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A Study on Half-Plane Laplace Problems with a Circular Hole

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ABSTRACT

This paper describes a numerical procedure for solving half-plane Laplace problems with a circular hole by using the null-field integral equation and degenerate kernels. The unknown boundary potential and flux are approximated by the truncated Fourier series. Degenerate kernels are utilized in the null-field integral equation. A linear algebraic system is obtained without boundary discretization. To avoid the integration along the infinite boundary of half-plane problem, image method is utilized. The present method is verified through two examples with the analytical solutions derived by Lebedev. In addition, the results of BEM and meshless method as well as exact solutions are also compared to show the accuracy and efficiency. This approach can be extended to problems with multiple circular holes without any difficulties.

Keywords: half-plane problem, image method, Laplace problem, Fourier series, degenerate kernel

INTRODUCTION

A number of problems in engineering involving infinite and half-plane domains, e.g. the soil-structure interaction, tunnel and concrete pipe design, have been studied by using finite element method (FEM), boundary element method (BEM), method of fundamental solution (MFS) and Trefftz method. As we know, FEM is a very efficient method in solving finite-domain problems, but it is not convenient to deal with infinite-domain problems. In this aspect, BEM is an efficient alternative which has been extensively used for solving infinite and half-plane problems [2-4].

To tackle the exterior problems containing circular holes by using FEM, special treatment should be addressed for truncating the unbounded domain. Due to this reason, BEM is a more efficient method to solve an infinite or half-plane problem. BEM with truncated boundary has been utilized to solve the half-plane problem [5-7]. To avoid the error due to truncated boundary, BEM in conjunction with the image concept has been employed.

In this paper, the BIEM by using image concept is utilized to solve half-plane problems with a circular hole. The unknown boundary potential and flux are approximated by using the truncated Fourier series [8, 9]. The Fourier coefficients can be determined by substituting the degenerate kernels in the null-field integral equation. Numerical results including Dirichlet and mixed-type cases are successfully given to illustrate the validity of the present approach. The accuracy and efficiency for the present method are also examined.

BOUNDARY INTEGRAL FORMULATION AND PROBLEM STATEMENT

Based on the boundary integral formulation for potential problems [1], we have

$$2\pi u(x)=\int_B T(s,x)u(s)dB(s)-\int_B U(s,x)t(s)dB(s),\ x\in D, \tag{1}$$

where s and x are the source and field points, respectively, D is the domain of interest, $u(s)$ and $t(s)$ denote the potential and its normal flux on the source point s, respectively, and $U(s,x)=\ln r$ is the fundamental solution which satisfies

$$\nabla^2 U(s,x)=\delta(x-s), \tag{2}$$

in which, $\delta(x-s)$ denotes the Dirac-delta function. $T(s,x)$ is defined by

$$T(s,x)=\frac{\partial U(s,x)}{\partial n_s}, \tag{3}$$

where n_s denotes the outward normal vector at the source point s.

By collocating x outside the domain, we can obtain the null-field integral equation as shown below

$$0=\int_B T(s,x)u(s)dB(s)-\int_B U(s,x)t(s)dB(s),\ x\in D^e. \tag{4}$$

The first half-plane problem with a circular hole is considered as shown in Fig. 1. The potential $u(x)$ satisfies the Laplace equation

$$\nabla^2 u(x)=0,\ x\in D, \tag{5}$$

subject to boundary conditions

$$u=0 \quad \text{on} \quad B_1, \tag{6}$$

$$u=1 \quad \text{on} \quad B_2. \tag{7}$$

For the second problem of mixed-type, the boundary conditions are

$$u=0 \quad \text{on} \quad B_1, \tag{8}$$

$$t=1 \quad \text{on} \quad B_2. \tag{9}$$

In order to avoid the boundary integral along B_1 in Eq. (4), the Green's function using the image point is obtained

$$U(s;x,x')=\ln|x-s|-\ln|x'-s|, \tag{10}$$

such that

$$U(s;x,x')|_{s\in B_1}=0. \tag{11}$$

Then, Eqs. (1) and (4) are reduced to

$$2\pi u(x)=\int_B T(s;x,x')u(s)dB(s)-\int_B U(s;x,x')t(s)dB(s),\ x\in D, \tag{12}$$

$$0=\int_B T(s;x,x')u(s)dB(s)-\int_B U(s;x,x')t(s)dB(s),\ x\in D^e \tag{13}$$

where $U(s;x,x')$ denotes the Green's function.

EXPANSION OF KERNELS AND BOUNDARY DENSITIES

Based on the separable properties, the U kernel function can be expanded into degenerate form as shown below

$$U(s,x)=\ln r=\begin{cases} U^I(s,x)=\ln R-\sum_{m=1}^{\infty}\frac{1}{m}(\frac{\rho}{R})^m\cos m(\theta-\phi),\ R>\rho, \\ U^E(s,x)=\ln\rho-\sum_{m=1}^{\infty}\frac{1}{m}(\frac{R}{\rho})^m\cos m(\theta-\phi),\ \rho>R, \end{cases} \tag{14}$$

where the superscripts I and E denote the interior and exterior cases, respectively, $x=(\rho,\phi)$ and $s=(R,\theta)$ are the polar coordinates. Based on the Fig. 3, we employ the image concept and obtain

$$U(s,x')=\ln r'=\begin{cases} U^I(s,x')=\ln R-\sum_{m=1}^{\infty}\frac{1}{m}(\frac{\rho'}{R})^m\cos m(\theta-\phi'),\ R>\rho', \\ U^E(s,x')=\ln\rho'-\sum_{m=1}^{\infty}\frac{1}{m}(\frac{R}{\rho'})^m\cos m(\theta-\phi'),\ \rho'>R, \end{cases} \tag{15}$$

where $x'=(\rho',\phi')$ is the image point outside the half-plane domain. The degenerate kernels for the Green's function are

$$\begin{aligned} U^I(s;x,x') &= \ln r-\ln r' \\ &= \ln R-\sum_{m=1}^{\infty}\frac{1}{m}(\frac{\rho}{R})^m\cos m(\theta-\phi)-\ln\rho'+\sum_{m=1}^{\infty}\frac{1}{m}(\frac{R}{\rho'})^m\cos m(\theta-\phi'),\ \rho<R<\rho' \end{aligned} \tag{16}$$

$$\begin{aligned} U^E(s;x,x') &= \ln r-\ln r' \\ &= \ln\rho-\sum_{m=1}^{\infty}\frac{1}{m}(\frac{R}{\rho})^m\cos m(\theta-\phi)-\ln\rho'+\sum_{m=1}^{\infty}\frac{1}{m}(\frac{R}{\rho'})^m\cos m(\theta-\phi'),\ R<\rho<\rho' \end{aligned} \tag{17}$$

After taking the normal derivative with respect to Eq. (14) and Eq. (15), the $T(x;s,s')$ kernel can be derived as

$$\begin{aligned} T^I(s;x,x') &= \frac{\partial U^I(s;x,x')}{\partial n_s}=-\frac{\partial U^I(s;x,x')}{\partial R} \\ &= -\frac{1}{R}-\sum_{m=1}^{\infty}(\frac{\rho^m}{R^{m+1}})\cos m(\theta-\phi)-\sum_{m=1}^{\infty}(\frac{R^{m-1}}{\rho'^m})\cos m(\theta-\phi'),\ \rho<R<\rho' \end{aligned} \tag{18}$$

$$\begin{aligned} T^E(s;x,x') &= \frac{\partial U^E(s;x,x')}{\partial n_s}=-\frac{\partial U^E(s;x,x')}{\partial R} \\ &= \sum_{m=1}^{\infty}(\frac{R^{m-1}}{\rho^m})\cos m(\theta-\phi)-\sum_{m=1}^{\infty}(\frac{R^{m-1}}{\rho'^m})\cos m(\theta-\phi'),\ R<\rho<\rho' \end{aligned} \tag{19}$$

The boundary densities can be expressed in Fourier series form as shown below

$$u(s)=f(\theta)=a_0+\sum_{n=1}^{\infty}(a_n\cos n\theta+b_n\sin n\theta),\ s\in B_2 \tag{20}$$

$$t(s)=g(\theta)=p_0+\sum_{n=1}^{\infty}(p_n\cos n\theta+q_n\sin n\theta),\ s\in B_2 \tag{21}$$

where the coefficients a_0, a_n and b_n are specified once $f(\theta)$ is given in the case 1, p_0, p_n and

q_n are the undetermined coefficients for the Dirichlet problem. In the case 2, $g(\theta)$ is given and the coefficients a_0, a_n and b_n need to be determined.

LINEAR ALGEBRAIC EQUATION

By substituting Eqs. (16) and (18) into Eq. (13) for the kernels and substituting Eqs. (20) and (21) for the boundary data, we can obtain a linear algebraic system as follows

$$[A]\{x\} = [B]\{y\} \tag{22}$$

where $[A]$ and $[B]$ are influence matrices, $\{x\}$ and $\{y\}$ denote the unknown and specified coefficient matrices, respectively.

By substituting Eqs. (17) and (19) and the determined Fourier coefficients into Eq. (12), we can obtain the potential in the domain of half-plane.

NUMERICAL EXAMPLES

In this section, we consider two exterior problems with a circular hole [10] as shown in Fig.1 and Fig. 2.

Case 1: The circle with radius $a=1$ lies below the ground at depth $h=5$, subject to the Dirichlet boundary conditions $u=0$ on B_1 and $u=1$ on B_2.

Case 2: The circle with radius $a=1$ lies below the ground at depth $h=5$, subject to the mixed-type boundary conditions $u=0$ on B_1 and $t=1$ on B_2.

To solve the problems, four approaches [11] including (a) single-layer MFS approach, (b) double-layer MFS approach, (c) BEM with truncated boundary, and (d) BEM with image concept as well as (f) Lebedev exact solution are given for comparison with the (e) present solutions. The node and mesh for the different methods are shown in Fig. 5. Boundary flux was obtained to converge well to the exact solution for five and ten terms of Fourier series as shown in Fig. 4. All numerical results are illustrated in Fig. 6 and Fig. 7. Good agreement is made.

CONCLUSIONS

A new method by using degenerate kernels, null-field integral equation and Fourier series is proposed to solve half-plane problems with a circular hole. Two cases, Dirichlet and mixed-type boundary conditions are considered. Numerical results agree well with those of single-layer MFS approach, double-layer MFS approach, BEM with truncated boundary and BEM with image concept as well as the Lebedev exact solution. Not only mesh and boundary elements are not required in the present formulation but also the convergence rate of the present method is faster than the four numerical approaches.

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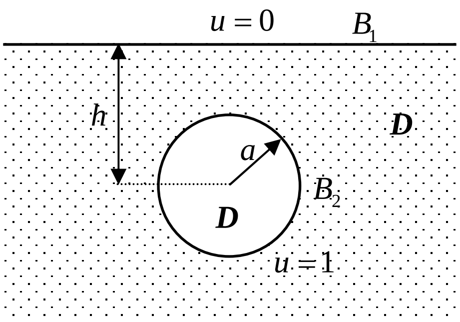
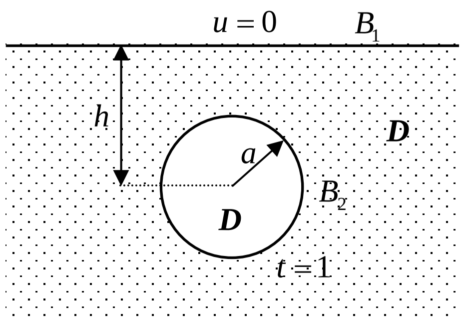


Analytical solution

$$u(x,y)=\frac{1}{\ln[(h+c)/a]}\cdot\ln\frac{\sqrt{(x^2-c^2+y^2)^2+4c^2y^2}}{(x-c)^2+y^2}$$

$$c=\sqrt{h^2-a^2}$$

Fig. 1 Half-plane problem-Dirichlet type



Analytical solution

$$u(\alpha,\beta)=\frac{Q}{k\pi\sinh\alpha_0}\left[\frac{\alpha}{2}+\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\frac{e^{-n\alpha_0}}{\cosh n\alpha_0}\sinh n\alpha\cos n\beta\right]$$

$$\cosh\alpha_0=h/a$$

Fig. 2 Half-plane problem-Mixed type

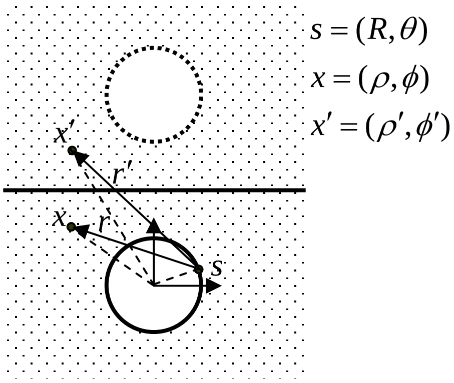


Fig. 3 Considered problem and auxiliary system

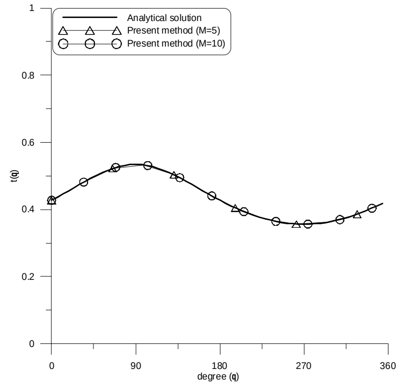
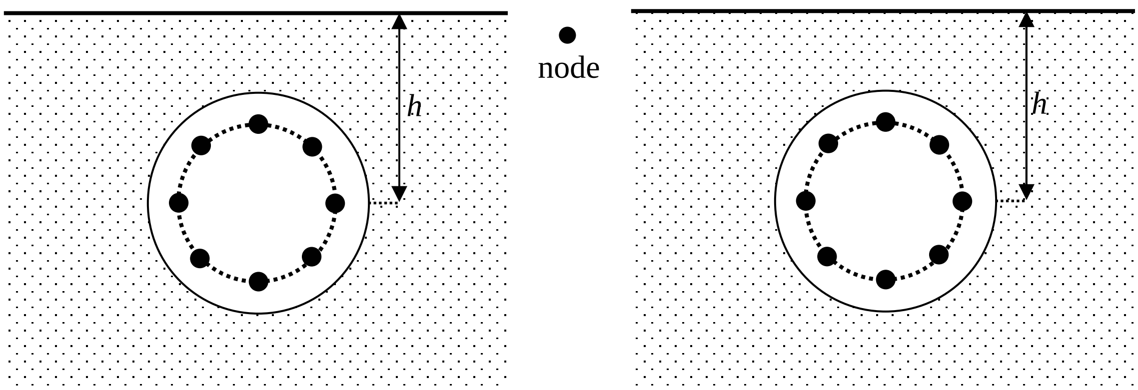


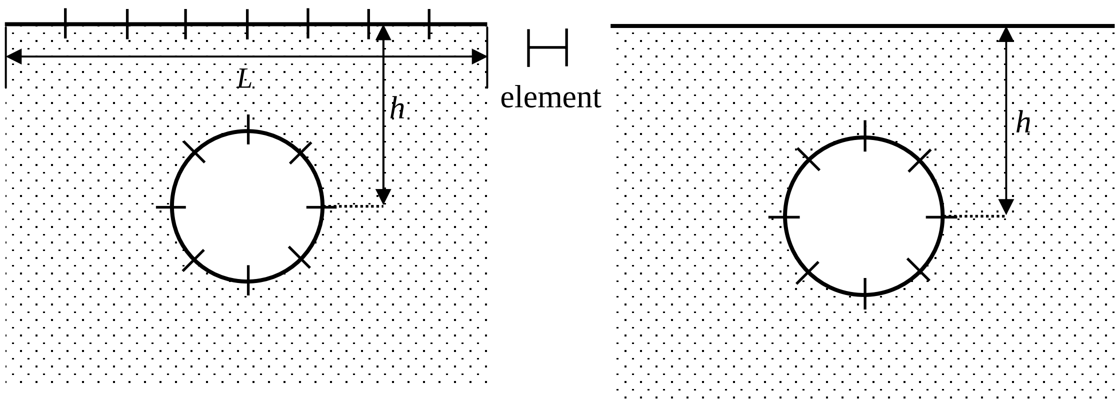
Fig. 4 The normal flux along the circular boundary

Fig. 5 Nodes and meshes



(a) MFS (Single-layer potential method)

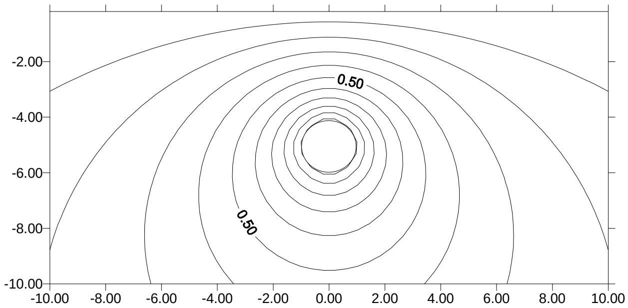
(b) MFS (Double-layer potential method)



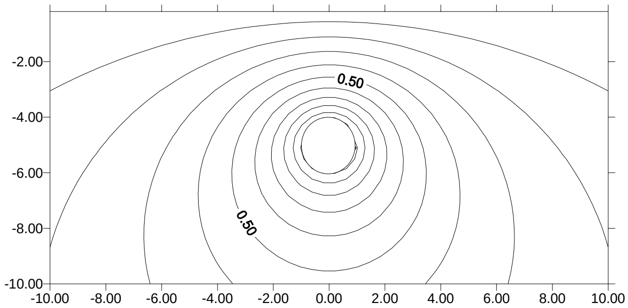
(c) BEM with truncated boundary

(d) BEM using image method

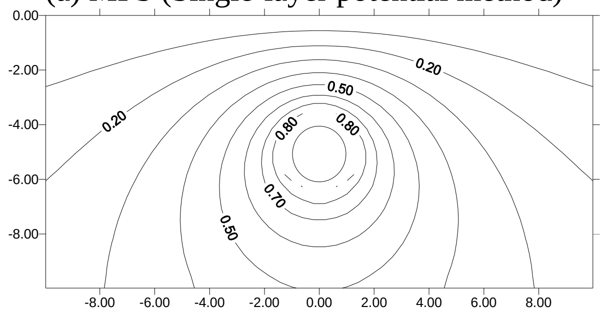
Fig. 6 Numerical results by using several methods (Dirichlet problem)



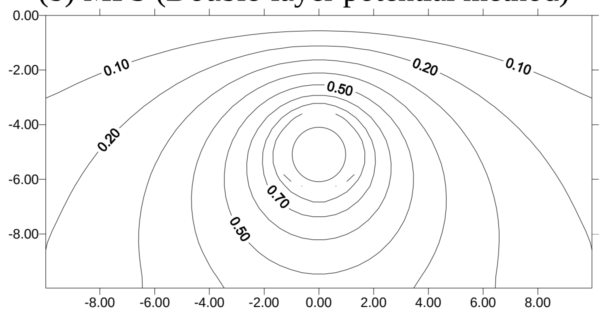
(a) MFS (Single-layer potential method)



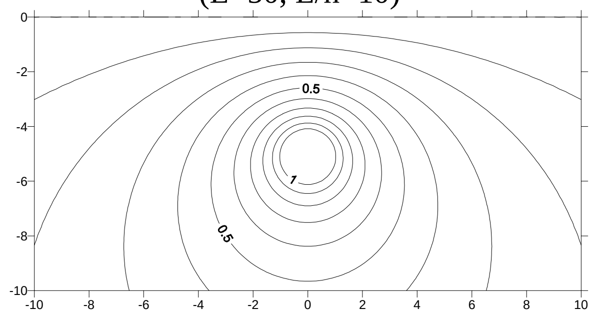
(b) MFS (Double-layer potential method)



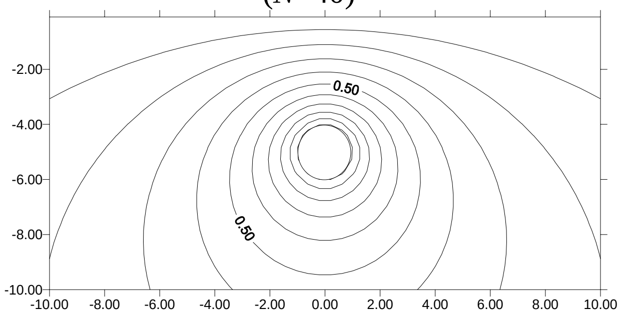
(c) BEM with truncated boundary
(L=50, L/h=10)



(d) BEM using image method
(*N*=40)

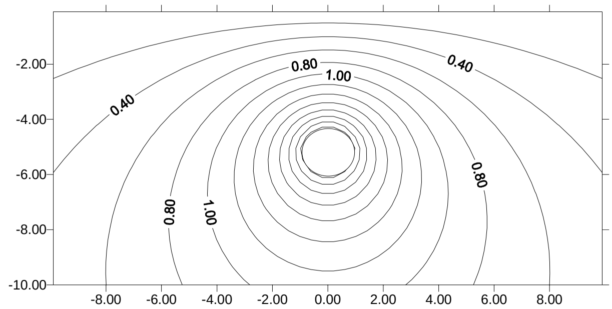


(e) Present method

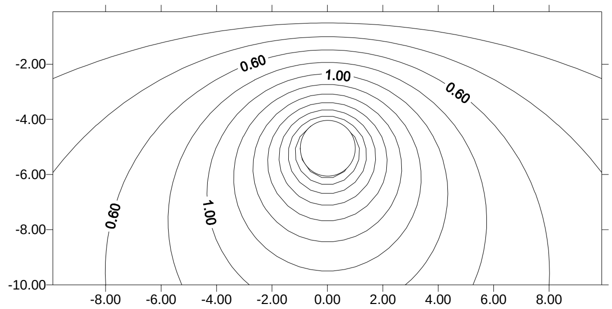


(f) Analytical solution (Lebedev)

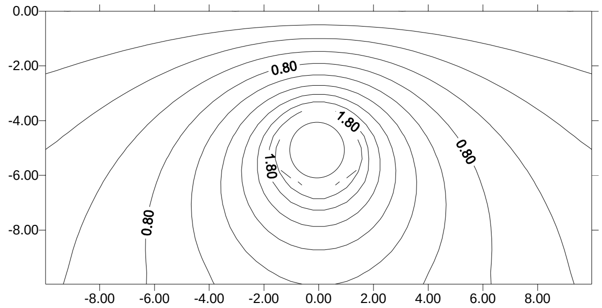
Fig 7 Numerical results by using several methods (Mixed-type problem)



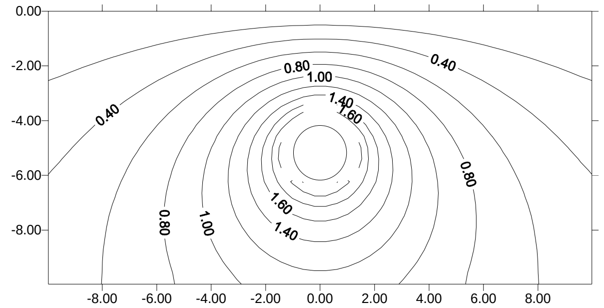
(a) MFS (Single-layer potential method)



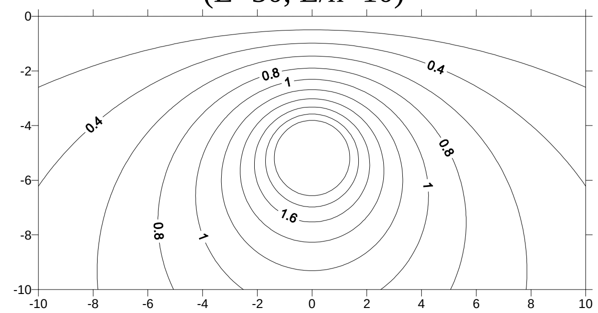
(b) MFS (Double-layer potential method)



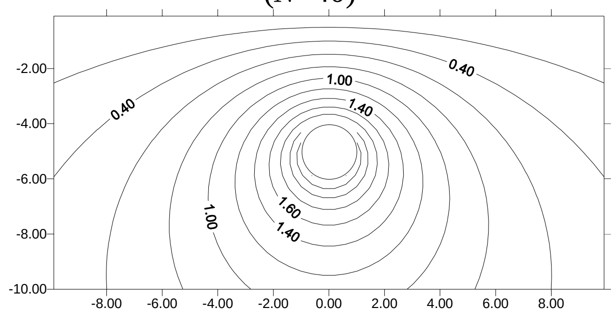
(c) BEM with truncated boundary
(L=50, L/h=10)



(d) BEM using image method
(N=40)



(e) Present method



(f) Analytical solution (Lebedev)

含圓洞半平面之拉普拉斯問題之研究

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摘要

本文以勢能理論為基礎，提出以退化核與傅立葉級數展開求解含孔洞半平面的問題，此方法可視為半解析法。邊界未知勢能與流通量使用有限項傅立葉級數來近似求得。利用退化核與傅立葉展開可導得一線性代數方法而無須對邊界離散。半平面無限邊界則採用映射法予以處理。文中以 Lebedev 導得解析解的兩個不同邊界條件的拉普拉斯問題進行測試。所得結果並與邊界元素法與無網格法作比較，驗證本方法的正確性。本文並可將單圓孔洞問題推廣至多圓孔洞問題。

關鍵字：半平面，映像法，拉普拉斯問題，傅立業級數，退化核