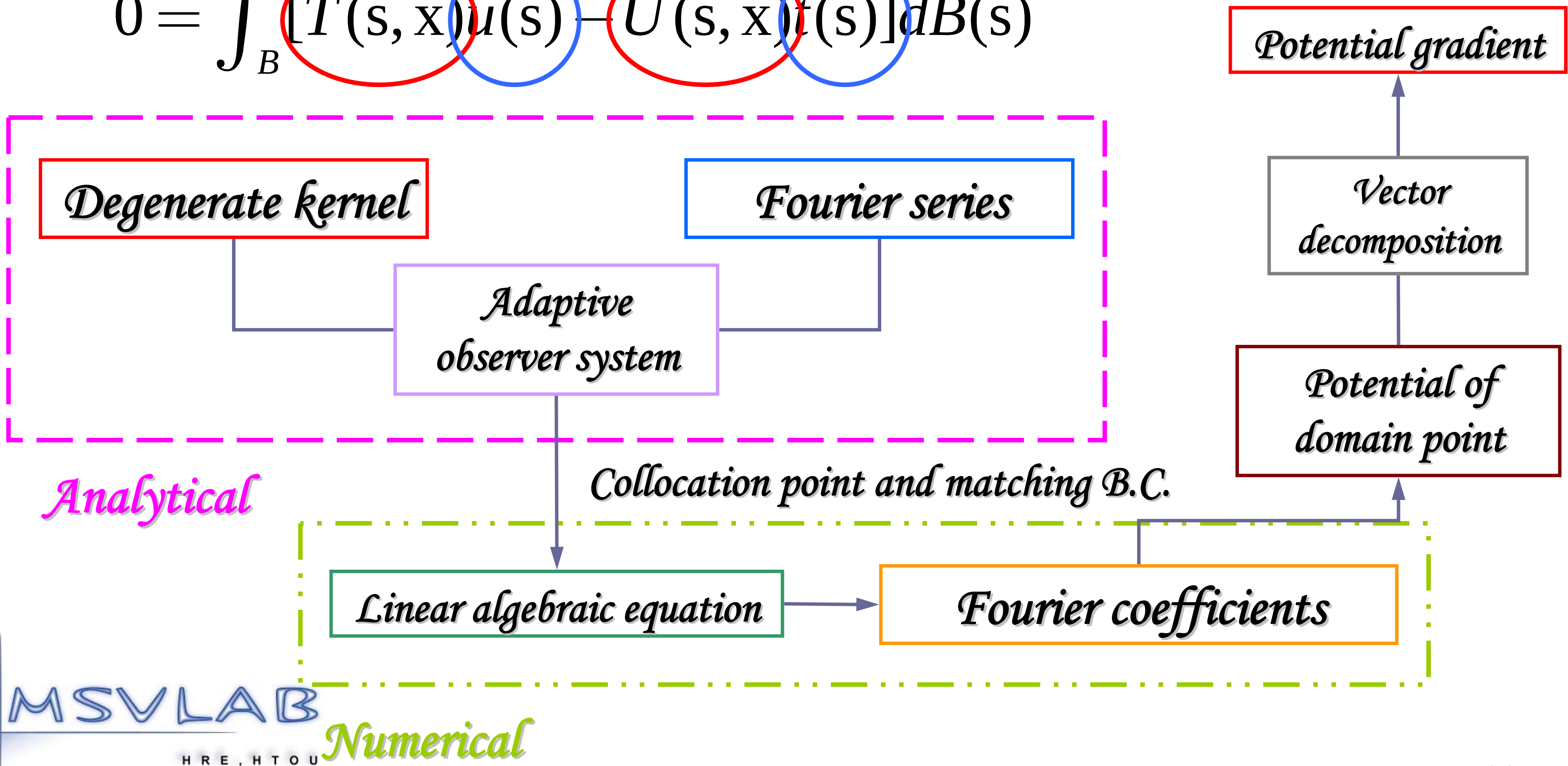
Flowchart of present method

$$0 = \int_B [T(s,x)u(s) - U(s,x)t(s)]dB(s)$$



Comparisons of conventional BEM and the present method

	Boundary density discretization	Auxiliary system	Formulation	Observer system	Singularity
Conventional BEM	Constant, Linear, (Algebraic Convergence)	Fundamental solution	Boundary integral equation	Fixed observer system	CPV, RPV and HPV
Present method	Fourier series Expansion (Exponential Convergence)	Degenerate kernel	Null-field integral equation	Adaptive observer system	No principal value

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Numerical examples

- *Laplace equation (EABE 2005, CMES 2005)*
- *Eigen problem*
- *Exterior acoustics*
- *Biharmonic equation (JAM, ASME 2005)*

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Laplace equation

- *Steady state heat conduction problems*
- *Electrostatic potential of wires*
- *Flow of an ideal fluid pass cylinders*
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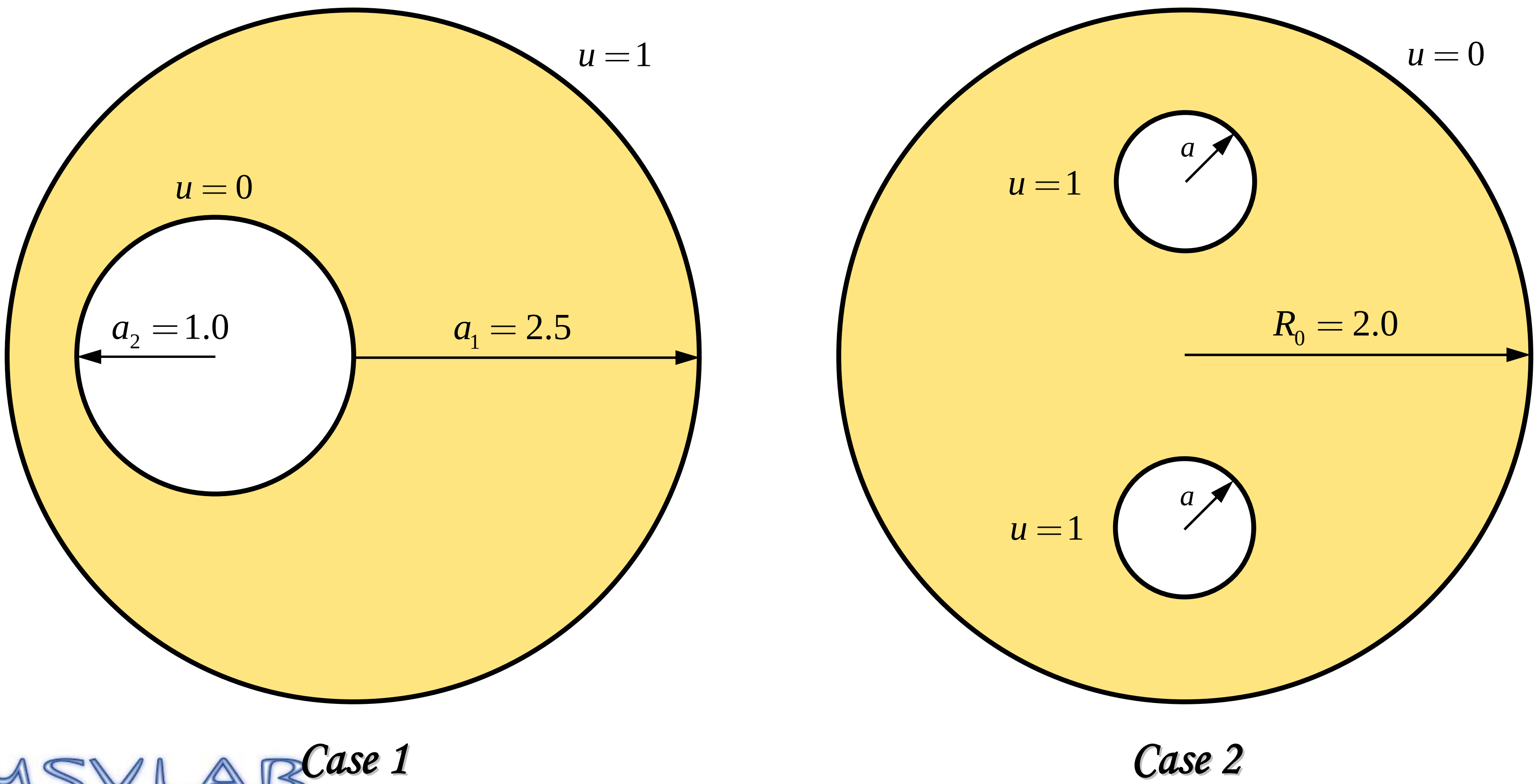
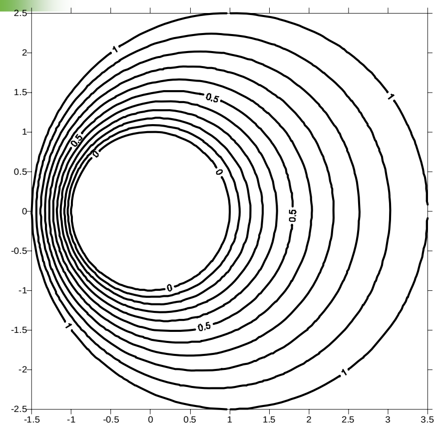
- *An infinite medium under antiplane shear*
- *Half-plane problems*

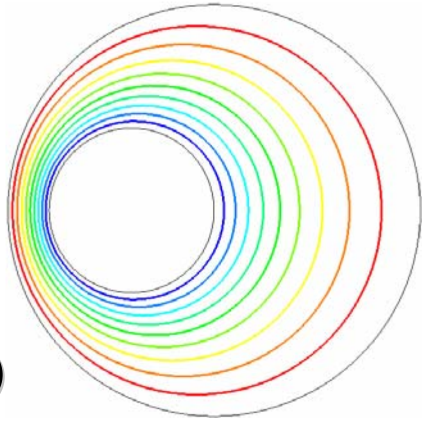
Steady state heat conduction problems



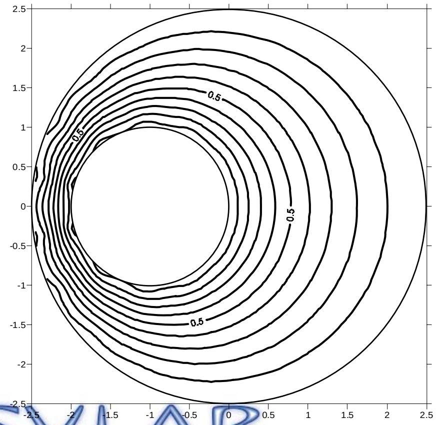
Case 1: Isothermal line



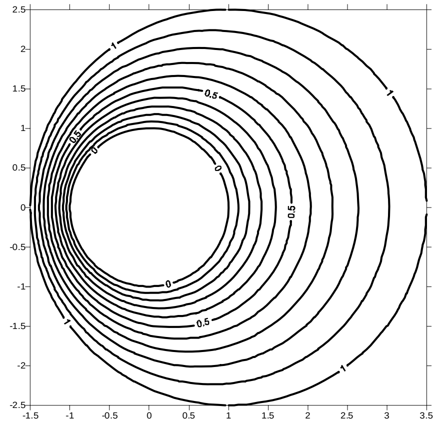
Exact solution (Carrier and Pearson)



FEM-ABAQUS (1854 elements)



BEM-BEPO2D (N=21)



Present method (M=10)

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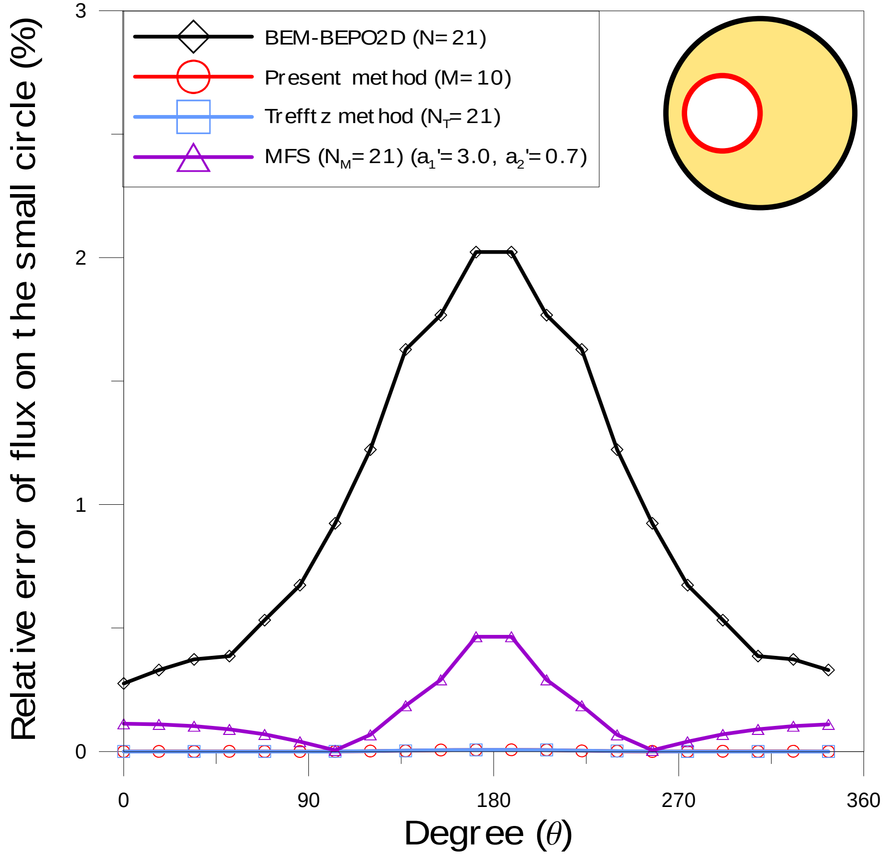
Relative error of flux on the small circle



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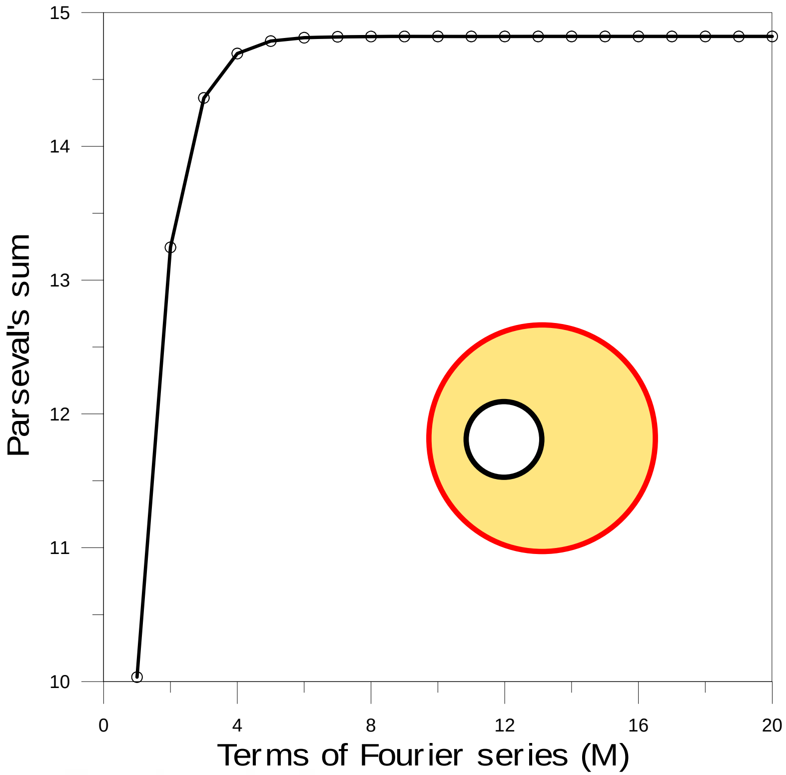
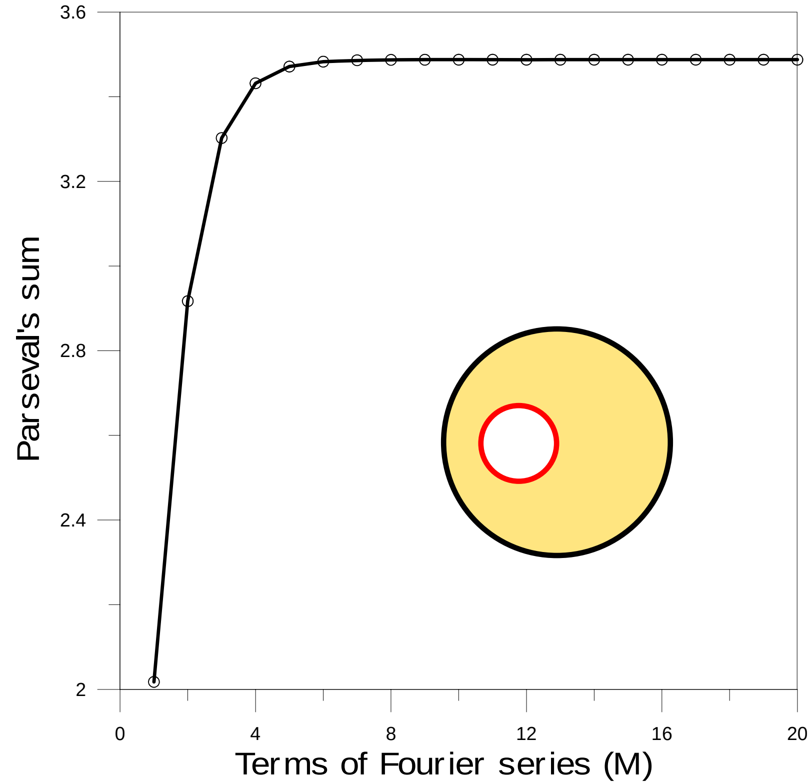
Convergence test - Parseval's sum for Fourier coefficients

Parseval's sum $\int_0^{2\pi} f^2(\theta)d\theta \doteqdot 2\pi a_0^2 + \pi\sum_{n=1}^{M}(a_n^2+b_n^2)$

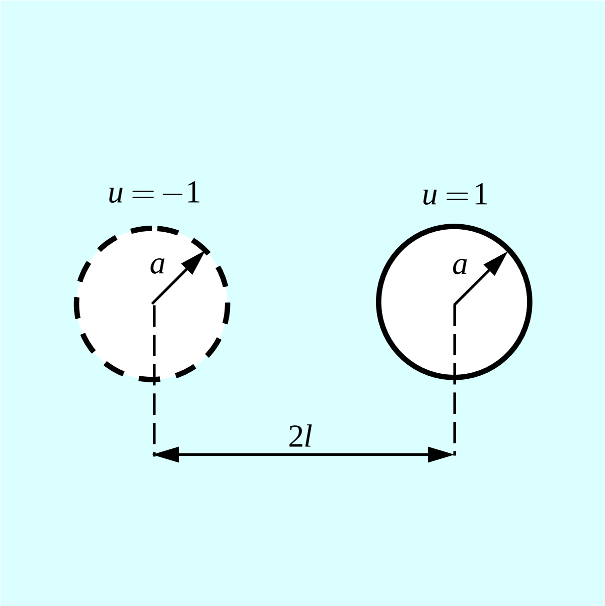




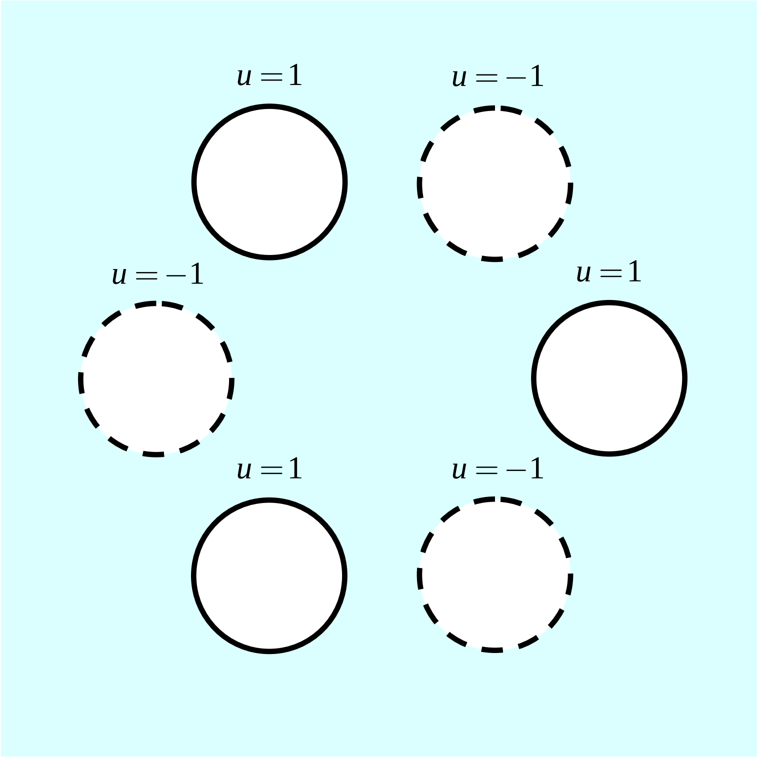
Laplace equation

- *Steady state heat conduction problems*
- *Electrostatic potential of wires*
- *Flow of an ideal fluid pass cylinders*
- *A circular bar under torque*
- *An infinite medium under antiplane shear*
- *Half-plane problems*

Electrostatic potential of wires

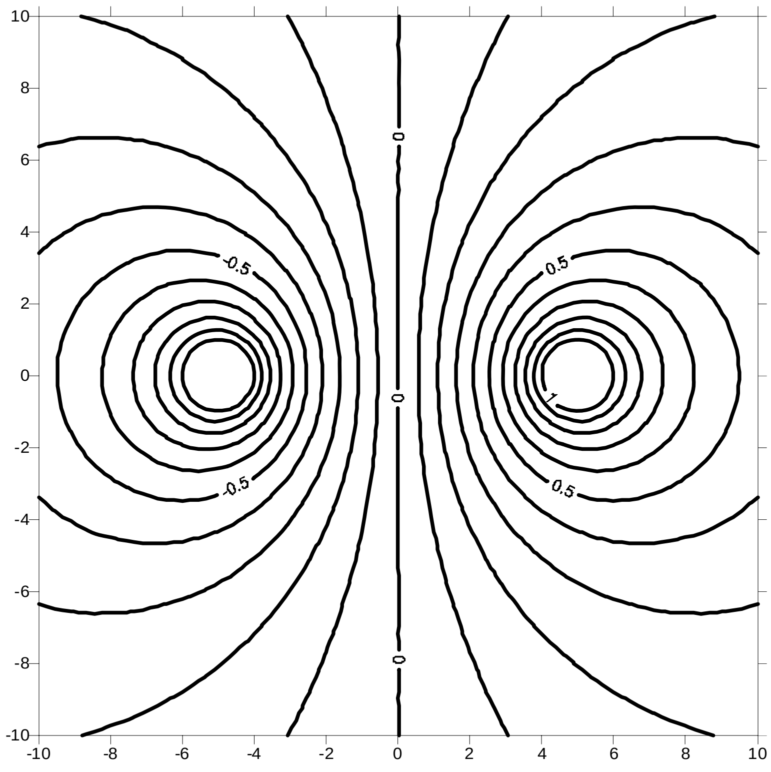


Two parallel cylinders held positive and negative potentials

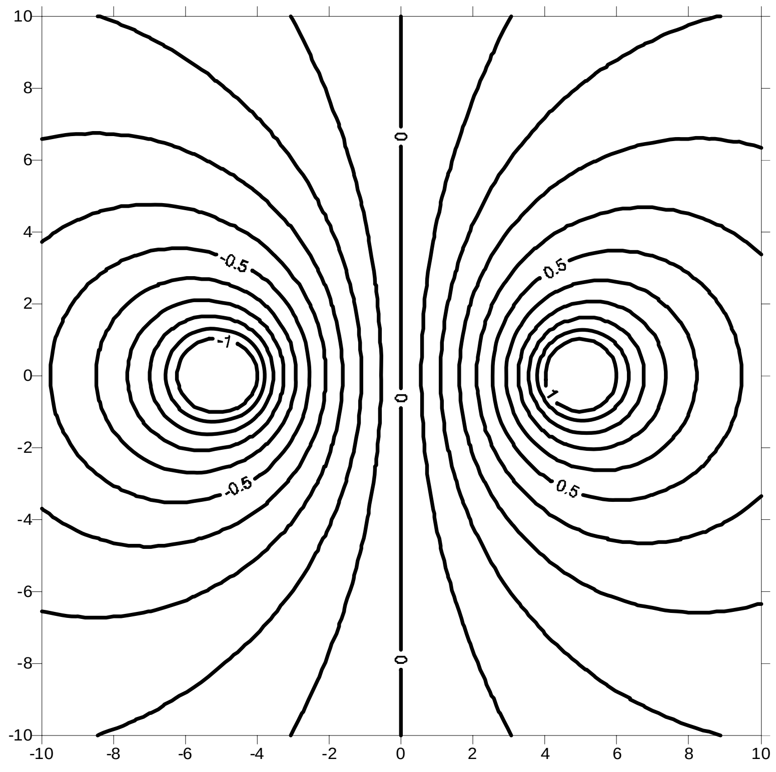


Hexagonal electrostatic potential

Contour plot of potential



Exact solution (Lebedev et al.)



Present method (M=10)

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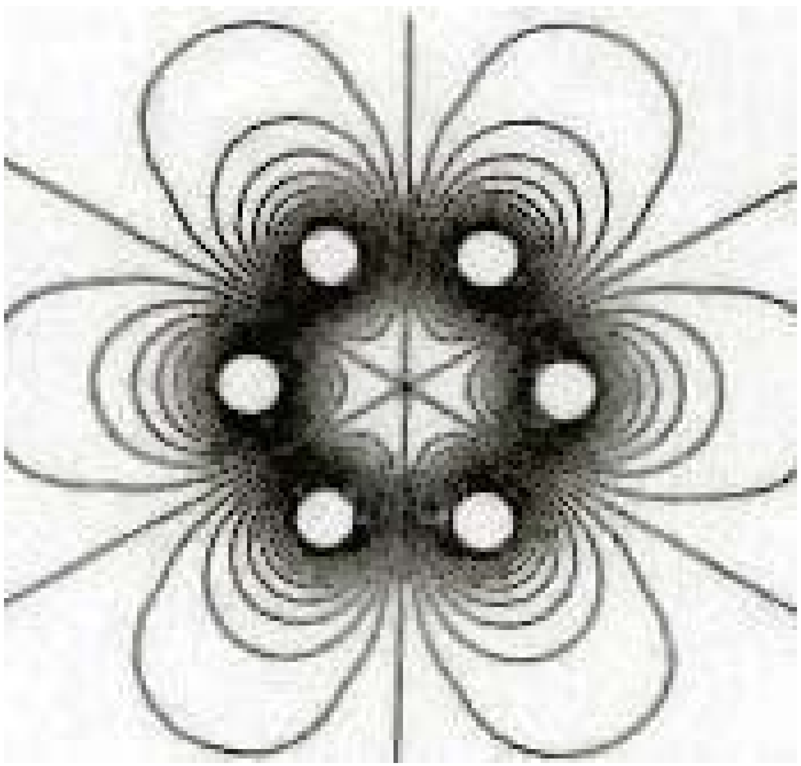
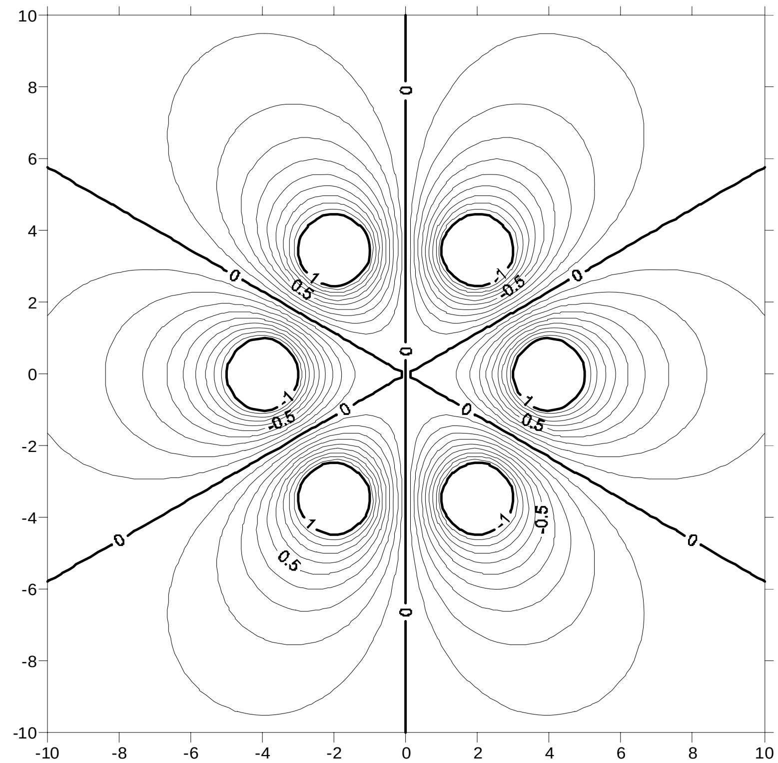
Contour plot of potential



Onishi's data (1991)



Present method (M=10)

Laplace equation

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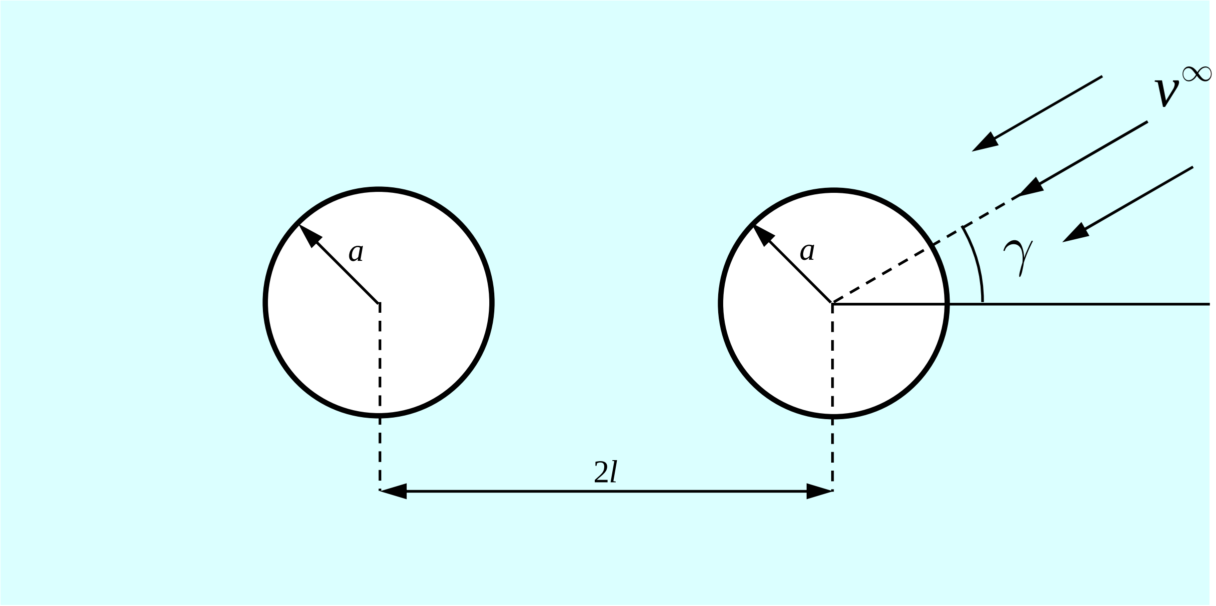
Flow of an ideal fluid pass two parallel cylinders



v^{∞} *is the velocity of flow far from the cylinders*

γ *is the incident angle*

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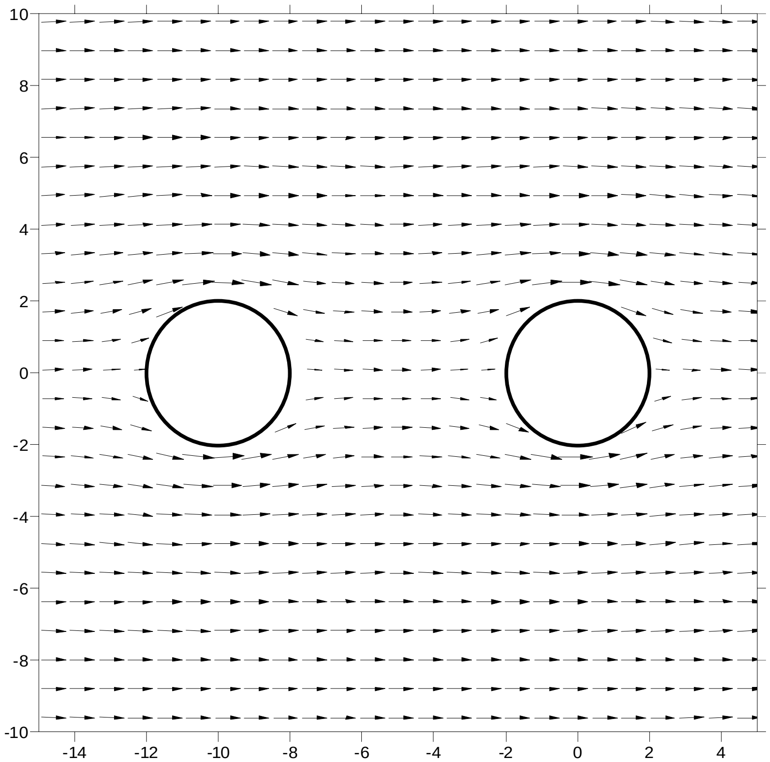
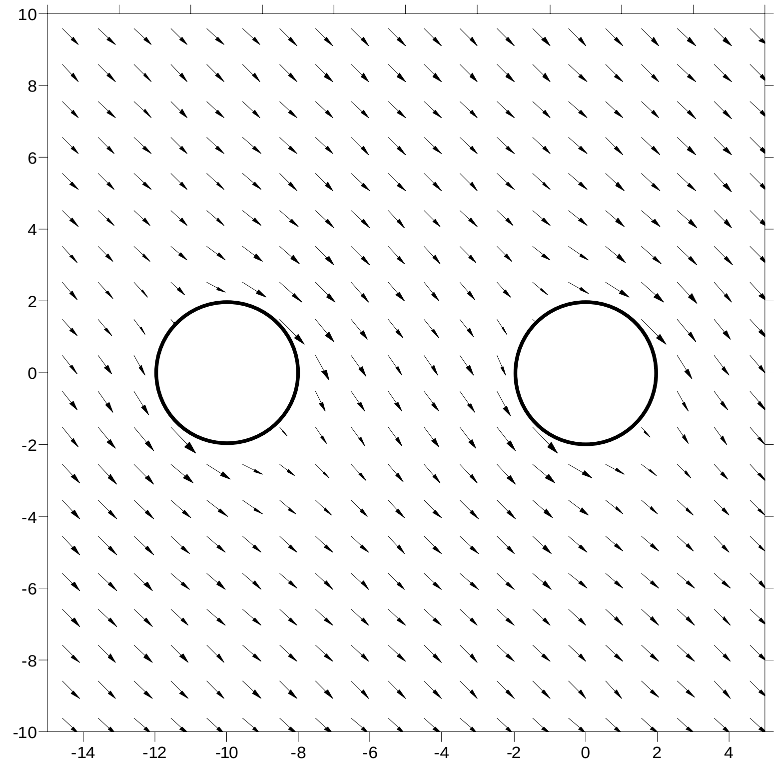
Velocity field in different incident angle

$\gamma = 180^\circ$



Present method (M=10)

$\gamma = 135^\circ$



Present method (M=10)

Laplace equation

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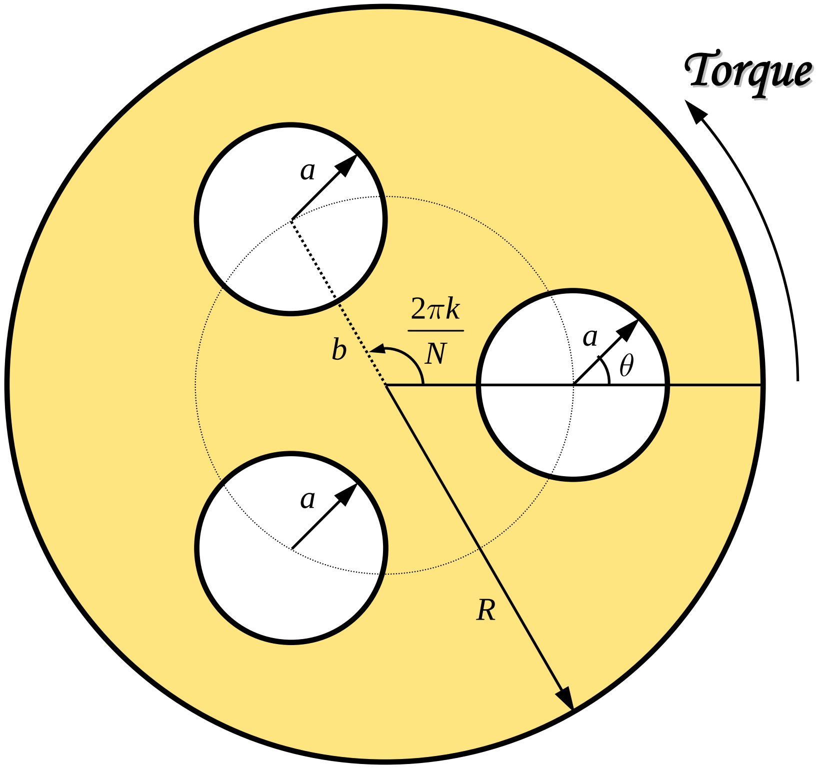
Torsion bar with circular holes removed



The warping function φ

$$\nabla^2 \varphi(x) = 0, \quad x \in D$$

Boundary condition

$$\frac{\partial \varphi}{\partial n} = x_k \sin\theta_k - y_k \cos\theta_k \quad \text{on} \quad B_k$$

where

$$x_i = b\cos\frac{2\pi i}{N}, \quad y_i = b\sin\frac{2\pi i}{N}$$

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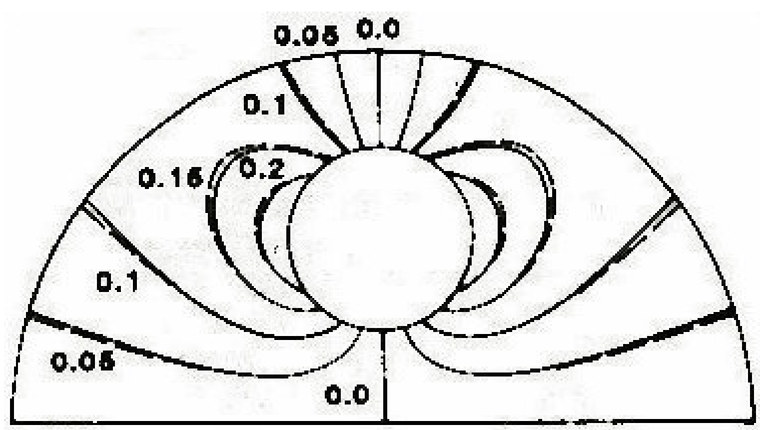
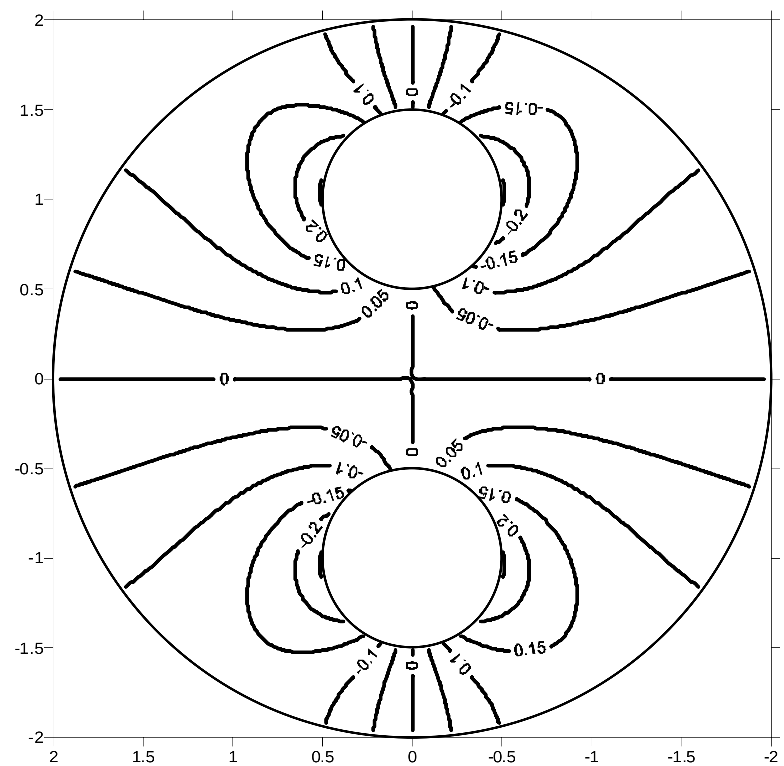
Axial displacement with two circular holes

Dashed line: exact solution
Solid line: first-order solution



Caulk's data (1983)
ASME Journal of Applied Mechanics



Present method (M=10)

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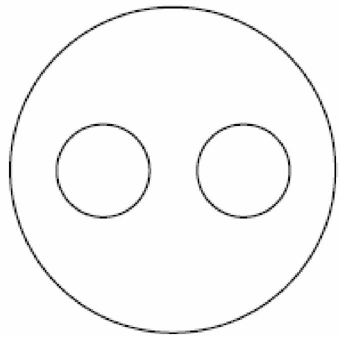
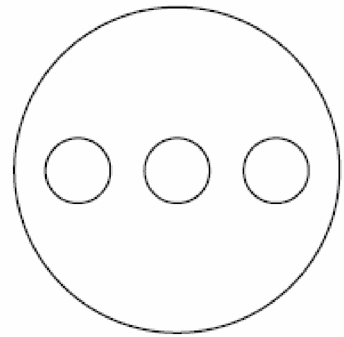
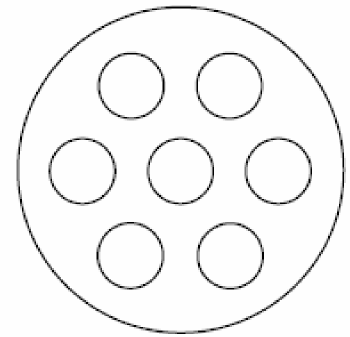
Torsional rigidity

	Case			
				
		$N=2, c/R=0$ $a/R=2/7, b/R=3/7$	$N=2, c/R=1/5$ $a/R=1/5, b/R=3/5$	$N=6, c/R=1/5$ $a/R=1/5, b/R=3/5$
$\frac{2G}{(\mu\pi R^4)}$	Caulk(First-order approximate)	0.8739	0.8741	0.7261
	Exact BIE formulation	0.8713	0.8732	0.7261
	Ling's results	0.8809	0.8093 ?	0.7305
	The present method	0.8712	0.8732	0.7245

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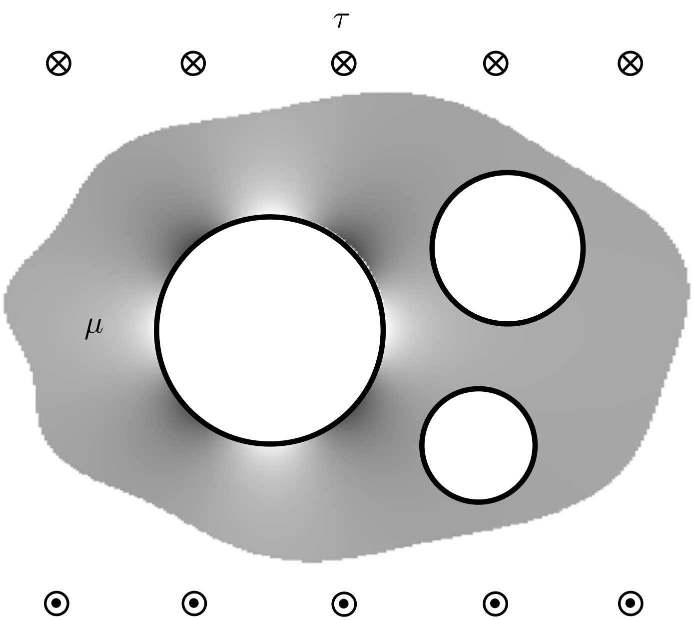
Infinite medium under antiplane shear



The displacement w^s

$$\nabla^2 w^s(x) = 0, \quad x \in D$$

Boundary condition

$$\frac{\partial w^s(x)}{\partial n} = \frac{\tau}{\mu}\sin\theta \quad \textit{on} \quad B_k$$

Total displacement

$$w = w^s + w^\infty$$

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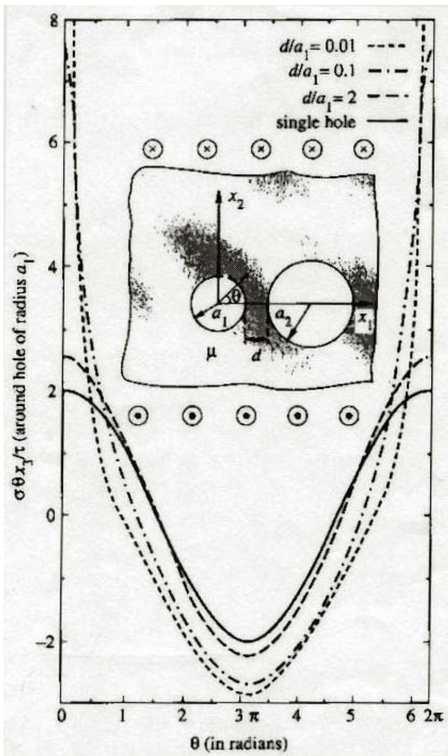
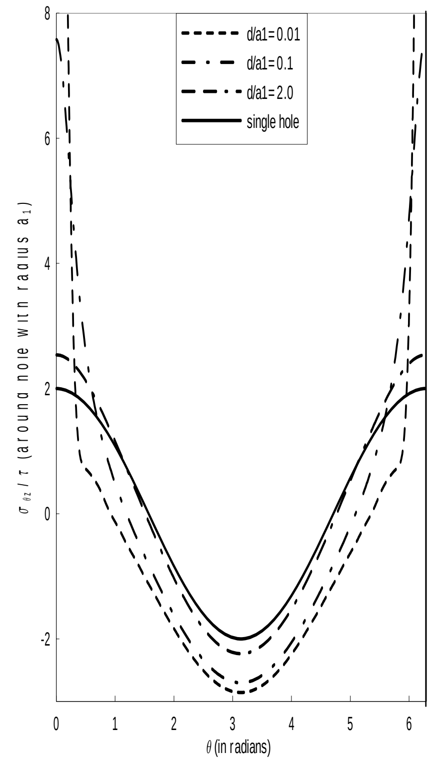
Shear stress $\sigma_{z\theta}$ around the hole of radius a_1 (x axis)



Honein's data (1992)

Quarterly of Applied Mathematics



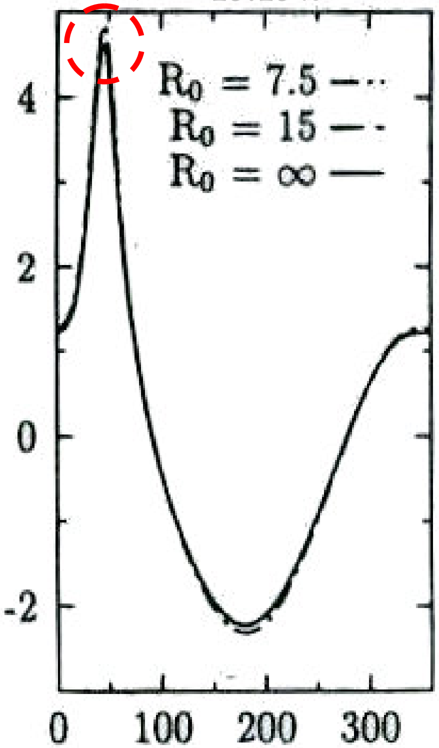
Present method (M=20)

Shear stress $\sigma_{z\theta}$ around the hole of radius a_1

Stress approach | *Analytical* | *Displacement approach*

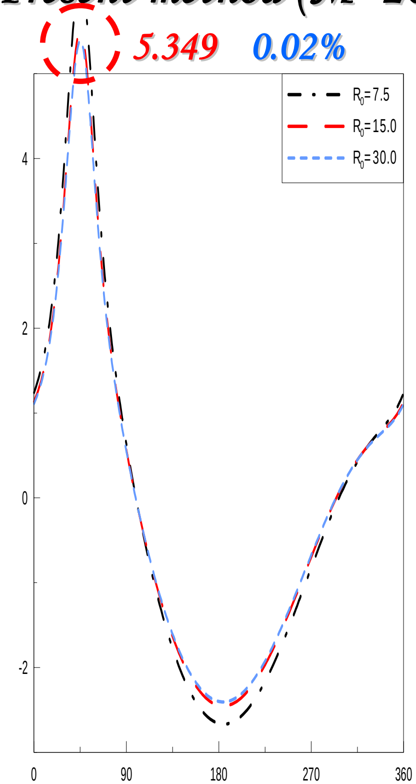
Steele's data (1992)

4.647 13.13%



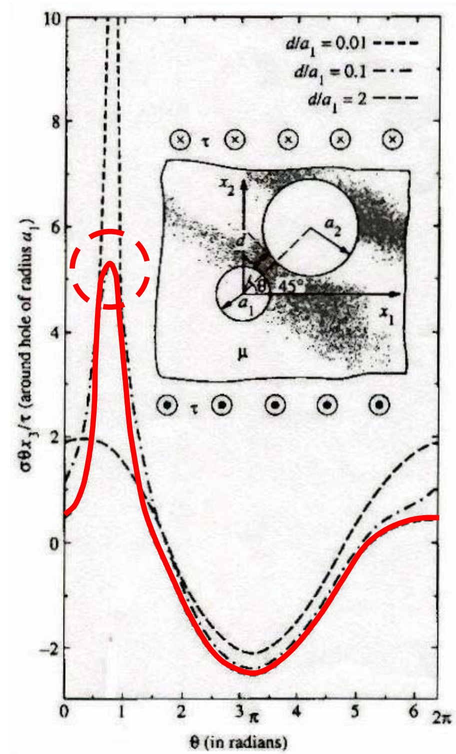
Present method (M=20)

5.349 0.02%



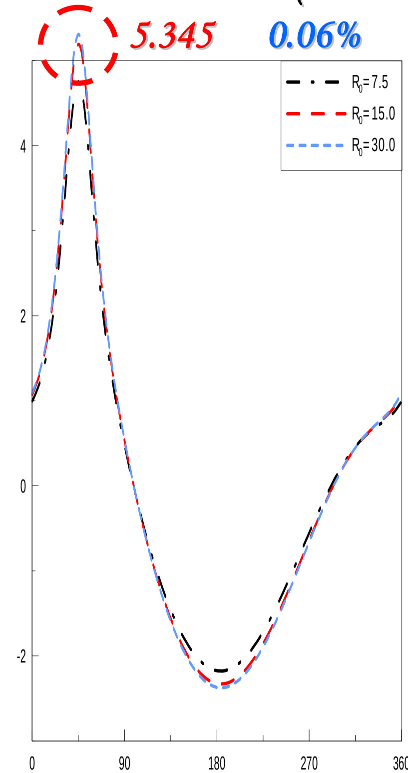
Honein's data (1992)

5.348



Present method (M=20)

5.345 0.06%



Laplace equation

- *Steady state heat conduction problems*
- *Electrostatic potential of wires*
- *Flow of an ideal fluid pass cylinders*
- *A circular bar under torque*
- *An infinite medium under antiplane shear*
- *Half-plane problems*

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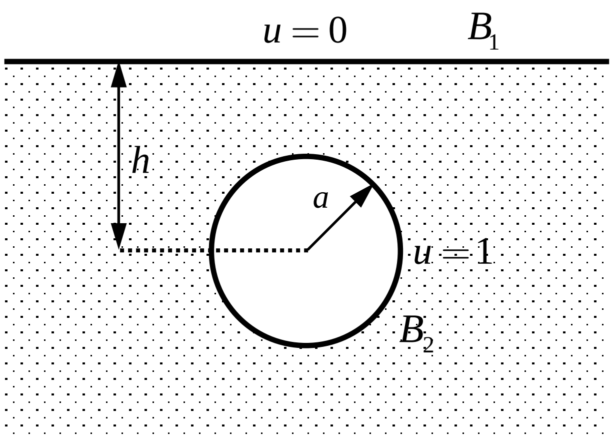
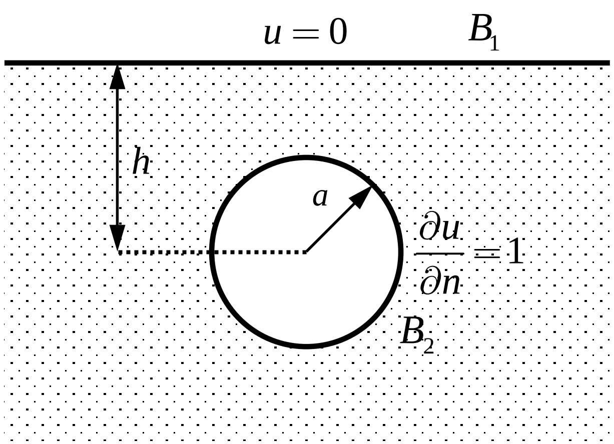
Half-plane problems



Dirichlet boundary condition
(Lebedev et al.)



Mixed-type boundary condition
(Lebedev et al.)

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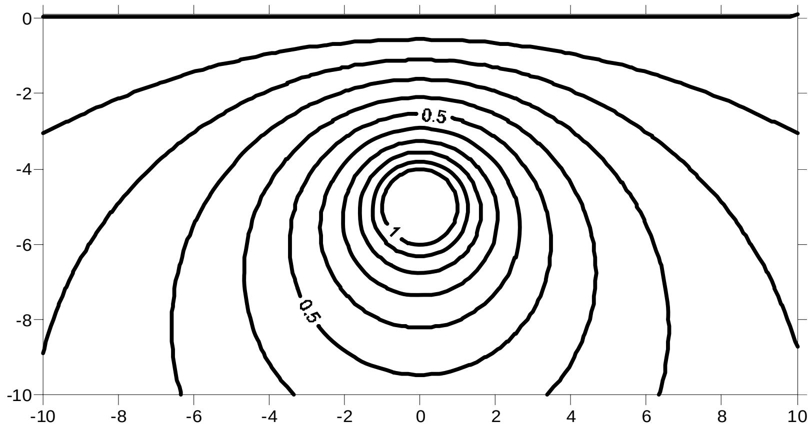
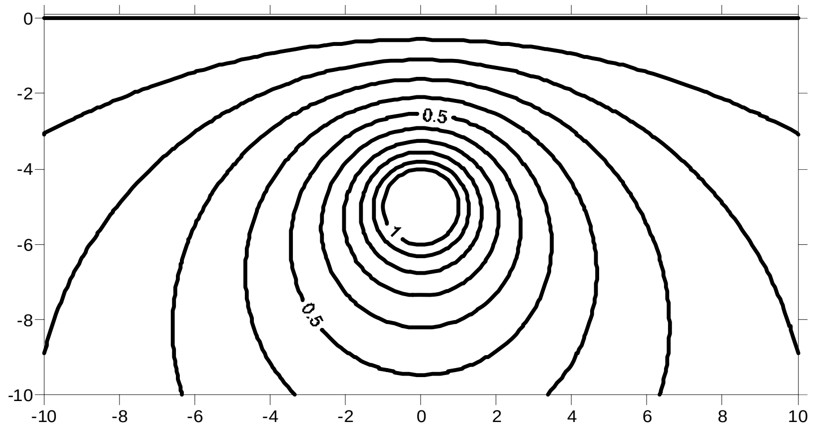
Dirichlet problem

Isothermal line



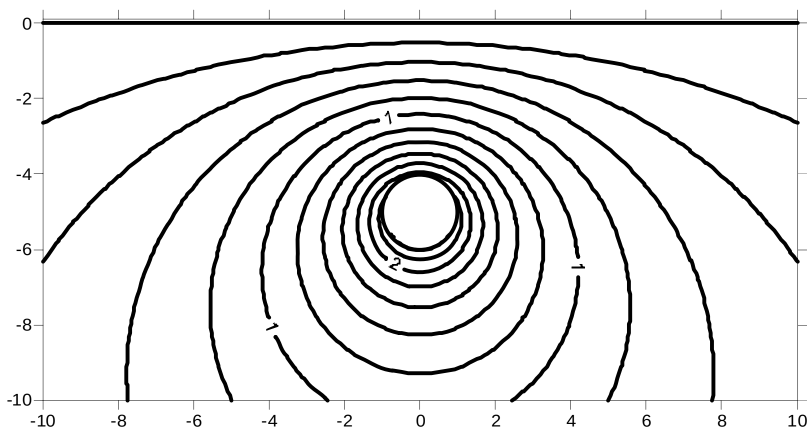
Exact solution (Lebedev et al.)



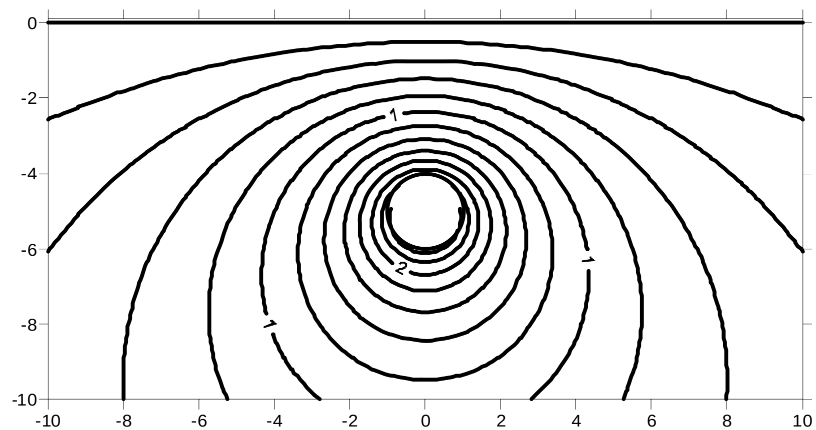
Present method (M=10)

Mixed-type problem

Isothermal line



Exact solution (Lebedev et al.)



Present method (M=10)

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Numerical examples

- *Laplace equation*
- *Eigen problem*
- *Exterior acoustics*
- *Biharmonic equation*

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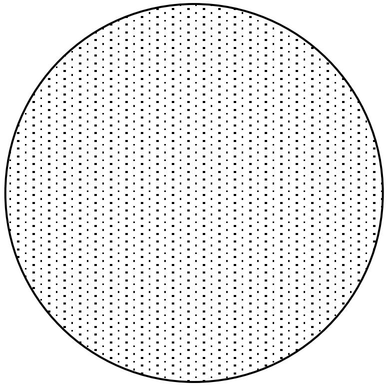
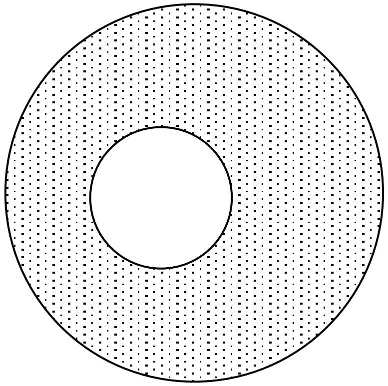
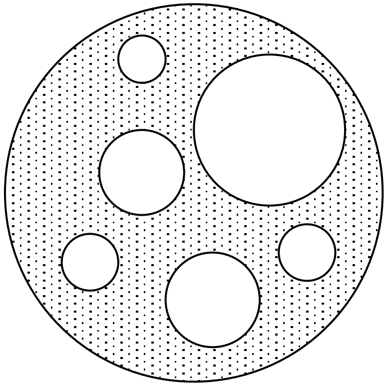
Problem statement



Simply-connected domain

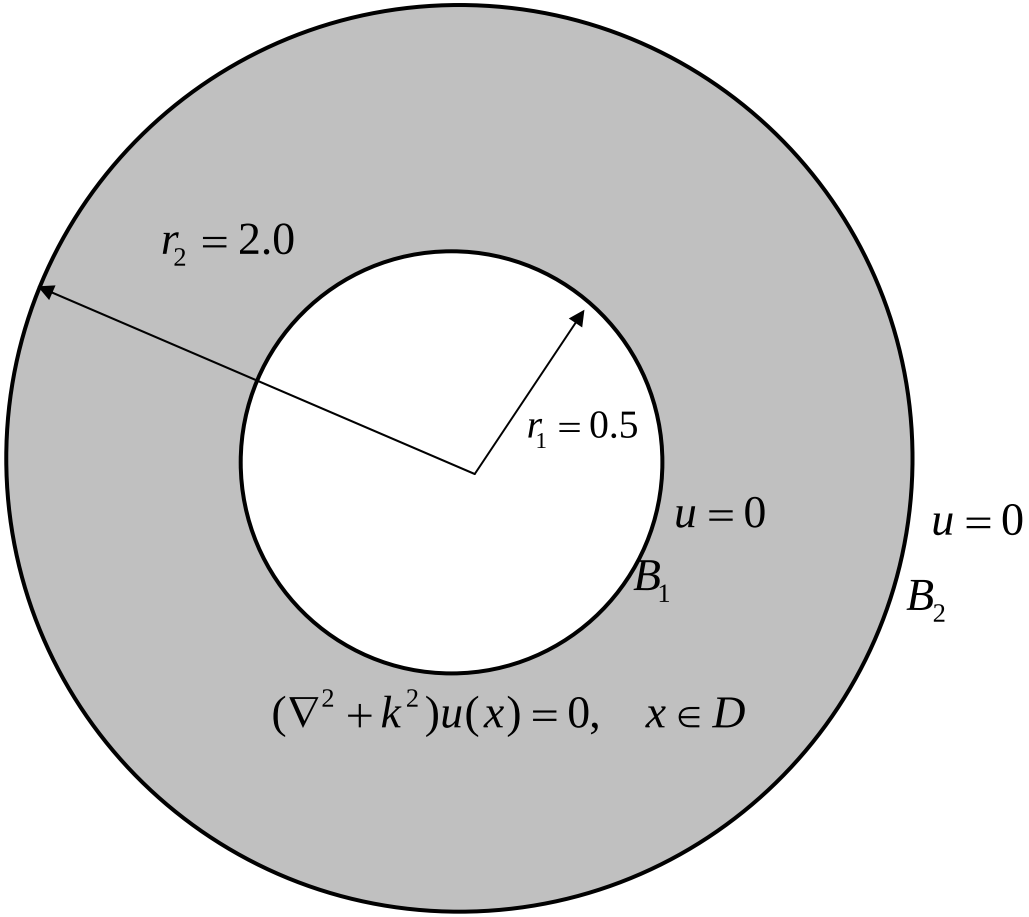


Doubly-connected domain



Multiply-connected domain

Example 1



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The former five true eigenvalues by using different approaches

	k_1	k_2	k_3	k_4	k_5
FEM (ABAQUS)	2.03	2.20	2.62	3.15	3.71
BEM (Burton & Miller)	2.06	2.23	2.67	3.22	3.81
BEM (CHIEF)	2.05	2.23	2.67	3.22	3.81
BEM (null-field)	2.04	2.20	2.65	3.21	3.80
BEM (fictitious)	2.04	2.21	2.66	3.21	3.80
Present method	2.05	2.22	2.66	3.21	3.80
Analytical solution[19]	2.05	2.23	2.66	3.21	3.80

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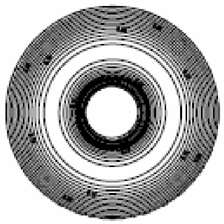
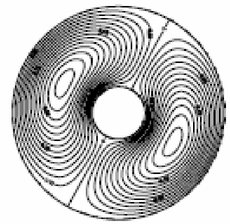
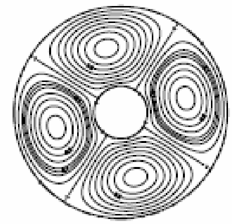
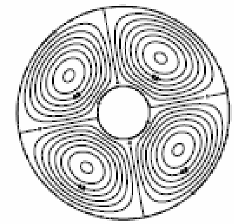
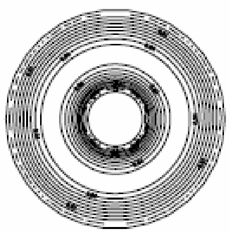
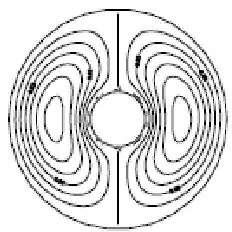
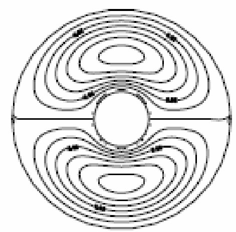
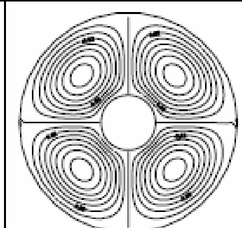
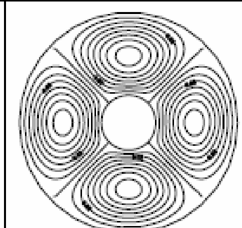
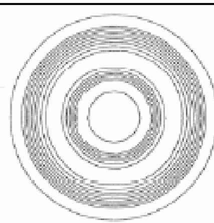
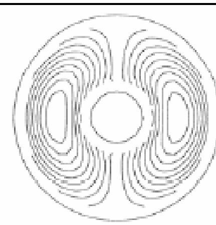
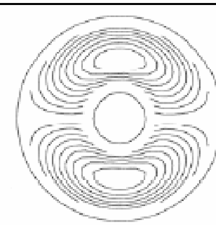
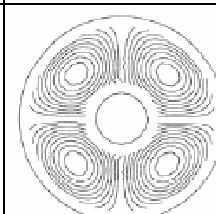
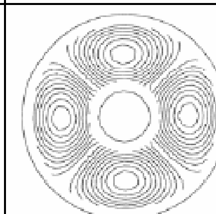
The former five eigenmodes by using present method, FEM and BEM

Method \ Mode	1	2	3	4	5
Present method					
	$k = 2.05$	$k = 2.22$	$k = 2.22$	$k = 2.66$	$k = 2.66$
BEM					
	$k = 2.06$	$k = 2.23$	$k = 2.23$	$k = 2.67$	$k = 2.67$
FEM					
	$k = 2.03$	$k = 2.20$	$k = 2.20$	$k = 2.62$	$k = 2.62$

Numerical examples

- *Laplace equation*
- *Eigen problem*
- *Exterior acoustics*
- *Biharmonic equation*

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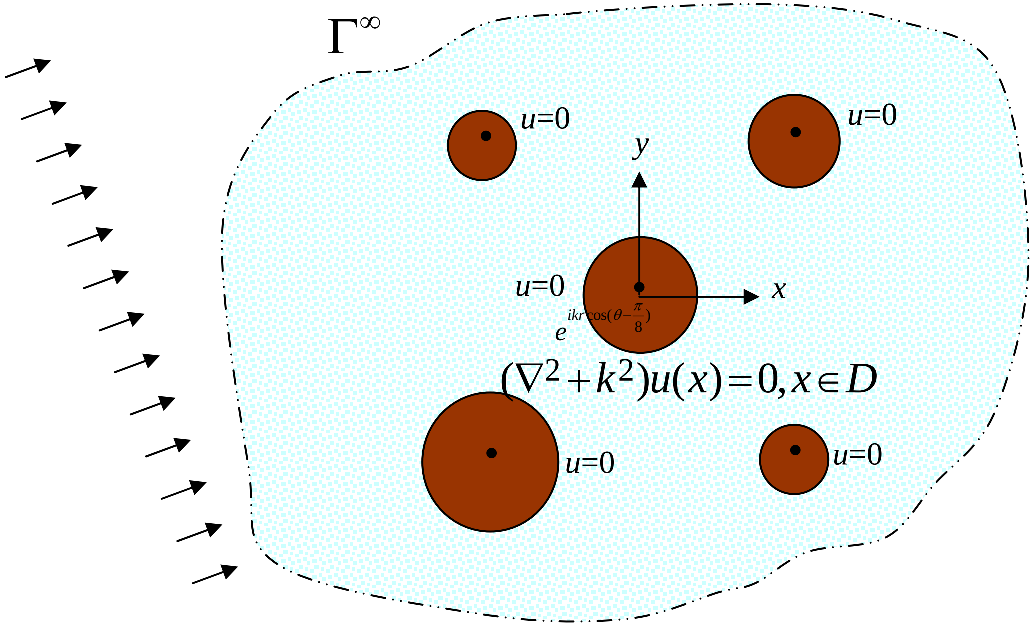
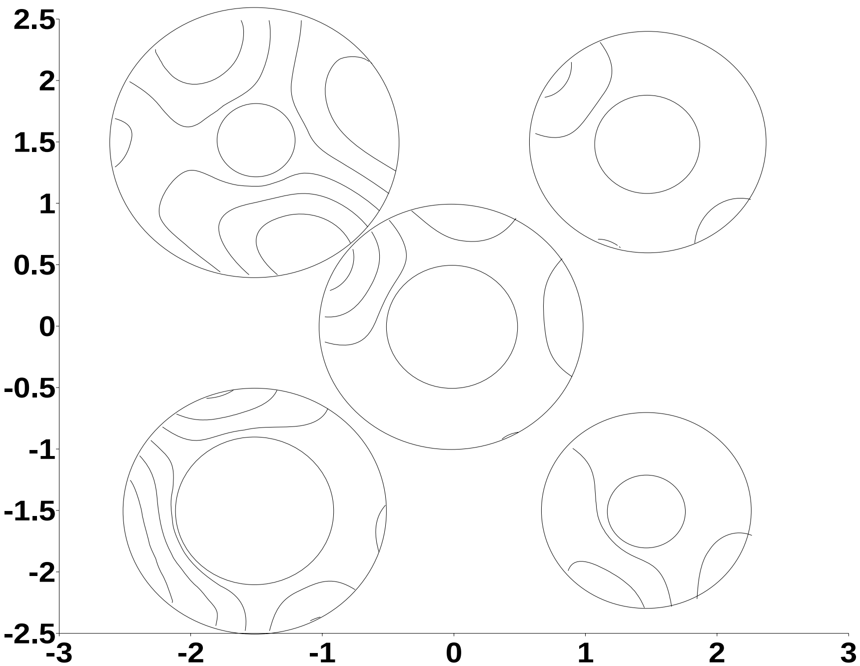
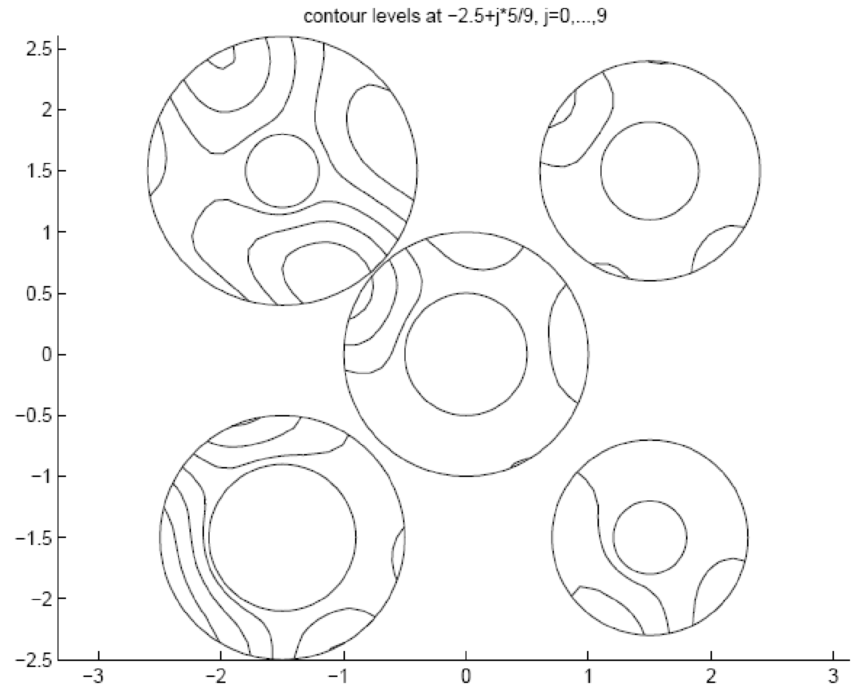
Sketch of the scattering problem (Dirichlet condition) for five cylinders



The contour plot of the real-part solutions of total field for $k = \pi$

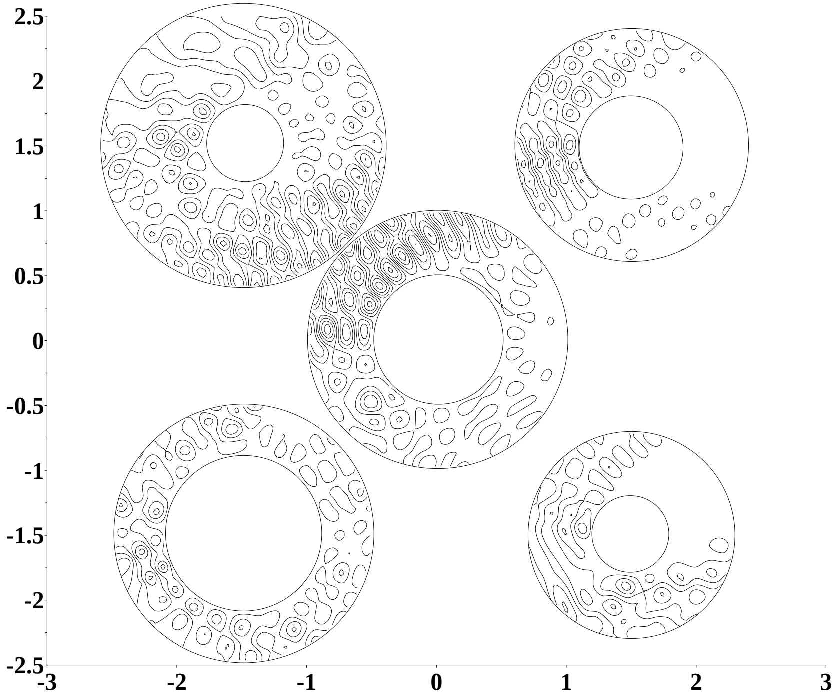


(a) Present method (M=20)

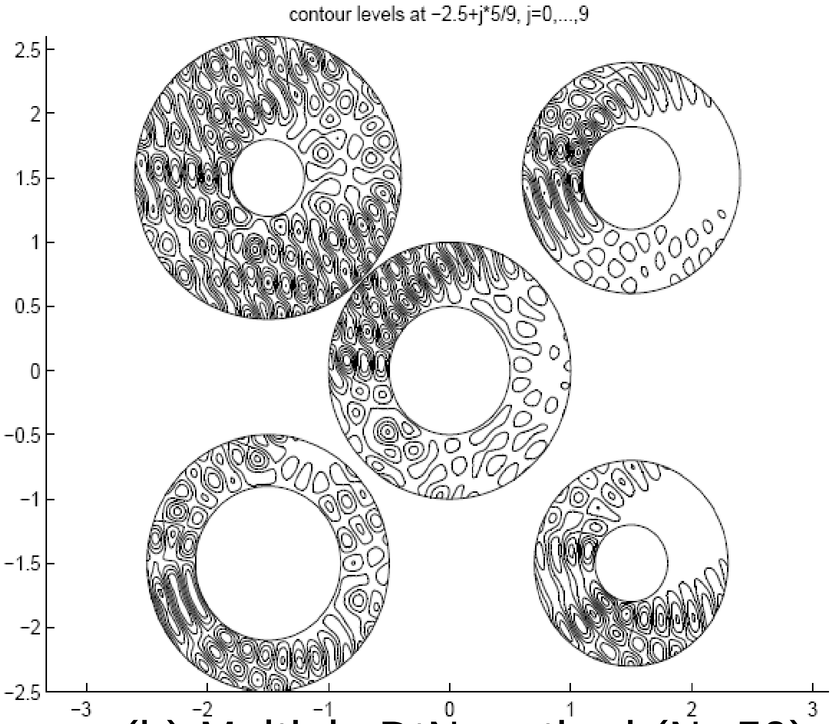


(b) Multiple DtN method (N=50)

The contour plot of the real-part solutions of total field for $k = 8\pi$



(a) Present method (M=20)



(b) Multiple DtN method (N=50)

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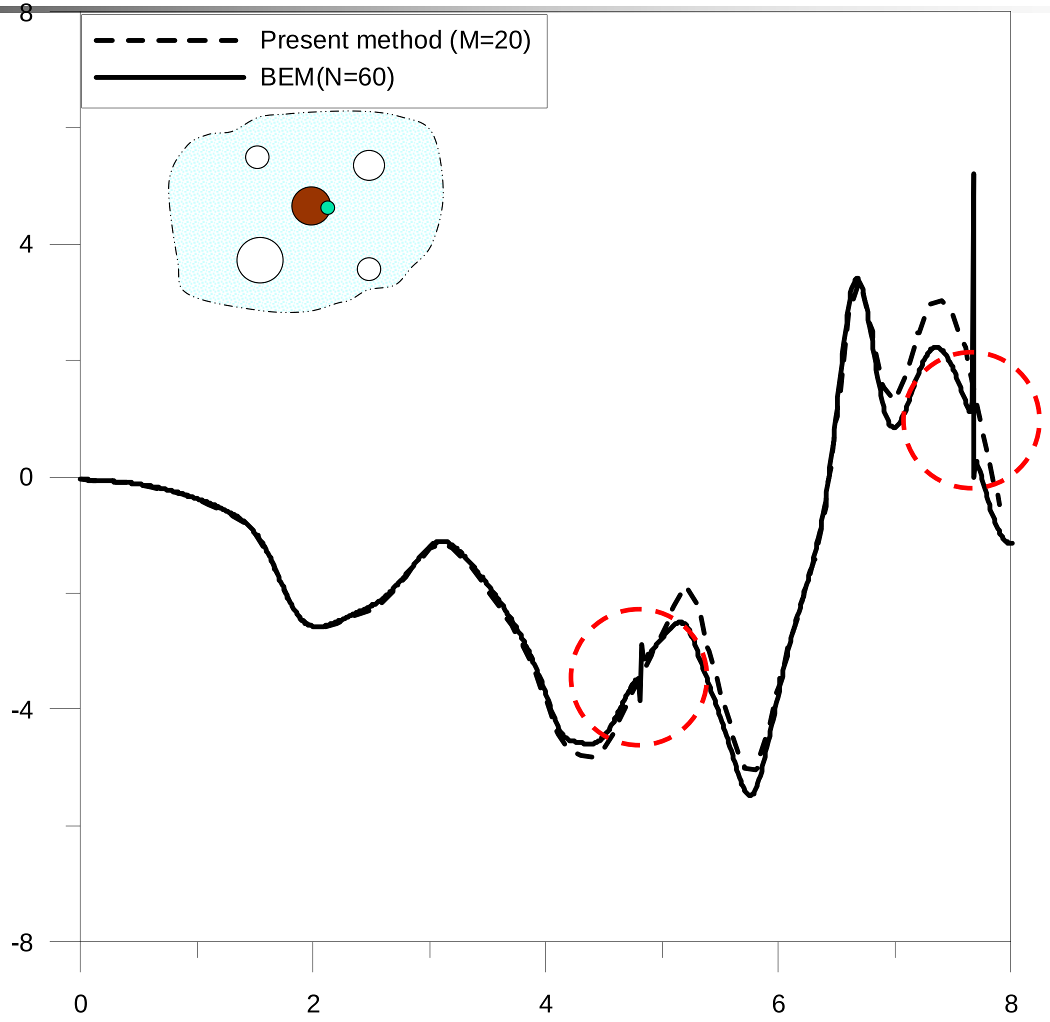
Fictitious frequencies



Numerical examples

- *Laplace equation*
- *Eigen problem*
- *Exterior acoustics*
- *Biharmonic equation*

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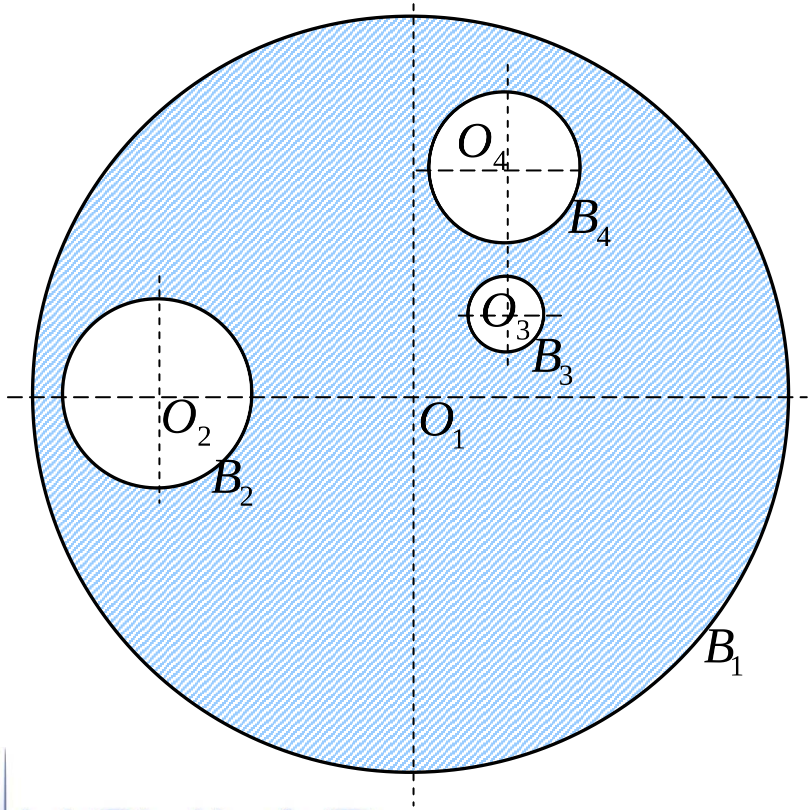
Plate problems



(Bird & Steele, 1991)

Geometric data:

$$O_1 = (0,0), R_1 = 20; \; O_2 = (-14,0), R_2 = 5;$$

$$O_3 = (5,3), R_3 = 2; \; O_4 = (5,10), R_4 = 4.$$

Essential boundary conditions:

$$u(s) = 0 \text{ and } \theta(s) = 0 \text{ on } B_1$$

$$u(s) = \sin\theta \text{ and } \theta(s) = 0 \text{ on } B_2$$

$$u(s) = -1 \text{ and } \theta(s) = 0 \text{ on } B_3$$

$$u(s) = 1 \text{ and } \theta(s) = 0 \text{ on } B_4$$

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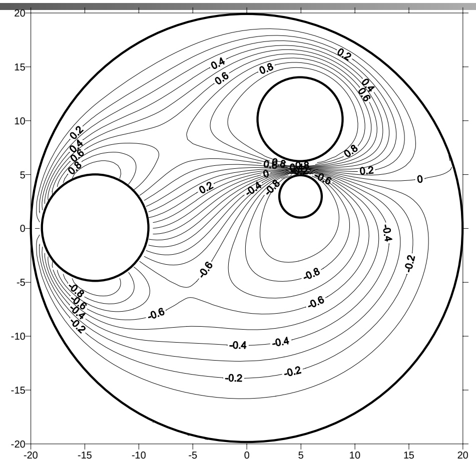
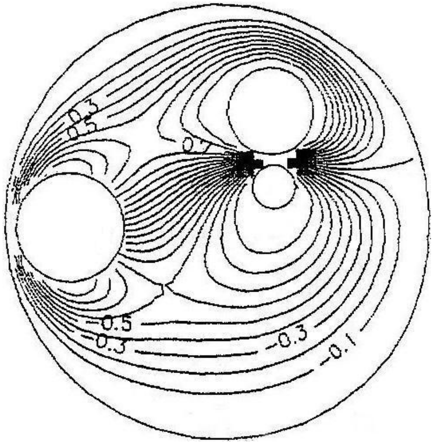
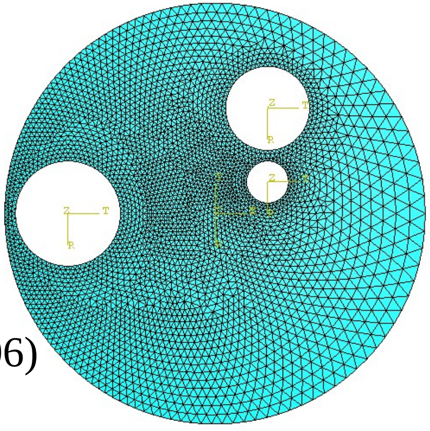
Contour plot of displacement



Present method (N=101)



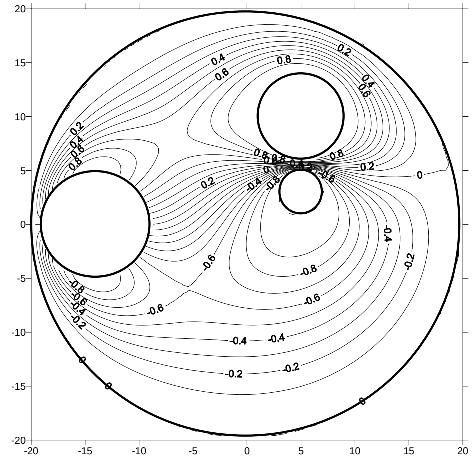
Bird and Steele (1991)



FEM mesh

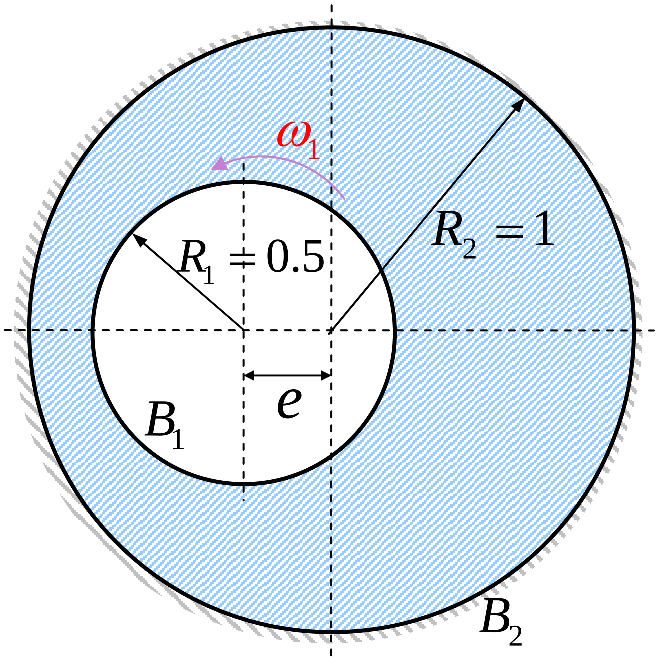
(No. of nodes=3,462,

No. of elements=6,606)



FEM (ABAQUS)

Stokes flow problem



Governing equation: $\nabla^4 u(x) = 0, \quad x \in \Omega$

Angular velocity: $\omega_1 = 1$

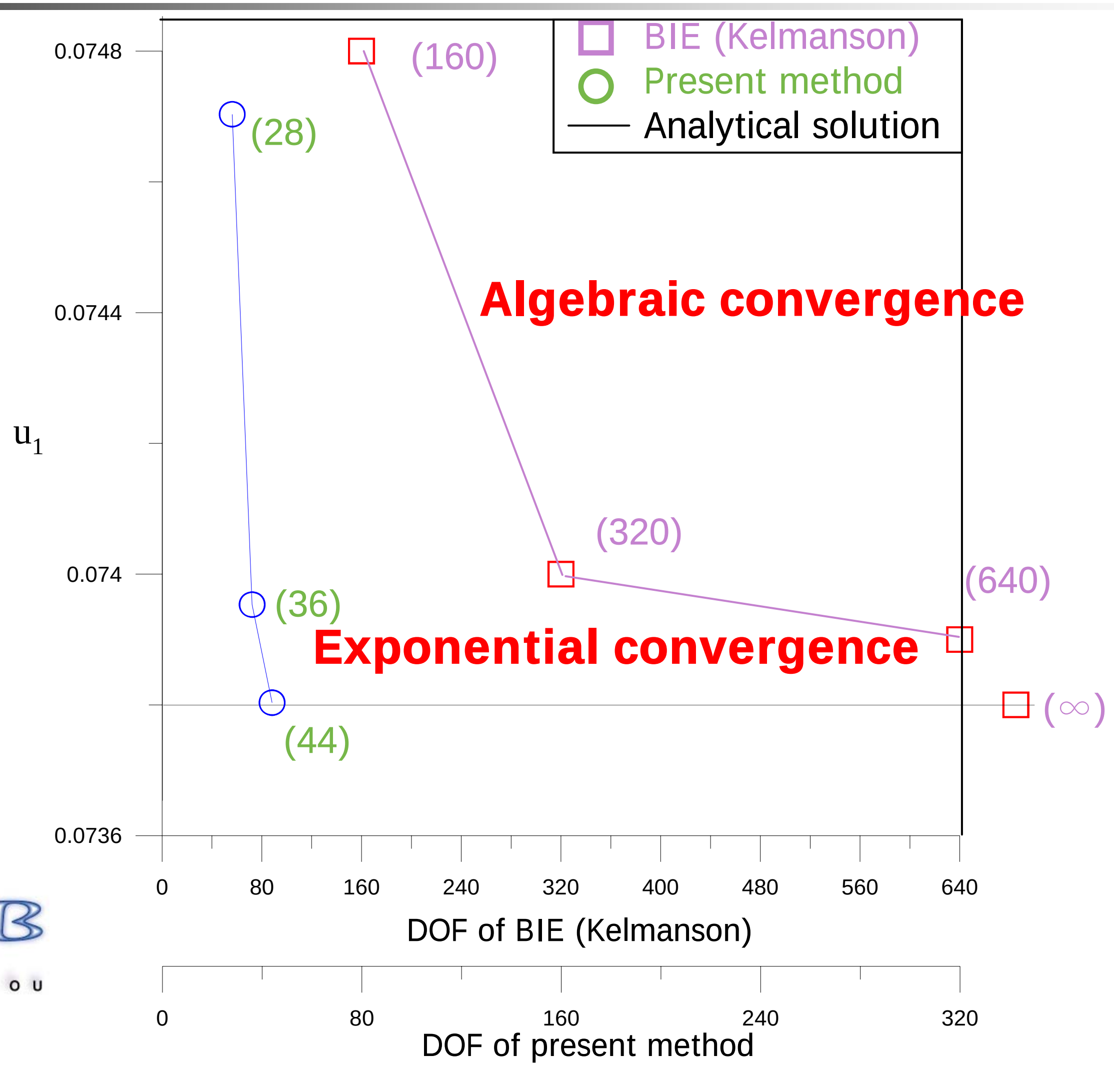
Boundary conditions:

$u(s) = u_1$ and $\theta(s) = 0.5$ on B_1

$u(s) = 0$ and $\theta(s) = 0$ on B_2 (Stationary)

Eccentricity: $\varepsilon = \dfrac{e}{(R_2 - R_1)}$

Comparison for $\varepsilon = 0.5$



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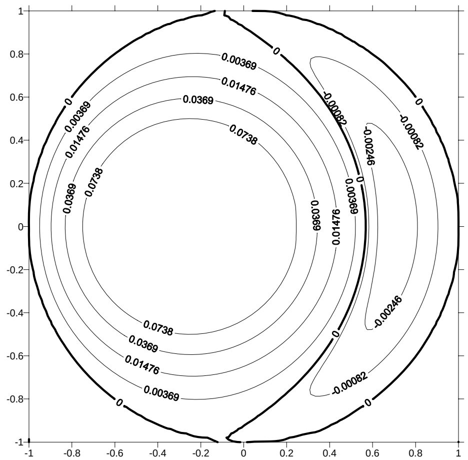
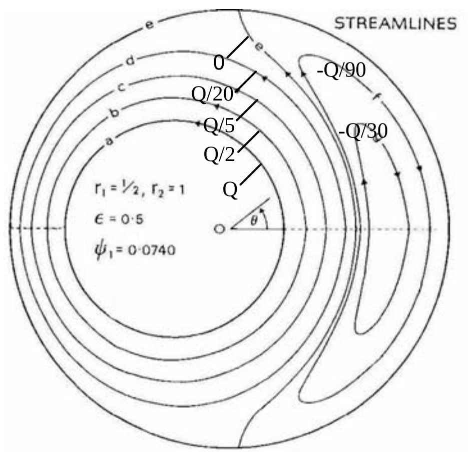
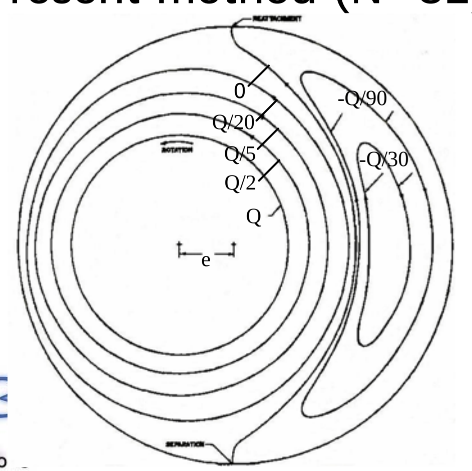
Contour plot of Streamline for $\varepsilon = 0.5$



Present method (N=81)



Kelmanson (Q=0.0740, n=160)



Kamal (Q=0.0738)

Outlines

- Motivation and literature review
- Mathematical formulation
  - Expansions of fundamental solution and boundary density
  - Adaptive observer system
  - Vector decomposition technique
  - Linear algebraic equation
- Numerical examples
- Conclusions

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Conclusions

- *A systematic approach using degenerate kernels, Fourier series and null-field integral equation has been successfully proposed to solve Laplace Helmholtz and Biharminic problems with circular boundaries.*
- *Numerical results agree well with available exact solutions, Caulk's data, Onishi's data and FEM (ABAQUS) for only few terms of Fourier series.*

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Conclusions

- *Engineering problems with circular boundaries which satisfy the Laplace Helmholtz and Biharminic problems can be solved by using the proposed approach in a more efficient and accurate manner.*
- *Free of boundary-layer effect*
- *Free of singular integrals*
- *Well posed*
- *Exponetial convergence*

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The End

Thanks for your kind attentions.

Your comments will be highly appreciated.

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