Construction of a curve by using the state equation of Frenet formula

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ABSTRACT

In this paper, the available formulae for the curvature of plane curve are reviewed not only for the time-like but also for the space-like parameter curve. Two ways to describe the curve are proposed. One is the straight way to obtain the Frenet formula according to the given curve of parameter form. The other is that we can construct the curve by solving the state equation of Frenet formula subject to the initial position, the initial tangent, normal and binormal vectors, and the given radius of curvature and torsion constant. The remainder theorem of the matrix and the Cayley–Hamilton theorem are both employed to solve the Frenet equation. We review the available formulae of the radius of curvature and examine their equivalence. Through the Frenet formula, the relation among different expressions for the radius of curvature formulae can be linked. Therefore, we can integrate the formulae in the engineering mathematics, calculus, mechanics of materials and dynamics. Besides, biproduct of two new and simpler formulae and the available four formulae in the textbook of the radius of curvature yield the same radius of curvature for the plane curve. Linkage of centrifugal force and radius of curvature is also addressed. A demonstrative example of the cycloid is given. Finally, we use the two new formulae to obtain the radius of curvature for four curves, namely a circle. The equivalence is also proved. Animation for 2D and 3D curves is also provided by using the Mathematica software to demonstrate the validity of the present approach.

KEYWORDS: radius of curvature, Frenet formula, inverse problem, cycloid

1. INTRODUCTION

Curve theory in the differential geometry is a study of 3D curves with an orthogonal frame. The Frenet formula is the governing equation of the curve if the curvature and torsion constant are given subject to the initial tangent, normal and binormal vectors. A note on natural coordinates and Frenet frames was studied [1]. Curvature along a curve, road and rail plays an important role in mathematics, civil engineering and mechanical engineering. In the textbook, one formula can be found in the engineering mathematics with the explicit formula of the curve $Y = Y(X)$. Two formulae for the curves in terms of time and space parameters can also be found. Although all the formulae in terms of the space parameter (arc length) look different, their equivalence must be true. In proving the equivalence of the formulae, we fortunately find two additional simpler formulae to determine the radius of curvature. For the direct problem, we can construct the Frenet formula according to the space-like parameter curve. For the inverse problem, we can construct the space-like parameter curve according to the Frenet formula subject to the initial position, the initial tangent, normal and binormal vectors, and the given radius of curvature and torsional constant [2–4]. The solution of state equation can be expressed in terms of the exponential of

a matrix [3, 5–8]. By employing the remainder theorem of the matrix and the Cayley–Hamilton theorem, the space-like curve can be obtained after solving the state equation of Frenet formula [2–4]. Not only the linkage between these formulae is addressed, but also two new formulae are obtained. An illustrative example of cycloid is given to show that the seven formulae yield the same radius of curvature. Besides, animation is also provided by using the Mathematica software. Centrifugal force plays an important role in rigid-body dynamics of particles. How to determine the centrifugal force by using the tangent velocity and the radius of curvature is also addressed in this paper [9].

In this paper, two new formulae are proposed, according to the Frenet formula, to determine the radius of curvature. The Frenet formula is also applied to relate several different formulae of radius of curvature for the plane curve in the textbook. At last, the Mathematica software is employed to animate 2D and 3D curves. The organization of this paper is as follows: Section 2 reviews the available radius of curvature formulae for the plane curve. Section 3 provides the equivalence of different forms of the radius of curvature. A link of vector calculus and dynamics is addressed. Section 4 shows an example of cycloid. Section 4.1 constructs the space-like curve by solving the state
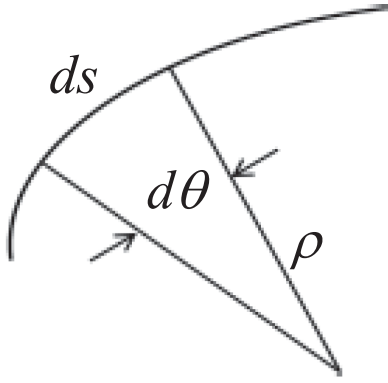
Figure 1 Schematic diagram of a curve.

equation of Frenet formula for the inverse problem [10]. Section 4.2 provides a spiral example (3D case). Section 5 provides a website that readers can link to see the animation for a cycloid, a circle and a spiral developed by the NTOU/MSV group. Finally, conclusions are provided in Section 6.

2. REVIEW OF THE RADIUS OF CURVATURE FOR A PLANE CURVE

2.1 Radius of curvature for the arc-length parameter curve

First, a space-like parameter curve is known as shown in Fig. 1. The radius of curvature can be derived from the calculus as follows:

$$\rho = \frac{ds}{d\theta}, \qquad (1)$$

where ρ is the radius of curvature, $d\theta$ is the infinitesimal radian and ds is an infinitesimal arc length as shown in Fig. 1, which can be written by the arc-length relationship of

$$(ds)^2 = (dX)^2 + (dY)^2 + (dZ)^2. \qquad (2)$$

Equation (2) is for the 3D case, while the arc length for the 2D case is reduced to

$$(ds)^2 = (dX)^2 + (dY)^2, \qquad (3)$$

where X and Y are the two components of the Cartesian coordinates and $Y = Y(X)$. The slope of the curve $Y = Y(X)$ at (X,Y) is

$$Y' = \frac{dY}{dX} = \tan\theta. \qquad (4)$$

By differentiating the two sides of Eq. (4), we have

$$\frac{d}{d\theta}\left(\frac{dY}{dX}\right) = \frac{dX}{d\theta}\frac{d}{dX}\left(\frac{dY}{dX}\right) = Y''\frac{dX}{d\theta} \qquad (5)$$

and

$$\frac{d\tan\theta}{d\theta} = \sec^2\theta = 1 + \tan^2\theta = 1 + (Y')^2. \qquad (6)$$

Thus, we have

$$Y''\frac{dX}{d\theta} = 1 + (Y')^2. \qquad (7)$$

Substitution of Eq. (3) into Eq. (1) yields

$$\rho = \sqrt{1 + (Y')^2}\,\frac{dX}{d\theta}. \qquad (8)$$

By substituting Eq. (7) into Eq. (8), we have

$$\rho = \sqrt{1 + (Y')^2}\,\frac{1 + (Y')^2}{Y''} = \frac{\left(1 + (Y')^2\right)^{3/2}}{Y''}. \qquad (9)$$

For the positive radius of curvature, we have

$$\rho = \frac{\left(1 + (Y')^2\right)^{3/2}}{|Y''|}. \qquad (10)$$

This is the explicit formula of the radius of curvature derived from the calculus.

2.2 Radius of curvature for time-like and space-like parameter curves

We consider the parameter curve $(X(t), Y(t))$ as a function of time, which is expressed as

$$X = X(t), \quad Y = Y(t), \quad \dot{X} = \frac{dX}{dt}, \quad \dot{Y} = \frac{dY}{dt}. \qquad (11)$$

By substituting Eq. (11) into Eq. (10), we have

$$\rho = \frac{\left(1 + Y'^2\right)^{3/2}}{|Y''|} = \frac{\left(1 + \left(\dot{Y}/\dot{X}\right)^2\right)^{3/2}}{\left|d\left(\dot{Y}/\dot{X}\right)/dX\right|}$$

$$= \frac{\left(1 + \left(\dot{Y}/\dot{X}\right)^2\right)^{3/2}}{\left|(dt/dX)[d(\dot{Y}/\dot{X})/dt]\right|} = \frac{\left(\dot{X}^2 + \dot{Y}^2\right)^{3/2}}{\left|\dot{X}\ddot{Y} - \ddot{X}\dot{Y}\right|}. \qquad (12)$$

This is the formula of radius of curvature in terms of the time-like parameter, which is also the formula in the mechanics of materials:

$$\rho = \frac{\left(\dot{X}^2 + \dot{Y}^2\right)^{3/2}}{\left|\dot{X}\ddot{Y} - \ddot{X}\dot{Y}\right|}. \qquad (13)$$

If the arc-length parameter s is a function of the time-like parameter t, it can be expressed as

$$s = s(t), \qquad X(t) = x(s), \qquad Y(t) = y(s), \qquad (14)$$

where x and y are position functions in terms of the arc-length parameter. By substituting the time parameter of Eq. (13) with the arc-length parameter of Eq. (14), we have

$$\rho = \frac{\left(\dot{X}^2 + \dot{Y}^2\right)^{3/2}}{\left|\dot{X}\ddot{Y} - \ddot{X}\dot{Y}\right|} = \frac{\left[\dot{s}^2\left(x'^2 + y'^2\right)\right]^{3/2}}{|\dot{s}^3|\,|x'y'' - x''y'|}$$

$$= \frac{|\dot{s}^3|\left(x'^2 + y'^2\right)^{3/2}}{|\dot{s}^3|\,|x'y'' - x''y'|} = \frac{\left(x'^2 + y'^2\right)^{3/2}}{|x'y'' - x''y'|}, \qquad (15)$$

where $\dot{s} = ds/dt$ and

$$(x'^2 + y'^2)^{3/2} = \left(\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2\right)^{3/2}$$

$$= \left(\left(\frac{ds}{ds}\right)^2\right)^{3/2} = 1. \qquad (16)$$

By substituting Eq. (16) into Eq. (15), an alternative and simple formula is obtained:

$$\rho = \frac{1}{|x'y'' - x''y'|}. \tag{17}$$

2.3 Derivation of the radius of curvature using the Frenet formula

If the time-like parameter representation is changed to the space-like parameter representation, we have

$$\underset{\sim}{r} = (X(t), Y(t), Z(t)) = (x(s), y(s), z(s)), \tag{18}$$

where $\underset{\sim}{r}$ is the position vector. The tangent vector t is defined as

$$t = \frac{d\underset{\sim}{r}}{ds}. \tag{19}$$

The unit normal vector $\underset{\sim}{v}$ should be orthogonal to $\underset{\sim}{t}$, such that

$$\underset{\sim}{v} = \frac{t'}{|\underset{\sim}{t}'|}, \tag{20}$$

where $\underset{\sim}{\tau}(s)$ is the unit tangent vector. According to Eq. (19), the unit tangent vector can be expressed as

$$\underset{\sim}{\tau}(s) = \frac{r'(s)}{|r'(s)|}. \tag{21}$$

By using the inner product of $\underset{\sim}{\tau}(s)$ and $\underset{\sim}{\tau}(s + ds)$, we have

$$\underset{\sim}{\tau}(s) \cdot \underset{\sim}{\tau}(s + ds) = |\underset{\sim}{\tau}(s)||\underset{\sim}{\tau}(s + ds)| \cos(d\theta) = \cos(d\theta). \tag{22}$$

Equation (22) can be written as

$$\underset{\sim}{\tau}(s) \cdot \underset{\sim}{\tau}(s + ds) = \cos(d\theta). \tag{23}$$

By employing Taylor's expansion of $\underset{\sim}{\tau}(s + ds)$ at s, we have

$$\underset{\sim}{\tau}(s) \cdot \underset{\sim}{\tau}(s + ds) = \underset{\sim}{\tau}(s) \cdot \sum_{n=0}^{\infty} \frac{\tau^{(n)}(s)}{n!} (ds)^n \tag{24}$$

and

$$\cos(d\theta) = 1 - \frac{1}{2!}(d\theta)^2 + \frac{1}{4!}(d\theta)^4 - \frac{1}{6!}(d\theta)^6 + \cdots. \tag{25}$$

According to the orthogonal property of $\underset{\sim}{\tau}$ and $\underset{\sim}{\tau}'$, we have

$$\underset{\sim}{\tau} \cdot \underset{\sim}{\tau}' = 0. \tag{26}$$

According to Eq. (26) and submitting Eqs (24) and (25) into Eq. (23), we could obtain

$$1 + \frac{1}{2}\underset{\sim}{\tau}(s) \cdot \underset{\sim}{\tau}''(s)(ds)^2 = 1 - \frac{1}{2}(d\theta)^2. \tag{27}$$

Equation (27) is reduced to

$$\underset{\sim}{\tau}(s) \cdot \underset{\sim}{\tau}''(s)(ds)^2 = -(d\theta)^2. \tag{28}$$

By differentiating Eq. (26), we have

$$\underset{\sim}{\tau} \cdot \underset{\sim}{\tau}'' = -\underset{\sim}{\tau}' \cdot \underset{\sim}{\tau}'. \tag{29}$$

By substituting Eq. (29) into Eq. (27), we have

$$-\underset{\sim}{\tau}' \cdot \underset{\sim}{\tau}'(ds)^2 = -(d\theta)^2. \tag{30}$$

Substitution of Eq. (1) into Eq. (30) yields

$$\left(\frac{ds}{d\theta}\right)^2 = \frac{1}{|\underset{\sim}{\tau}'|^2}. \tag{31}$$

According to $\rho\, d\theta = ds$, Eq. (31) yields

$$\rho = \frac{1}{|\underset{\sim}{\tau}'|}. \tag{32}$$

From Eq. (32), the radius of curvature is obtained as shown below:

$$\rho = \frac{1}{|\underset{\sim}{\tau}'|} = \frac{1}{\sqrt{(x'')^2 + (y'')^2 + (z'')^2}}. \tag{33}$$

Then, the radius of curvature for the 3D case [Eq. (33)] is reduced to the 2D result:

$$\rho = \frac{1}{\sqrt{(x'')^2 + (y'')^2}}. \tag{34}$$

In the literature [9, 10], the Frenet formula for the 3D curve is given below:

$$\begin{pmatrix} \tau' \\ v' \\ \beta' \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{\rho} & 0 \\ -\frac{1}{\rho} & 0 & \frac{1}{\sigma} \\ 0 & -\frac{1}{\sigma} & 0 \end{pmatrix} \begin{pmatrix} \tau \\ v \\ \beta \end{pmatrix}, \tag{35}$$

where $\underset{\sim}{\beta}$ is the binormal vector and is defined as $\underset{\sim}{\beta} = \underset{\sim}{\tau} \times \underset{\sim}{v}$, and σ is the torsion constant and can be determined by $\sigma = 1/|\underset{\sim}{\beta}'|$.

3. EQUIVALENCE OF THE DIFFERENT FORMULAE OF RADIUS OF CURVATURE

3.1 Frenet formula

Although Eq. (34) derived by the Frenet equation and Eq. (17) derived from the radius of curvature of mechanics of material are both expressed by using the space-like (arc-length) parameter, they look different. However, the result should be equivalent. We will prove the equivalence by using two methods as shown below.

Method I:
By employing the Frenet formula, we have

$$\frac{1}{|x'y'' - x''y'|} = \frac{1}{|\underset{\sim}{\tau} \times \underset{\sim}{\tau}'|} = \frac{1}{|\underset{\sim}{\tau}||\underset{\sim}{\tau}'|\sin(\pi/2)}$$

$$= \frac{1}{|\underset{\sim}{\tau}'|} = \frac{1}{(1/\rho)|\underset{\sim}{v}|} = \frac{1}{1/\rho} = \rho, \tag{36}$$

where the unit tangent vector $\underset{\sim}{\tau} = (x', y')$. Then, Eq. (34) can be derived as

$$\frac{1}{\sqrt{(x'')^2 + (y'')^2}} = \frac{1}{|\underset{\sim}{\tau}'|} = \frac{1}{(1/\rho)|\underset{\sim}{v}|} = \frac{1}{1/\rho} = \rho, \tag{37}$$

where $\underset{\sim}{\tau}' = (x'', y'')$. By using the Frenet formula, Eqs (17) and (34) are proved to be equivalent.

Method II:

Since $x(s)$ and $y(s)$ represent the space-like (arc-length) parameter curve, we have

$$(x')^2 + (y')^2 = 1. \tag{38}$$

By differentiating both sides of Eq. (38) with respect to s, we have

$$2x'x'' + 2y'y'' = 0. \tag{39}$$

We can assume the proportional relation from Eq. (39) to have

$$\frac{y''}{-x'} = \frac{x''}{y'} = k, \tag{40}$$

where k is a proportional function. Equation (40) yields

$$y'' = -kx', \qquad x'' = ky'. \tag{41}$$

By substituting Eq. (41) into Eqs (17) and (34), respectively, we have

$$\frac{1}{|x'y'' - x''y'|} = \frac{1}{|x'(-k)x' - ky'y'|}$$
$$= \frac{1}{|k|\left|(x')^2 + (y')^2\right|} = \frac{1}{|k|} \tag{42}$$

and

$$\frac{1}{\sqrt{(x'')^2 + (y'')^2}} = \frac{1}{\sqrt{k^2(x')^2 + k^2(y')^2}}$$
$$= \frac{1}{|k|\sqrt{(x')^2 + (y')^2}} = \frac{1}{|k|}. \tag{43}$$

The equivalence of Eqs (17) and (34) can be proved by using Eqs (42) and (43) after setting $|k| = 1/\rho$, respectively. In other words, the proportional function is the curvature, which is the inverse of radius, ρ. It also indicates that Eq. (40) presents two simpler formulae for the radius of curvature. From Eq. (42) or Eq. (43), we have

$$\rho = \frac{1}{|k|}. \tag{44}$$

On the basis of the arc-length expression, the two new and neat formulae for the radius of curvature can be derived from Eqs (40) and (44) as

$$\rho = \left|\frac{x'}{y''}\right| \tag{45}$$

and

$$\rho = \left|\frac{y'}{x''}\right|. \tag{46}$$

3.2 Dynamics and mechanics of materials

When a curve is expressed as a time-like parameter model, its position vector can be expressed as

$$\underset{\sim}{r}(t) = X(t)\,\underset{\sim}{i} + Y(t)\,\underset{\sim}{j}. \tag{47}$$

Its tangent velocity vector is

$$\underset{\sim}{V}(t) = \frac{\mathrm{d}\underset{\sim}{r}(t)}{\mathrm{d}t} = \dot{X}(t)\,\underset{\sim}{i} + \dot{Y}(t)\,\underset{\sim}{j}, \tag{48}$$

where $\underset{\sim}{V}(t)$ is the tangent velocity vector, and the speed function V_0 can be expressed as

$$V_0 = \frac{\mathrm{d}s}{\mathrm{d}t}. \tag{49}$$

The unit tangent vector $\underset{\sim}{e}_t$ can be expressed as

$$\underset{\sim}{e}_t = \frac{\underset{\sim}{V}}{V_0} = \frac{\dot{X}\,\underset{\sim}{i} + \dot{Y}\,\underset{\sim}{j}}{\sqrt{\dot{X}^2 + \dot{Y}^2}}. \tag{50}$$

The unit normal vector $\underset{\sim}{e}_n$ is

$$\underset{\sim}{e}_n = \frac{\dot{Y}\,\underset{\sim}{i} - \dot{X}\,\underset{\sim}{j}}{\sqrt{\dot{X}^2 + \dot{Y}^2}}. \tag{51}$$

According to the vector calculus, the acceleration $\underset{\sim}{a}$ can be expressed as

$$\underset{\sim}{a}(t) = \frac{\mathrm{d}^2\underset{\sim}{r}(t)}{\mathrm{d}t^2} = \ddot{X}\,\underset{\sim}{i} + \ddot{Y}\,\underset{\sim}{j}. \tag{52}$$

By decomposing into tangent and normal components, we have

$$\underset{\sim}{a} = \frac{\mathrm{d}}{\mathrm{d}t}\underset{\sim}{V}(t) = \dot{V}_0\,\underset{\sim}{e}_t + V_0\left(\frac{\mathrm{d}}{\mathrm{d}t}\underset{\sim}{e}_t\right) \tag{53}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\underset{\sim}{e}_t = \frac{\mathrm{d}\underset{\sim}{e}_t}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}t} = \left(\frac{\underset{\sim}{e}_n}{\rho}\right)V_0. \tag{54}$$

Equation (53) can be written as

$$\underset{\sim}{a} = \dot{V}_0\,\underset{\sim}{e}_t + \frac{V_0^2}{\rho}\,\underset{\sim}{e}_n. \tag{55}$$

The magnitude of normal acceleration is

$$\frac{V_0^2}{\rho} = \underset{\sim}{a} \cdot \underset{\sim}{e}_n. \tag{56}$$

According to the dynamics, we have

$$\rho = \frac{V_0^2}{\underset{\sim}{a} \cdot \underset{\sim}{e}_n} = \frac{\left(\dot{Y} + \dot{X}\right)^{3/2}}{\sqrt{\ddot{X}\dot{Y} - \dot{X}\ddot{Y}}}. \tag{57}$$

Equation (57) illustrates the equivalence of the formula in dynamics and mechanics of materials of Eq. (12). In summary, it can be seen that the formula of the radius of curvature is equivalent as shown in Fig. 2.

4. A CYCLOID EXAMPLE (2D CASE)

We verify the new formulae, Eqs (45) and (46), and compare with Eqs (17) and (34) by a cycloid example (see Fig. 3). The time-like parameter curve is expressed as follows:

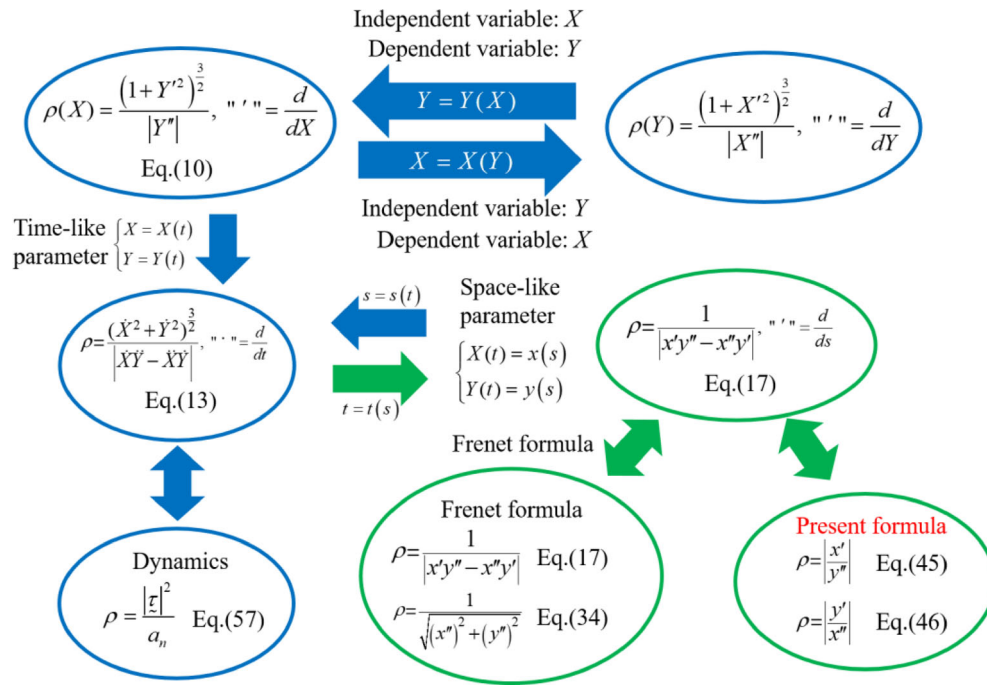$$\begin{cases} X(t) = t - \sin t, \\ Y(t) = 1 - \cos t. \end{cases} \tag{58}$$

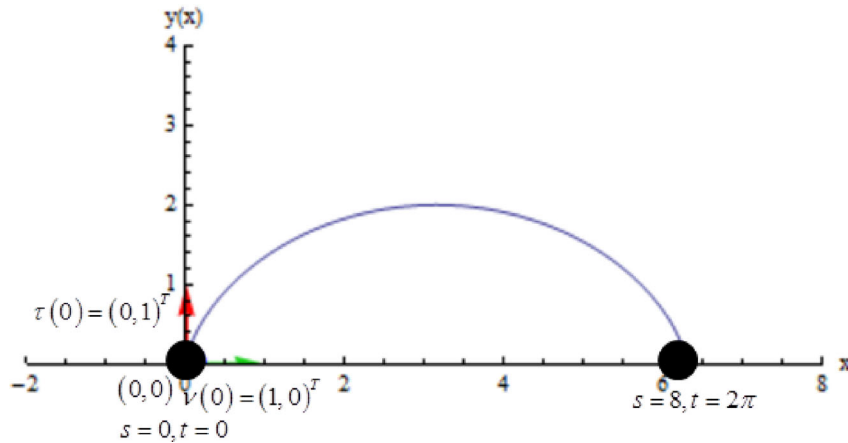**Figure 2** Linkage of different formulae for the radius of curvature.

Figure 3 A cycloid of $\rho(s) = \sqrt{8s - s^2}$, $0 \le s \le 8$, subject to the initial position $(X(t), Y(t)) = (0, 0)$, the initial normal vector $\underset{\sim}{\tau}(0) = (0, 1)$ and the initial binormal vector $\underset{\sim}{\nu}(0) = (1, 0)$.

According to Eq. (3), the relationship between the arc length and the time can be expressed as follows:

$$s = \int \sqrt{\left(\frac{dX}{dt}\right)^2 + \left(\frac{dY}{dt}\right)^2}\, dt = -4 \cos\left(\frac{t}{2}\right) + c. \quad (59)$$

If the initial condition $s = 0$ is corresponding to $t = 0$, we have $c = 4$. The arc-length parameter description for the cycloid is expressed as

$$\begin{cases} x(s) = 2 \cos^{-1}\left(\frac{4-s}{4}\right) - \sin\left(2 \cos^{-1}\left(\frac{4-s}{4}\right)\right), \\ y(s) = 1 - \cos\left(2 \cos^{-1}\left(\frac{4-s}{4}\right)\right). \end{cases} \quad (60)$$

Now, we go through the four formulae of the arc parameter for the radius of curvature. By using Eq. (17), we can obtain the radius of curvature as

$$\rho(s) = \frac{1}{|x'y'' - x''y'|}$$

$$= \frac{1}{\left|\left[(4-s)^2/16\sqrt{8s-s^2}\right] + \left[(8s-s^2)/16\sqrt{8s-s^2}\right]\right|}$$

$$= \sqrt{8s - s^2}. \quad (61)$$

According to Eq. (34), we obtain the radius of curvature as

$$\rho(s) = \frac{1}{\sqrt{x''^2 + y''^2}}$$

$$= \frac{1}{\left|\sqrt{[(16 - 8s + s^2)/16(8s - s^2)] + (1/16)}\right|}$$

$$= \sqrt{8s - s^2}. \tag{62}$$

Based on Eqs (45) and (46), we obtain the same radius of curvature as

$$\rho(s) = \left|\frac{x'}{y''}\right| = \left|\frac{(8s - s^2)/4\sqrt{8s - s^2}}{-1/4}\right| = \sqrt{8s - s^2} \tag{63}$$

and

$$\rho(s) = \left|\frac{y'}{x''}\right| = \left|\frac{(4 - s)/4}{(4 - s)/4\sqrt{8s - s^2}}\right| = \sqrt{8s - s^2}. \tag{64}$$

The two new formulae of radius of curvature for the cycloid in terms of the arc-parameter representation are consistent with the formulae of radius of curvature of Frenet and mechanics of material. This verifies the correctness of the new formulae of radius of curvature. In total, seven formulae yield the same radius of curvature as shown in Table 1 for four curves, i.e. a circle, a cycloid, an astroid and a cardioid. Here, we also give an alternative formula $\rho = (1 + (x')^2)^{3/2}/x''$ if $x = x(y)$ is given instead of $y = y(x)$.

4.1 Construction of the space-like parameter curve using the Frenet formula (2D case)

Although we can straightforwardly obtain the radius of curvature $\rho(s)$ from the time parameter curve of Eq. (58), we intend to solve the Frenet equation to reconstruct the arc-length parameter curve. We use the remainder theorem of the matrix to solve the state equation:

$$\left\{\begin{matrix} \underset{\sim}{\tau}' \\ \underset{\sim}{\nu}' \end{matrix}\right\} = [A(s)] \left\{\begin{matrix} \underset{\sim}{\tau} \\ \underset{\sim}{\nu} \end{matrix}\right\}, \tag{65}$$

subject to the initial condition, $\underset{\sim}{\tau}(0) = (0, 1)$, $\underset{\sim}{\nu}(0) = (1, 0)$ and

$$[A(s)] = \begin{bmatrix} 0 & \frac{1}{\rho(s)} \\ -\frac{1}{\rho(s)} & 0 \end{bmatrix}. \tag{66}$$

It could be easily obtained that the eigenvalues of the state matrix, $[A(s)]$, are $i/\rho(s)$ and $-i/\rho(s)$. The curve of cycloid is shown in Fig. 3 and its radius of curvature is arc-length dependent: $\rho(s) = \sqrt{8s - s^2}$, $0 \le s \le 8$. Equation (66) is a state equation for $(\underset{\sim}{\tau}, \underset{\sim}{\nu})^{\mathrm{T}}$. The state solution is obtained by

$$\left\{\begin{matrix} \underset{\sim}{\tau}(s) \\ \underset{\sim}{\nu}(s) \end{matrix}\right\} = \exp^{\int_0^s [A(w)]\mathrm{d}w} \left\{\begin{matrix} \underset{\sim}{\tau}(0) \\ \underset{\sim}{\nu}(0) \end{matrix}\right\}. \tag{67}$$

According to the remainder theorem of the function of real number and the eigenvalue functions of $[A(s)]$, the corresponding exponential function is shown below:

$$\mathrm{e}^{us} = \left(u^2 + \frac{1}{\rho^2}\right) Q(u) + p(s) \left(u - \frac{i}{\rho}\right) + q(s), \tag{68}$$

where i is the imaginary unit. If ρ is not a function of arc length, i.e. the state matrix is a constant matrix, then the matrix function could be expressed as

$$\mathrm{e}^{[u]s} = \left([u]^2 + \frac{1}{\rho^2}\right) Q([u]) + p(s) \left([u] - \frac{i}{\rho}\right) + q(s). \tag{69}$$

If u is a function of arc length, then we have

$$\mathrm{e}^{\int_0^s u(w)\mathrm{d}w} = \left(u^2(s) + \frac{1}{\rho^2(s)}\right) Q(u(s))$$

$$+ p(s) \left(u(s) - \frac{i}{\rho(s)}\right) + q(s). \tag{70}$$

By using the remainder theorem of the matrix, we can express the matrix function of $\mathrm{e}^{\int_0^s [A(w)]\mathrm{d}w}$ as

$$\mathrm{e}^{\int [A(s)]\mathrm{d}s} = \left([A(s)]^2 + \left(\frac{1}{\rho(s)}[I]\right)^2\right) Q(A(s))$$

$$+ p(s) \left([A(s)] - \frac{i}{\rho(s)}[I]\right) + q(s)[I], \tag{71}$$

where $[I]$ is the identity matrix. The two functions, $p(s)$ and $q(s)$, can be obtained by way of Eq. (70) as

$$p(s) = \frac{8s - s^2}{4}, \quad q(s) = \frac{4 - s}{4} + i\frac{\sqrt{8s - s^2}}{4}. \tag{72}$$

By introducing the Cayley–Hamilton theorem into Eq. (71), we have

$$\mathrm{e}^{\int_0^s [A(w)]\mathrm{d}w} = p(s) \left([A(s)] - \frac{[I]}{\rho(s)}i\right) + q(s)[I]$$

$$= \begin{bmatrix} -\frac{p(s)}{\rho(s)}i & \frac{p(s)}{\rho(s)} \\ -\frac{p(s)}{\rho(s)} & -\frac{p(s)}{\rho(s)}i \end{bmatrix} + \begin{bmatrix} q(s) & 0 \\ 0 & q(s) \end{bmatrix}. \tag{73}$$

By substituting $p(s)$ and $q(s)$ of Eq. (72) into Eq. (73), we have

$$\mathrm{e}^{\int_0^s [A(w)]\mathrm{d}w} = \begin{bmatrix} \frac{4-s}{4} & \frac{\sqrt{8s-s^2}}{4} \\ -\frac{\sqrt{8s-s^2}}{4} & \frac{4-s}{4} \end{bmatrix}. \tag{74}$$

Therefore, we obtain

$$\underset{\sim}{\tau}(s) = \left(\frac{\sqrt{8s - s^2}}{4}, \frac{4 - s}{4}\right). \tag{75}$$

By integrating $\underset{\sim}{\tau}(s)$, we have

$$\begin{cases} x(s) = 2\cos^{-1}\left(\frac{4-s}{4}\right) - \frac{(4-s)\sqrt{8s-s^2}}{8}, \\ y(s) = \frac{8s-s^2}{8}, \end{cases} \tag{76}$$

subject to the initial position of $(X(t), Y(t)) = (0, 0)$. By employing Eq. (59), Eq. (76) can be replaced by using the time-like parameter model:

$$\begin{cases} X(t) = t - \sin t, \\ Y(t) = 1 - \cos t. \end{cases} \tag{77}$$

Table 1 Formulae of a radius of curvature for a circle, a cycloid, an astroid and a cardioid.

	Formula for the radius of curvature	Circle	Cycloid	Astroid	Cardioid
		(figure, arrow labeled a)	*(figure)* $\begin{cases}X(t)=a(t-\sin t)\\ Y(t)=a(1-\cos t)\end{cases}$	*(figure)* $x^{\frac{2}{3}}+y^{\frac{2}{3}}=1$	*(figure)* $r=a(1-\sin\theta)$
Explicit y–x: $Y=Y(X)$	$\rho=\dfrac{(1+Y'^2)^{3/2}}{\lvert Y''\rvert}$ [Eq. (10)]	$\rho=1$	×	$\rho(X)=3X^{1/3}\sqrt{1-X^{2/3}}$	×
Explicit x–y: $X=X(Y)$	$\rho=\dfrac{(1+X'^2)^{3/2}}{\lvert X''\rvert}$	$\rho(Y)=1$	$\rho(Y)=\sqrt{8Y}$	$\rho(Y)=3Y^{1/3}\sqrt{1-Y^{2/3}}$	×
$s=s(t)$		$s=t$	$s=4-4\cos\left(\tfrac{t}{2}\right)$	$s=-\tfrac{3}{4}\cos(2t)$	$s(t)=\begin{cases}4\cos\left(\tfrac{\pi}{4}-\tfrac{t}{2}\right)-2\sqrt{2}, & 0<t<\tfrac{\pi}{2}\\[4pt] -4\cos\left(\tfrac{t}{2}-\tfrac{\pi}{4}\right)-2\sqrt{2}+8, & \tfrac{\pi}{2}<t<2\pi\end{cases}$
$t=t(s)$		$t=s$	$t=2\cos^{-1}\left(\tfrac{4-s}{4}\right)$	$t=\tfrac{1}{2}\cos^{-1}\left(\tfrac{-4s}{3}\right)$	$t(s)=\begin{cases}\tfrac{\pi}{2}-2\cos^{-1}\left(\tfrac{s+2\sqrt{2}}{4}\right), & 0<s<4-2\sqrt{2}\\[4pt] \tfrac{\pi}{2}+2\cos^{-1}\left(\tfrac{8-2\sqrt{2}-s}{4}\right), & 4-2\sqrt{2}<s<8\end{cases}$
Time-like: $\begin{cases}X=X(t)\\ Y=Y(t)\end{cases}$	$\rho=\dfrac{(\dot X^2+\dot Y^2)^{3/2}}{\lvert\dot X\ddot Y-\ddot X\dot Y\rvert}$ [Eq. (13)]	$\rho=1$	$\rho(t)=\sqrt{8(1-\cos t)}$	$\rho(t)=3\sin(t)\cos(t)$	$\rho(t)=\tfrac{2}{3}\sqrt{2-2\sin(t)}$
Space-like: $\begin{cases}x=x(s)\\ y=y(s)\end{cases}$	$\rho=\dfrac{1}{\lvert x'y''-x''y'\rvert}$ [Eq. (17)]	$\rho=1$	$\rho(s)=\sqrt{8s-s^2}$	$\rho(s)=3\sqrt{\tfrac{1}{4}-\tfrac{4s^2}{9}}$	$\rho(s)=\begin{cases}\dfrac{\sqrt{-s^2-4\sqrt{2}s+8}}{3}, & 0<s<4-2\sqrt{2}\\[6pt] \dfrac{\sqrt{-s^2-4\sqrt{2}s+16s+32\sqrt{2}-56}}{3}, & 4-2\sqrt{2}<s<8\end{cases}$
	$\rho=\dfrac{1}{\sqrt{(x'')^2+(y'')^2}}$ [Eq. (34)]	$\rho=1$	$\rho(s)=\sqrt{8s-s^2}$	$\rho(s)=3\sqrt{\tfrac{1}{4}-\tfrac{4s^2}{9}}$	$\rho(s)=\begin{cases}\dfrac{\sqrt{-s^2-4\sqrt{2}s+8}}{3}, & 0<s<4-2\sqrt{2}\\[6pt] \dfrac{\sqrt{-s^2-4\sqrt{2}s+16s+32\sqrt{2}-56}}{3}, & 4-2\sqrt{2}<s<8\end{cases}$
	$\rho=\left\lvert\dfrac{x'}{y''}\right\rvert$ [Eq. (45)]	$\rho=1$	$\rho(s)=\sqrt{8s-s^2}$	$\rho(s)=3\sqrt{\tfrac{1}{4}-\tfrac{4s^2}{9}}$	$\rho(s)=\begin{cases}\dfrac{\sqrt{-s^2-4\sqrt{2}s+8}}{3}, & 0<s<4-2\sqrt{2}\\[6pt] \dfrac{\sqrt{-s^2-4\sqrt{2}s+16s+32\sqrt{2}-56}}{3}, & 4-2\sqrt{2}<s<8\end{cases}$
	$\rho=\left\lvert\dfrac{y'}{x''}\right\rvert$ [Eq. (46)]	$\rho=1$	$\rho(s)=\sqrt{8s-s^2}$	$\rho(s)=3\sqrt{\tfrac{1}{4}-\tfrac{4s^2}{9}}$	$\rho(s)=\begin{cases}\dfrac{\sqrt{-s^2-4\sqrt{2}s+8}}{3}, & 0<s<4-2\sqrt{2}\\[6pt] \dfrac{\sqrt{-s^2-4\sqrt{2}s+16s+32\sqrt{2}-56}}{3}, & 4-2\sqrt{2}<s<8\end{cases}$

× denotes not available.

This method of determining the space-like parameter curve with the nonconstant coefficient matrix uses the remainder theorem for the matrix. We use the matrix function, the matrix remainder theorem and the Cayley–Hamilton theorem of matrix. The analytical solution for the space-like parameter curve is obtained.

All the different formulae of radius of curvature in the textbook and two new formulae agree well. In Table 1, we verify the consistency of the seven formulae by using a circle, an astroid, a cardioid and a cycloid. Besides, the equivalence of each formula is proved. Finally, animation with the position vector (r), the unit tangent vector (τ) and the unit normal vector (v) is also implemented. In Table 2, the details for constructing the curves of cycloid are given by using the Frenet formula.

4.2 A spiral example (3D case)

Here, we give a 3D spiral as an example. The governing equation of curve for Frenet formula is given as shown below:

$$S' = [A]\, S, \tag{78}$$

where

$$S' = \left\{ \begin{array}{c} \tau'(s) \\ v'(s) \\ \beta'(s) \end{array} \right\}, \quad [A] = \begin{bmatrix} 0 & \frac{1}{2} & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \\ 0 & -\frac{1}{2} & 0 \end{bmatrix},$$

$$S = \left\{ \begin{array}{c} \tau(s) \\ v(s) \\ \beta(s) \end{array} \right\}.$$

Here, $[A]$ is a constant matrix instead of a nonconstant matrix in a cycloid. The initial position is given by

$$\begin{cases} x(0) = -1, \\ y(0) = 0, \\ z(0) = 0. \end{cases} \tag{79}$$

The initial tangent, normal and binormal vectors are

$$\begin{cases} \tau(0) = \left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \\ v(0) = (1, 0, 0), \\ \beta(0) = \left(0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \end{cases} \tag{80}$$

respectively. The state vector of X can be obtained as

$$X = e^{[A]s} \left\{ \begin{array}{c} \tau(0) \\ v(0) \\ \beta(0) \end{array} \right\}, \tag{81}$$

where $e^{[A]s}$ is obtained by employing the similar transformation as shown below:

$$e^{[A]s} = \begin{bmatrix} -1 & -1 & 1 \\ -i\sqrt{2} & i\sqrt{2} & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} e^{(i/\sqrt{2})s} & 0 & 0 \\ 0 & e^{(i/\sqrt{2})s} & 0 \\ 0 & 0 & e^{(i/\sqrt{2})s} \end{bmatrix}$$

$$\begin{bmatrix} \frac{-1}{4} & \frac{i}{2\sqrt{2}} & \frac{1}{4} \\ \frac{-1}{4} & \frac{-i}{2\sqrt{2}} & \frac{1}{4} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}. \tag{82}$$

Equation (82) yields

$$e^{[A]s} = \begin{bmatrix} \cos^2\left(\frac{s}{2\sqrt{2}}\right) & \frac{\sin(s/\sqrt{2})}{\sqrt{2}} & \sin^2\left(\frac{s}{2\sqrt{2}}\right) \\ \frac{-\sin(s/\sqrt{2})}{\sqrt{2}} & \cos\left(\frac{s}{\sqrt{2}}\right) & \frac{\sin(s/\sqrt{2})}{\sqrt{2}} \\ \sin^2\left(\frac{s}{2\sqrt{2}}\right) & \frac{-\sin(s/\sqrt{2})}{\sqrt{2}} & \cos^2\left(\frac{s}{2\sqrt{2}}\right) \end{bmatrix}. \tag{83}$$

Equation (83) can also be derived by using the remainder theorem of the matrix and the Cayley–Hamilton theorem. By substituting Eqs (80) and (83) into Eq. (81), we have

$$X = \left\{ \begin{array}{c} \tau(s) \\ v(s) \\ \beta(s) \end{array} \right\} = \begin{bmatrix} \cos^2\left(\frac{s}{2\sqrt{2}}\right) & \frac{\sin(s/\sqrt{2})}{\sqrt{2}} & \sin^2\left(\frac{s}{2\sqrt{2}}\right) \\ \frac{-\sin(s/\sqrt{2})}{\sqrt{2}} & \cos\left(\frac{s}{\sqrt{2}}\right) & \frac{\sin(s/\sqrt{2})}{\sqrt{2}} \\ \sin^2\left(\frac{s}{2\sqrt{2}}\right) & \frac{-\sin(s/\sqrt{2})}{\sqrt{2}} & \cos^2\left(\frac{s}{2\sqrt{2}}\right) \end{bmatrix}$$

$$\left\{ \begin{array}{c} 0\,i + \frac{1}{\sqrt{2}}j + \frac{1}{\sqrt{2}}k \\ 1\,i + 0\,j + 0\,k \\ 0\,i + \frac{-1}{\sqrt{2}}j + \frac{1}{\sqrt{2}}k \end{array} \right\}. \tag{84}$$

After rearranging, we have

$$X = \left\{ \begin{array}{c} \tau(s) \\ v(s) \\ \beta(s) \end{array} \right\} = \left\{ \begin{array}{c} \frac{\sin\left(\frac{s}{\sqrt{2}}\right)}{\sqrt{2}}i + \frac{\cos\left(\frac{s}{\sqrt{2}}\right)}{\sqrt{2}}j + \frac{1}{\sqrt{2}}k \\ \cos\left(\frac{s}{\sqrt{2}}\right)i - \sin\left(\frac{s}{\sqrt{2}}\right)j + 0\,k \\ \frac{-\sin(s/\sqrt{2})}{\sqrt{2}}i + \frac{-\cos(s/2\sqrt{2})+\sin(s/2\sqrt{2})}{\sqrt{2}}j \\ + \frac{\cos(s/2\sqrt{2})-\sin(s/2\sqrt{2})}{\sqrt{2}}k \end{array} \right\}. \tag{85}$$

By integrating $\tau(s)$ in Eq. (85) to obtain $(x(s), y(s), z(s))$, we have

$$x(s) = \int \frac{\sin(s/\sqrt{2})}{\sqrt{2}}ds = -\cos\left(\frac{s}{\sqrt{2}}\right) + c_1, \tag{86}$$

$$y(s) = \int \frac{\cos(s/\sqrt{2})}{\sqrt{2}}ds = \sin\left(\frac{s}{\sqrt{2}}\right) + c_2, \tag{87}$$

$$z(s) = \int \frac{1}{\sqrt{2}}ds = \frac{s}{\sqrt{2}} + c_3. \tag{88}$$

By substituting Eqs (86)–(88) into the initial position of Eq. (79), we have $c_1 = c_2 = c_3 = 0$. Therefore, the space-like description curve (one length) $(x(s), y(s), z(s))$ yields

$$(x(s), y(s), z(s)) = \left(-\cos\left(\frac{s}{\sqrt{2}}\right), \sin\left(\frac{s}{\sqrt{2}}\right), \frac{s}{\sqrt{2}}\right). \tag{89}$$

Table 2 Constructing the curve by solving the Frenet formula.

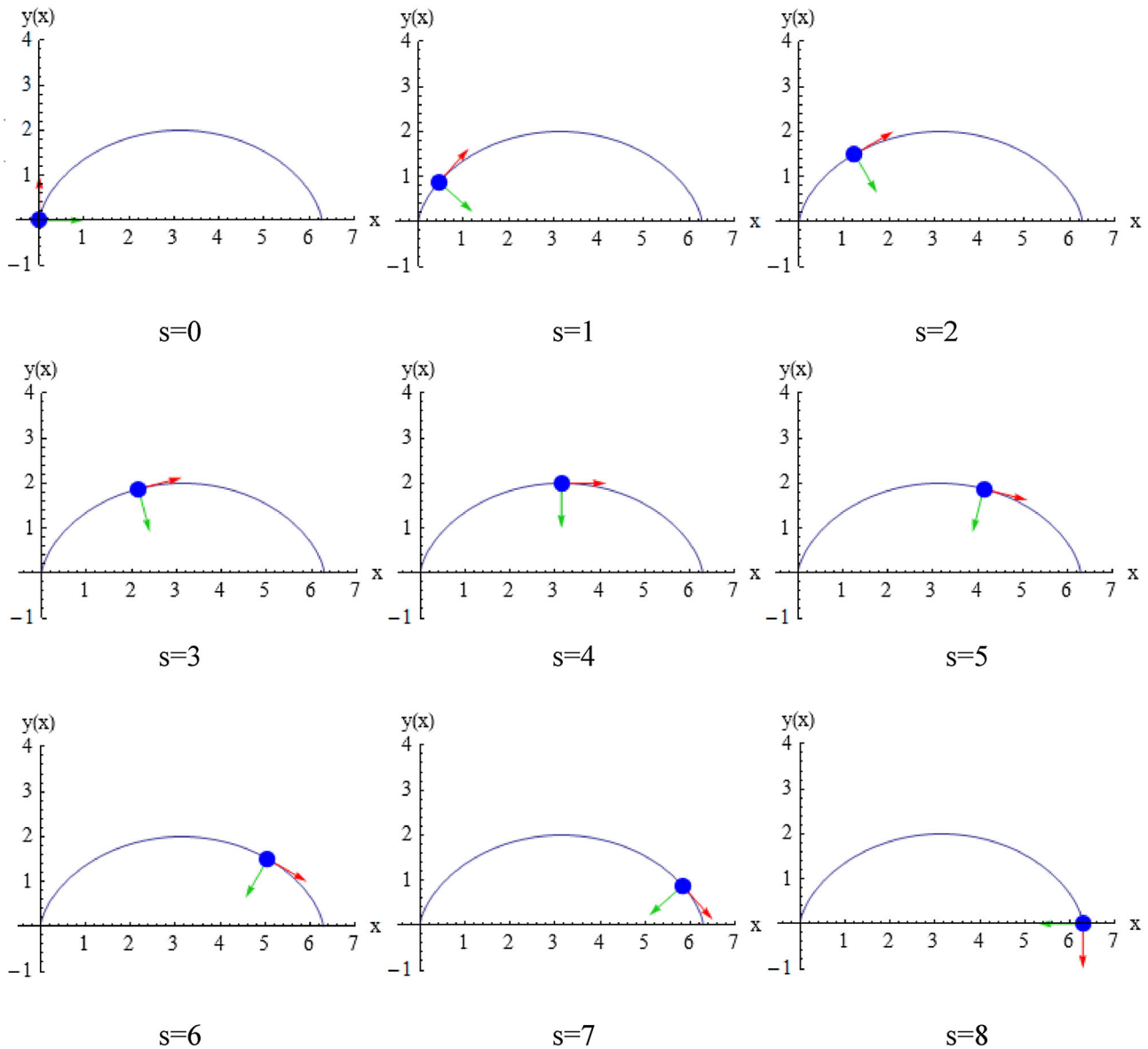|	Circle	Cycloid	Spiral
$S' = [A]S$	$S = \begin{pmatrix} \tau \\ \nu \end{pmatrix}$	$S = \begin{pmatrix} \tau \\ \nu \end{pmatrix}$	$S = \begin{pmatrix} \tau \\ \nu \\ \beta \end{pmatrix}$
	$[A] = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$	$[A] = \begin{bmatrix} 0 & \frac{1}{\sqrt{8s-s^2}} \\ \frac{-1}{\sqrt{8s-s^2}} & 0 \end{bmatrix}$	$[A] = \begin{bmatrix} 0 & \frac{1}{2} & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \\ 0 & -\frac{1}{2} & 0 \end{bmatrix}$
Initial position $\begin{pmatrix} x(0) \\ y(0) \\ z(0) \end{pmatrix}$	$\begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} x(0) \\ y(0) \\ z(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$
Initial tangent vector τ, initial normal vector ν and initial binomial vector β: $\begin{pmatrix} \tilde\tau(0) \\ \tilde\nu(0) \\ \tilde\beta(0) \end{pmatrix}$	$\begin{pmatrix} \tilde\tau(0) \\ \tilde\nu(0) \end{pmatrix} = \begin{pmatrix} (0,1) \\ (1,0) \end{pmatrix}$	$\begin{pmatrix} \tilde\tau(0) \\ \tilde\nu(0) \end{pmatrix} = \begin{pmatrix} (0,1) \\ (1,0) \end{pmatrix}$	$\begin{pmatrix} \tilde\tau(0) \\ \tilde\nu(0) \\ \tilde\beta(0) \end{pmatrix} = \begin{pmatrix} (0, \frac{1}{\sqrt2}, \frac{1}{\sqrt2}) \\ (-1, 0, 0) \\ (0, -\frac{1}{\sqrt2}, \frac{1}{\sqrt2}) \end{pmatrix}$
$e^{\int_0^s A(u)\,du}$	$\begin{bmatrix} \cos(s) & \sin(s) \\ -\sin(s) & \cos(s) \end{bmatrix}$	$\begin{bmatrix} \frac{4-s}{4} & \frac{\sqrt{8s-s^2}}{4} \\ -\frac{\sqrt{8s-s^2}}{4} & \frac{4-s}{4} \end{bmatrix}$	$\begin{bmatrix} \frac{1+\cos(s/\sqrt2)}{2} & \frac{\sin(s/\sqrt2)}{\sqrt2} & \frac{1-\cos(s/\sqrt2)}{2} \\ -\frac{\sin(s/\sqrt2)}{\sqrt2} & \cos\frac{s}{\sqrt2} & \frac{\sin(s/\sqrt2)}{\sqrt2} \\ \frac{1-\cos(s/\sqrt2)}{2} & -\frac{\sin(s/\sqrt2)}{\sqrt2} & \frac{1+\cos(s/\sqrt2)}{2} \end{bmatrix}$
$\begin{pmatrix} \tilde\tau(s) \\ \tilde\nu(s) \\ \tilde\beta(s) \end{pmatrix} = e^{\int_0^s A(u)\,du} \begin{pmatrix} \tilde\tau(0) \\ \tilde\nu(0) \\ \tilde\beta(0) \end{pmatrix}$	$\begin{pmatrix} \tilde\tau(s) \\ \tilde\nu(s) \end{pmatrix} = \begin{pmatrix} (\sin(s), \cos(s)) \\ (\cos(s), -\sin(s)) \end{pmatrix}$	$\begin{pmatrix} \tilde\tau(s) \\ \tilde\nu(s) \end{pmatrix} = \begin{pmatrix} (\frac{\sqrt{8s-s^2}}{4}, \frac{4-s}{4}) \\ (\frac{4-s}{4}, -\frac{\sqrt{8s-s^2}}{4}) \end{pmatrix}$	$\begin{pmatrix} \tilde\tau(s) \\ \tilde\nu(s) \\ \tilde\beta(s) \end{pmatrix} = \begin{pmatrix} (-\frac{\sin(s/\sqrt2)}{\sqrt2}, \cos\frac{s}{\sqrt2}, \frac{1}{\sqrt2}) \\ (-\cos\frac{s}{\sqrt2}, -\sin\frac{s}{\sqrt2}, 0) \\ (\frac{\sin(s/\sqrt2)}{\sqrt2}, -\cos\frac{s}{\sqrt2}, \frac{1}{\sqrt2}) \end{pmatrix}$
$\begin{pmatrix} x(s) \\ y(s) \end{pmatrix}$	$\begin{cases} x(s) = 1 - \cos(s) \\ y(s) = \sin(s) \end{cases}$	$\begin{cases} x(s) = 2\cos^{-1}(\frac{4-s}{4}) - \frac{(4-s)\sqrt{8s-s^2}}{8} \\ y(s) = \frac{8s-s^2}{8} \end{cases}$	$\begin{cases} x(s) = \cos\frac{s}{\sqrt2} \\ y(s) = \sin\frac{s}{\sqrt2} \\ z(s) = \frac{s}{\sqrt2} \end{cases}$
$s(t)$	$s = t$	$s = 4 - 4\cos\left(\frac{t}{2}\right)$	$s = \sqrt2 t$
$\begin{pmatrix} X(t) \\ Y(t) \end{pmatrix}$	$\begin{cases} X(t) = 1 - \cos(s) \\ Y(t) = \sin(s) \end{cases}$	$\begin{cases} X(t) = t - \sin(t) \\ Y(t) = 1 - \cos(t) \end{cases}$	$\begin{cases} X(t) = \cos t \\ Y(t) = \sin t \\ Z(t) = t \end{cases}$
Curve			

Figure 4 Animation of the cycloid curve in each step by the space-like parameter.

5. ANIMATION

To view the trajectory of curves, the animation for a circle (2D), a cycloid and a spiral curve (3D) is performed as shown at the website. Since the article is presented in a paper form, the animation action is disassembled as piecewisely shown in Fig. 4 for the 2D cycloid. Figure 5 shows the 3D spiral curve step by step. The animation can be found at the following website.

Cycloid by Y.T. Chou:
http://msvlab.hre.ntou.edu.tw/univ/cycloid.avi.
Circle by Y.H. Shih:
http://msvlab.hre.ntou.edu.tw/univ/track%202D%20change.avi
Spiral curve by J.H. Dai: http://msvlab.hre.ntou.edu.tw/univ/track%203D%20change.avi.

6. CONCLUSIONS

In this paper, the Frenet formula was used to successfully link all the different formulae of radius of curvature for the 2D curve in the textbook. According to the Frenet formula, two new and simpler expressions of the radius of curvature were derived. Illustrative examples of a cycloid, a circle, an astroid and a cardioid were given to show the same radius using the seven formulae. In addition, the initial conditions (position, normal, binormal and tangent vectors) and the given radius of curvature were used to construct the space-like parameter curve by solving the state equation of Frenet formula. A cycloid was used as an example to provide a reference for teachers and students in the engineering field. By solving the state equation of Frenet formula, another 3D spiral curve was demonstrated to obtain the curve. Finally, the
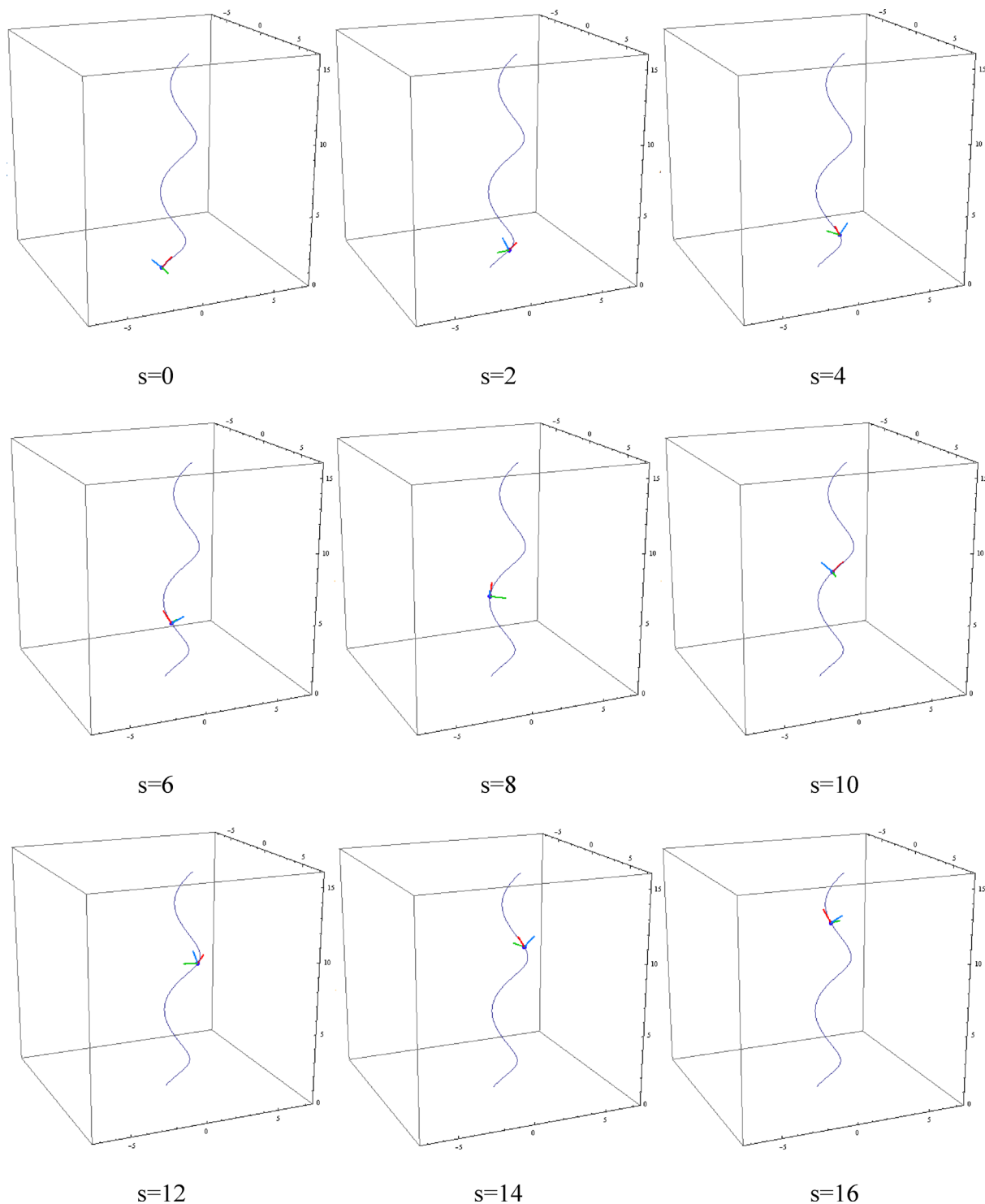
Figure 5 Animation of the spiral curve in each step by the space-like parameter.

animation was also provided for a circle, a cycloid and a spiral curve by using the Mathematica software.

DATA AVAILABILITY

Some or all data, models, or code that support the findings of this study are available from the corresponding author upon reasonable request.

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