Proceedings of Symposium on Advances of Mechanics
In Honor of President Robert R. Hwang
May 12, 2006, Keelung, Taiwan

NEW METHODS FOR ELASTIC TORSION OF BAR WITH ARBITRARY SHAPE OF CROSS-SECTION

Chein-Shan Liu
Department of Mechanical and Mechatronic Engineering, National Taiwan Ocean University, Keelung, Taiwan
E-mail: csliu@mail.ntou.edu.tw

ABSTRACT

The solutions of the torsion problem of uniform bars with arbitrary cross-section are presented by the first kind Fredholm integral equation along a selected circle. Then we consider the Lavrentiev regularization and Tikhonov regularization to treat the ill-posed problem. The termwise separable property of kernel function allows us to obtain a two-point boundary value solution. Numerical examples show the effectiveness of the new methods in providing excellent estimates of the unknown data from the given boundary conditions.

I. INTRODUCTION

The problem of elastic torsion is a classical one in the theory of elasticity [1,2]. This problem can be formulated either in terms of the Neumann boundary value problem of the Laplace equation for the warping function, or the Dirichlet boundary value problem of the Laplace equation for the conjugate warping function or as the Dirichlet boundary value problem of the Poisson equation for the stress function. It seems that the second formulation is a good starting point of this paper.

Although the exact solutions have been found for some simple bars with popular cross-section shapes like as circle, ellipse, rectangle and triangle, in general, for a given shape of bar the finding of closed-form functions of its torsion, shear stresses as well as rigidity is not an easy task.

Indeed, the explicit solutions are the exception, and if one were to choose an arbitrary shape of bar then the geometric nonlinearity commences and then typically the numerical solution would be required. Series solutions, different coordinate systems, special functions and complex variables have all been used in the elastic torsion problems.

The most widely used numerical methods are finite difference, finite element and boundary element methods. For a complicated shape of the cross-section we usually require a large number of nodes and elements to match the geometrical shape. For the uniform bar with polygon cross-section there were much advanced methods to treat it as reviewed by Hassenpflug [3]. For the solutions of complex torsion problems the boundary collocation method was also applied by many people as can be seen in the paper by Kolodziej and Fraska [4]; on the other hand there also appeared the complex polynomial method and the complex variable boundary element method as advocated by Aleynikov and Stromov [5].

II. THE FREDHOLM INTEGRAL EQUATION

It is known that the torsion of an elastic prismatic rod comes to Dirichlet's problem for the Poisson equation

$$\Delta\phi(x,y) = -2, \quad (x,y) \in \Omega, \tag{1}$$

$$\phi(x,y) = 0, \quad (x,y) \in \Gamma, \tag{2}$$

where ϕ is the stress function and Γ is the contour which enclosed the rod cross-section in a plane domain $(x,y) \in \Omega$.

If one introduces the conjugate warping function $u(x,y)$ [2]

$$u(x,y) = \phi(x,y) + \frac{1}{2}(x^2 + y^2), \tag{3}$$

then one will obtain the following Dirichlet's problem for the Laplace equation

$$\Delta u(x,y) = 0, \quad (x,y) \in \Omega, \tag{4}$$

$$u(x,y) = \frac{x^2 + y^2}{2}, \quad (x,y) \in \Gamma. \tag{5}$$

The warping function v together with u constitutes an analytic complex function and they satisfy the Cauchy-Riemann equations

$$v_x = u_y, \quad v_y = -u_x \tag{6}$$

If the bar is along the z-direction and assume that the shear modulus is G and β is the twist angle per unit length, then the shear stresses of the cross section in the x and y-directions are respectively,

$$\tau_{xz} = G\beta(u_y - y), \quad \tau_{yz} = -G\beta(u_x - x). \tag{7}$$

It is easy to derive

$$\sqrt{\tau_{xz}^2 + \tau_{yz}^2} = G\beta\sqrt{(u_r - r)^2 - u_\theta^2 / r^2}. \tag{8}$$

In this paper we consider new methods to solve the problem which consists of the Laplace equation and Cauchy data at a non-circular regular boundary:

$$\Delta u = u_{rr} + \frac{1}{r}u_{rr} + \frac{1}{r^2}u_{\theta\theta} = 0, \tag{9}$$

$$u(\rho, \theta) = h(\theta), \quad 0 \le \theta \le 2\pi, \tag{10}$$

where $h(\theta)$ is a given function, and $\rho = \rho(\theta)$ is a given regular contour describing the shape of the rod. The contour Γ in the polar coordinates is given by $\Gamma = \{(r, \theta) \mid r = \rho(\theta),\ 0 \le \theta \le 2\pi\}$.

We replace Eq. (10) by the following boundary condition:

$$u(R_0, \theta) = f(\theta), \quad 0 \le \theta \le 2\pi, \tag{11}$$

where $f(\theta)$ is an unknown function to be determined, and R_0 is a given positive constant, such that the disk $D = \{(r, \theta) \mid r \le R_0,\ 0 \le \theta \le 2\pi\}$ can cover the entire domain Ω of the considered problem. The advantage of this replacement is that we have a closed-form solution in the form of the Poisson integral:

$$u(r, \theta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{R_0^2 - r^2}{R_0^2 - 2R_0 r\cos(\theta - \xi) + r^2} f(\xi)d\xi. \tag{12}$$

R_0 can be viewed as the radius of an artificial circle, and $f(\theta)$ is an unknown function on this artificial circle. Indeed, a suitable selection of the parameter R_0 maybe improve the quality of the new numerical method.

By utilizing the technique of separation of variables we can also write a series expansion of $u(r, \theta)$ satisfying Eqs. (9) and (11):

$$u(r, \theta) = \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k r^k \cos k\theta + b_k r^k \sin k\theta), \tag{13}$$

where

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(\xi)d\xi, \tag{14}$$

$$a_k = \frac{1}{\pi R_0^k}\int_0^{2\pi} f(\xi)\cos k\xi d\xi, \tag{15}$$

$$b_k = \frac{1}{\pi R_0^k}\int_0^{2\pi} f(\xi)\sin k\xi d\xi. \tag{16}$$

By imposing the condition (10) on Eq. (13) we obtain

$$\frac{a_0}{2} + \sum_{k=1}^{\infty}[a_k \rho^k \cos k\theta + b_k \rho^k \sin k\theta] = h(\theta). \tag{17}$$

Substituting Eqs. (14)-(16) into Eq. (17) leads to the first kind Fredholm integral equation:

$$\int_0^{2\pi} K(\theta, \xi) f(\xi)d\xi = h(\theta), \tag{18}$$

where

$$K(\theta, \xi) = \frac{1}{2\pi} + \sum_{k=1}^{\infty}\{B_k[\cos k\theta\cos k\xi + \sin k\theta \sin k\xi]\} \tag{19}$$

is the kernel function, and

$$B_k(\theta) := \frac{\rho^k(\theta)}{\pi R_0^k}. \tag{20}$$

There were two different boundary integral equation methods for the Laplace equation [6]. One is the double-layer method and the other is the Green's boundary formula. Both of these two methods and the details on numerical methods of them are described by Jaswon and Symm [7]. Our method is different from that two methods, and the new method is more easy to handle because it is a boundary integral equation on a given artificial circle, instead of on the contour Γ. As we know that in the open literature there has no body to connect the irregular torsion problem to this type integral equation.

III. TWO-POINT BOUNDARY VALUES SOLUTION

In order to obtain $f(\theta)$ we have to solve the first kind Fredholm integral equation (18). This however is a rather difficult task, since this integral equation is highly ill-posed.

We assume that there exists an regularized parameter α, such that Eq. (18) can be regularized by

$$\alpha f(\theta) + \int_0^{2\pi} K(\theta, \xi) f(\xi)d\xi = h(\theta), \tag{21}$$

which is known as one of the second type Fredholm integral equation. The above regularization method to obtain a regularized solution by solving the singularly perturbed operator equation is usually called the Lavrentiev regularization method [8].

We assume that the kernel function can be approximated by m terms with

$$K(\theta, \xi) = \frac{1}{2\pi} + \sum_{k=1}^{m}\{B_k[\cos k\theta\cos k\xi + \sin k\theta \sin k\xi]\}. \tag{22}$$

This assumption is for the convenience of our derivation but is not essential.

By inspection we have

$$K(\theta, \xi) = \mathbf{P}(\theta) \cdot \mathbf{Q}(\xi), \tag{23}$$

where **P** and **Q** are 2m+1-vectors given by

$$\mathbf{P} := \begin{bmatrix} \frac{1}{2\pi} \\ B_1\cos\theta \\ B_1\sin\theta \\ B_2\cos 2\theta \\ B_2\sin 2\theta \\ \vdots \\ B_m\cos m\theta \\ B_m\sin m\theta \end{bmatrix}, \quad \mathbf{Q} := \begin{bmatrix} 1 \\ \cos\xi \\ \sin\xi \\ \cos 2\xi \\ \sin 2\xi \\ \vdots \\ \cos m\xi \\ \sin m\xi \end{bmatrix}, \tag{24}$$

and the dot between **P** and **Q** denotes the inner product, which is sometime written as $\mathbf{P}^{\mathrm{T}}\mathbf{Q}$, where the superscript T signifies the transpose.

With the aid of Eq. (23), Eq. (21) can be decomposed as

$$\alpha f(\theta)+\int_0^{\theta}\mathbf{P}^{\mathrm{T}}(\theta)\mathbf{Q}(\xi)f(\xi)d\xi+\int_0^{2\pi}\mathbf{P}^{\mathrm{T}}(\theta)\mathbf{Q}(\xi)f(\xi)d\xi=h(\theta). \tag{25}$$

Let us define

$$\mathbf{u}_1(\theta):=\int_0^{\theta}\mathbf{Q}(\xi)f(\xi)d\xi, \tag{26}$$

$$\mathbf{u}_2(\theta):=\int_{2\pi}^{\theta}\mathbf{Q}(\xi)f(\xi)d\xi, \tag{27}$$

and then Eq. (25) can be expressed as

$$\alpha f(\theta)+\mathbf{P}^{\mathrm{T}}(\theta)[\mathbf{u}_1(\theta)-\mathbf{u}_2(\theta)]=h(\theta). \tag{28}$$

If $\mathbf{u}_1$ and $\mathbf{u}_2$ can be solved, we can calculate $f(\theta)$.

Taking the differential of Eqs. (26) and (27) with respect to θ, we obtain

$$\mathbf{u}_1'(\theta)=\mathbf{Q}(\theta)f(\theta), \tag{29}$$

$$\mathbf{u}_2'(\theta)=\mathbf{Q}(\theta)f(\theta). \tag{30}$$

Inserting Eq. (28) for $f(\theta)$ into the above two equations, we obtain

$$\alpha\mathbf{u}_1'(\theta)=\mathbf{Q}(\theta)\mathbf{P}^{\mathrm{T}}(\theta)[\mathbf{u}_2(\theta)-\mathbf{u}_1(\theta)]+h(\theta)\mathbf{Q}(\theta), \tag{31}$$

$$\alpha\mathbf{u}_2'(\theta)=\mathbf{Q}(\theta)\mathbf{P}^{\mathrm{T}}(\theta)[\mathbf{u}_2(\theta)-\mathbf{u}_1(\theta)]+h(\theta)\mathbf{Q}(\theta), \tag{32}$$

$$\mathbf{u}_1(0)=\mathbf{0},,\quad \mathbf{u}_2(2\pi)=\mathbf{0}, \tag{33}$$

where the last two conditions follow from Eqs. (26) and (27) readily. The above equations constitute a two-point boundary value problem.

IV. AN ANALYTICAL SOLUTION

In this section we will find a closed-form solution of $f(\theta)$. From Eqs. (29) and (30) it can be seen that $\mathbf{u}_1'=\mathbf{u}_2'$, which means that

$$\mathbf{u}_1=\mathbf{u}_2+\mathbf{c}, \tag{34}$$

where **c** is a constant vector to be determined. By using the final condition in Eq. (33) we find that

$$\mathbf{u}_1(2\pi)=\mathbf{u}_2(2\pi)+\mathbf{c}=\mathbf{c}. \tag{35}$$

Substituting Eq. (34) into (31) we have

$$\alpha\mathbf{u}_1'(\theta)=-\mathbf{Q}(\theta)\mathbf{P}^{\mathrm{T}}(\theta)\mathbf{c}+h(\theta)\mathbf{Q}(\theta), \tag{36}$$

Integrating and using the initial condition in Eq. (33) it follows that

$$\mathbf{u}_1(\theta)=\frac{-1}{\alpha}\int_0^{\theta}\mathbf{Q}(\xi)\mathbf{P}^{\mathrm{T}}(\xi)d\xi\mathbf{c}+\frac{1}{\alpha}\int_0^{\theta}h(\xi)\mathbf{Q}(\xi)d\xi. \tag{37}$$

Taking $\theta=2\pi$ in the above equation and imposing the condition (35), one obtains a governing equation for **c**:

$$\mathbf{R}\mathbf{c}=\int_0^{2\pi}h(\xi)\mathbf{Q}(\xi)d\xi, \tag{38}$$

where

$$\mathbf{R}:=\alpha\mathbf{I}_{2m+1}+\int_0^{2\pi}\mathbf{Q}(\xi)\mathbf{P}^{\mathrm{T}}(\xi)d\xi. \tag{39}$$

It is straightforward to write

$$\mathbf{c}=\mathbf{R}^{-1}\int_0^{2\pi}h(\xi)\mathbf{Q}(\xi)d\xi. \tag{40}$$

On the other hand, from Eqs. (28) and (35) we have

$$\alpha f(\theta)=h(\theta)-\mathbf{P}(\theta)\cdot\mathbf{c}. \tag{41}$$

Inserting Eq. (40) into the above equation, we obtain

$$\alpha f(\theta)=h(\theta)-\mathbf{P}(\theta)\cdot\mathbf{R}^{-1}\int_0^{2\pi}h(\xi)\mathbf{Q}(\xi)d\xi. \tag{42}$$

Even we have solved $f(\theta)$ with a closed-form in the above, the computations in Eq. (42) require a lot of $(2m+1)\times(2m+2)$ integrations and an inverse of $(2m+1)\times(2m+1)$ matrix. However, for most cases we can derive a closed-form representation of Eq. (42) when the boundary data $h(\theta)$ is expanded by a finite number of Fourier coefficients.

In order to demonstrate this process to derive a closed-form solution let us consider a simple case with

$$h(\theta)=A+B\cos 2\theta. \tag{43}$$

The analysis for other case can be carried out similarly.

For this case we have

$$\int_0^{2\pi}h(\xi)\mathbf{Q}(\xi)d\xi=\begin{bmatrix}2\pi A\\0\\0\\\pi B\\0\\\vdots\\0\end{bmatrix}, \tag{44}$$

and in the computation of the matrix **R** we only need to consider m = 4 because the numbers are all zero after the first eight terms in the fourth column of **R**, which is found to be

$$\mathbf{R}=\begin{bmatrix}R_{11}&0&0&R_{14}&0&0&0&R_{18}&0\\0&R_{22}&0&0&0&R_{26}&0&0&0\\0&0&R_{33}&0&0&0&R_{37}&0&0\\0&0&0&R_{44}&0&0&0&R_{48}&0\\0&0&0&0&R_{55}&0&0&0&R_{59}\\0&R_{62}&0&0&0&R_{66}&0&0&0\\0&0&R_{73}&0&0&0&R_{77}&0&0\\0&0&0&R_{84}&0&0&0&R_{88}&0\\0&0&0&0&R_{95}&0&0&0&R_{99}\end{bmatrix}. \tag{45}$$

In order to calculate

$$\mathbf{R}^{-1}\int_0^{2\pi}h(\xi)\mathbf{Q}(\xi)d\xi$$

we require to find the inverse of **R**; however, this task becomes easy when we note that there are required only the first and the fourth columns in $\mathbf{R}^{-1}$, since there are only the first and the fourth components of $\int_0^{2\pi} \mathrm{h}(\xi)\mathbf{Q}(\xi)d\xi$ nonzero. Thus we can write the first and the fourth columns of $\mathbf{R}^{-1}$ as to be

$$\mathbf{R}^{-1} = \begin{bmatrix} \text{first} & \text{fourth} \\ \frac{1}{R_{11}} & \frac{-R_{14}}{R_{11}R_{44}} \\ 0 & 0 \\ 0 & 0 \\ 0 & \frac{1}{R_{44}} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & \frac{-R_{84}}{R_{44}R_{88}-R_{48}R_{84}} \\ 0 & 0 \end{bmatrix}. \tag{46}$$

Inserting Eqs. (46), (44) and (24) into Eq. (42), we thus obtain

$$\mathrm{f}(\theta) = \frac{1}{\alpha}\mathrm{h}(\theta) - \frac{A}{\alpha R_{11}} + \frac{BR_{14}}{2\alpha R_{11}R_{44}} - \frac{B\pi}{\alpha R_{44}}B_2(\theta)\cos 2\theta + \frac{B\pi R_{84}}{\alpha(R_{44}R_{88}-R_{48}R_{84})}B_4(\theta)\cos 4\theta. \tag{47}$$

For a given $\mathrm{h}(\theta)$, one may employ the above equation to calculate $\mathrm{f}(\theta)$ very efficiently.

V. THE TIKHONOV REGULARIZATION

In this section we provide a new numerical method for Eq. (18). The range $[0, 2\pi]$ is divided into n-1 subintervals with $\Delta\theta = 2\pi / n - 1$ and $\theta_i = (i-1)\Delta\theta$. The numerical value of $\mathrm{f}(\theta)$ at the i-th grid point is denoted by $f_i = f(\theta_i)$, and there are totally n unknowns $f_1, f_2, \ldots, f_n$. We apply the trapezoidal quadrature on the integral term in Eq. (18), which results in

$$\Delta\theta\left[\frac{1}{2}K(i,1)f_1 + \sum_{j=2}^{n-1}K(i,j)f_j + \frac{1}{2}K(i,n)f_n\right] = h_i, \tag{48}$$

where $K(i,j) := K(\theta_i, \theta_j)$ and $h_i = h(\theta_i)$.

The above system can be rearranged into a linear system equation:

$$\mathbf{Ax} = \mathbf{b}, \tag{49}$$

where

$$\mathbf{A} := \Delta\theta\begin{bmatrix} \frac{1}{2}K(1,1) & K(1,2) & \cdots & \frac{1}{2}K(1,n) \\ \frac{1}{2}K(2,1) & K(2,2) & \cdots & \frac{1}{2}K(2,n) \\ \vdots & \vdots & \vdots & \vdots \\ \frac{1}{2}K(n,1) & K(n,2) & \cdots & \frac{1}{2}K(n,n) \end{bmatrix}, \tag{50}$$

$$\mathbf{b} := \begin{bmatrix} h_1 \\ h_1 \\ \vdots \\ h_n \end{bmatrix}, \tag{51}$$

and $\mathbf{x} = (f_1, f_2, \ldots, f_n)^T$.

There are several methods to deal with Eq. (49) when $\mathbf{A}$ is ill-conditioned. Here we consider an iterative regularization method of the Tikhonov type for Eq. (49) by investigating the long term behavior of the following equation [9]:

$$\dot{\mathbf{x}} = -\alpha\mathbf{x} + \mathbf{A}^T(\mathbf{b} - \mathbf{Ax}) =: \mathbf{f}(\mathbf{x}), \tag{52}$$

until $\|\dot{\mathbf{x}}\|$ is smaller enough than a given criterion, where we introduce an independent variable t, and the superimposed dot denotes the differential with respect to t. The fixed point, i.e. $\mathbf{f}(\mathbf{x}) = 0$, of the above equation is the solution of Eq. (49). When t approach infinity we expect $\mathbf{x}$ tends to the solution of Eq. (49).

A nonstandard group preserving scheme (NGPS) for Eq. (52) has been developed by Liu [10,11],

$$\mathbf{x}_{k+1} = \mathbf{x}_k + \frac{4\|\mathbf{x}_k\|^2 + 2\phi\,\mathbf{f}_k\cdot\mathbf{x}_k}{4\|\mathbf{x}_k\|^2 - \phi^2\|\mathbf{f}_k\|^2}\phi\,\mathbf{f}_k, \tag{53}$$

where

$$\phi(\Delta t) := \frac{1-\exp(-\rho_0\Delta t)}{\rho_0}, \tag{54}$$

and ρ_0 can be a number not smaller than the Lipschitz constant of Eq. (52):

$$L = \|\mathbf{A}\| \geq \max\{|\lambda_i| : i = 1, 2, \ldots, n\}. \tag{55}$$

VI. INTEGRATION OF Eq. (36)

When one applies a numerical integration method on Eq. (36), one can utilize it to find the solution of $\mathbf{c}$ by iterations. The process is starting from an initially given $\mathbf{c}$, then calculate Eq. (36) to obtain $\mathbf{u}_1(1) = \mathbf{c}$, and then using it again as the new $\mathbf{c}$ to integrate Eq. (36), until the process converges according to a given stopping criterion. If $\mathbf{c}$ is available, we can return Eq. (41) to calculate $f^{\alpha}(\theta)$. Then in terms of the Poisson integral equation (12) we can calculate $u(r,\theta)$ inside the domain Ω.

Here we require to check that does the regularized solution $f^{\alpha}(\theta)$ satisfy Eq. (18) by using its discretized form (49).

If Eq. (49) can not be satisfied, we can calculate the new f by

$$f_i = f_i^{\alpha} + \frac{e\mathbf{b}_2\cdot\mathbf{b}_3}{\|\mathbf{b}_3\|^2}, \tag{56}$$

where e is a selected constant and

$$\mathbf{b}_2 = \mathbf{b} - \mathbf{Ax}^{\alpha}, \tag{57}$$

$$\mathbf{b}_3 = \mathbf{A1}, \tag{58}$$

in which $\mathbf{1} = (1, 1, \ldots, 1)^{\mathrm{T}}$.

VII. NUMERICAL EXAMPLES

1. An elliptical cross section

At first let us consider the torsion of the rod whose cross section is an ellipse with semiaxes a and b, and the contour in the polar coordinates is described by

$$\rho(\theta) = \sqrt{x^2 + y^2} = \sqrt{a^2 \cos^2 \theta + b^2 \sin^2 \theta}. \tag{59}$$

For this case we have a boundary condition

$$u(\rho, \theta) = h(\theta) = \frac{1}{2}(x^2 + y^2) = \frac{1}{2}(a^2 \cos^2 \theta + b^2 \sin^2 \theta)$$

$$= \frac{1}{4}(a^2 + b^2) + \frac{1}{4}(a^2 - b^2)\cos 2\theta. \tag{60}$$

The exact solution of u is known to be (Timoshenko and Goodier, 1961)

$$u(x, y) = \frac{1}{2}(x^2 + y^2) + \frac{a^2 b^2}{a^2 + b^2}\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right) = \frac{a^2 b^2}{a^2 + b^2}$$

$$+ \frac{a^2 - b^2}{2(a^2 + b^2)} r^2 \cos 2\theta. \tag{61}$$

For this example we have applied the numerical method in Section VI. The parameters used in this calculation are $m = 11$, $a = 2$, $b = 1$, $R_0 = 2.1$, $\alpha = 0.62$ and $e = 0.736$, and three iterations are used to calculate $\mathbf{c}$ starting from an initial $\mathbf{c} = \mathbf{0}$. The numerical result of $f(\theta)$ is compared with the exact one, which is obtained from Eq. (61) by inserting $r = R_0$, in Fig. 1(a).

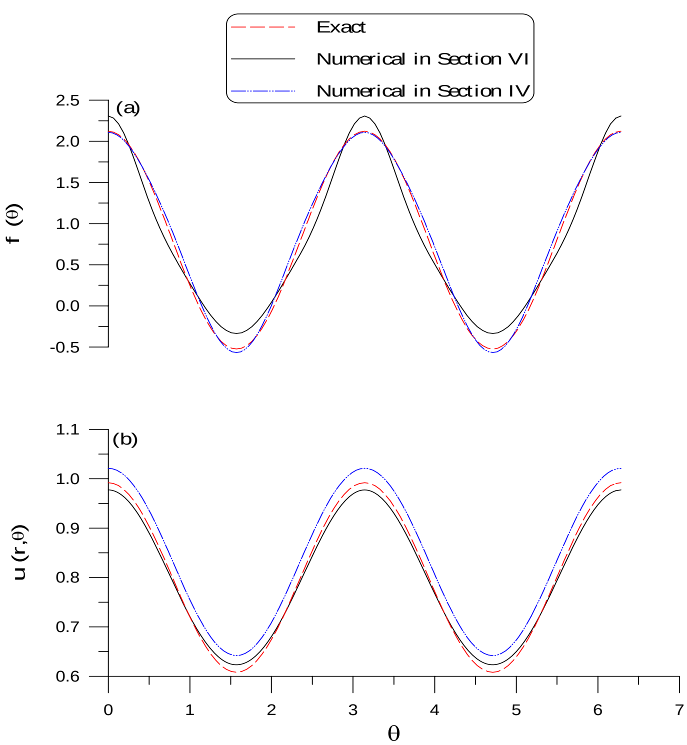

Fig. 1 For the ellipse torsion problem we compare exact solutions and numerical solutions for $f(\theta)$ in (a) and $u(r,\theta)$ in (b).

At a given radius $r = 0.8$ we compare the exact solution with the numerical solution as shown in Fig. 1(b). It can be seen that the numerical solution is rather accurate.

Now, we apply the method in Section IV to this example. Inserting Eq. (60) for $h(\theta)$, $A = (a^2 + b^2)/4$ and $B = (a^2 - b^2)/4$ into Eq. (47) we can obtain $f(\theta)$. We have fixed $\alpha = 0.425$ and compared the above numerical result with the exact one in Fig. 1(a), and the numerical result of u is also compared with the exact one in Fig. 1(b). Even it is slightly less accurate than the one by using the numerical method in Section VI, the computations of this method is very effective and time saving.

2. An epitrochoid-shape bar

Let us consider the torsion of the rod whose cross section is epitrochoid-shaped described by

$$x(\theta) = (a+b)\cos\theta - \cos(a+b)/b\theta, \tag{62}$$

$$y(\theta) = (a+b)\sin\theta - \sin(a+b)/b\theta, \tag{63}$$

$$\rho(\theta) = \sqrt{x^2 + y^2} = \sqrt{(a+b)^2 + 1 - 2(a+b)\cos a/b\theta}. \tag{64}$$

For this case we have a boundary condition

$$u(\rho, \theta) = h(\theta) = \frac{1}{2}(x^2 + y^2)$$

$$= \frac{1}{2}[(a+b)^2 + 1 - 2(a+b)\cos a/b\theta]. \tag{65}$$

For this problem we have no closed-form solution. Let $a = 2$ and $b = 1$ we can apply the method in Section IV to this problem. Inserting Eq. (65) for $h(\theta)$, $A = [(a+b)^2)+1]/2$ and $B = -(a+b)$ into Eq. (47) we can obtain $f(\theta)$. In the calculation we have fixed $R_0 = 8.5$ and $\alpha = 0.0089$. Let $\mathbf{b}_1 = \mathbf{A}^{\mathrm{T}}\mathbf{b}$ and $\mathbf{b}_2 = \mathbf{A}^{\mathrm{T}}\mathbf{A}\mathbf{x}$. We have compared $\mathbf{b}_1$ and $\mathbf{b}_2$ in Fig. 2(a), and the iso-warping values are plotted in Fig. 2(b) for $u = 8, 9, 10, 12$.

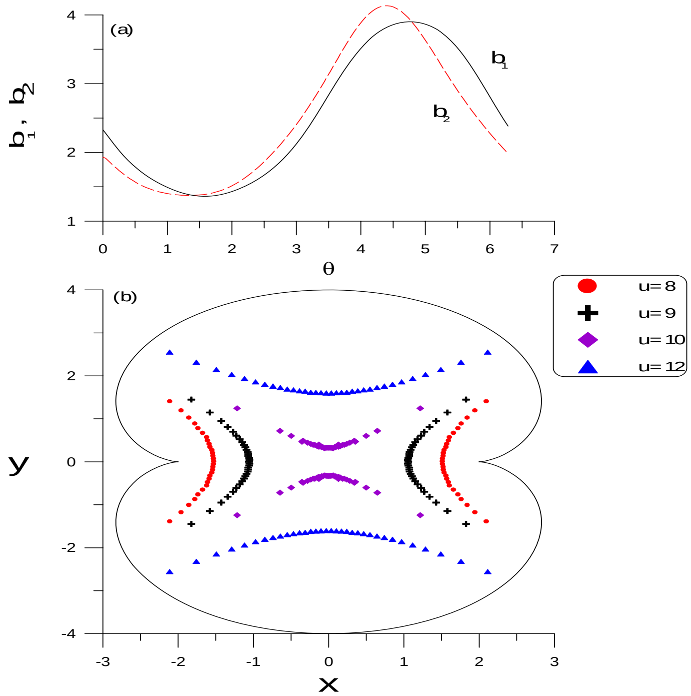

Fig. 2 For an epitrochoid bar the numerical result of b_2 is compared with the exact b_1 in (a), and (b) plotting the contour levels of conjugate warping function.

3. A kite-shaped bar

Then let us consider the torsion of the rod whose cross section is kite-shaped described by

$$x(\theta) = 0.6\cos\theta + 0.3\cos 2\theta - 0.2, \tag{66}$$

$$y(\theta) = 0.6\sin\theta, \tag{67}$$

$$\rho(\theta) = \sqrt{x^2 + y^2}$$
$$= \sqrt{(0.6\cos\theta + 0.3\cos 2\theta - 0.2)^2 + 0.36\sin^2\theta}. \tag{68}$$

For this case we have a boundary condition

$$u(\rho,\theta) = h(\theta) = \frac{1}{2}(x^2 + y^2)$$
$$= \frac{1}{2}[(0.6\cos\theta + 0.3\cos 2\theta - 0.2)^2 + 0.36\sin^2\theta]. \tag{69}$$

For this problem we have no closed-form solution. We have applied the numerical method in Section V to this problem with $m = 11$, $\Delta t = 1$, , $\rho_0 = 1$, $\alpha = 0.0001$ and $R_0 = 3$. Starting from an initial value of $\mathbf{x}$, we have applied 500 iterations to calculate $\mathbf{x}$. Let $\mathbf{b}_1 = \mathbf{A}^T\mathbf{b}$ and $\mathbf{b}_2 = \mathbf{A}^T\mathbf{A}\mathbf{x}$. We compare $\mathbf{b}_1$ and $\mathbf{b}_2$ in Fig. 3(a), which can be seen rather better. The iso-warping values are plotted in Fig. 3(b) for u = 0.45, 0.5, 0.6, 0.75. The pattern is rather complex.

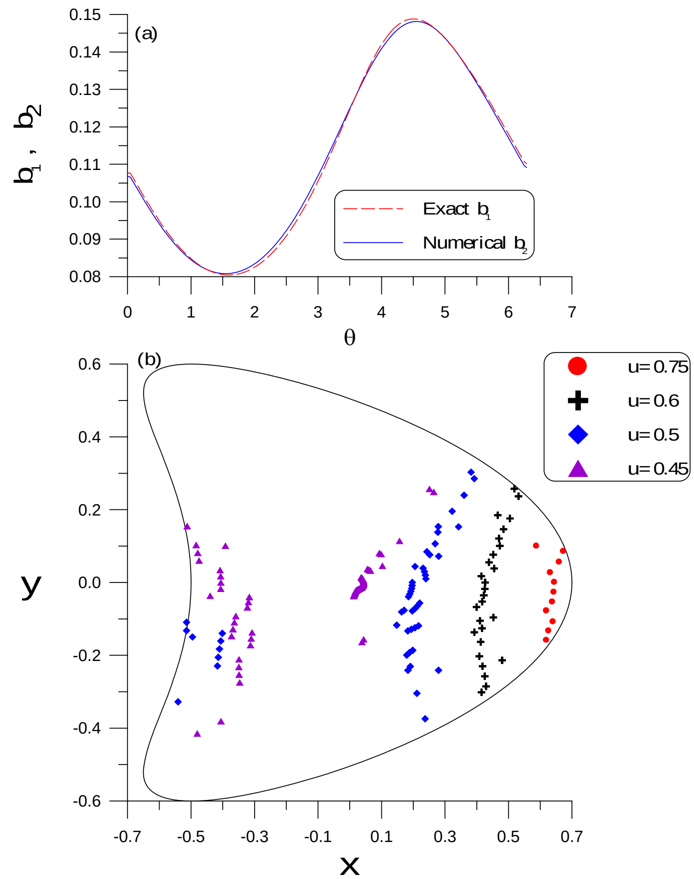

Fig. 3 For a kite-shaped bar the numerical result of b_2 is compared with the exact b_1 in (a), and (b) plotting the contour levels of conjugate warping function.

VIII. CONCLUSIONS

In this paper we have proposed three new methods to calculate the solutions of elastic torsion problems with arbitrary shape of the cross-section. It was demonstrated that when the boundary data has only few Fourier terms, we can find an analytical solution of the boundary data on an artificial circle in the regularized sense, and thus by the Poisson integral we can calculate the conjugate warping function at any point inside the domain of the considered problem. The numerical examples show that the effectiveness of these methods and the accuracy is rather better.

NOMENCLATURE

ϕ	stress function
Ω, Γ	plane domain and contour of bar
(x, y)	Cartesian coordinates
(r, θ)	polar coordinates
$\rho(\theta), h(\theta)$	boundary curve and data
u	conjugate warping function
v	warping function

τ_{xz}, τ_{yz}	shear stresses
R_0	the radius of artificial circle
$f(\theta)$	unknown data on artificial circle
D	the disk with radius R_0
a_0, a_k, b_k	Fourier coefficients
$K(\theta, \xi)$	kernel function
B_k	coefficients defined in Eq. (20)
α	regularized parameter
P, **Q**	kernel vectors
$\mathbf{u}_1$, $\mathbf{u}_2$	vectors defined in Eqs. (26) and (27)
c	constant vector of the difference of $\mathbf{u}_1$ and $\mathbf{u}_2$
R	a regularized matrix of kernel function
A	a discretized matrix of kernel function
b	a discretized of boundary data
f	vector field
x	unknown vector of data f
$\phi(\Delta t)$	a generalized time step
L	Lipshitz constant

REFERENCES

1. Timoshenko, S. P., and Goodier, J. M. (1961) *Theory of Elasticity*, McGraw-Hill, New York.
2. Little, R. W. (1973) *Elasticity*, Prentice-Hill, New Jersey.
3. Hassenpflug, W. C. (2003) Torsion of uniform bars with polygon cross-section, *Comp. Math. Appl.*, Vol. 46, pp. 313-392.
4. Kolodziej, J. A. and Fraska, A. (2005) Elastic torsion of bars possessing regular polygon in cross-section using BCM, *Comp. Struct.*, Vol. 84, pp. 78-91.
5. Aleynikov, S. M. and Stromov, A. V. (2004) Comparison of complex methods for numerical solutions of boundary problems of the Laplace equation, *Eng. Anal. Boundary Elem.*, Vol. 28, pp. 615-622.
6. Chen, J. T. and Wu, C. S. (2006) Alternative derivations for the Poisson integral formula, *Int. J. Math. Educ. Sci. Tech.*, Vol. 37, pp. 165-185.
7. Jaswon, M. A. and Symm, G. T. (1977) *Integral Equation Methods in Potential Theory and Elastostatics*, Academic, New York.
8. Lavrentiev, M. M. (1967) *Some Improperly Posed Problems of Mathematical Physics*, Springer, New York.
9. Tikhonov, A. N. and Arsenin, V. Y. (1977) *Solutions of Ill-Posed Problems*, John Wiley, New York.
10. Liu, C.-S. (2001) Cone of non-linear dynamical system and group preserving schemes, *Int. J. Non-Linear Mech.*, Vol. 36, pp. 1047-1068.
11. Liu, C.-S. (2005) Nonstandard group-preserving schemes for very stiff ordinary differential equations, *CMES: Computer Modeling in Engineering & Sciences*, Vol. 9, pp. 255-272.

ANECDOTES

本人能獲得國科會傑出研究獎，有很大部份是來自黃校長的勉勵。黃校長在公忙之餘也經常關心本人的研究工作。更曾經兩度親筆致函本人表現出黃校長的關心及期望。在黃校長積極推動下，海洋大學的整體研究環境有很大改善，讓學校的教授能夠更專心於作學術研究。