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Regularized meshless method for antiplane shear problems with multiple inclusions

正規化無網格法求解反平面含多夾雜問題

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Abstract

In this paper, we employ the regularized meshless method (RMM) to solve antiplane shear problem with multiple inclusions. The solution is represented by a distribution of double layer potentials. The troublesome singularity in the MFS is avoided and the diagonal terms of influence matrices are determined. The coupled problem considerably reduces to two problems. One is the exterior problem for matrix with hole subject to a far-displacement field, the other is the interior problem for inclusion. The two boundary data between matrix and inclusion satisfy the continuous conditions in the interface between the inclusion and antiplane matrix. The linear algebraic system is obtained by matching boundary conditions and continuity conditions. Finally, the numerical results demonstrate the accuracy of the solutions after compared with analytical solutions and the Laurent series expansion method. Good agreements are obtained.

Keywords: antiplane shear, inclusion, regularized meshless method, MFS, shear stress.

摘要

本文係利用正規化無網格法求解反平面含多夾雜問題，其解是由雙層勢能表示。傳統的基本解法是利用避開奇異行為去決定影響矩陣的對角線項，正規化無網格法是利用奇異扣除奇異的方法去正規化奇異行為及決定影響矩陣的對角線項。本問題可以拆解成兩個邊界值問題，一個是材料含空洞的外域問題受到無窮遠的位移場作用，另一個是夾雜的內域問題。而介於材料與夾雜間的接合面則須滿足連續條件。配合邊界條件與連續條件之後就可以決定線性代數系統。最後，我們將數值結果與 Laurent 級數展開法比較，驗證其正確性，我們將獲得正確的結果。

關鍵字: 反平面剪力，夾雜，正規化無網格法，基本解法，剪應力。

1. Introduction

It is known that most engineering materials contain some defects in the form of cracks, inclusions, or second-phase particles. In 1992, Honein [1] derived the solution for double circular cylindrical elastic inclusions perfectly bonded to an elastic matrix of infinite extent, under anti-plane shear. The solution was obtained via iterations of Möbius transformations. Later, Gong [2] derived the general solution for multiple circular inclusions under antiplane shear in 1995. The general solutions are obtained by using antiplane complex potentials and the Laurent series expansion method [2]. In this study, we implement a systematic method for multiple inclusions under anti-plane shear.

The method of fundamental solutions (MFS) [3] is one of the meshless methods [4] and belongs to a boundary method of boundary value problems [5], which can be viewed as a discrete type of indirect boundary element method [6, 7]. In the MFS, the solution is approximated by a set of fundamental solutions of the governing equations which are expressed in terms of sources located outside the physical domain. The unknown coefficients in the linear combination of the fundamental solutions are determined by matching the boundary condition. The method is relatively easy to implement. It is adaptive in the sense that it can take into account sharp changes in the solution and in the geometry of the domain and can easily incorporate complex boundary conditions. A survey of the MFS and related method over the last thirty years has been found [3]. However, the MFS is still not a popular method because of the debatable artificial boundary distance of source location in numerical implementation especially for a complicated geometry. The diagonal coefficients of influence matrices are divergent in conventional case when the fictitious boundary approaches the physical boundary. In spite of its gain of singularity free, the influence matrices become ill-posed when the fictitious boundary is far away from the physical boundary. It results in an ill-posed problem since the condition number for the influence matrix becomes very large.

Recently, we developed a modified MFS, namely regularized meshless method (RMM), to overcome the drawback of MFS for solving the simply and multiply-connected Laplace problem [8, 9]. The method eliminates the well-known drawback of equivocal artificial boundary. The subtracting and adding-back technique [8] can regularize the singularity and hypersingularity of the kernel functions. This method can simultaneously distribute the observation and source points on the physical boundary even using the singular kernels instead of non-singular kernels. The diagonal terms of the influence matrices can be extracted out by using the proposed technique.

In this paper, the RMM is provided to solve the antiplane shear problem with multiple inclusions. A general-purpose program is developed to solve the antiplane shear problems for arbitrary number of inclusions. The results are compared with analytical solutions and the Laurent series expansion method [2]. Furthermore, the stress concentration for different shear modulus ratio will be studied through several examples to show the validity of our method.

2. Governing equation and boundary conditions

Consider inclusions embedded in another matrix of infinitely domain as shown in Fig. 2. The inclusions and the matrix have different elastic properties. The matrix is subject to a remote antiplane shear, $\sigma_{zy} = \tau_{\infty}$. The displacement field of the antiplane deformation is defined as:

$$u = v = 0\,, \quad w = w(x, y)\,, \tag{1}$$

where w is a function of x and y. For a linear elastic body, the stress components are

$$\sigma_{13} = \sigma_{31} = \mu \frac{\partial w}{\partial x}\,, \tag{2}$$

$$\sigma_{23} = \sigma_{32} = \mu \frac{\partial w}{\partial y}\,, \tag{3}$$

where μ is the shear modulus. The equilibrium equation can be simplified to

$$\frac{\partial \sigma_{31}}{\partial x} + \frac{\partial \sigma_{32}}{\partial y} = 0\,. \tag{4}$$

Thus, we have

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = \nabla^2 w = 0\,. \tag{5}$$

The continuity conditions across the matrix-inclusion interface is written as

$$w^m = w^i\,, \tag{6}$$

$$\mu_0 \frac{\partial w^m}{\partial n} = -\mu_1 \frac{\partial w^i}{\partial n}\,, \tag{7}$$

where the superscripts *i* and *m* denote inclusion and material.

3. Conventional method of fundamental solutions

By employing the RBF technique [7], the representation of the solution in Eq. (5) for multiple inclusions problem as shown in Fig. 2, can be approximated in terms of the strengths α_j of the singularities at s_j as

$$\begin{aligned} w(x_i) &= \sum_{j=1}^{N} T(s_j, x_i)\alpha_j \\ &= \sum_{j=1}^{N_1} T(s_j, x_i)\alpha_j + \sum_{j=N_1+1}^{N_1+N_2} T(s_j, x_i)\alpha_j + \cdots + \sum_{j=N_1+N_2+\cdots+N_{m-1}+1}^{N} T(s_j, x_i)\alpha_j\,, \end{aligned} \tag{8}$$

$$\frac{\partial w(x_i)}{\partial n_{x_i}} = \sum_{j=1}^{N} M(s_j, x_i)\alpha_j$$

$$= \sum_{j=1}^{N_1} M(s_j, x_i)\alpha_j + \sum_{j=N_1+1}^{N_1+N_2} M(s_j, x_i)\alpha_j + \cdots + \sum_{j=N_1+N_2+\cdots+N_{m-1}+1}^{N} M(s_j, x_i)\alpha_j \,, \tag{9}$$

where $T(s_j, x_i)$ is RBF, x_i and s_j represent *i*th observation point and *j*th source point, respectively, α_j are the *j*th unknown coefficients (strength of the singularity), $N_1, N_2, \cdots, N_m$ are the numbers of source points on m numbers of boundaries of inclusions, respectively, while N is the total numbers of source points $(N = N_1 + N_2 + \cdots + N_m)$ and $M(s_j, x_i) = \dfrac{\partial T(s_j, x_i)}{\partial n_{x_i}}$. The coefficients $\{\alpha_j\}_{j=1}^{N}$ are determined so that BCs are satisfied at the boundary points. The distributions of source points and observation points are shown in Fig. 1 (a) for the MFS. The chosen bases are the double layer potentials [7] as

$$T(s_j, x_i) = \frac{((x_i - s_j), n_j)}{r_{ij}^2} \,, \tag{10}$$

$$M(s_j, x_i) = \frac{2((x_i - s_j), n_j)((x_i - s_j), \overline{n_i})}{r_{ij}^4} - \frac{(n_j, \overline{n_i})}{r_{ij}^2} \,, \tag{11}$$

where (,) is the inner product of two vectors, r_{ij} is $|s_j - x_i|$, n_j is the normal vector at s_j, and $\overline{n_i}$ is the normal vector at x_i.

It is noted that the double layer potentials have both singularity and hypersingularity when source and field points coincide, which lead to difficulty in the conventional MFS. The fictitious distance between the fictitious (auxiliary) boundary (B') and the physical boundary (B), d, shown in Fig. 1 (a) needs to be chosen deliberately. To overcome the abovementioned shortcoming, s_j is distributed on the physical boundary as shown in Fig. 1 (b), by using the proposed regularized technique as written in next Section.

4. Regularized meshless method

The antiplane shear problem with multiple inclusions is decomposed into two problems as shown in Figs. 3 and 4. One is the exterior problem for matrix with hole subject to a far-displacement field, the other is the interior problem for inclusions. The two boundary data between matrix and inclusion satisfy the continuous conditions in Eqs.(6) and (7). Furthermore, the exterior problem for the matrix can be decomposed into two problems as shown in Fig. 4. One is the matrix with no hole subject to a far-displacement field, the other is the matrix with hole. The representations of the two solutions for interior problem ($w(x_i^I)$) and exterior problem ($w(x_i^O)$) are formulated by using the RMM as follows:

(1) Interior problem (*Inclusions*)

When the collocation point x_i approaches the source point s_j, the kernels in Eqs. (8) and (9) become singular. Eqs. (8) and (9) for the multiply-inclusions problem as plotted in Fig. 2(b) need to be regularized by using the regularization of subtracting and adding-back technique [8] as follows:

$$\begin{aligned} w(x_i^I) &= \sum_{j=1}^{N_1} T(s_j^I, x_i^I)\alpha_j + \cdots + \sum_{j=N_1+\cdots+N_{p-1}+1}^{N_1+\cdots+N_p} T(s_j^I, x_i^I)\alpha_j + \cdots \\ &+ \sum_{j=N_1+\cdots+N_{m-2}+1}^{N_1+\cdots+N_{m-1}} T(s_j^I, x_i^I)\alpha_j + \sum_{j=N_1+\cdots+N_{m-1}+1}^{N} T(s_j^I, x_i^I)\alpha_j \\ &- \sum_{j=N_1+\cdots+N_{p-1}+1}^{N_1+\cdots+N_p} T(s_j^I, x_i^I)\alpha_i, \quad x_i^I \in B_p, \quad p = 1, \ 2, \ 3, \cdots, \ m. \end{aligned} \tag{12}$$

where x_i^I is located on the boundaries B_p ($p = 1, \ 2, \ 3, \cdots, \ m$), and

$$\sum_{j=N_1+\cdots+N_{p-1}+1}^{N_1+\cdots+N_p} T(s_j^I, x_i^I) = 0, \quad x_i^I \in B_p, \quad p = 1, \ 2, \ 3, \cdots, \ m. \tag{13}$$

Therefore, we can obtain

$$\begin{aligned} w(x_i^I) &= \sum_{j=1}^{N_1} T(s_j^I, x_i^I)\alpha_j + \cdots + \sum_{j=N_1+\cdots+N_{p-1}+1}^{i-1} T(s_j^I, x_i^I)\alpha_j \\ &+ \sum_{j=i+1}^{N_1+\cdots+N_p} T(s_j^I, x_i^I)\alpha_j + \cdots + \sum_{j=N_1+\cdots+N_{m-2}+1}^{N_1+\cdots+N_{m-1}} T(s_j^I, x_i^I)\alpha_j \\ &+ \sum_{j=N_1+\cdots+N_{m-1}+1}^{N} T(s_j^I, x_i^I)\alpha_j - \left[\sum_{j=N_1+\cdots+N_{p-1}+1}^{N_1+\cdots+N_p} T(s_j^I, x_i^I) - T(s_i^I, x_i^I) \right]\alpha_i, \\ & x_i^I \in B_p, \quad p = 1, \ 2, \ 3, \cdots, \ m. \end{aligned} \tag{14}$$

Similarly, the boundary flux is obtained as

$$\begin{aligned} \frac{\partial w(x_i^I)}{\partial n_{x_i^I}} &= \sum_{j=1}^{N_1} M(s_j^I, x_i^I)\alpha_j + \cdots + \sum_{j=N_1+\cdots+N_{p-1}+1}^{N_1+\cdots+N_p} M(s_j^I, x_i^I)\alpha_j + \cdots \\ &+ \sum_{j=N_1+\cdots+N_{m-2}+1}^{N_1+\cdots+N_{m-1}} M(s_j^I, x_i^I)\alpha_j + \sum_{j=N_1+\cdots+N_{m-1}+1}^{N} M(s_j^I, x_i^I)\alpha_j \\ &- \sum_{j=N_1+\cdots+N_{p-1}+1}^{N_1+\cdots+N_p} M(s_j^I, x_i^I)\alpha_i, \quad x_i^I \in B_p, \quad p = 1, \ 2, \ 3, \cdots, \ m. \end{aligned} \tag{15}$$

in which

$$\sum_{j=N_1+\cdots+N_{p-1}+1}^{N_1+\cdots+N_p} M(s_j^I, x_i^I) = 0, \quad x_i^I \in B_p, \quad p = 1, \ 2, \ 3, \cdots, \ m. \tag{16}$$

Therefore, we obtain

$$\frac{\partial w(x_i^I)}{\partial n_{x_i^I}} = \sum_{j=1}^{N_1} M(s_j^I, x_i^I)\alpha_j + \cdots + \sum_{j=N_1+\cdots+N_{p-1}+1}^{i-1} M(s_j^I, x_i^I)\alpha_j$$
$$+ \sum_{j=i+1}^{N_1+\cdots+N_p} M(s_j^I, x_i^I)\alpha_j + \cdots + \sum_{j=N_1+\cdots+N_{m-2}+1}^{N_1+\cdots+N_{m-1}} M(s_j^I, x_i^I)\alpha_j \quad (17)$$
$$+ \sum_{j=N_1+\cdots+N_{m-1}+1}^{N} M(s_j^I, x_i^I)\alpha_j - \left[\sum_{j=N_1+\cdots+N_{P-1}+1}^{N_1+\cdots+N_p} M(s_j^I, x_i^I) - M(s_i^I, x_i^I)\right]\alpha_i,$$
$$x_i^I \in B_p, \quad p = 1,\ 2,\ 3, \cdots,\ m.$$

(2) Exterior problem (*Matrix*)

When the observation point x_i^O locates on the boundaries B_p ($p = 1,\ 2,\ 3, \cdots,\ m$), Eq. (12) becomes

$$w(x_i^O) = \sum_{j=1}^{N_1} T(s_j^O, x_i^O)\alpha_j + \cdots + \sum_{j=N_1+\cdots+N_{P-1}+1}^{N_1+\cdots+N_p} T(s_j^O, x_i^O)\alpha_j + \cdots$$
$$+ \sum_{j=N_1+\cdots+N_{m-2}+1}^{N_1+\cdots+N_{m-1}} T(s_j^O, x_i^O)\alpha_j + \sum_{j=N_1+\cdots+N_{m-1}+1}^{N} T(s_j^O, x_i^O)\alpha_j \quad (18)$$
$$- \sum_{j=N_1+\cdots+N_{P-1}+1}^{N_1+\cdots+N_p} T(s_j^I, x_i^I)\alpha_i, \quad x_i^{O\ or\ I} \in B_p, \quad p = 1,\ 2,\ 3, \cdots,\ m.$$

Where x_i^O is also located on the boundaries B_p ($p = 1,\ 2,\ 3, \cdots,\ m$) and

$$\sum_{j=N_1+\cdots+N_{m-1}+1}^{N} T(s_j^I, x_i^I)\alpha_i = 0, \quad x_i^I \in B_p, \quad p = 1,\ 2,\ 3, \cdots,\ m. \quad (19)$$

Hence, we obtain

$$w(x_i^O) = \sum_{j=1}^{N_1} T(s_j^O, x_i^O)\alpha_j + \cdots + \sum_{j=N_1+\cdots+N_{p-1}+1}^{i-1} T(s_j^O, x_i^O)\alpha_j$$
$$+ \sum_{j=i+1}^{N_1+\cdots+N_p} T(s_j^O, x_i^O)\alpha_j + \cdots + \sum_{j=N_1+\cdots+N_{m-2}+1}^{N_1+\cdots+N_{m-1}} T(s_j^O, x_i^O)\alpha_j \quad (20)$$
$$+ \sum_{j=N_1+\cdots+N_{m-1}+1}^{N} T(s_j^O, x_i^O)\alpha_j - \left[\sum_{j=N_1+\cdots+N_{P-1}+1}^{N_1+\cdots+N_p} T(s_j^I, x_i^I) - T(s_i^O, x_i^O)\right]\alpha_i,$$
$$x_i^{O\ or\ I} \in B_p, \quad p = 1,\ 2,\ 3, \cdots,\ m.$$

Similarly, the boundary flux is obtained as

$$\frac{\partial w(x_i^O)}{\partial n_{x_i^O}} = \sum_{j=1}^{N_1} M(s_j^O, x_i^O)\alpha_j + \cdots + \sum_{j=N_1+\cdots+N_{p-1}+1}^{N_1+\cdots+N_p} M(s_j^O, x_i^O)\alpha_j + \cdots \quad (21)$$

$$+\sum_{j=N_1+\cdots+N_{m-2}+1}^{N_1+\cdots+N_{m-1}} M(s_j^O, x_i^O)\alpha_j + \sum_{j=N_1+\cdots+N_{m-1}+1}^{N} M(s_j^O, x_i^O)\alpha_j$$

$$-\sum_{j=N_1+\cdots+N_{P-1}+1}^{N_1+\cdots+N_p} M(s_j^I, x_i^I)\alpha_i\,,\quad x_i^{O\ or\ I} \in B_p\,,\quad p=1,\ 2,\ 3,\cdots,\ m\,.$$

in which

$$\sum_{j=N_1+\cdots+N_{m-1}+1}^{N} M(s_j^I, x_i^I) = 0\,,\quad x_i^I \in B_p\,,\quad p=1,\ 2,\ 3,\cdots,\ m\,. \tag{22}$$

Hence, we obtain

$$\begin{aligned}\frac{\partial w(x_i^O)}{\partial n_{x_i^O}} &= \sum_{j=1}^{N_1} M(s_j^O, x_i^O)\alpha_j + \cdots + \sum_{j=N_1+\cdots+N_{p-1}+1}^{i-1} M(s_j^O, x_i^O)\alpha_j \\ &+ \sum_{j=i+1}^{N_1+\cdots+N_p} M(s_j^O, x_i^O)\alpha_j + \cdots + \sum_{j=N_1+\cdots+N_{m-2}+1}^{N_1+\cdots+N_{m-1}} M(s_j^O, x_i^O)\alpha_j \\ &+ \sum_{j=N_1+\cdots+N_{m-1}+1}^{N} M(s_j^O, x_i^O)\alpha_j - \left[\sum_{j=N_1+\cdots+N_{P-1}+1}^{N_1+\cdots+N_p} M(s_j^I, x_i^I) - M(s_i^O, x_i^O)\right]\alpha_i\,, \\ & x_i^{O\ or\ I} \in B_p\,,\quad p=1,\ 2,\ 3,\cdots,\ m\,.\end{aligned} \tag{23}$$

The detailed derivations of Eqs. (13), (16), (19) and (22) are given in the reference [8].

5. Construction of influence matrices for inclusion problems under antiplane shear

By superimposing the original system to two systems and matching continuity conditions as shown in Figs. 3-5, the linear algebraic system for antiplane shear problems can be obtained as:

$$\begin{bmatrix} -[T_w^I] & [T_w^O] \\ \dfrac{\mu^i}{\mu^m}[M_w^I] & [M_w^O] \end{bmatrix} \begin{Bmatrix} \{\alpha^i\} \\ \{\alpha^m\} \end{Bmatrix} = \begin{Bmatrix} -\{w^\infty\} \\ -\left\{\dfrac{\partial w^\infty}{\partial n}\right\} \end{Bmatrix}, \tag{24}$$

where w^∞ denotes the out-of-plane elastic displacement at infinity. Following Eq. (24), the unknown densities ($\{\alpha^i\}$ and $\{\alpha^m\}$) are obtained and the field solution can be solved by using Eq. (8)

6. Numerical example

Fig. 5 shows the matrix imbedded three inclusions under antiplane shear. The geometry conditions is $d = 2r_1$. It is interesting to note that a uniform stress field results when the shear modulus is the same for the inclusion and the matrix. Therefore, the stress concentrations $\sigma_{Z\theta}$ in the matrix around the interface of the first inclusion are shown in Figs. 6 (a)~(d), respectively. From Fig. 6 (a), it is obvious that the case of holes ($\mu_1/\mu_0 = \mu_2/\mu_0 = \mu_3/\mu_0 = 0.0$) leads to the maximum stress concentration at $\theta = 0^\circ$. Because of the interaction effects, it is larger than 2 of a single hole [1].

The stress component $\sigma_{Z\theta}$ vanishes in the case of approximate to rigid inclusions ($\mu_1/\mu_0 = \mu_2/\mu_0 = \mu_3/\mu_0 = 5.0$). The agreement result compared with the Laurent series expansion method.

7. Conclusions

In this study, we employed the RMM approach to solve for antiplane shear problems with multiple inclusions. Only the boundary nodes on the physical boundary are required. The major difficulty of the coincidence of the source and collocation points in the conventional MFS is then circumvented. Therefore, the controversy of the fictitious boundary outside the physical domain by using the conventional MFS no longer exists. Although it results in the singularity and hypersingularity due to the use of double layer potential, the finite values of the diagonal terms for the influence matrices have been extracted out by employing the regularization technique. The numerical results by applying the developed program agreed very well with the Laurent series expansion method.

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References

1. Honein E, Honein T and Herrmann G. On two circular inclusions in harmonic problems. Quarterly of Applied Mathematics 1992; **50:** 479-499.
2. Gong SX. Antiplane interaction among multiple circular inclusions. Mechanics Research Communications 1995; **22:** 257-262.
3. Kupradze VD and Aleksidze MA. The method of functional equations for the approximate solution of certain boundary value problems. U.S.S.R. Computational Mathematics and Mathematical Physics 1964; **4**: 199-205.
4. Chen W and Tanaka M. A meshfree integration-free and boundary-only RBF technique. Computers & Mathematics with Applications 2002; **43**: 379-391.
5. Chen JT, Chang MH, Chen KH and Lin SR. The boundary collocation method with meshless concept for acoustic eigenanalysis of two-dimensional cavities using radial basis function. Journal of Sound and Vibration 2002; **257**: 667-711.
6. Kang SW and Lee JM. Application of free vibration analysis of membranes using the non-dimensional dynamic influence function. Journal of Sound and Vibration 2000; **234**, 455–470.
7. Cheng AHD. Particular solutions of Laplacian, Helmholtz-type, and polyharmonic operators involving higher order radial basis functions. Engineering Analysis with Boundary Elements 2000; **24**: 531-538.

8. Young DL, Chen KH and Lee CW. Novel meshless method for solving the potential problems with arbitrary domain. Journal of Computational Physics 2005; **209**: 290-321.

9. Chen KH, Kao JH, Chen JT, Young DL and Lu MC, Regularized meshless method for multiply-connected-domain Laplace problems, Engineering Analysis with Boundary Elements 2006; Accepted.

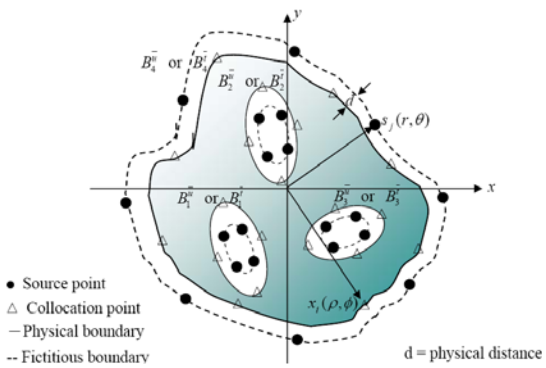

Fig. 1 (a) Conventional MFS.

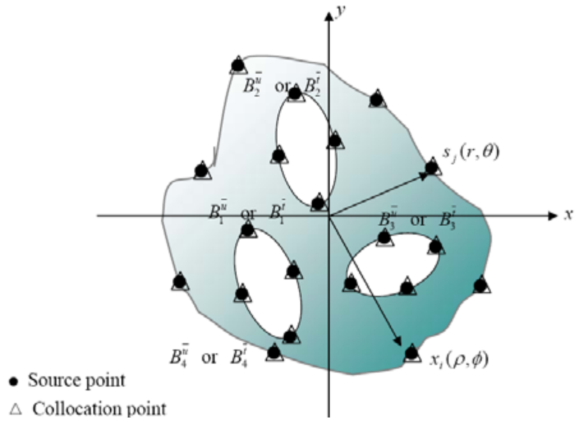

Fig. 1 (b) Regularized meshless method.

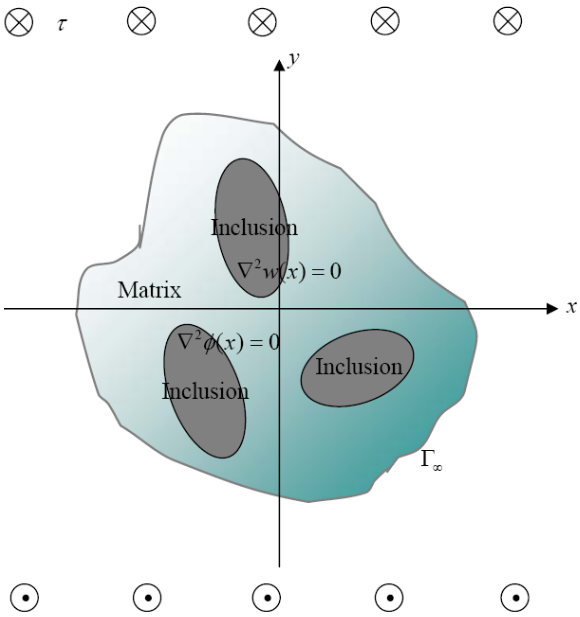

Fig. 2 Problem sketch for multiple inclusions problem under remote shear.

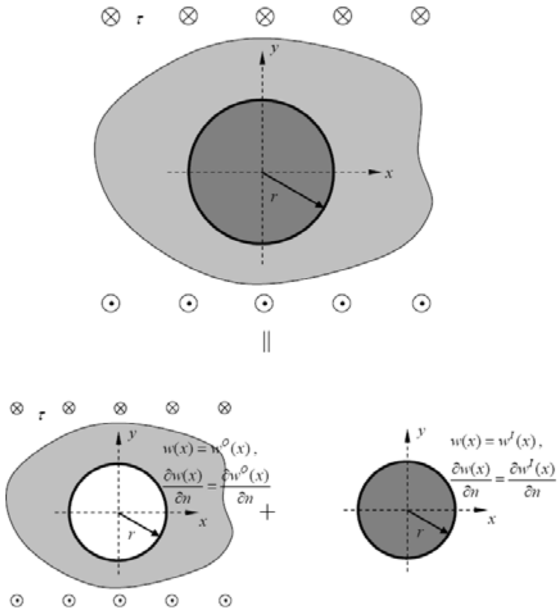

Fig. 3 Decomposition of the problem.

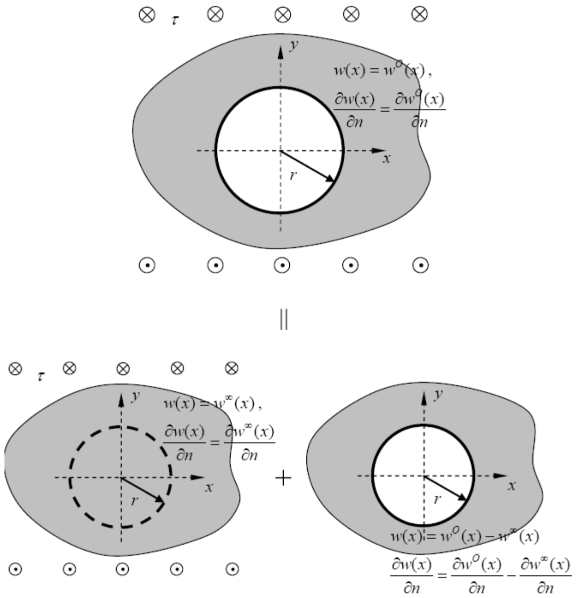

Fig. 4 Decomposition of the problem of Fig. 3 (a).

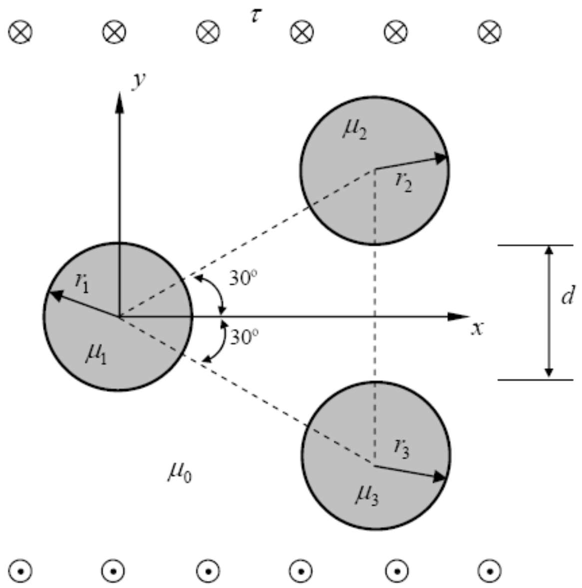

Fig. 5 Problem sketch of three inclusions under antiplane shear.

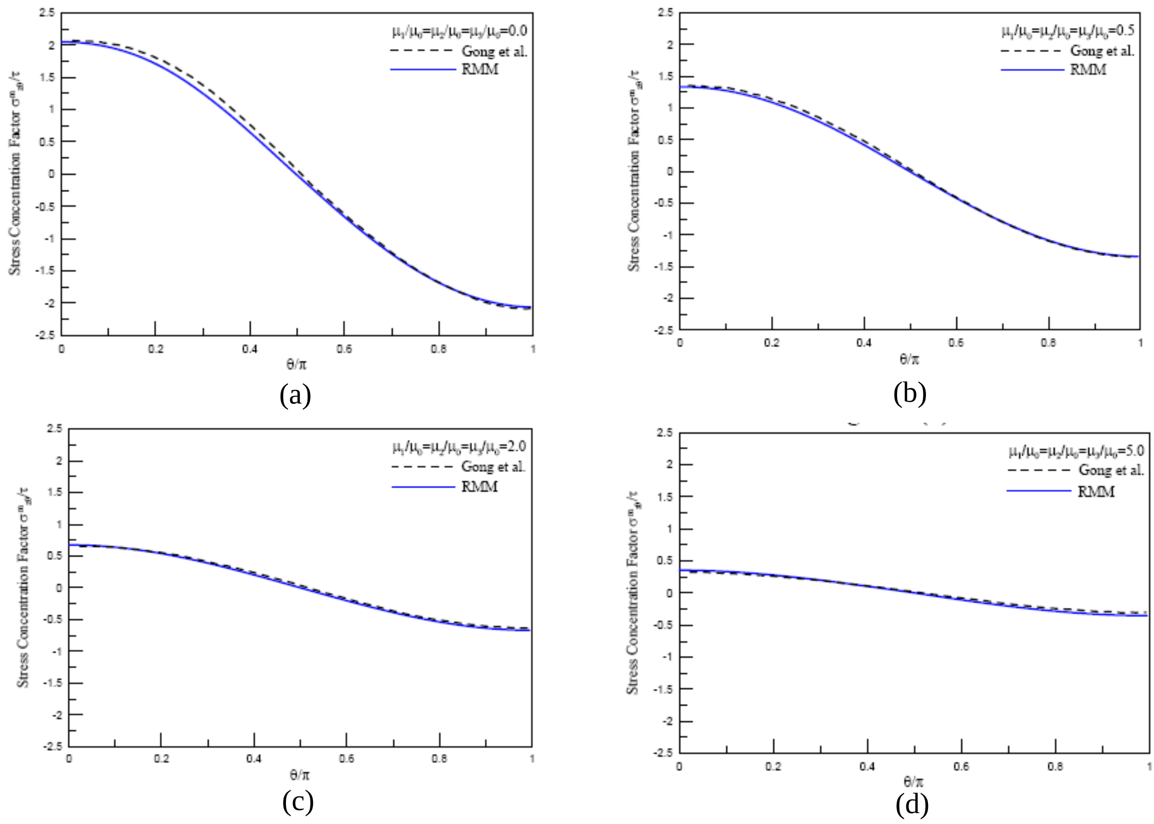

Fig. 6 Stress concentration factor $\sigma_{z\theta}^{m}/\tau$ along the boundaries of both the left inclusion and matrix for various different shear modulus ratios.