Interaction of Water Waves With Vertical Cylinders Using the Dual Boundary Element Method

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Abstract

The scattering of water waves by bottom-mounted vertical circular cylinders is solved by using the dual boundary element method. Both the nonuniqueness problems due to near-trapped mode and fictitious frequency are examined. The critical wave number for the near-trapped mode is numerically detected. It is found that near-trapped mode is a physical phenomenon and fictitious frequency stems from the numerical instability. The numerical oscillation near the fictitious frequency is observed by using BEM. The Burton and Miller approach is employed to deal with the problem of irregular frequencies. A numerical example of water wave-structure interaction by vertical circular cylinders was demonstrated to see the validity of the present formulation.

Keywords: scattering, water wave, dual boundary element method, near-trapped mode, irregular frequency

1. INTRODUCTION

For designing the offshore platforms mounted on the sea bed, such as oil platforms which consist of a number of legs, it is important to understand the interaction between the vertical cylinders and plane wave. For the problem of plane waves impinging on vertical cylinders, a closed-form solution of force on a single vertical cylinder was derived by MacCamy and Fuchs [1]. The similar analysis extended to two cylinders was investigated by Spring and Monkmeyer [2]. They used the addition theorem to analytically derive the scattered-wave solution. Not only equal size but also unequal size of the cylinders subject to the incident wave of arbitrary angle was analyzed. They claimed that their method is a direct approach, since they formulated the problem by using a linear algebraic system and the solution is obtained easily from a single matrix inversion. A different method presented by Twersky [3] is called the multiple-scattering approach. In his approach, he took one cylinder at a time and scattering coefficient was solved sequentially. Besides, the boundary conditions which they solved are also different. Simon [4] as well as McIver and Evans [5] proposed an approximate solution based on the assumption that the cylinders are widely spaced. Later, Linton and Evans [6] also used the same approximate method which proposed earlier by Spring and Monkmeyer [2]. The main contribution was to provide a simple formula for the potential on the surfaces of the cylinders which makes the computation of forces much more straightforward. However, their results of four cylinders [6] were incorrect and corrigendum was given in Linton and Evans [7], Chen, Lee and Lin [8].

Trapped modes and near-trapped modes appear in different fields, such as string vibration, hydraulic engineering, earthquake engineering, ocean engineering and physics as described below. （1）String vibration：A string vibration can be modeled by a wave equation. When this system is subject to an incident wave, the spring may have a near-trapped mode under a certain spring layout and incident wave number. （2）Hydraulic engineering：In hydraulic engineering, certain harmonic waves may exist at a depth discontinuity, but are unable to propagate from shallow to deep water [9, 10]. This phenomenon also belongs to one kind of near-trapped modes. （3）

Earthquake engineering：In earthquake engineering, surface wave may seriously result in damage for structures. For a thin-layer inclusion in a half-space medium, trapped wave may be present, e.g., Love wave [11] or Stonely wave. （4）Ocean engineering：In ocean engineering, the construction of offshore platform is subjected to wave loads all the year. Duclos and Clèment [12] and Williams and Li [13] proposed a simplified model of linear theory to simulate the interaction between cylinders subject to the incident wave. In this analysis, a specific distance between cylinders in companion with a certain wave number may cause a near-trapped mode. Discussions on this topic will be addressed in this paper. （5）Quantum physics：The trapped mode, previously mentioned, occurs in physics as well as engineering. The bound state in a square-well potential in quantum mechanics is another case of trapped modes.

We will study the near-trapped mode by using dual boundary element approach. The near-trapped modes in physics were observed in a consistent way of other works [14]. The dual boundary integral equation approach was successfully employed to solve the scattering problems of plane wave[15], membrane with degenerate boundary [16] and vibration of plate [17]and membrane with multiple connected domain problems[18].

By using the dual boundary element method to solve the water wave problem, two kinds of peaks for the resultant force on the cylinder versus the wave number are observed. One is the near-trapped modes, another is the irregular frequency. The existence of the irregular frequencies represents the drawback of the boundary integral equation method, Ohmatsu [19] presented a combined integral equation method (CIEM) which was similar to the CHIEF-block method for acoustics proposed by Wu and Seybert [20]. Many researchers on exterior acoustics also encountered the problem of the fictitious frequency. To the authors' best knowledge, no paper simultaneously discussed the near-trapped mode in physics and fictitious frequency in mathematics. When near-trapped modes and fictitious frequencies both appear, how to recognize the source of nonuniqueness becomes an interesting and important issue.

Regarding the fictitious frequency, two ideas of Burton and Miller (B&M) method [21] and CHIEF [22] approach, have been proposed to deal with this problem. The former one needs hypersingular formulation while the latter one may take risk once the CHIEF point falls in the node of corresponding interior modes. Based on the circulant properties and degenerate kernels, by using the CHIEF method [23], an analytical and numerical experiment in discrete system of a cylinder is achieved. Dokumaci [24] and Juhl [25] both pointed out that numerical oscillation becomes serious as the number of boundary elements increases.

In this paper, we focus on the dual boundary element method to solve the scattering of water waves. Both the near-trapped mode (nonuniqueness in physics) and the fictitious frequency (nonuniqueness in mathematics) are addressed and extracted out in this paper by using the Burton and Miller method. In Section 2, we introduce the formulation of dual boundary integral equation for the Helmholtz problem. The formulation of Burton and Miller method is formulated in Section 3. An example is demonstrated in Section 4 for the near-trapped mode by using Burton and Miller method.

2. Problem statement and integral formulation

2.1 Problem statement

Now we assume *N* vertical cylinders mounted at $z=-h$ upward to the free surface as shown in Figure 1. The governing equation of the water wave problem is the Laplace equation

$$\nabla^2\Phi(x,y,z;t)=0,\quad (x,y,z)\in D, \tag{1}$$

where ∇^2 and *D* are the Laplacian operator and the domain of interest, respectively, and $\Phi(x,y,z,t)$ is the velocity potential which satisfies the boundary conditions of seabed, kinematic boundary condition at free surface and dynamic boundary condition at free surface as shown below:

$$-\frac{\partial\Phi}{\partial n}=0,\quad z=-h(x,y), \tag{2}$$

$$-\Phi_z=H_t-\Phi_x H_x-\Phi_y H_y,\quad z=H(x,y,t), \tag{3}$$

$$-\Phi_t+gz+\frac{1}{2}(\Phi_x^2+\Phi_y^2+\Phi_z^2)=B(t),\quad z=H(x,y,t), \tag{4}$$

in which g is the gravity acceleration, $H(x,y,t)$ is the free-surface elevation and $B(t)$ is the Bernoulli constant. Based on the linear water wave theory and using the technique of separation variable for space and time, we have

$$\Phi(x,y,z,t)=\mathrm{Re}\{\phi(x,y)f(z)e^{-i\omega t}\} \tag{5}$$

where

$$f(z)=\frac{-igA}{\omega}\frac{\cosh k(z+h)}{\cosh kh} \tag{6}$$

in which ω is the angular frequency, k represents the wave number and equals to ω over wave speed, $H(x,y,t)$ can be defined by

$$H(x,y,t)=\mathrm{Re}\{\eta(x,y)e^{-i\omega t}\} \tag{7}$$

where

$$\eta(x,y)=A\phi(x,y) \tag{8}$$

and A represents the amplitude of incident wave of angle β as shown below:

$$\phi_I(x,y)=e^{ik(x\cos\beta+y\sin\beta)}\equiv e^{ikr\cos(\theta-\beta)}. \tag{9}$$

Substituting Eq.(5) into Eq.(1), we have

$$(\nabla^2+k^2)\phi(x,y)=0,\quad (x,y)\in D. \tag{10}$$

Rigid cylinders yield the Neumann boundary conditions as shown below:

$$\frac{\partial\phi(x,y)}{\partial n}=0,\ \ (x,y)\in B. \tag{11}$$

The dispersion relationship is

$$k\tanh kh=\frac{\omega^2}{g}. \tag{12}$$

The dynamic pressure can be obtained by

$$p=-\rho\frac{\partial\Phi}{\partial t}=\rho gA\frac{\cosh k(z+h)}{\cosh kh}\phi(x,y)e^{-i\omega t}. \tag{13}$$

The two components of the first-order force X^j on the jth cylinder are given by integrating the pressure over the circular boundary as shown below:

$$X^j=-\frac{\rho gAa_j}{k}\tanh kh\int_0^{2\pi}\phi(x,y)\begin{Bmatrix}\cos\theta_j\\ \sin\theta_j\end{Bmatrix}d\theta_j, \tag{14}$$

where a_j denotes the radius of the jth cylinder.

2.2 Dual boundary integral equations

The integral equation for the domain point can be derived from the third Green's identity, we have

$$2\pi u(x)=\int_B T(s,x)u(s)\,dB(s)-\int_B U(s,x)t(s)\,dB(s),\ \ x\in D, \tag{15}$$

$$2\pi t(x)=\int_B M(s,x)u(s)\,dB(s)-\int_B L(s,x)t(s)\,dB(s),\ \ x\in D, \tag{16}$$

where s and x are the source and field points, respectively, D is the domain of interest, $t(s)=\frac{\partial u(s)}{\partial n_s}$, n_s and n_x denote the outward normal vectors at the source point s and field point x, respectively. The kernel function,

$U(s,x) = -\frac{\pi i}{2} H_0^{(1)}(kr)$, is the fundamental solution which satisfies

$$\nabla^2 U(s,x) = 2\pi\delta(x-s), \tag{17}$$

where $\delta(x-s)$ denotes the Dirac-delta function, $H_n^{(1)}(kr) = J_n(kr) + iY_n(kr)$ is the *n*-th order Hankel function of the first kind, J_n is the *n*-th order Bessel function of the first kind, Y_n is the *n*-th order Bessel function of the second kind, $r = |x-s|$, $i^2 = -1$. The other kernel functions, *T*(*s*,*x*), *L*(*s*,*x*), and *M*(*s*,*x*), are defined by

$$T(s,x) = \frac{\partial U(s,x)}{\partial n_s}, \tag{18}$$

$$L(s,x) = \frac{\partial U(s,x)}{\partial n_x}, \tag{19}$$

$$M(s,x) = \frac{\partial^2 U(s,x)}{\partial n_s \partial n_x}. \tag{20}$$

By moving the field point to the boundary, Eqs.(15) and (16) reduce to

$$\pi u(x) = C.P.V.\int_B T(s,x)u(s)\,dB(s) - R.P.V.\int_B U(s,x)t(s)\,dB(s), x \in B, \tag{21}$$

$$\pi t(x) = H.P.V.\int_B M(s,x)u(s)\,dB(s) - C.P.V.\int_B L(s,x)t(s)\,dB(s), x \in B, \tag{22}$$

where *R.P.V.*, *C.P.V.* and *H.P.V.* denote the Riemann principal value (Riemann sum), Cauchy principal value and Hadamard principal value (or Hadamard finite part), respectively. Once the field point *x* locates outside the domain ($x \in D^c$), we obtain the dual null-field integral equations as shown below

$$0 = \int_B T(s,x)u(s)\,dB(s) - \int_B U(s,x)t(s)\,dB(s), \quad x \in D^c, \tag{23}$$

$$0 = \int_B M(s,x)u(s)\,dB(s) - \int_B L(s,x)t(s)\,dB(s), \quad x \in D^c, \tag{24}$$

where D^c is the complementary domain. Equations (15), (16), (23) and (24) are conventional formulations where the point can not be located on the real boundary. Singularity occurs and concept of principal values is required once Eqs.(21) and (22) are considered. The flux *t*(*s*) is the directional derivative of *u*(*s*) along the outer normal direction at *s*. For the interior point, *t*(*x*) is artificially defined. For example, $t(x) = \partial u(s)/\partial x_1$, if $\mathbf{n} = (1,0)$ and $t(x) = \partial u(x)/\partial x_2$, if $\mathbf{n} = (0,1)$ where (x_1, x_2) is the coordinate of the field point *x*.

3. BURTON AND MILLER METHOD

Burton and Miller method combined the singular integral equation and its normal derivative (hypersingular integral equation) with an imaginary constant. This method was valid for all wave numbers.

$$[[T] + \frac{i}{k}[M]]\{u\} = [[U] + \frac{i}{k}[L]]\{t\}. \tag{25}$$

4. ILLUSTRATIVE EXAMPLES

We consider the water wave problem by an array of four bottom-mounted vertical rigid circular cylinders with the same radius a located at the vertices of a square (0, 0), (b, b), (2b, 0), (b, -b), as shown in Figure 2. By considering the incident wave in the direction of 45 degrees Figure 3 show the free-surface elevation at point p versus the wave number by using the boundary element method and the Burton and Miller method, respectively. It's interesting to find that two kinds of peak occur in the Figure 3. One kind of peak appears at the corresponding wave number which happens to be the zeros of Bessel function $J_n(kr)$, e.g., 3.8317 ($J_{1,1}$), 5.1356 ($J_{2,1}$), 5.5201 ($J_{0,2}$) and 6.3802 ($J_{3,1}$), etc.… The other peak occurs at the wavenumber of k=4.082 which is physically-realizable as a near-trapped mode. Figure 3 also shows that the peak due to fictitious (irregular) frequency is suppressed to be smooth after using the Burton and Miller method, while the peak value still exists there. It's explained that fictitious frequencies occur when we employ BEM to solve exterior Helmholtz problems. It belongs to numerical resonance instead of physical phenomenon. The maximum free-surface elevation amplitude is plotted in Figure 4. It agrees well with that of the plane wave BEM by Perrey-Debain *et al.* [26]. Figure 5 shows the forces in the direction of wave advance versus the wavenumber (ka). It can be found that the peak force on cylinders 1 and 3 is about 14 times force of an isolated cylinder at the wavenumber ka=4.082. This phenomenon is the physical ringing. The peak value due to the near-trapped mode is compared well with those of Evans and Porter [14]. Peak value due to the near-trapped mode (physics) and fictitious frequency (mathematics) are both observed in the present paper since BEM was utilized. Fictitious frequency is not present in the work of Evans and Porter [14] since they employed the Twersky's method.

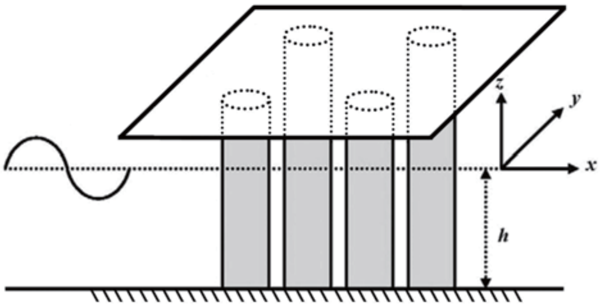

Fig. 1 *Problem statements of water waves with an array of vertical cylinders*

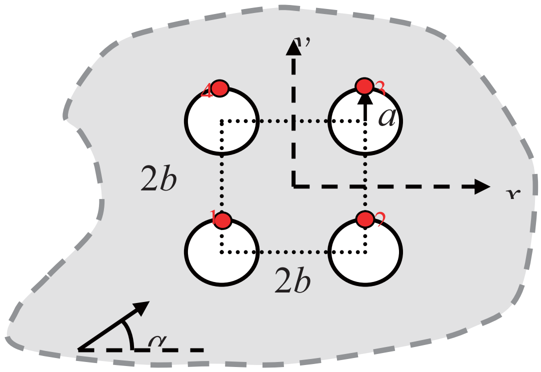

Fig. 2 *Interaction of an incident water wave with four cylinders*

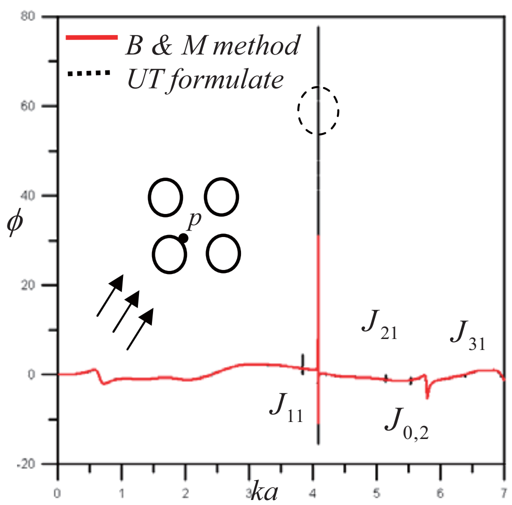

Fig. 3 *Free-surface elevation versus wavenumbe by using the UT formulation and B & M*

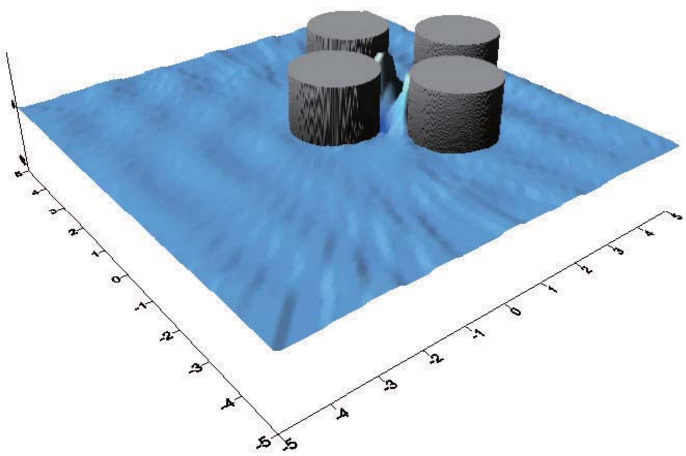

Fig. 4 *Contour of the maximum free-surface elevation amplitude*

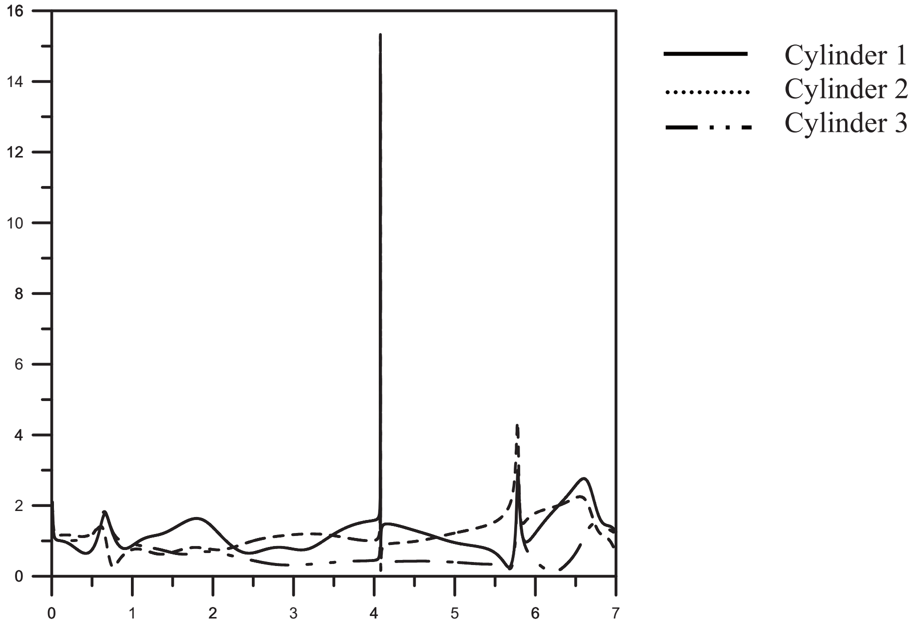

Fig.5 *The force ratio in the direction of wave advance versus wavenumber*

CONCLUSIONS

In this paper, we applied a dual BEM for water wave scattering problems with four circular cylinders. Discussions on the physical phenomena of near-trapped mode as well as the numerical oscillation due to fictitious frequency in BEM were both addressed. The fictitious frequency is present and can be suppressed by using the Burton and Miller method. In the meanwhile, the peak due to the near-trapped mode also occurs and its value keeps a constant.

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