Lectures on the Trefftz method, MFS (method of fundamental solutions) and image method for Laplace problems

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(LectureYoung.ppt)

Outline

- Introduction
- Methods
- Analytical and numerical solutions
  - Image method
  - Trefftz method
- Illustrative examples
- Mathematical equivalence the Trefftz method and MFS
- Semi-analytical solution
- Numerical results
- Conclusions
- References

The simplest Trefftz method

Solve $y''(x) = 0$

subject to $y(0) = a,\ \ y(1) = b$

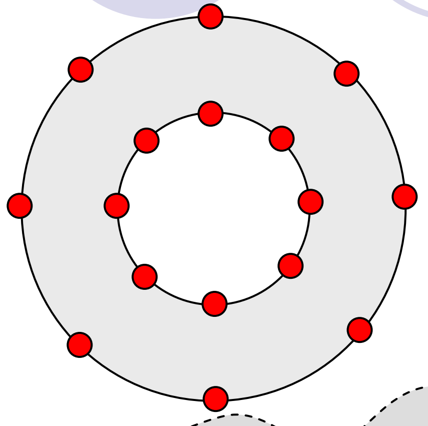
$$\Rightarrow y(x) = \sum_{i=1}^{2} c_i \psi_i(x)$$

$$\left.\begin{array}{l} \psi_1(x) = 1 \\ \psi_2(x) = x \end{array}\right\} \Rightarrow \textbf{ Trefftz base}$$

Trefftz method

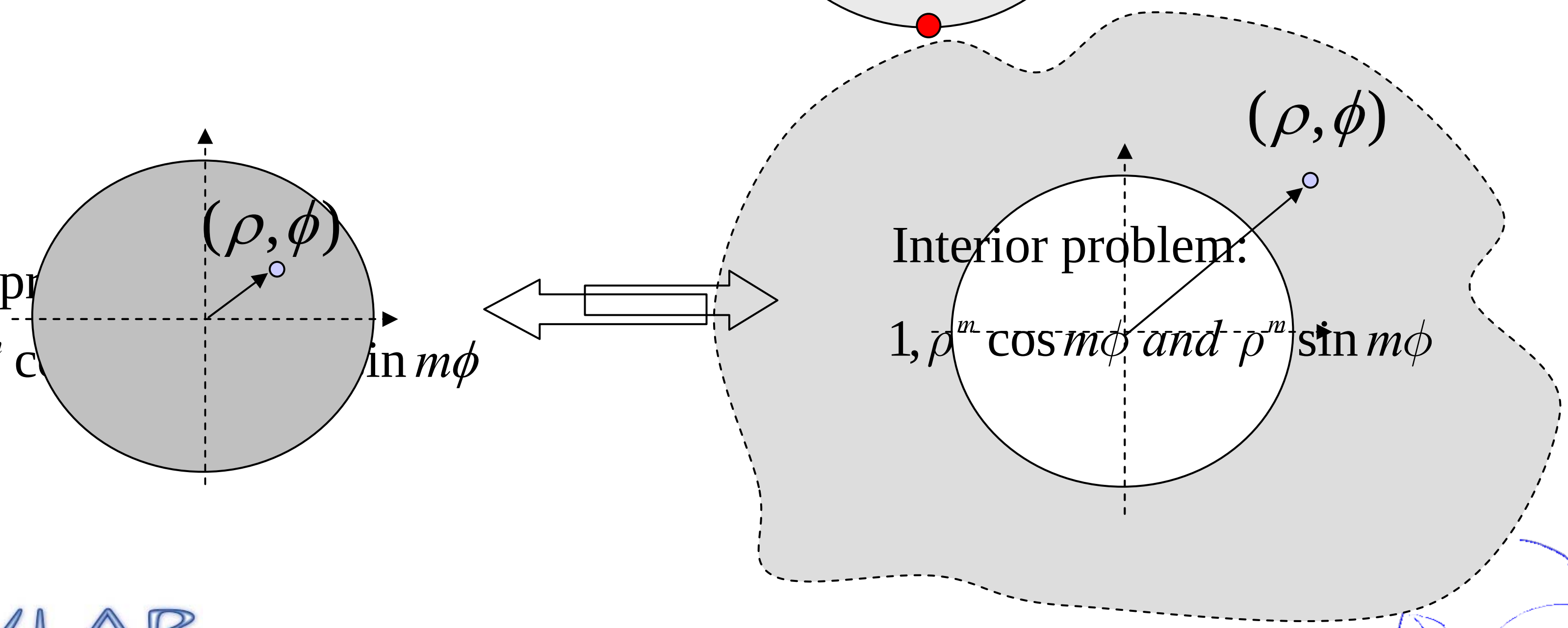
$$u(x)=\sum_{j=1}^{M}c_j\Phi_j$$

Φ_j is the j^{th} T-complete function



exterior pr

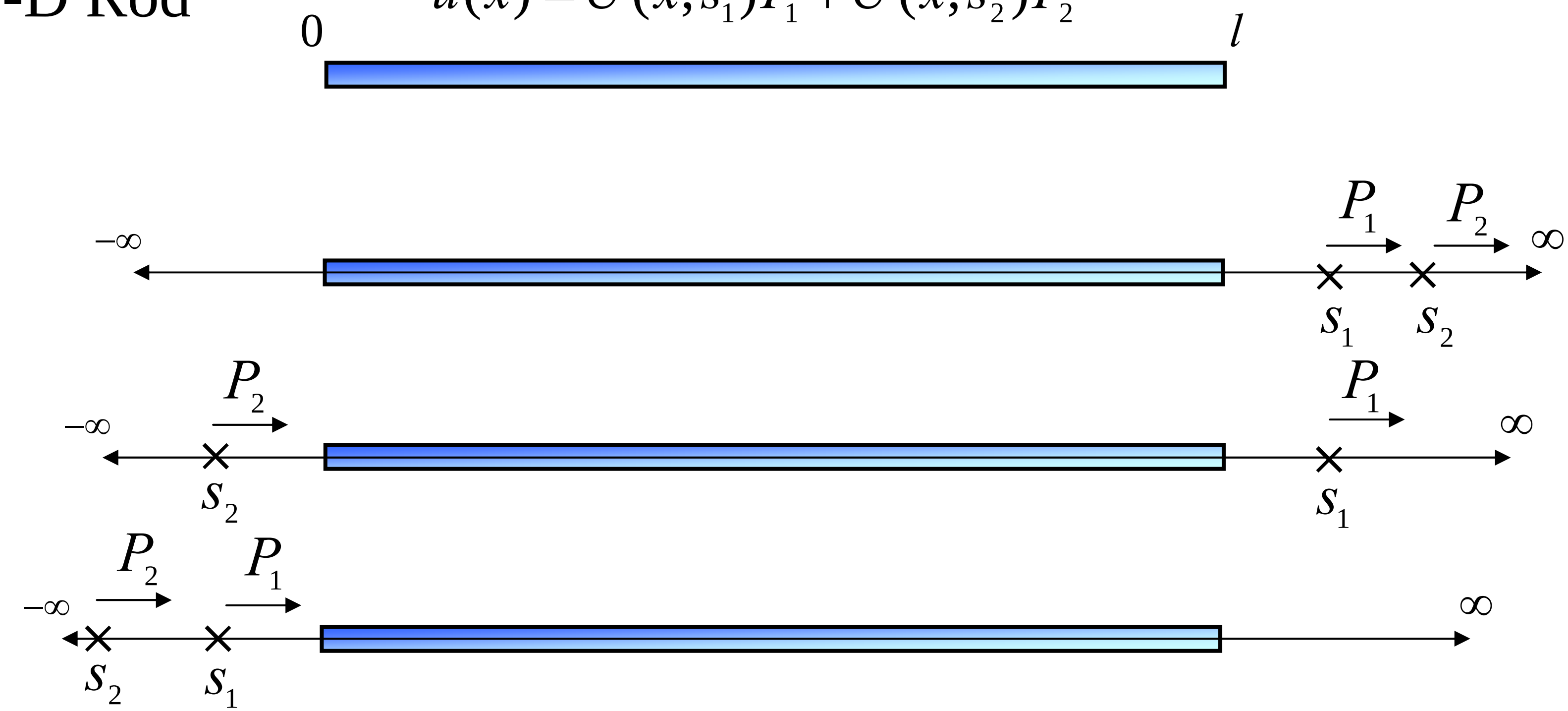
$\ln\rho,\ \rho^{-m}$ c … in $m\phi$



The simplest MFS

1-D Rod

$$u(x) = U(x, s_1)P_1 + U(x, s_2)P_2$$



where $U(x,s)$ is the fundamental solution.

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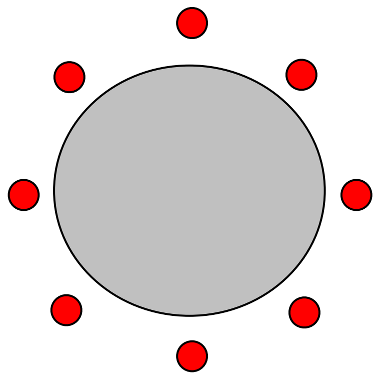
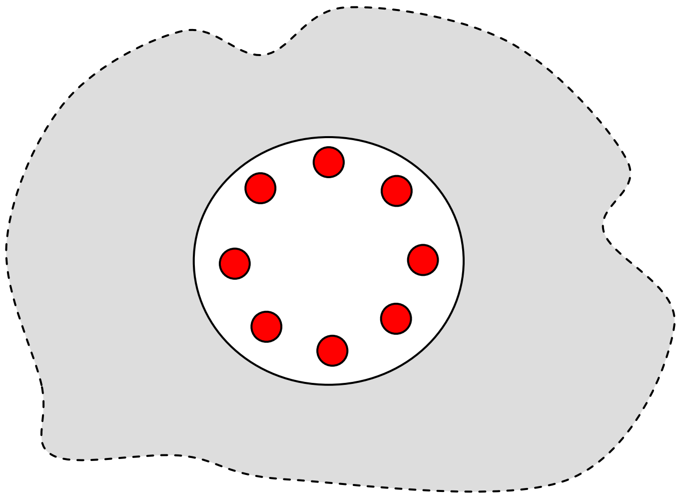
Method of Fundamental Solution (MFS)

$$u(x) = \sum_{j=1}^{N} w_j U(x, s_j)$$

Interior problem



exterior problem



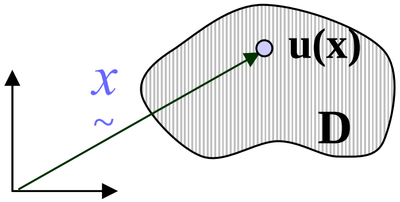
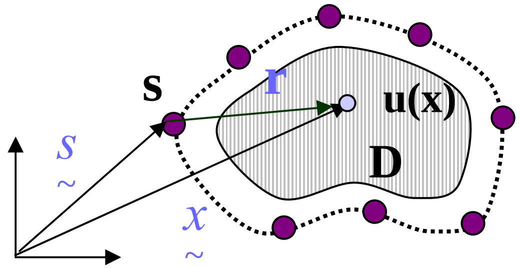
$$U(x,s) = \ln r,\ r = \left| x - s_j \right|,\ j \in N$$

MFS: Theoretical Result

Alex Cheng's comments

- Bogomolny — Error analysis
- Schaback — Base equivalence
- Jeng-Tzong Chen — Solution equivalence
- Zi-Cai Li, et al. — Effective condition number
- **D. L. Young and K. H. Chen** — MFS to the real boundary

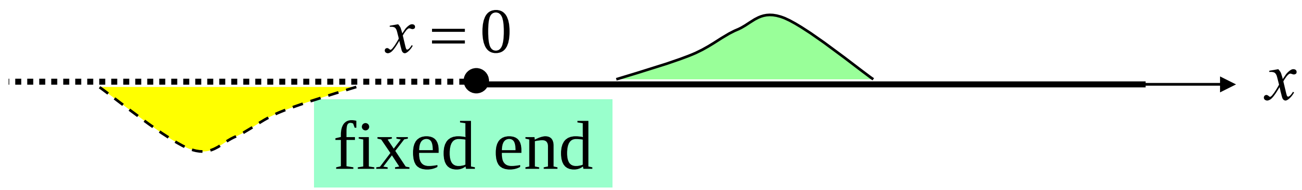
Trefftz method and MFS

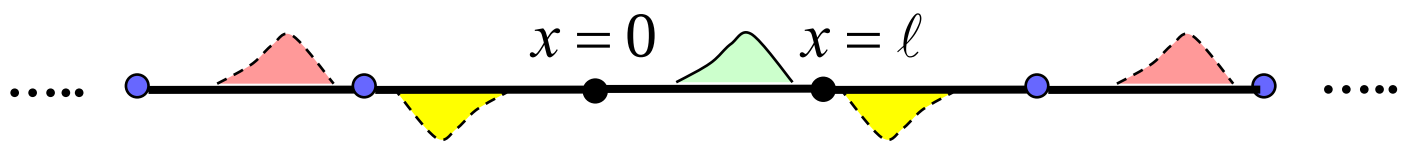
Method	Trefftz method	MFS
Definition	$u(x)=\sum_{j=1}^{M} c_j \Phi_j$	$u(x)=\sum_{j=1}^{N} w_j U(x,s_j)$
Figure caption		
Base	Φ_j , (T-complete function)	$U(x,s)=\ln r$, $r=\|x-s\|$
G. E.	$\nabla^2 u(x)=0$	$\nabla^2 u(x)=0$
Match B. C.	*Determine* c_j	*Determine* w_j

M is the number of complete functions
N is the number of source points in the MFS

The simplest image method

1-D String





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Optimal source location

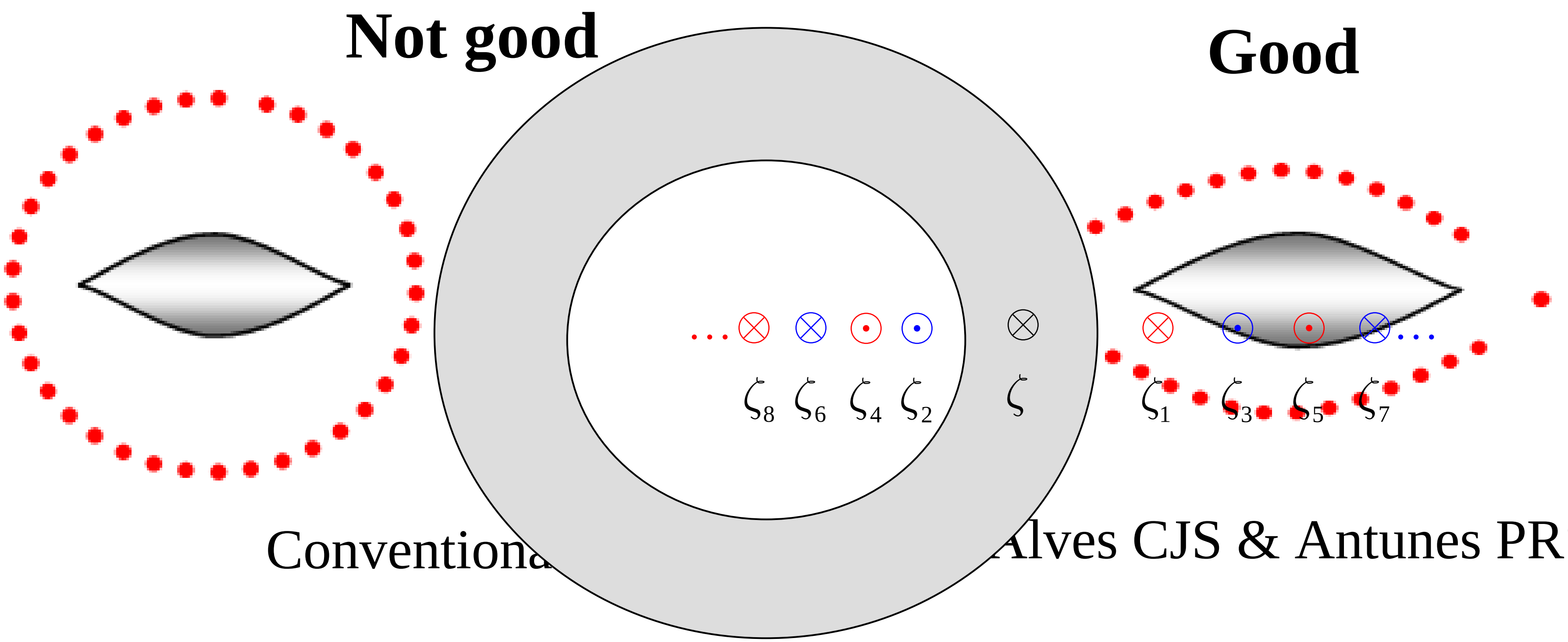
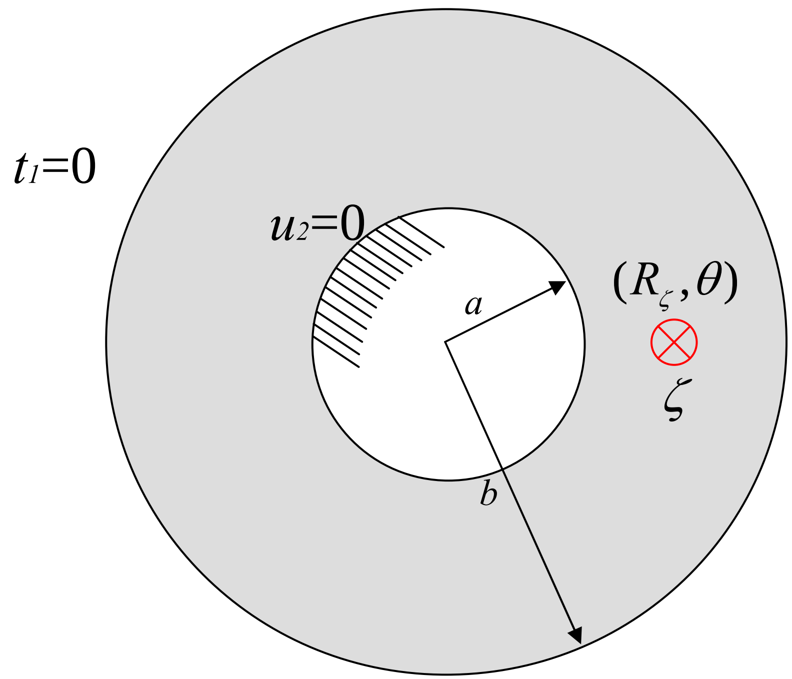


Image method ⟹ MFS (special case)

Illustrative examples

Case 1 Fixed-free annular problem



Governing equation :

$$\nabla^2 G(x,\zeta) = \delta(x-\zeta),\ x \in \Omega$$

Boundary condition :

Fixed-free boundary

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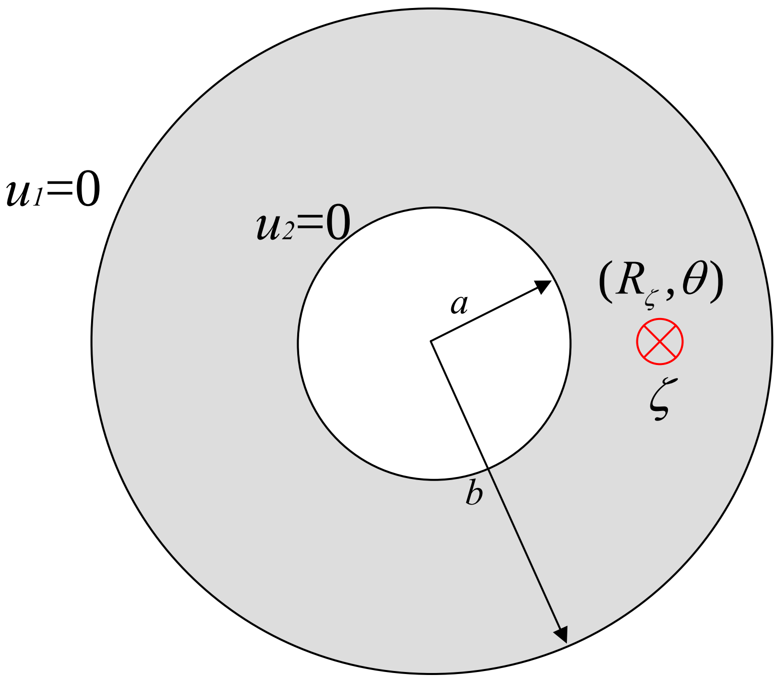
Illustrative examples

Case 2 Fixed-fixed annular problem



Governing equation :

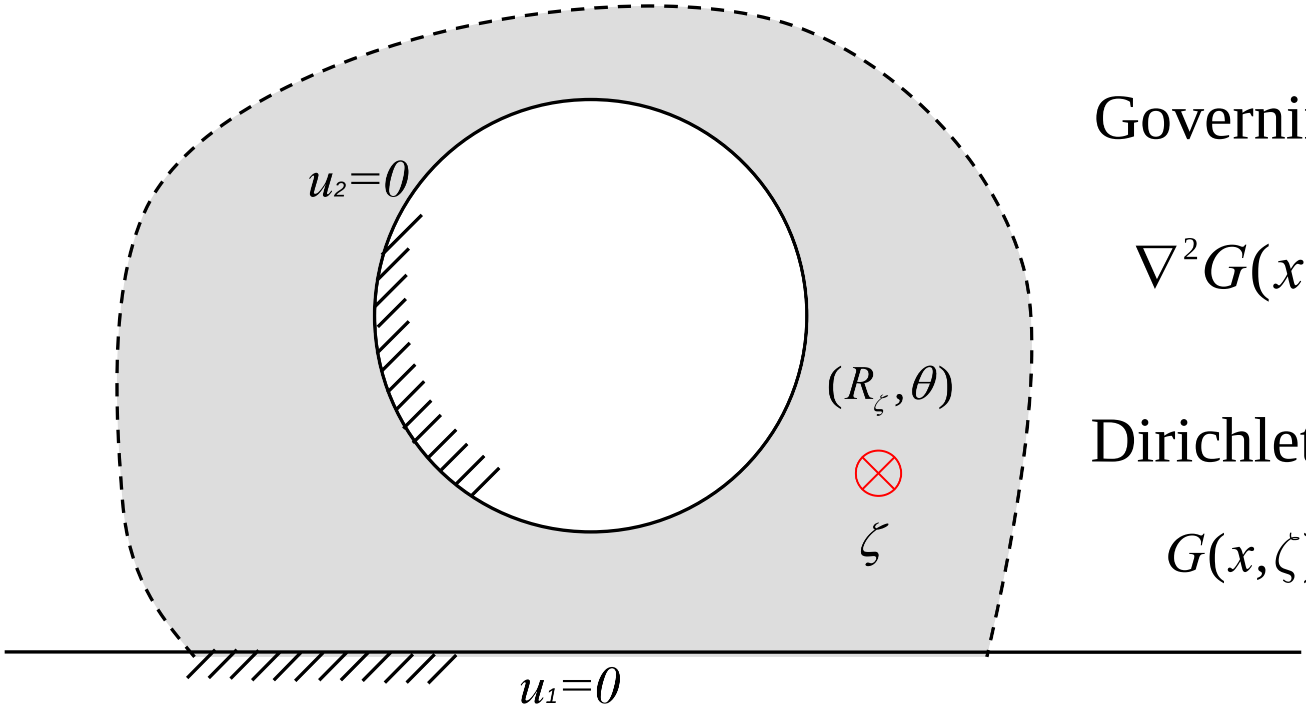
$$\nabla^2 G(x,\zeta)=\delta(x-\zeta),\ x\in\Omega$$

Boundary condition :

Fixed-fixed boundary

Illustrative examples

Case 3 Half-plane problem



Governing equation :

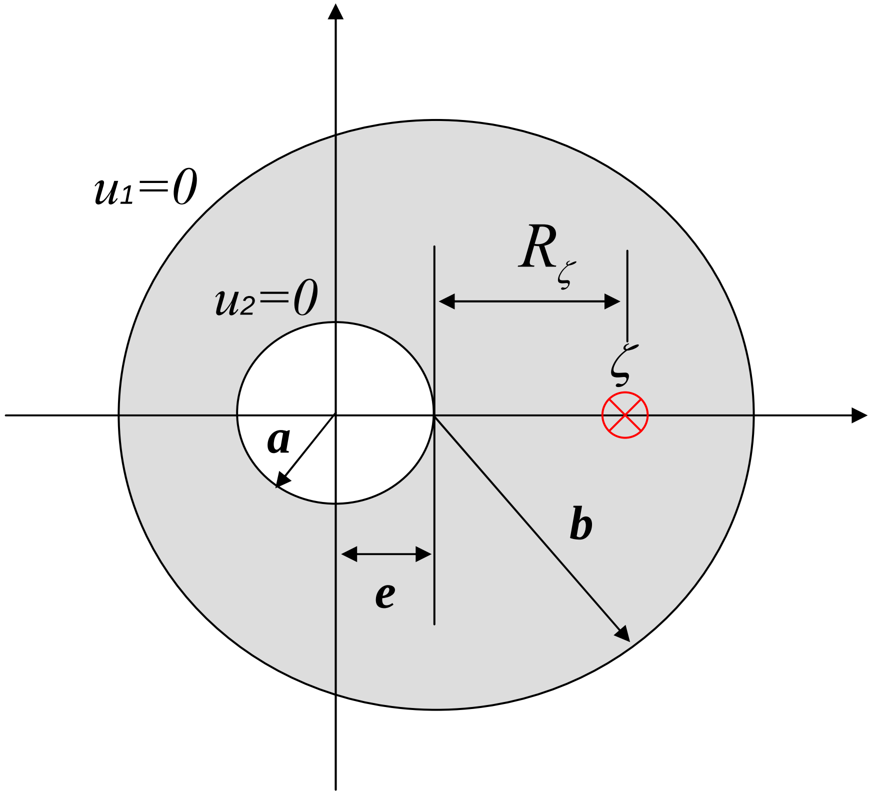
$$\nabla^2 G(x,\zeta)=\delta(x-\zeta),\ x\in\Omega$$

Dirichlet boundary condition :

$$G(x,\zeta)=0,\ x\in B$$

Illustrative examples

Case 4 Eccentric problem



Governing equation :

$$\nabla^2 G(x,\zeta) = \delta(x-\zeta),\ x \in \Omega$$

Dirichlet boundary condition :

$$G(x,\zeta) = 0,\ x \in B$$

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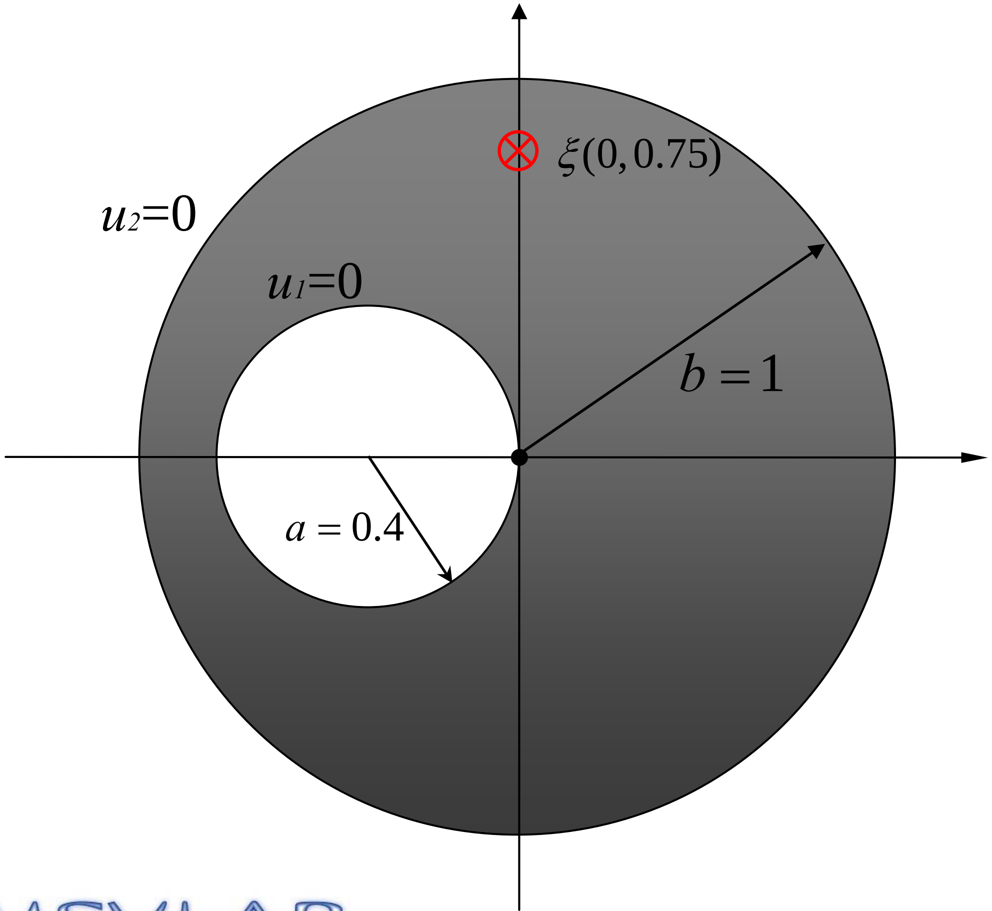
Illustrative examples

Case 5 Eccentric problem



Governing equation :

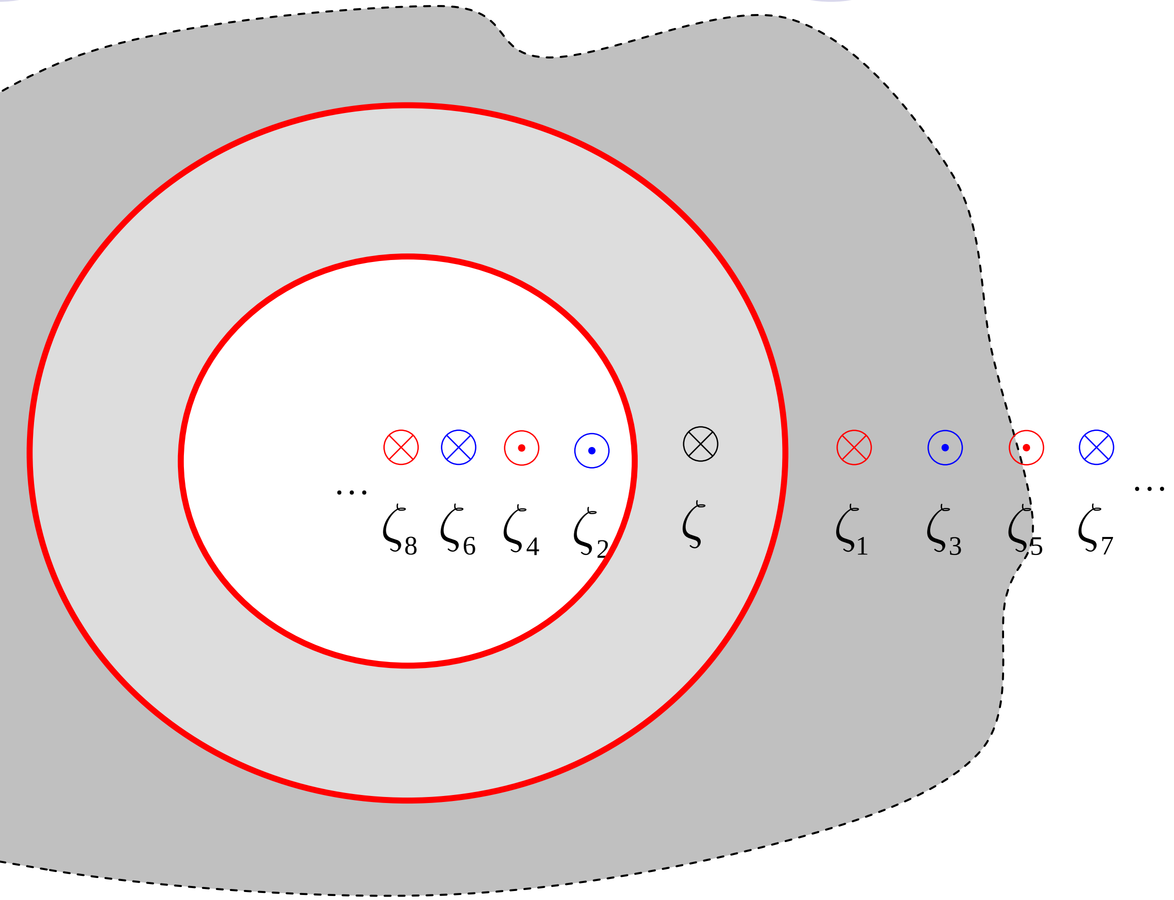
$$\nabla^2 G(x,\zeta)=\delta(x-\zeta),\ x\in\Omega$$

Dirichlet boundary condition :

$$G(x,\zeta)=0,\ x\in B$$

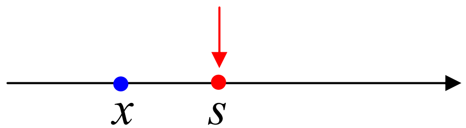
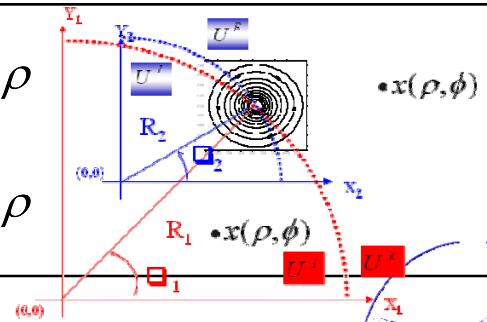
Image location: case 1

inside	outside
$\odot\zeta_2(R_2,\theta)$	$\otimes\zeta_1(R_1,\theta)$
$\odot\zeta_4(R_4,\theta)$	$\odot\zeta_3(R_3,\theta)$
$\otimes\zeta_6(R_6,\theta)$	$\odot\zeta_5(R_5,\theta)$
$\otimes\zeta_8(R_8,\theta)$	$\otimes\zeta_7(R_7,\theta)$
$\vdots$	$\vdots$
$\odot\zeta_{8i-6}(R_{8i-6},\theta)$	$\otimes\zeta_{8i-7}(R_{8i-7},\theta)$
$\odot\zeta_{8i-4}(R_{8i-4},\theta)$	$\odot\zeta_{8i-5}(R_{8i-5},\theta)$
$\otimes\zeta_{8i-2}(R_{8i-2},\theta)$	$\odot\zeta_{8i-3}(R_{8i-3},\theta)$
$\otimes\zeta_{8i}(R_{8i},\theta)$	$\otimes\zeta_{8i-1}(R_{8i-1},\theta)$



Addition theorem & degenerate kernel

Addition theorem	Subtraction theorem
$e^{x+s} = e^{x} \cdot e^{s}$	$e^{x-s} = e^{x} / e^{s}$
$\cos(x+s) = \cos x \cos s - \sin x \sin s$	$\cos(x-s) = \cos x \cos s + \sin x \sin s$
$\sin(x+s) = \sin x \cos s + \cos x \sin s$	$\sin(x-s) = \sin x \cos s - \cos x \sin s$

	Degenerate kernel for Laplace problem
1-D	$r = \begin{cases} x-s, \; if \;\; x > s \\ s-x, \; if \;\; x < s \end{cases}$ 
2-D	$\ln r = \begin{cases} \ln R - \sum_{m=1}^{\infty} \frac{1}{m} (\frac{\rho}{R})^{m} \cos m(\theta-\phi), \; R \geq \rho \\ \ln \rho - \sum_{m=1}^{\infty} \frac{1}{m} (\frac{R}{\rho})^{m} \cos m(\theta-\phi), \; R < \rho \end{cases}$ 

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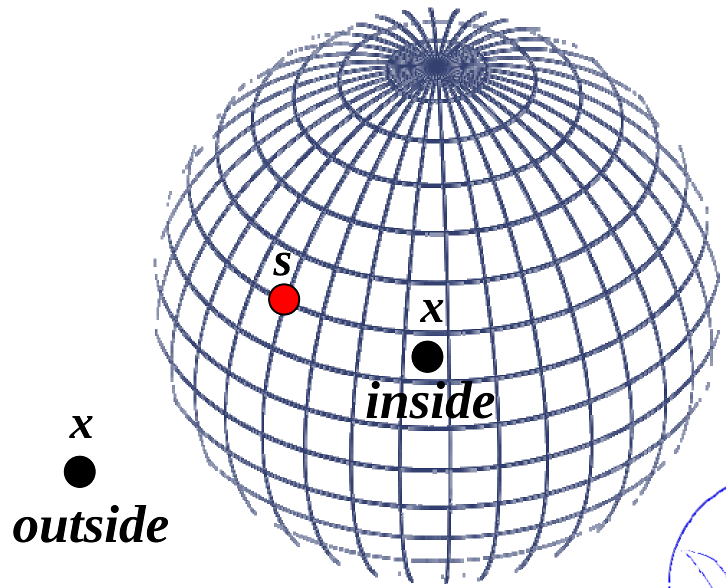
3-D degenerate kernel

$$\frac{-1}{r}=\begin{cases} U^{i}=\dfrac{-1}{R}-\displaystyle\sum_{n=1}^{\infty}\sum_{m=0}^{n}\varepsilon_{m}\frac{(n-m)!}{(n+m)!}\cos\left(m(\bar{\phi}-\phi)\right)P_{n}^{m}(\cos\bar{\theta})P_{n}^{m}(\cos\theta)\frac{\rho^{n}}{R^{n+1}},\ R\geq\rho \\ U^{e}=\dfrac{-1}{\rho}-\displaystyle\sum_{n=1}^{\infty}\sum_{m=0}^{n}\varepsilon_{m}\frac{(n-m)!}{(n+m)!}\cos\left(m(\bar{\phi}-\phi)\right)P_{n}^{m}(\cos\bar{\theta})P_{n}^{m}(\cos\theta)\frac{R^{n}}{\rho^{n+1}},\ R<\rho \end{cases}$$

$$\varepsilon_{m}=\begin{cases}1,\ m=0\\ 2,\ m=1,2,\ldots,\infty\end{cases}$$

$$\begin{cases}x=(\rho,\phi,\theta)\\ s=(R,\bar{\phi},\bar{\theta})\end{cases}$$



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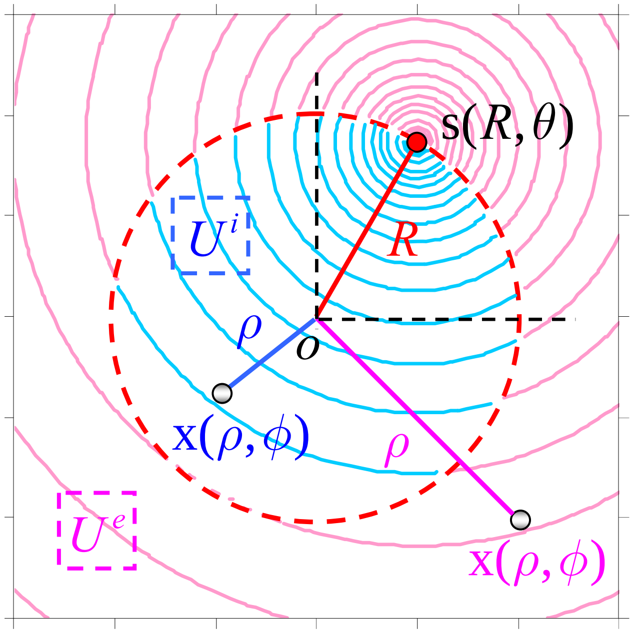
2-D Degenerate kernal

$$\mathrm{Re}\{\ln(z_x - z_s)\} = \ln r$$

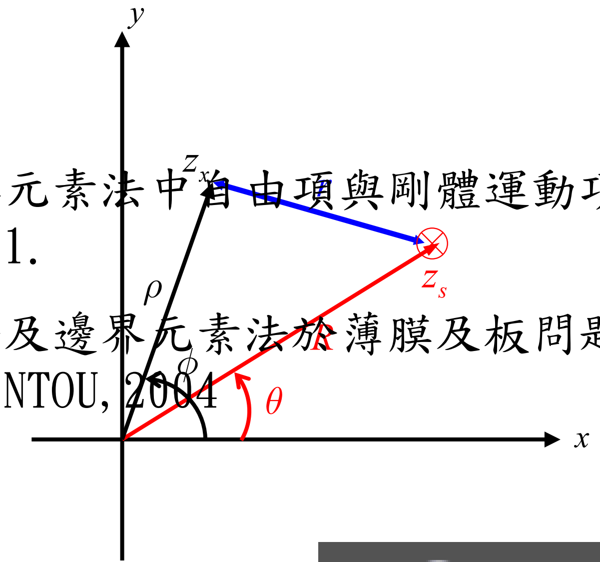
$$\ln r = \begin{cases} \ln R - \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{\rho}{R}\right)^m \cos m(\theta - \phi), & R \geq \rho \\ \ln \rho - \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{R}{\rho}\right)^m \cos m(\theta - \phi), & R < \rho \end{cases}$$



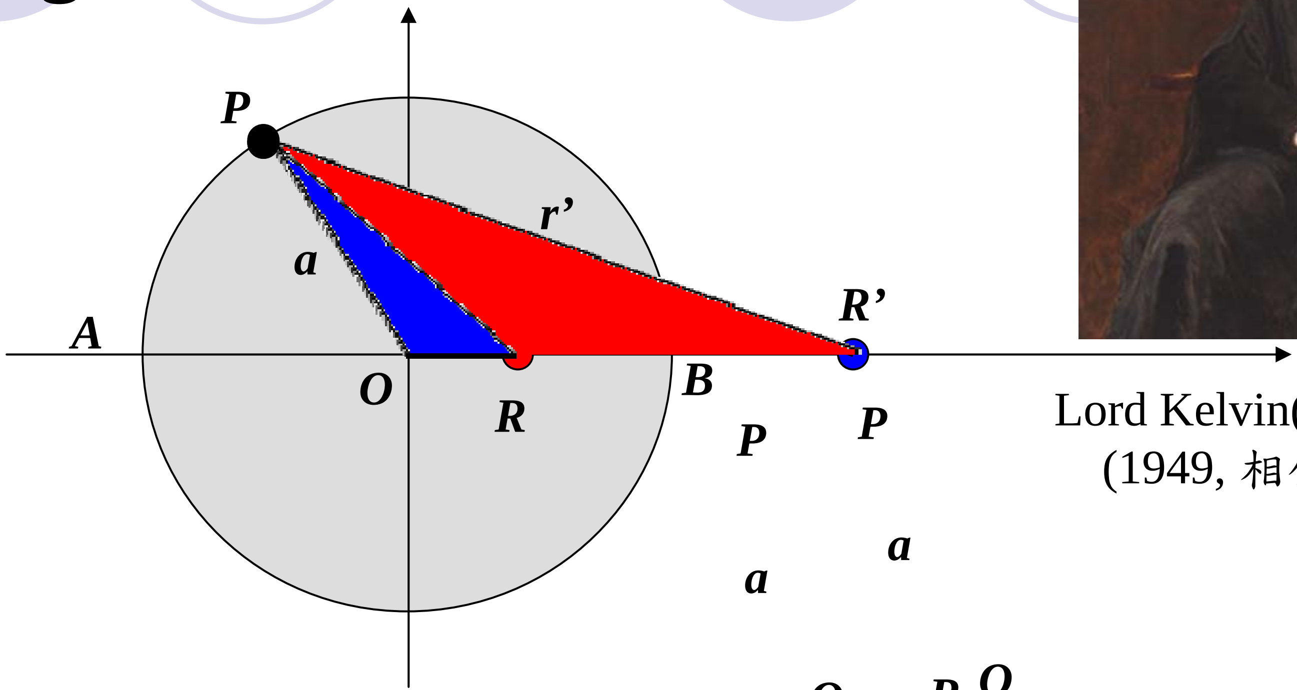
References:

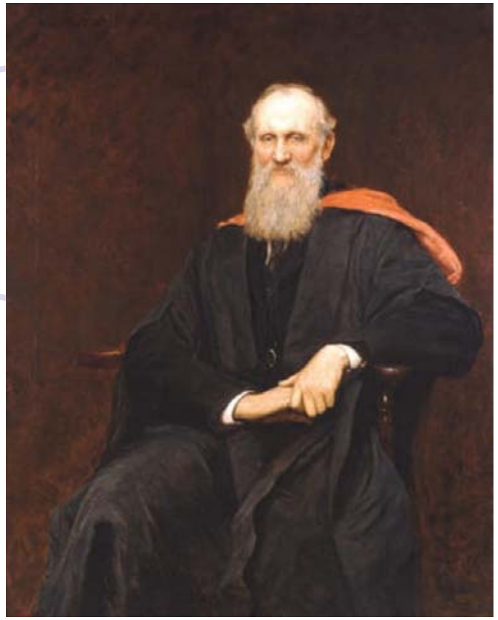
陳韋誌, 對偶邊界元素法中自由項與剛體運動項之研究, NTOU, 2001.

吳清森, 無網格法及邊界元素法於薄膜及板問題之退化尺度分析, NTOU, 2004



Conventional method to determine the image location





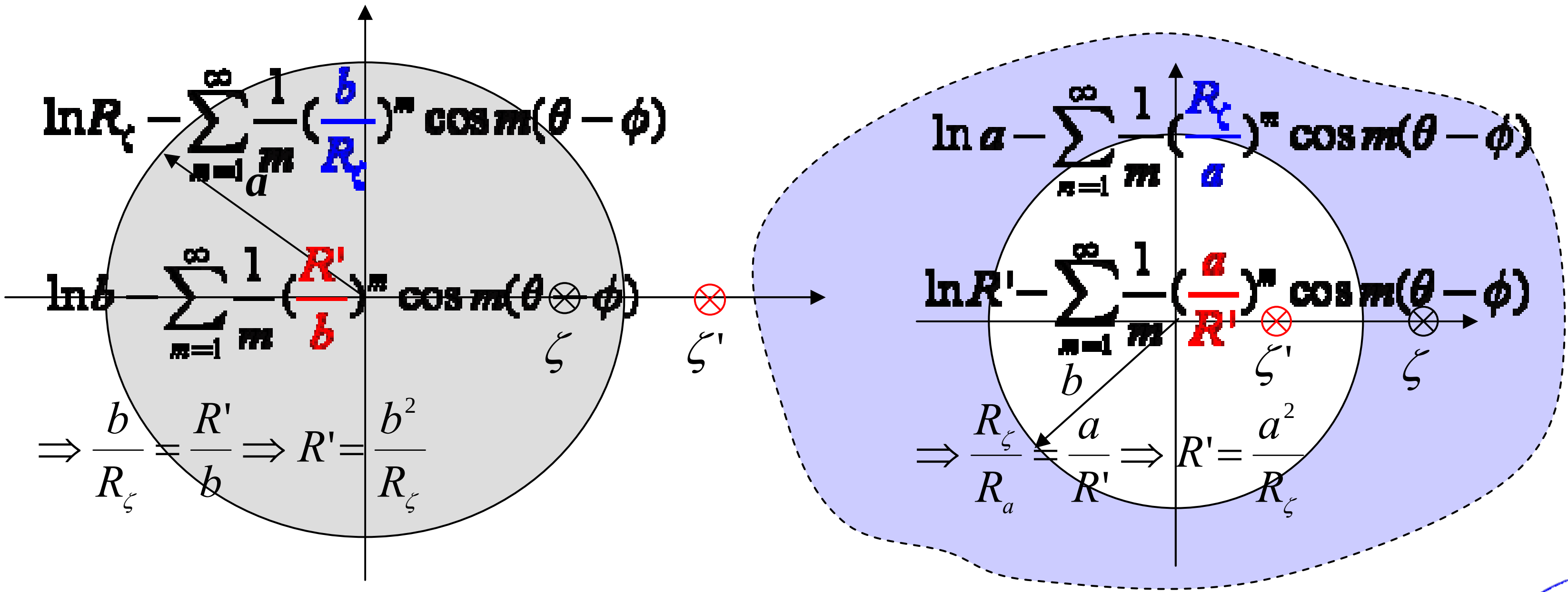
Lord Kelvin(1824~1907)
(1949, 相似三角形)

$$\frac{RP}{R'P}=\frac{a+OR}{a+OR'}=\frac{a-OR}{OR'-a} \Leftrightarrow \frac{OR}{a}=\frac{a}{OR'} \Rightarrow OR'=\frac{a^2}{OR}$$

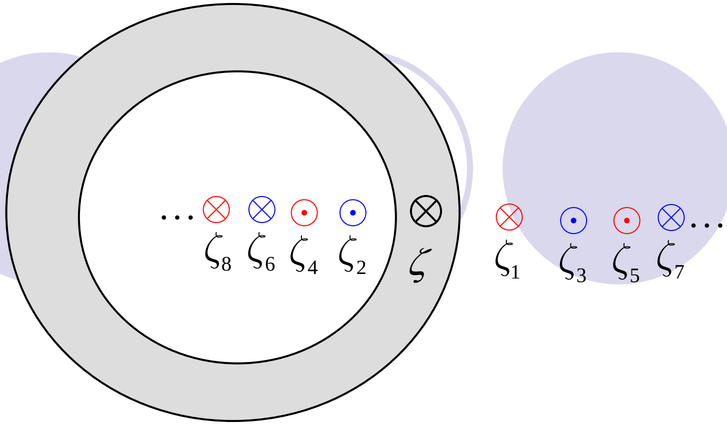
Greenberg (1971, 取巧法)

Chen and Wu-新觀點 (2006)

$$U(x,\zeta)=\begin{cases}\ln\rho-\sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{R_\zeta}{\rho}\right)^m\cos m(\theta-\phi)\\ \ln R_\zeta-\sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{\rho}{R_\zeta}\right)^m\cos m(\theta-\phi)\end{cases}$$



MFS-Image group



$$R_1=\frac{b^2}{R_\zeta}, R_9=\frac{b^2}{R_\zeta}\frac{b^4}{a^4}\ldots\ldots\ldots\ldots R_{8i-7}=\frac{b^2}{R_\zeta}(\frac{b^4}{a^4})^{i-1}$$

$$R_2=\frac{a^2}{R_\zeta}, R_{10}=\frac{a^2}{R_\zeta}\frac{a^4}{b^4}\ldots\ldots\ldots\ldots R_{8i-6}=\frac{a^2}{R_\zeta}(\frac{a^4}{b^4})^{i-1}$$

$$R_3=\frac{b^2R_\zeta}{a^2}, R_{11}=\frac{b^2R_\zeta}{a^2}\frac{b^4}{a^4}\ldots\ldots R_{8i-5}=\frac{b^2R_\zeta}{a^2}(\frac{b^4}{a^4})^{i-1}$$

$$R_4=\frac{a^2R_\zeta}{b^2}, R_{12}=\frac{a^2R_\zeta}{b^2}\frac{a^4}{b^4}\ldots\ldots R_{8i-4}=\frac{a^2R_\zeta}{b^2}(\frac{a^4}{b^4})^{i-1}$$

$$R_5=\frac{b^4}{a^2R_\zeta}, R_{13}=\frac{b^4}{a^2R_\zeta}\frac{b^4}{a^4}\ldots\ldots R_{8i-3}=\frac{b^4}{a^2R_\zeta}(\frac{b^4}{a^4})^{i-1}$$

$$R_6=\frac{a^4}{b^2R_\zeta}, R_{14}=\frac{a^4}{b^2R_\zeta}\frac{a^4}{b^4}\ldots\ldots R_{8i-2}=\frac{a^4}{b^2R_\zeta}(\frac{a^4}{b^4})^{i-1}$$

$$R_7=\frac{b^4R_\zeta}{a^4}, R_{15}=\frac{b^4R_\zeta}{a^4}\frac{b^4}{a^4}\ldots\ldots R_{8i-1}=\frac{b^4R_\zeta}{a^4}(\frac{b^4}{a^4})^{i-1}$$

$$R_8=\frac{a^4R_\zeta}{b^4}, R_{16}=\frac{a^4R_\zeta}{b^4}\frac{a^4}{b^4}\ldots\ldots R_{8i}=\frac{a^4R_\zeta}{b^4}(\frac{a^4}{b^4})^{i-1}$$

$$G_m(x,\zeta)=\frac{1}{2\pi}\{\ln|x-\zeta|+\lim_{N\to\infty}[\sum_{i=1}^{N}(\ln|x-\zeta_{8i-7}| -\ln|x-\zeta_{8i-6}|-\ln|x-\zeta_{8i-5}|-\ln|x-\zeta_{8i-4}| -\ln|x-\zeta_{8i-3}|+\ln|x-\zeta_{8i-2}|+\ln|x-\zeta_{8i-1}| +\ln|x-\zeta_{8i}|)]\}$$

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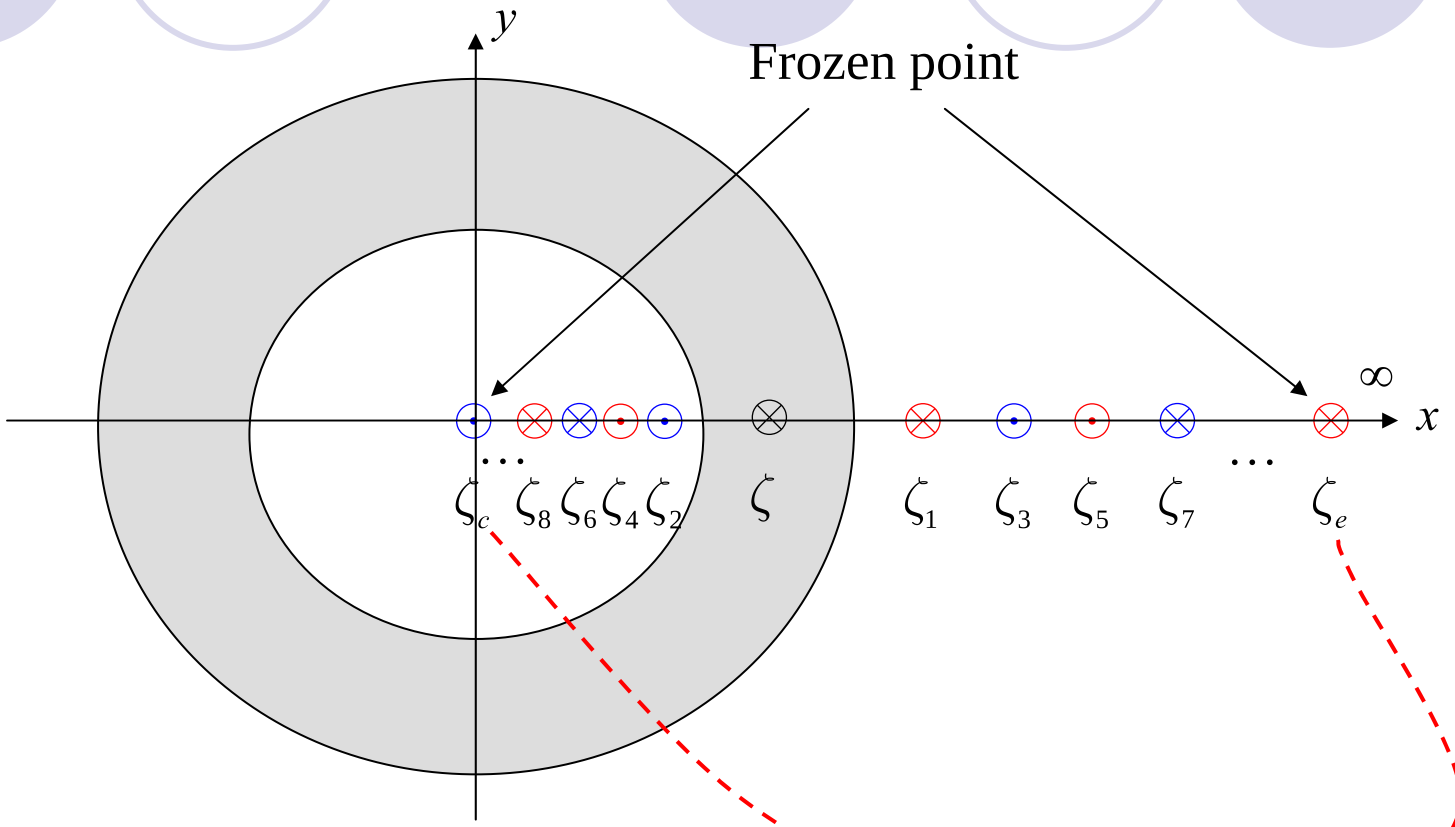
Analytical derivation



$$G(x,\zeta)=\frac{1}{2\pi}\{\ln|x-\zeta|+\lim_{N\to\infty}[\sum_{i=1}^{N}(\ln|x-\zeta_{8i-7}|-\ln|x-\zeta_{8i-6}|-\ln|x-\zeta_{8i-5}|-\ln|x-\zeta_{8i-4}|$$
$$-\ln|x-\zeta_{8i-3}|+\ln|x-\zeta_{8i-2}|+\ln|x-\zeta_{8i-1}|+\ln|x-\zeta_{8i}|)+c(N)\ln\rho+e(N)]\}$$

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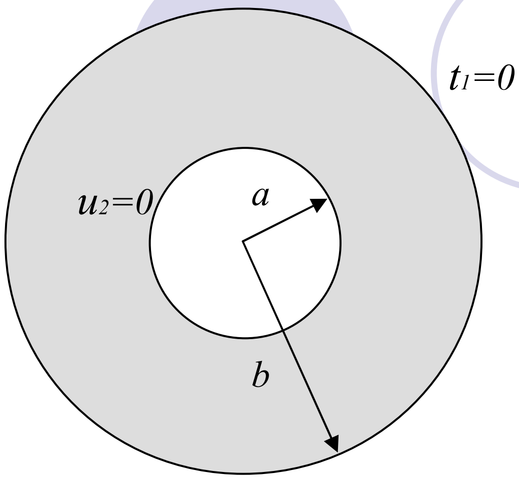
Numerical solution



$$\begin{cases} \rho = a \Rightarrow G(x_a,\zeta)=0 \\ \rho = b \Rightarrow G(x_b,\zeta)=0 \end{cases}$$

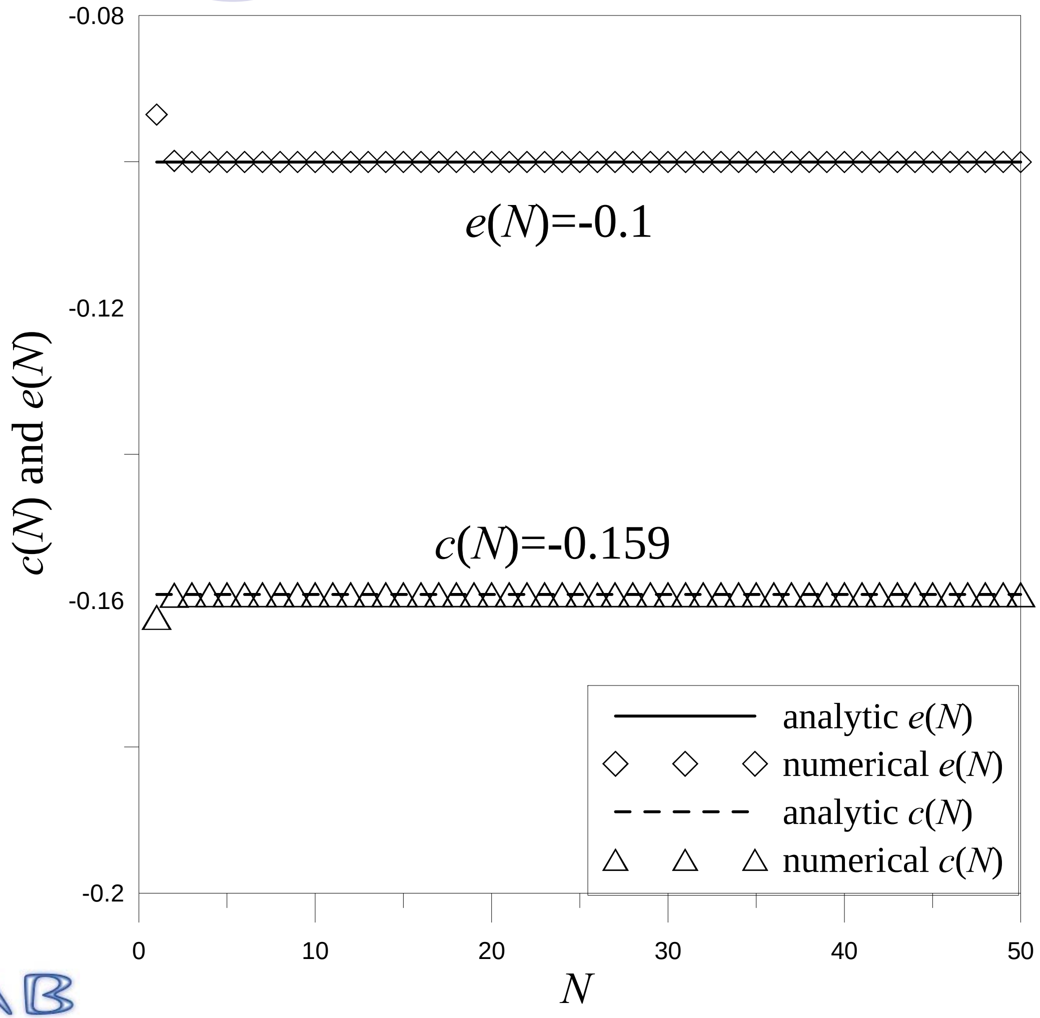
$$G(x,\zeta)\big|_{x=x_a} \Rightarrow \frac{1}{2\pi}\{\ln|x-\zeta| + \lim_{N\to\infty}[\sum_{i=1}^{N}(\ln|x-\zeta_{8i-7}| - \ln|x-\zeta_{8i-6}| - \ln|x-\zeta_{8i-5}| - \ln|x-\zeta_{8i-4}| - \ln|x-\zeta_{8i-3}| + \ln|x-\zeta_{8i-2}| + \ln|x-\zeta_{8i-1}| + \ln|x-\zeta 8i|) + c(N)\ln x + e(N)]\}\big|_{x=x_a} = 0$$

$$\frac{\partial G(x,\zeta)}{\partial x}\bigg|_{x=x_b} \Rightarrow \frac{1}{2\pi}\frac{\partial}{\partial x}\bigg|_{x=x_b}\{\ln|x-\zeta| + \lim_{N\to\infty}[\sum_{i=1}^{N}(\ln|x-\zeta_{8i-7}| - \ln|x-\zeta_{8i-6}| - \ln|x-\zeta_{8i-5}| - \ln|x-\zeta_{8i-4}| - \ln|x-\zeta_{8i-3}| + \ln|x-\zeta_{8i-2}| + \ln|x-\zeta_{8i-1}| + \ln|x-\zeta 8i|) + c(N)\ln x + e(N)]\}\big|_{x=x_b} = 0$$

$$\Downarrow$$

$$\begin{Bmatrix} \left[\ln|x-s| - \sum_{i=1}^{N}\left(\ln|x-\zeta_{8i-7}| + \cdots \ln|x-\zeta_{8i}|\right)\right]\Big|_{x=x_a} \\ \left\{\frac{\partial}{\partial x}\left[\ln|x-s| - \sum_{i=1}^{N}\left(\ln|x-\zeta_{8i-7}| + \cdots \ln|x-\zeta_{8i}|\right)\right]\right\}\Big|_{x=x_b} \end{Bmatrix} + \begin{bmatrix} (\ln x)\big|_{x=x_a} & 1 \\ \frac{\partial}{\partial x}(\ln x)\Big|_{x=x_b} & \frac{\partial}{\partial x}(1) \end{bmatrix} \begin{Bmatrix} c(N) \\ e(N) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

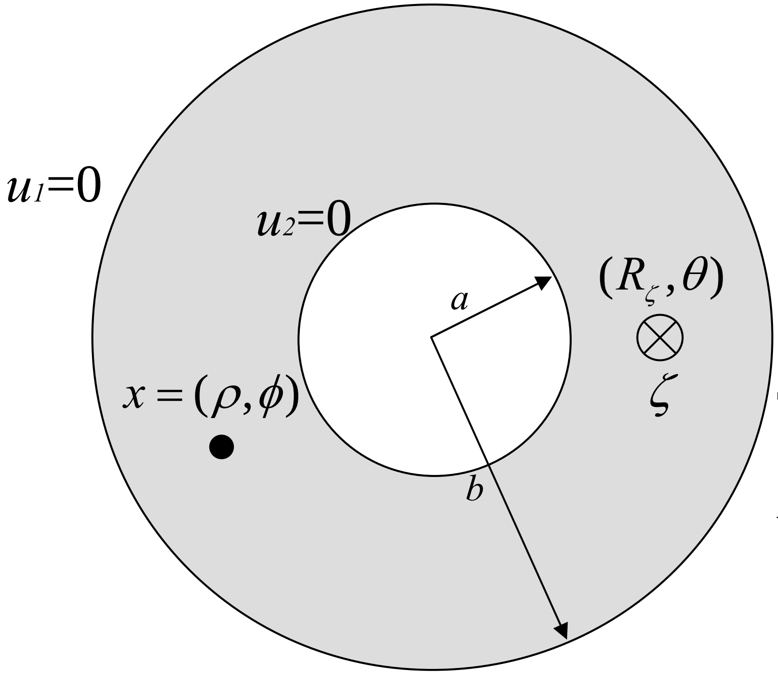
Numerical and analytic ways to determine *c*(*N*) and *e*(*N*)



$$e(N) = \frac{\ln a - \ln R_\zeta}{2\pi}$$

$$c(N) = \frac{-1}{2\pi}$$

Case 2: analytical derivation the annular case subject to Dirichlet BCs



Analytical solution

$$G(x,\zeta) = \frac{1}{2\pi}\{\ln|x-\zeta| - \lim_{N\to\infty}\Big[\sum_{i=1}^{N}(\ln|x-\zeta_{4i-3}| + \ln|x-\zeta_{4i-2}| - \ln|x-\zeta_{4i-1}| - \ln|x-\zeta_{4i}|) - (N\ln\frac{R_\zeta^{\,2}}{a^2} + \ln b\frac{\ln a - \ln R_\zeta}{\ln a - \ln b}) - \frac{\ln b - \ln R_\zeta}{(\ln b - \ln a)} ln\rho\Big]\}$$

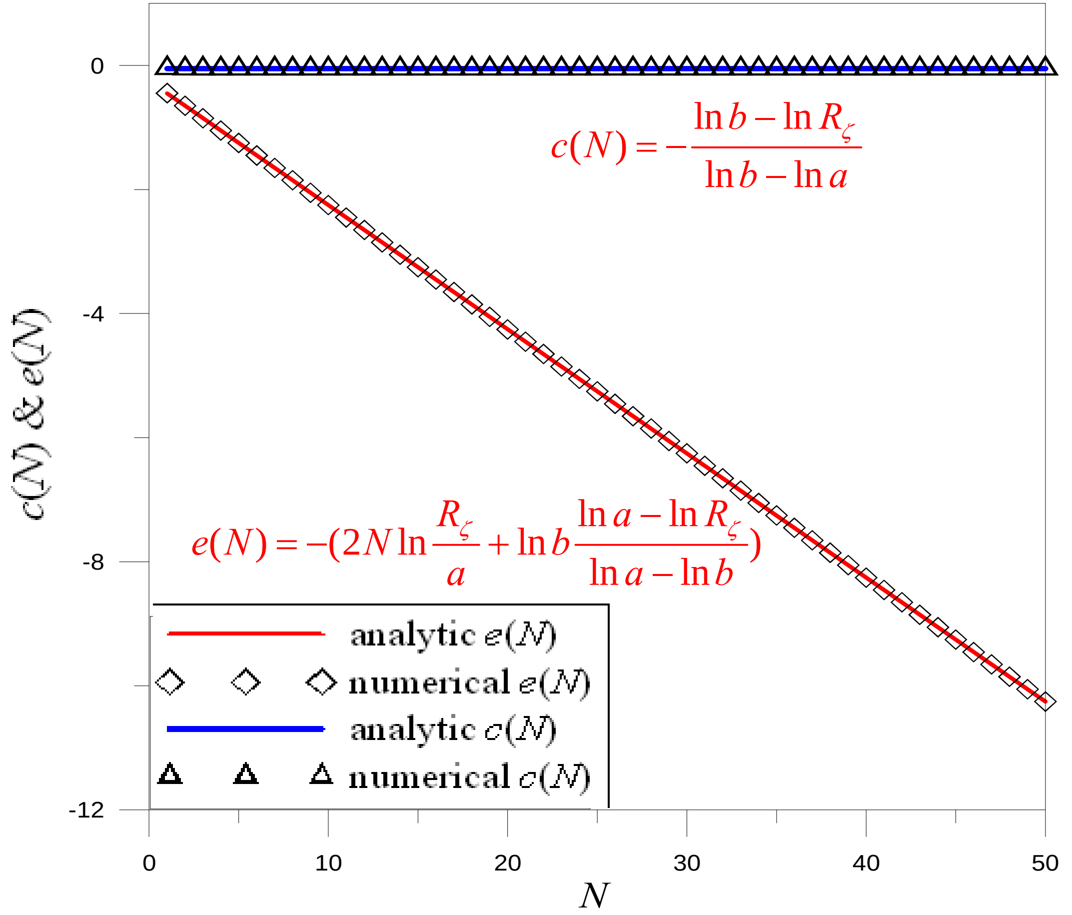
Reviewer comment of EABE :

The N in Eq. to tend to infinity , but in Eq. N is still present as a finite number. Really, what is $2N\ln(\frac{R_\zeta}{a})$ when $N\to\infty$?

Reply : **Euler constant**

$$\lim_{N\to\infty}\left(\sum_{m=1}^{N}\frac{1}{m} - \ln N\right) = \gamma$$

Numerical and analytic ways to determine $c(N)$ and $e(N)$



Values of $c(N)$ and $e(N)$ for the fixed-fixed case.

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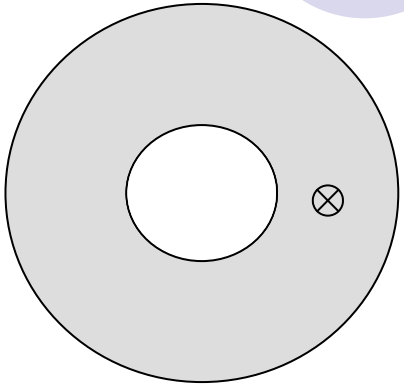
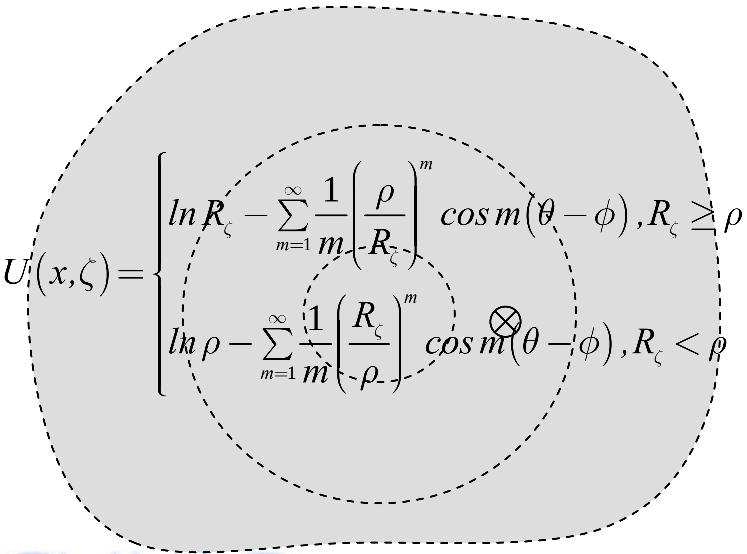
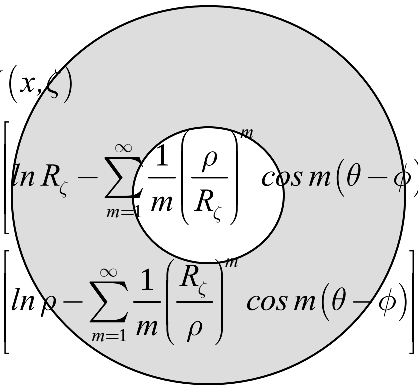
Trefftz Method: case 1

PART 1





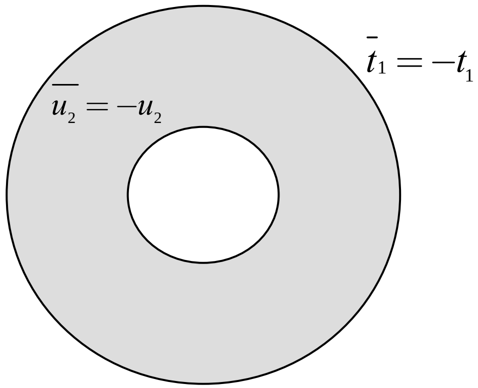
$$U(x,\zeta)=\begin{cases} \ln R_\zeta-\sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{\rho}{R_\zeta}\right)^m\cos m(\theta-\phi), & R_\zeta\ge\rho \\ \ln\rho-\sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{R_\zeta}{\rho}\right)^m\cos m(\theta-\phi), & R_\zeta<\rho \end{cases}$$



$$2\pi u(x)=U(x,\zeta)$$

$$u(x)=\begin{cases} \frac{1}{2\pi}\left[\ln R_\zeta-\sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{\rho}{R_\zeta}\right)^m\cos m(\theta-\phi)\right], & R_\zeta\ge\rho \\ \frac{1}{2\pi}\left[\ln\rho-\sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{R_\zeta}{\rho}\right)^m\cos m(\theta-\phi)\right], & R_\zeta<\rho \end{cases}$$

Boundary value problem



$$G_T(x,s) = \sum_{j=1}^{N_T} c_j \Phi_j,$$

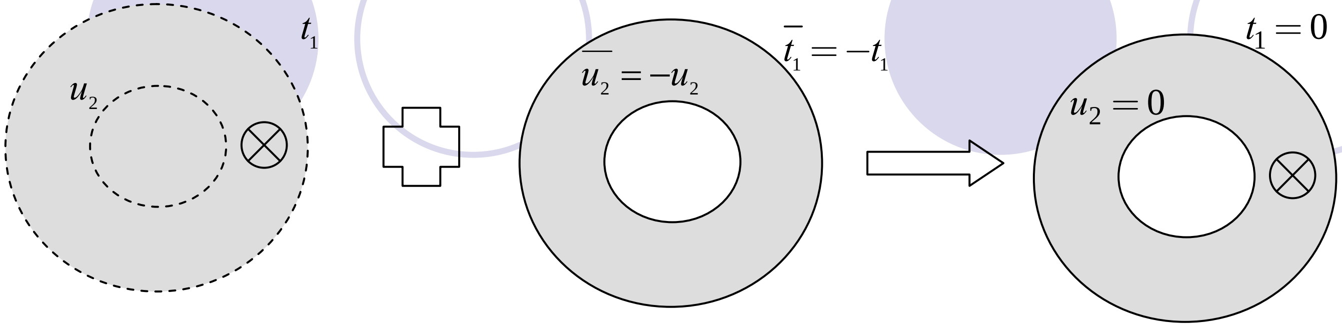
interior $\rightarrow 1, \rho^m \cos m\phi, \rho^m \sin m\phi$

exterior $\rightarrow \ln\rho, \rho^{-m}\cos m\phi, \rho^{-m}\sin m\phi$

$$\Downarrow$$

PART 2 $$G_T(x,\zeta) = p_0 + \bar{p}_0 \ln\rho + \sum_{m=1}^{\infty}\left[(p_m\rho^m + \bar{p}_m\rho^{-m})\cos m\phi + (q_m\rho^m + \bar{q}_m\rho^{-m})\sin m\phi\right] = -u(x)$$

$$\begin{Bmatrix} p_0 \\ \bar{p}_0 \end{Bmatrix} = \frac{1}{2\pi}\begin{Bmatrix} \ln a - \ln R_\zeta \\ -1 \end{Bmatrix} \begin{Bmatrix} p_m \\ \bar{p}_m \end{Bmatrix} = \begin{Bmatrix} \dfrac{\cos m\theta\left[a^m(\dfrac{a}{R_\zeta})^m - R_\zeta{}^m\right]}{2m\pi\left(b^{2m}+a^{2m}\right)} \\ \dfrac{a^m b^m \cos m\theta\left[b^m(\dfrac{a}{R_\zeta})^m + a^m(\dfrac{R_\zeta}{b})^m\right]}{2m\pi\left(b^{2m}+a^{2m}\right)} \end{Bmatrix} \begin{Bmatrix} q_m \\ \bar{q}_m \end{Bmatrix} = \begin{Bmatrix} \dfrac{\sin m\theta\left[a^m(\dfrac{a}{R_\zeta})^m - R_\zeta{}^m\right]}{2m\pi\left(b^{2m}+a^{2m}\right)} \\ \dfrac{a^m b^m \sin m\theta\left[b^m(\dfrac{a}{R_\zeta})^m + a^m(\dfrac{R_\zeta}{b})^m\right]}{2m\pi\left(b^{2m}+a^{2m}\right)} \end{Bmatrix}$$



PART 1 + PART 2 :

$$G(x,\zeta) = u + \bar{u}$$

$$u(x) = \begin{cases} \dfrac{1}{2\pi}\left[\ln R_\zeta - \sum_{m=1}^{\infty} \dfrac{1}{m}\left(\dfrac{\rho}{R_\zeta}\right)^m \cos m(\theta - \phi)\right], & R_\zeta \geq \rho \\ \dfrac{1}{2\pi}\left[\ln \rho - \sum_{m=1}^{\infty} \dfrac{1}{m}\left(\dfrac{R_\zeta}{\rho}\right)^m \cos m(\theta - \phi)\right], & R_\zeta < \rho \end{cases}$$

$$\bar{u}(x) = \frac{1}{2\pi}\left\{p_0 + \bar{p}_0 \ln \rho + \sum_{m=1}^{\infty}\left[(p_m \rho^m + \bar{p}_m \rho^{-m})\cos m\phi + (q_m \rho^m + \bar{q}_m \rho^{-m})\sin m\phi\right]\right\}$$

⇓

$$G(x,\zeta) = \frac{1}{2\pi}\left\{\ln|x - \zeta| + p_0 + \bar{p}_0 \ln \rho + \sum_{m=1}^{\infty}\left[(p_m \rho^m + \bar{p}_m \rho^{-m})\cos m\phi + (q_m \rho^m + \bar{q}_m \rho^{-m})\sin m\phi\right]\right\}$$

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Mathematical equivalence of solutions derived by using the Trefftz method and MFS for annular case

MFS(Image method)

$$G(x,\zeta)=\frac{1}{2\pi}\Big\{\ln|x-\zeta|+\lim_{N\to\infty}\sum_{i=1}^{N}\{\ln|x-\zeta_{8i-7}|-\ln|x-\zeta_{8i-6}|-\ln|x-\zeta_{8i-5}|$$

$$-\ln|x-\zeta_{8i-4}|-\ln|x-\zeta_{8i-3}|+\ln|x-\zeta_{8i-2}|+\ln|x-\zeta_{8i-1}|+\ln|x-\zeta_{8i}|\}$$

$$+\ln a-\ln R_{\zeta}-\ln\rho\Big\},\ a\le\rho\le b$$

The same

Trefftz method

$$\begin{Bmatrix} p_0 \\ \bar{p}_0 \end{Bmatrix}=\frac{1}{2\pi}\begin{bmatrix} \ln a-\ln R_{\zeta} \\ -1 \end{bmatrix}$$

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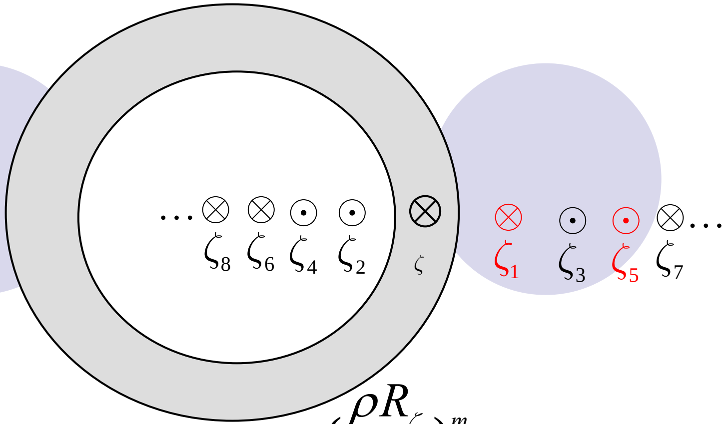
Trefftz method series expansion

$$\begin{Bmatrix} p_m \\ \bar{p}_m \end{Bmatrix} = \begin{bmatrix} \dfrac{\cos m\theta \left[a^m \left(\dfrac{a}{R_\zeta} \right)^m - R_\zeta^{\ m} \right]}{2m\pi\left(b^{2m}+a^{2m}\right)} \\ \dfrac{a^m b^m \cos m\theta \left[b^m \left(\dfrac{a}{R_\zeta} \right)^m + a^m \left(\dfrac{R_\zeta}{b} \right)^m \right]}{2m\pi\left(b^{2m}+a^{2m}\right)} \end{bmatrix} \qquad \begin{Bmatrix} q_m \\ \bar{q}_m \end{Bmatrix} = \begin{bmatrix} \dfrac{\sin m\theta \left[a^m \left(\dfrac{a}{R_\zeta} \right)^m - R_\zeta^{\ m} \right]}{2m\pi\left(b^{2m}+a^{2m}\right)} \\ \dfrac{a^m b^m \sin m\theta \left[b^m \left(\dfrac{a}{R_\zeta} \right)^m + a^m \left(\dfrac{R_\zeta}{b} \right)^m \right]}{2m\pi\left(b^{2m}+a^{2m}\right)} \end{bmatrix}$$

Without loss of generality

$$\theta = 0 \quad \Longrightarrow \quad \left(\frac{(\frac{a^2}{R_\zeta})^m \rho^m}{b^{2m}+a^{2m}} - \frac{R_\zeta^{\ m}\rho^m}{b^{2m}+a^{2m}} + \frac{(\frac{b^2a^2}{R_\zeta})^m \rho^{-m}}{b^{2m}+a^{2m}} + \frac{a^{2m}R_\zeta^{\ m}\rho^{-m}}{b^{2m}+a^{2m}} \right) \cos(m\phi)$$

Image method series expansion



$$\begin{cases} \otimes\zeta_{8i-7} \Rightarrow -\left[(\frac{\rho}{R_1})^m + (\frac{\rho}{R_9})^m + \cdots \right] = -\left[(\frac{\rho R_\zeta}{b^2})^m + (\frac{\rho R_\zeta}{b^2}\frac{a^4}{b^4})^m + \cdots \right] = -\frac{(\frac{\rho R_\zeta}{b^2})^m}{1-(\frac{a}{b})^{4m}} \\ \odot\zeta_{8i-3} \Rightarrow (\frac{\rho}{R_5})^m + (\frac{\rho}{R_{13}})^m + \cdots = (\frac{a^2\rho R_\zeta}{b^4})^m + (\frac{a^2\rho R_\zeta}{b^4}\frac{a^4}{b^4})^m + \cdots = \frac{(\frac{a^2\rho R_\zeta}{b^4})^m}{1-(\frac{a}{b})^{4m}} \end{cases}$$

$$\otimes\zeta_{8i-7} + \odot\zeta_{8i-3} = -\frac{(\frac{\rho R_\zeta}{b^2})^m}{1-(\frac{a}{b})^{4m}} + \frac{(\frac{a^2\rho R_\zeta}{b^4})^m}{1-(\frac{a}{b})^{4m}} = -\frac{\rho^m R_\zeta^{\ m}(b^{2m}-a^{2m})}{(b^{2m}-a^{2m})(b^{2m}+a^{2m})} = \boxed{-\frac{\rho^m R_\zeta^{\ m}}{(b^{2m}+a^{2m})}}$$

Trefftz series expansion

$$\left(\frac{(\frac{a^2}{R_\zeta})^m \rho^m}{b^{2m}+a^{2m}} \boxed{-\frac{R_\zeta^{\ m}\rho^m}{b^{2m}+a^{2m}}} + \frac{(\frac{b^2 a^2}{R_\zeta})^m \rho^{-m}}{b^{2m}+a^{2m}} + \frac{a^{2m} R_\zeta^{\ m} \rho^{-m}}{b^{2m}+a^{2m}} \right) \cos(m\phi)$$

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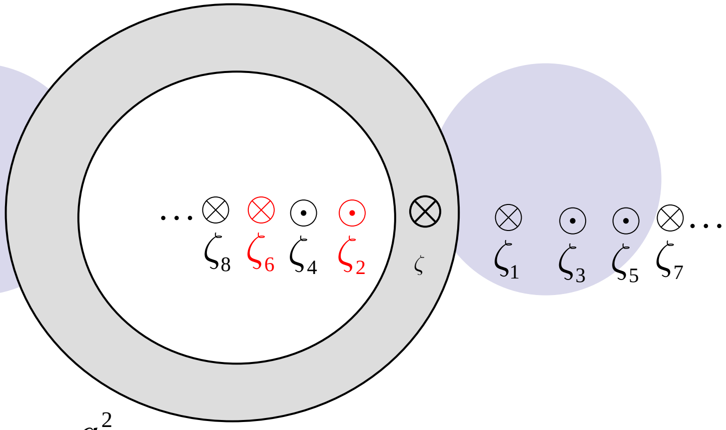
Image method series expansion



$$\begin{cases} \odot\zeta_{8i-6} \Rightarrow (\frac{R_2}{\rho})^m + (\frac{R_{10}}{\rho})^m + \cdots = (\frac{a^2}{R_\zeta \rho})^m + (\frac{a^2}{R_\zeta \rho}\frac{a^4}{b^4})^m + \cdots = \dfrac{(\frac{a^2}{R_\zeta \rho})^m}{1-(\frac{a}{b})^{4m}} \\ \otimes\zeta_{8i-2} \Rightarrow -\left[(\frac{R_6}{\rho})^m + (\frac{R_{14}}{\rho})^m + \cdots\right] = -\left[(\frac{a^4}{b^2 R_\zeta \rho})^m + (\frac{a^4}{b^2 R_\zeta \rho}\frac{a^4}{b^4})^m + \cdots\right] = -\dfrac{(\frac{a^4}{b^2 R_\zeta \rho})^m}{1-(\frac{a}{b})^{4m}} \end{cases}$$

$$\odot\zeta_{8i-6} + \otimes\zeta_{8i-2} = \frac{(\frac{a^2}{R_\zeta \rho})^m}{1-(\frac{a}{b})^{4m}} - \frac{(\frac{a^4}{b^2 R_\zeta \rho})^m}{1-(\frac{a}{b})^{4m}} = \frac{(\frac{a^2 b^2}{R_\zeta \rho})^m (b^{2m}-a^{2m})}{(b^{2m}-a^{2m})(b^{2m}+a^{2m})} = \boxed{\frac{(\frac{a^2 b^2}{R_\zeta \rho})^m}{(b^{2m}+a^{2m})}}$$

Trefftz series expansion

$$\left(\frac{(\frac{a^2}{R_\zeta})^m \rho^m}{b^{2m}+a^{2m}} - \frac{R_\zeta^{\,m} \rho^m}{b^{2m}+a^{2m}} + \boxed{\frac{(\frac{b^2 a^2}{R_\zeta})^m \rho^{-m}}{b^{2m}+a^{2m}}} + \frac{a^{2m} R_\zeta^{\,m} \rho^{-m}}{b^{2m}+a^{2m}} \right) \cos(m\phi)$$

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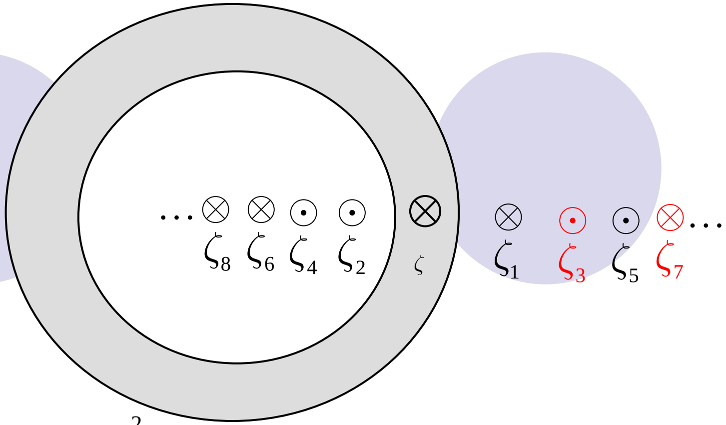
Image method series expansion



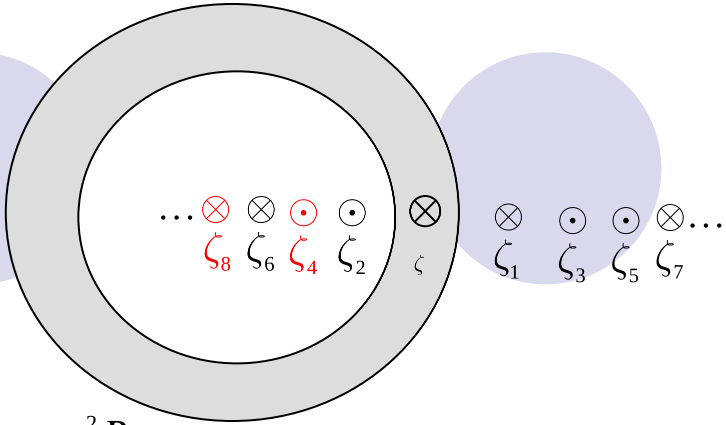
$$\begin{cases} \odot\zeta_{8i-5} \Rightarrow (\frac{\rho}{R_3})^m + (\frac{\rho}{R_{11}})^m + \cdots = (\frac{a^2\rho}{b^2R_\zeta})^m + (\frac{a^2\rho}{b^2R_\zeta}\frac{a^4}{b^4})^m + \cdots = \dfrac{(\frac{a^2\rho}{b^2R_\zeta})^m}{1-(\frac{a}{b})^{4m}} \\ \otimes\zeta_{8i-1} \Rightarrow -\left[(\frac{\rho}{R_7})^m + (\frac{\rho}{R_{15}})^m + \cdots\right] = -\left[(\frac{a^4\rho}{b^4R_\zeta})^m + (\frac{a^4\rho}{b^4R_\zeta}\frac{a^4}{b^4})^m + \cdots\right] = -\dfrac{(\frac{a^4\rho}{b^4R_\zeta})^m}{1-(\frac{a}{b})^{4m}} \end{cases}$$

$$\odot\zeta_{8i-5} + \otimes\zeta_{8i-1} = \frac{(\frac{a^2\rho}{b^2R_\zeta})^m}{1-(\frac{a}{b})^{4m}} - \frac{(\frac{a^4\rho}{b^4R_\zeta})^m}{1-(\frac{a}{b})^{4m}} = \frac{(\frac{a^2\rho}{R_\zeta})^m(b^{2m}-a^{2m})}{(b^{2m}-a^{2m})(b^{2m}+a^{2m})} = \boxed{\frac{(\frac{a^2\rho}{R_\zeta})^m}{(b^{2m}+a^{2m})}}$$

Trefftz series expansion

$$\left(\boxed{\frac{(\frac{a^2}{R_\zeta})^m\rho^m}{b^{2m}+a^{2m}}} - \frac{R_\zeta^m\rho^m}{b^{2m}+a^{2m}} + \frac{(\frac{b^2a^2}{R_\zeta})^m\rho^{-m}}{b^{2m}+a^{2m}} + \frac{a^{2m}R_\zeta^m\rho^{-m}}{b^{2m}+a^{2m}}\right)\cos(m\phi)$$

Image method series expansion



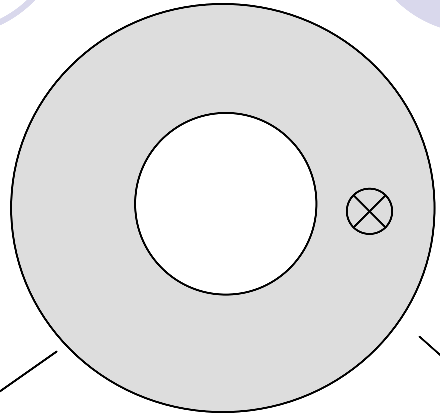
$$\begin{cases} \odot\zeta_{8i-4} \Rightarrow (\frac{R_4}{\rho})^m + (\frac{R_{12}}{\rho})^m + \cdots = (\frac{a^2 R_\zeta}{b^2 \rho})^m + (\frac{a^2 R_\zeta}{b^2 \rho}\frac{a^4}{b^4})^m + \cdots = \dfrac{(\frac{a^2 R_\zeta}{b^2 \rho})^m}{1-(\frac{a}{b})^{4m}} \\ \otimes\zeta_{8i} \Rightarrow -\left[(\frac{R_8}{\rho})^m + (\frac{R_{16}}{\rho})^m + \cdots\right] = -\left[(\frac{a^4 R_\zeta}{b^4 \rho})^m + (\frac{a^4 R_\zeta}{b^4 \rho}\frac{a^4}{b^4})^m + \cdots\right] = -\dfrac{(\frac{a^4 R_\zeta}{b^4 \rho})^m}{1-(\frac{a}{b})^{4m}} \end{cases}$$

$$\odot\zeta_{8i-4} + \otimes\zeta_{8i} = \frac{(\frac{a^2 R_\zeta}{b^2 \rho})^m}{1-(\frac{a}{b})^{4m}} - \frac{(\frac{a^4 R_\zeta}{b^4 \rho})^m}{1-(\frac{a}{b})^{4m}} = \frac{(\frac{a^2 R_\zeta}{\rho})^m (b^{2m}-a^{2m})}{(b^{2m}-a^{2m})(b^{2m}+a^{2m})} = \boxed{\frac{(\frac{a^2 R_\zeta}{\rho})^m}{(b^{2m}+a^{2m})}}$$

Trefftz series expansion

$$\left(\frac{(\frac{a^2}{R_\zeta})^m \rho^m}{b^{2m}+a^{2m}} - \frac{R_\zeta^{\,m} \rho^m}{b^{2m}+a^{2m}} + \frac{(\frac{b^2 a^2}{R_\zeta})^m \rho^{-m}}{b^{2m}+a^{2m}} + \boxed{\frac{a^{2m} R_\zeta^{\,m} \rho^{-m}}{b^{2m}+a^{2m}}} \right) \cos(m\phi)$$

Mathematical equivalence of solutions derived by using the Trefftz method and MFS



Trefftz method

$$1, \rho^m \cos m\phi, \rho^m \sin m\phi$$
$$\ln \rho, \rho^{-m} \cos m\phi, \rho^{-m} \sin m\phi$$
$$m = 0, 1, 2, 3, \cdots, \infty$$

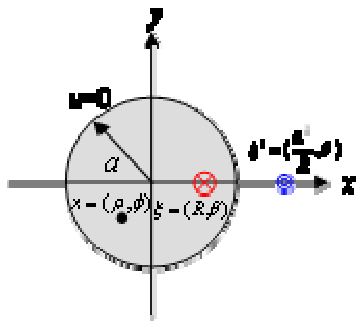
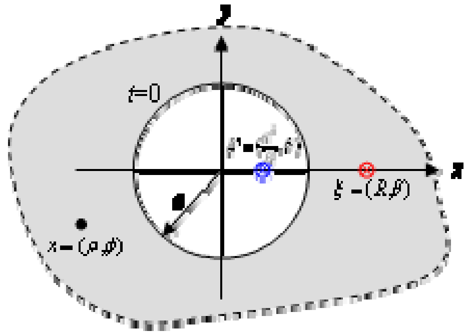
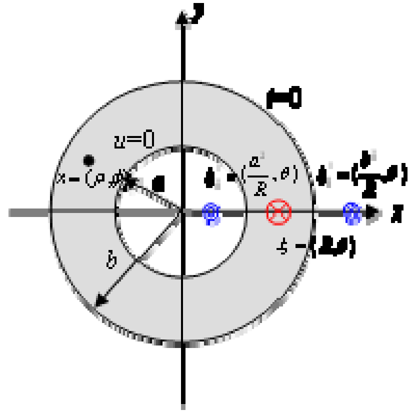
MFS

$$\ln |x - s_j|, \; j \in N$$

Equivalence

addition theorem

Image method of interior, exterior and annular problems

	Interior Dirichlet problem	Exterior Neumann problem	Annular problem
Problems			
Solution of image parts	$G(x,\xi;\xi')=\frac{1}{2\pi}[\ln\lvert x-\xi\rvert-\ln\lvert x-\xi'\rvert]$	$G(x,\xi;\xi')=\frac{1}{2\pi}[\ln\lvert x-\xi\rvert+\ln\lvert x-\xi'\rvert]$	$G(x,\xi;\xi')=\frac{1}{2\pi}\{\ln\lvert x-\xi\rvert$ $+\lim_{N\to\infty}\sum_{i=1}^{N}[\ln\lvert x-\xi_{8i-7}\rvert-\ln\lvert x-\xi_{8i-6}\rvert$ $-\ln\lvert x-\xi_{8i-5}\rvert-\ln\lvert x-\xi_{8i-4}\rvert$ $-\ln\lvert x-\xi_{8i-3}\rvert+\ln\lvert x-\xi_{8i-2}\rvert$ $+\ln\lvert x-\xi_{8i-1}\rvert+\ln\lvert x-\xi_{8i}\rvert]\}, a<\rho<b$
Extra terms of complementary solutions	$\frac{1}{2\pi}[(\ln a-\ln R)\cdot 1]$	$\frac{1}{2\pi}[(\ln\rho)\cdot(-1)]$	$\frac{1}{2\pi}[(\ln a-\ln R)\cdot 1+(-1)\cdot\ln\rho]$

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Mathematical equivalence for the solution

工程數學裡一個簡單的問題:

Solve $y''(x) - y(x) = 0$

$$\Rightarrow \begin{cases} y(x) = c_1 e^{x} + c_2 e^{-x} \\ y(x) = b_1 \cosh x + b_2 \sinh x \end{cases}$$

Equivalence

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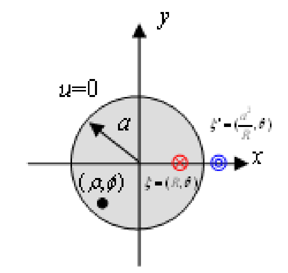
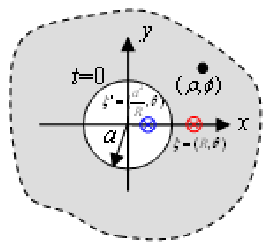
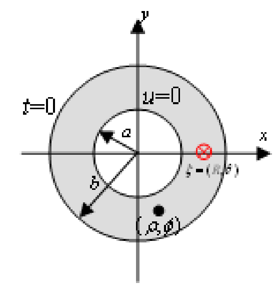
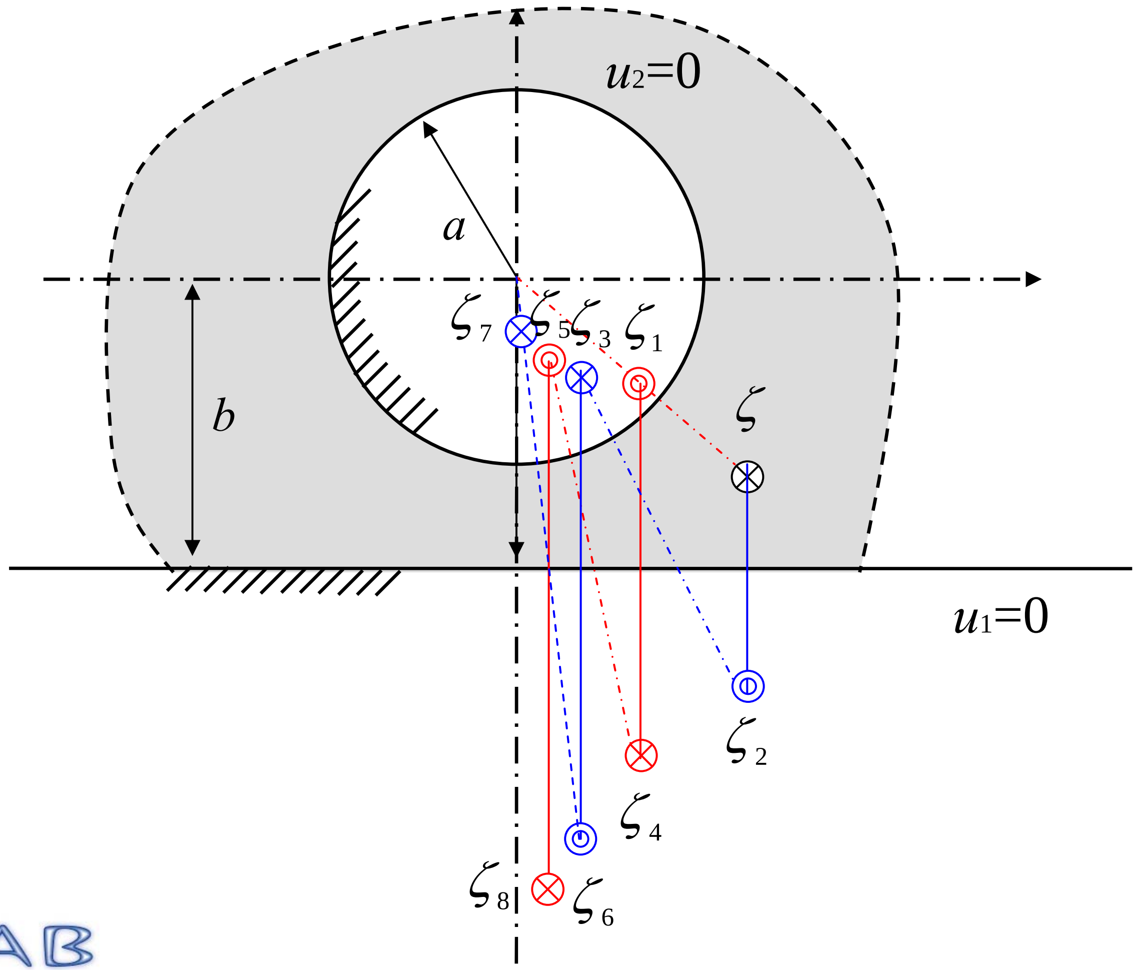
Trefftz & MFS

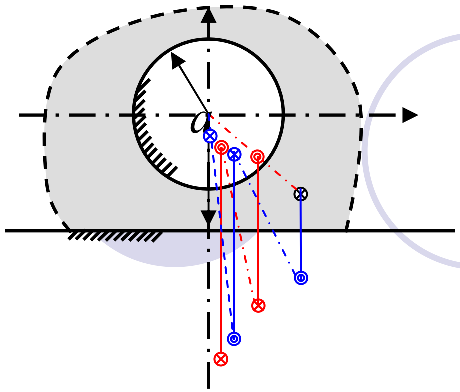
解的數學等效性

Problems \ Methods		Trefftz method	Special MFS (image method)
Interior Dirichlet case		$G(x,\xi)=\frac{\ln\lvert x-\xi\rvert}{2\pi}-\frac{\ln a}{2\pi}+\sum_{m=1}^{\infty}\left[\frac{R^m\rho^m\cos m\theta}{2m\pi a^{2m}}\cos m\phi+\frac{R^m\rho^m\sin m\theta}{2m\pi a^{2m}}\sin m\phi\right]$	$G(x,\xi)=\frac{1}{2\pi}[\ln\lvert x-\xi\rvert-\ln\lvert x-\xi'\rvert-\ln a+\ln R]$
	Constant term	$\frac{-1}{2\pi}\ln a$	$\frac{-1}{2\pi}\ln a$
	Series terms	$\sum_{m=1}^{\infty}\frac{R^m\rho^m\cos m\theta}{2m\pi a^{2m}}\cos m\phi+\frac{R^m\rho^m\sin m\theta}{2m\pi a^{2m}}\sin m\phi$	$\sum_{m=1}^{\infty}\frac{R^m\rho^m\cos m\theta}{2m\pi a^{2m}}\cos m\phi+\frac{R^m\rho^m\sin m\theta}{2m\pi a^{2m}}\sin m\phi$
Exterior Neumann case		$G(x,\xi)=\frac{\ln\lvert x-\xi\rvert}{2\pi}-\sum_{m=1}^{\infty}\left[\frac{a^{2m}\cos m\theta}{2m\pi R^m\rho^m}\cos(m\phi)+\frac{a^{2m}\sin m\theta}{2m\pi R^m\rho^m}\cos(m\phi)\right]$	$G(x,\xi)=\frac{1}{2\pi}[\ln\lvert x-\xi\rvert+\ln\lvert x-\xi'\rvert-\ln\rho]$
	Constant term	Constant=0	Constant=0
	Series terms	$\sum_{m=1}^{\infty}\frac{-a^{2m}\cos m\theta}{2m\pi R^m\rho^m}\cos m\phi-\frac{a^{2m}\sin m\theta}{2m\pi R^m\rho^m}\sin m\phi$	$\sum_{m=1}^{\infty}\frac{-a^{2m}\cos m\theta}{2m\pi R^m\rho^m}\cos m\phi-\frac{a^{2m}\sin m\theta}{2m\pi R^m\rho^m}\sin m\phi$
Annular case		$G(x,\xi)=\frac{\ln\lvert x-\xi\rvert}{2\pi}+\frac{\ln a-\ln R-\ln\rho}{2\pi}+\sum_{m=1}^{\infty}\left(\frac{(\frac{a^2\rho}{R})^m}{b^{2m}+a^{2m}}-\frac{(R\rho)^m}{b^{2m}+a^{2m}}+\frac{(\frac{b^2a^2}{R\rho})^m}{b^{2m}+a^{2m}}+\frac{(\frac{a^2R}{\rho})^m}{b^{2m}+a^{2m}}\right)\cos m(\theta-\phi)$	$G(x,\xi)=\frac{1}{2\pi}\{\ln\lvert x-\xi\rvert+\lim_{l\to\infty}\sum_{i=1}^{\infty}[\ln\lvert x-\xi_{8i-7}\rvert-\ln\lvert x-\xi_{8i-6}\rvert-\ln\lvert x-\xi_{8i-5}\rvert-\ln\lvert x-\xi_{8i-4}\rvert-\ln\lvert x-\xi_{8i-3}\rvert+\ln\lvert x-\xi_{8i-2}\rvert+\ln\lvert x-\xi_{8i-1}\rvert+\ln\lvert x-\xi_{8i}\rvert]+\ln a-\ln R-\ln\rho\}$
 source point ⊗ image point ⊗ ◎	Constant term	$p_0+\bar{p}_0\ln\rho=\frac{1}{2\pi}(\ln a-\ln R-\ln\rho)$	$\frac{1}{2\pi}(\ln a-\ln R-\ln\rho)$
	Series terms	$\left(\frac{(\frac{a^2\rho}{R})^m}{b^{2m}+a^{2m}}-\frac{(R\rho)^m}{b^{2m}+a^{2m}}+\frac{(\frac{b^2a^2}{R\rho})^m}{b^{2m}+a^{2m}}+\frac{(\frac{a^2R}{\rho})^m}{b^{2m}+a^{2m}}\right)\cos(m\theta)\cos(m\phi)+\left(\frac{(\frac{a^2\rho}{R})^m}{b^{2m}+a^{2m}}-\frac{(R\rho)^m}{b^{2m}+a^{2m}}+\frac{(\frac{b^2a^2}{R\rho})^m}{b^{2m}+a^{2m}}+\frac{(\frac{a^2R}{\rho})^m}{b^{2m}+a^{2m}}\right)\sin(m\theta)\sin(m\phi)$	$\left(\frac{(\frac{a^2\rho}{R})^m}{b^{2m}+a^{2m}}-\frac{(R\rho)^m}{b^{2m}+a^{2m}}+\frac{(\frac{b^2a^2}{R\rho})^m}{b^{2m}+a^{2m}}+\frac{(\frac{a^2R}{\rho})^m}{b^{2m}+a^{2m}}\right)\cos(m\theta)\cos(m\phi)+\left(\frac{(\frac{a^2\rho}{R})^m}{b^{2m}+a^{2m}}-\frac{(R\rho)^m}{b^{2m}+a^{2m}}+\frac{(\frac{b^2a^2}{R\rho})^m}{b^{2m}+a^{2m}}+\frac{(\frac{a^2R}{\rho})^m}{b^{2m}+a^{2m}}\right)\sin(m\theta)\sin(m\phi)$

Semi-analytical solution: case 3



MFS-Image group



$$\otimes\zeta \rightarrow \begin{cases} \ln R_\zeta - \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{a}{R_\zeta}\right)^m \cos m(\theta - \phi),\ a < R_\zeta \\ \ln \rho - \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{R_\zeta}{\rho}\right)^m \cos m(\theta - \phi),\ \rho < R_\zeta \end{cases}$$

$$\odot\zeta_1 \rightarrow \begin{cases} \ln a - \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{R_1}{a}\right)^m \cos m(\theta_1 - \phi),\ a > R_1 \\ \ln \rho - \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{R_1}{\rho}\right)^m \cos m(\theta_1 - \phi),\ \rho > R_1 \end{cases}$$

$$\frac{R_1}{a} = \frac{a}{R_\zeta} \Rightarrow R_1 = \frac{a^2}{R_\zeta}$$

$$\otimes\zeta_3 \rightarrow \begin{cases} \ln a - \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{R_3}{a}\right)^m \cos m(\theta_2 - \phi),\ a > R_3 \\ \ln \rho - \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{R_3}{\rho}\right)^m \cos m(\theta_2 - \phi),\ \rho > R_3 \end{cases}$$

$$\frac{R_3}{a} = \frac{a}{R_2} \Rightarrow R_3 = \frac{a^2}{R_2}$$

$\vdots$

$$\odot\zeta_2 \rightarrow \begin{cases} \ln R_2 - \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{a}{R_2}\right)^m \cos m(\theta_2 - \phi),\ a < R_2 \\ \ln R_2 - \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{\rho}{R_2}\right)^m \cos m(\theta_2 - \phi),\ \rho < R_2 \end{cases}$$

$$\theta_2 = \tan^{-1} \frac{2b - R_\zeta \sin\theta_1}{R_\zeta \cos\theta_1},\quad R_2 = \frac{R_\zeta \cos\theta_1}{\cos\theta_2}$$

$$\otimes\zeta_4 \rightarrow \begin{cases} \ln R_4 - \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{a}{R_4}\right)^m \cos m(\theta_3 - \phi),\ a < R_4 \\ \ln R_4 - \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{\rho}{R_4}\right)^m \cos m(\theta_3 - \phi),\ \rho < R_4 \end{cases}$$

$$\theta_3 = \tan^{-1} \frac{2b - R_1 \sin\theta_1}{R_1 \cos\theta_1},\quad R_4 = \frac{R_1 \cos\theta_1}{\cos\theta_3}$$

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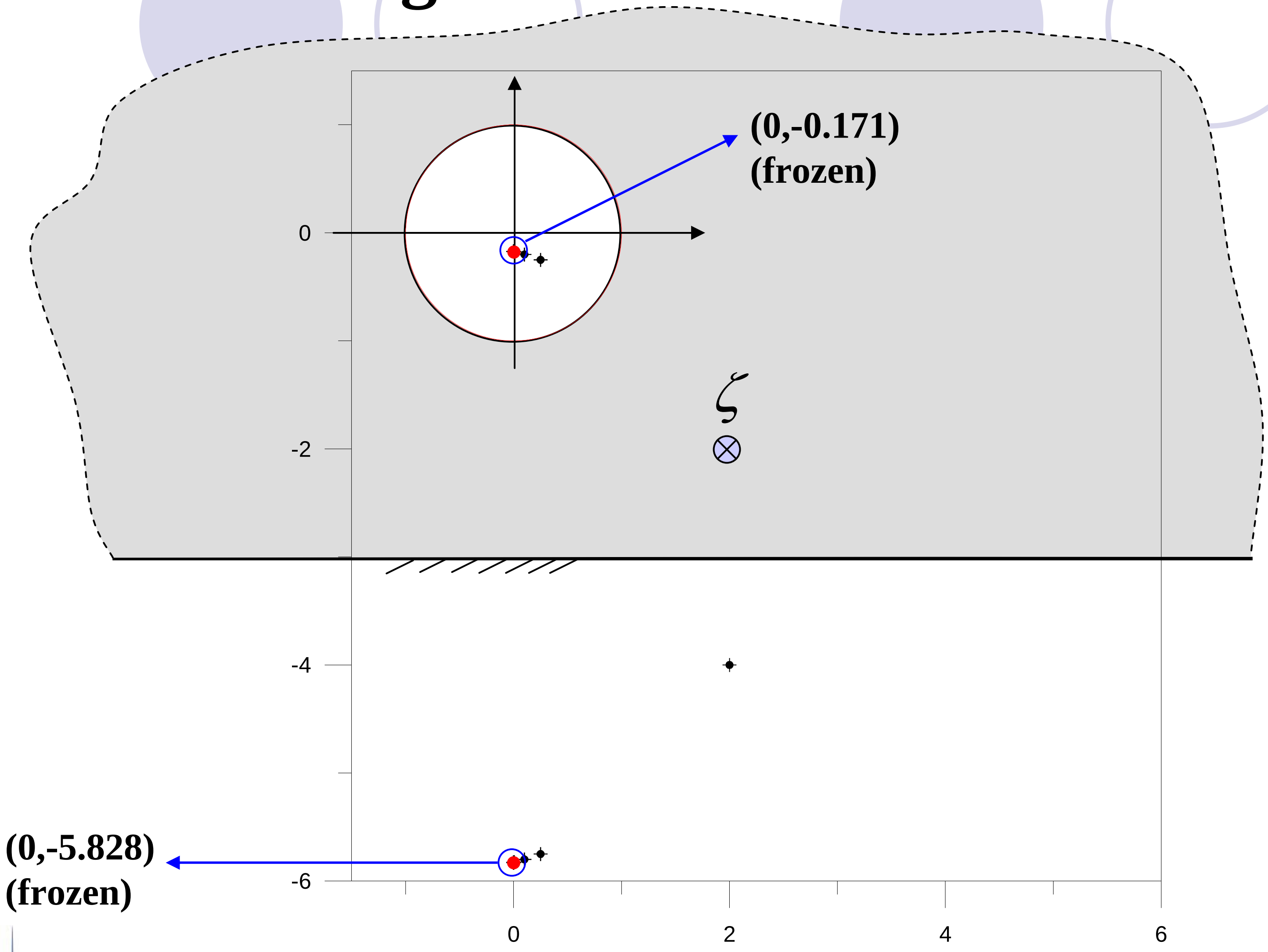
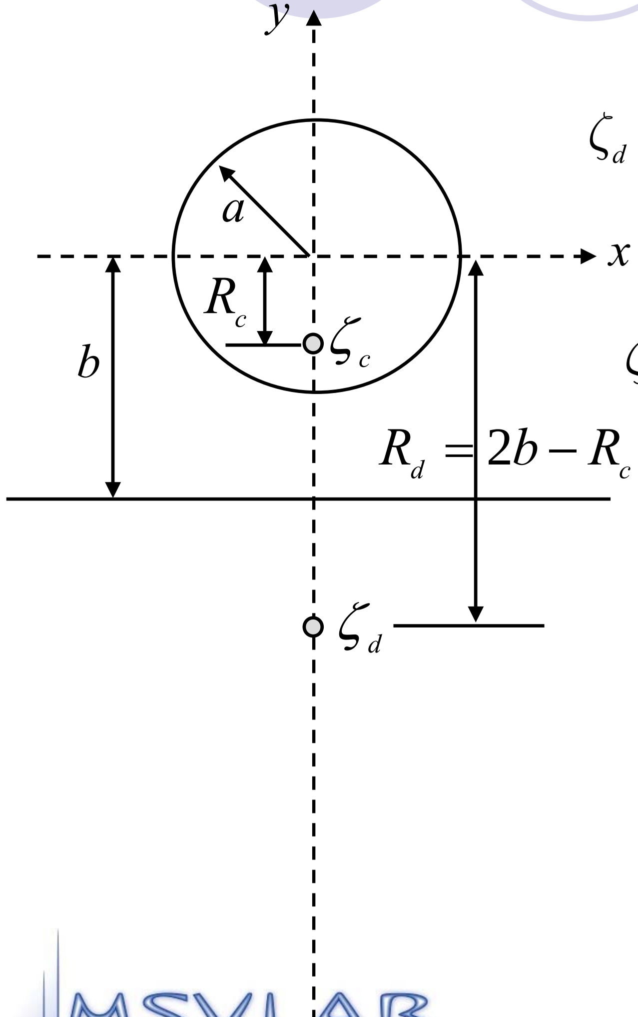
Frozen image location



Successive images (20 points)

Analytical derivation of location for the two frozen points



$$\zeta_d \rightarrow \ln|x-\zeta_d| = \ln R_d - \sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{a}{R_d}\right)^m \cos m(\psi_d-\phi),\ a<R_d$$

$$\zeta_c \rightarrow \ln|x-\zeta_c| = \ln a - \sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{R_c}{a}\right)^m \cos m(\psi_d-\phi),\ a>R_c$$

$$\Downarrow$$

$$\frac{R_c}{a} = \frac{a}{R_d} \Rightarrow R_c = \frac{a^2}{R_d}$$

$$\therefore R_c = \frac{a^2}{2b-R_c}$$

$$\Rightarrow R_c^2 - 6R_c + 1 = 0$$

$$\Rightarrow R_c = 3 \pm 2\sqrt{2}$$ **(0.171 & 5.828)**

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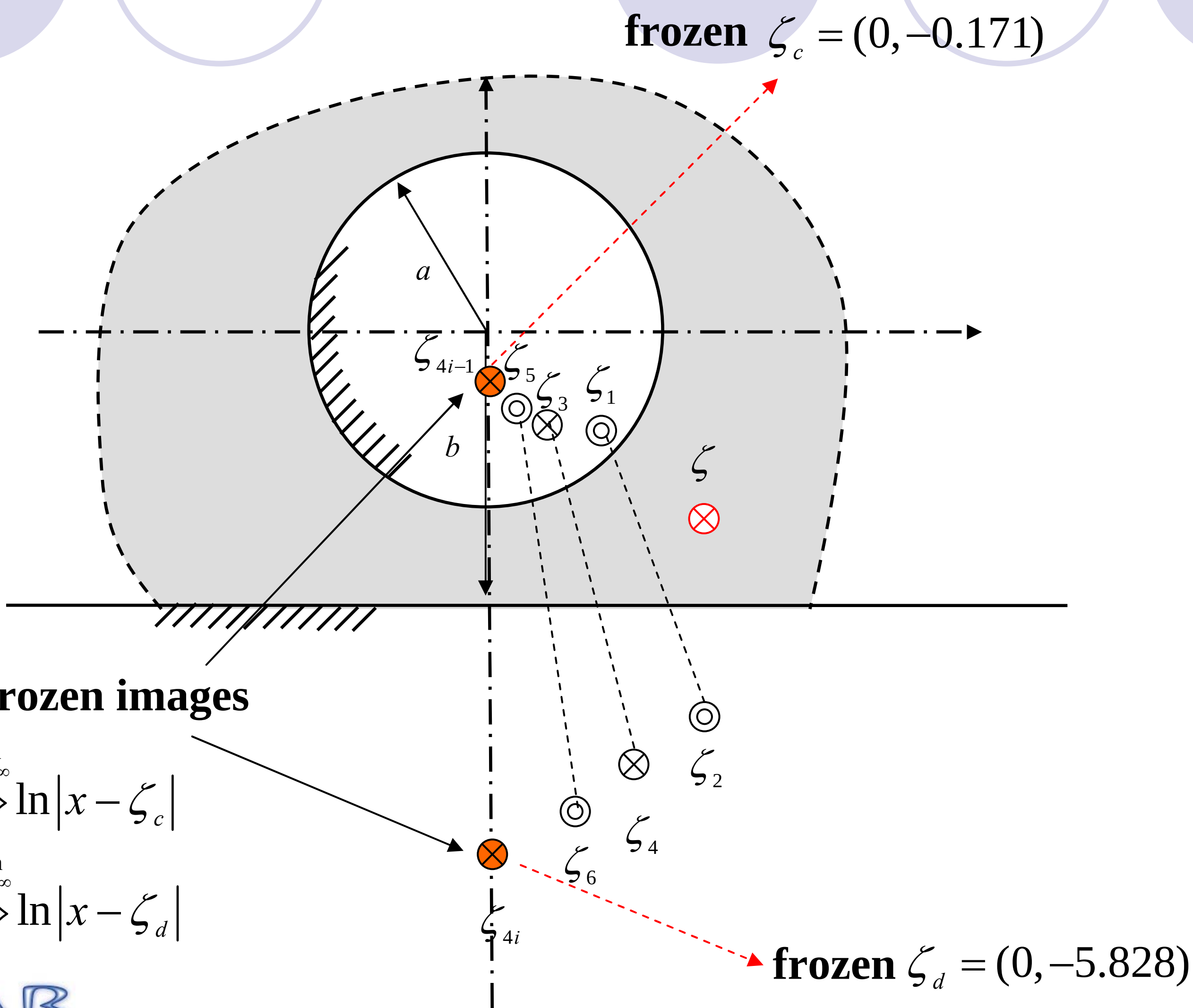
Series of images

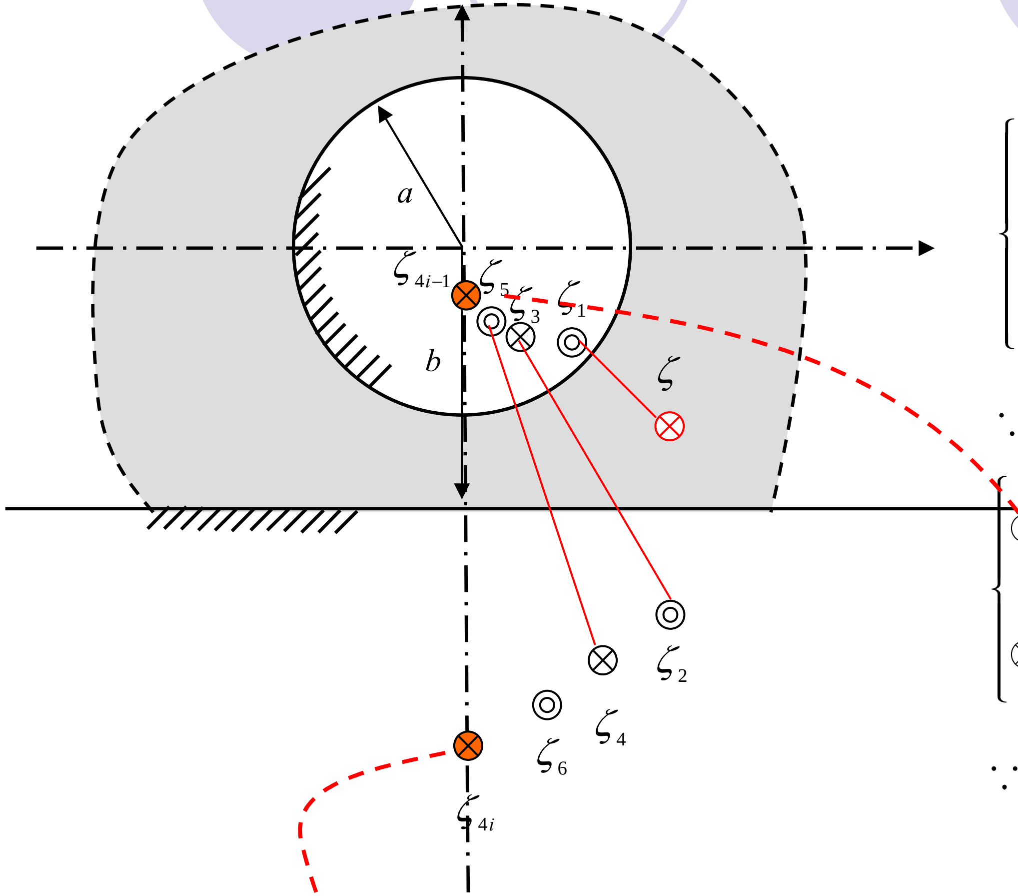


The final two frozen images

$$\odot \zeta_{4i-3}, i \in N \overset{\lim_{N\to\infty}}{\Rightarrow} \ln|x - \zeta_c|$$

$$\odot \zeta_{4i-2}, i \in N \overset{\lim_{N\to\infty}}{\Rightarrow} \ln|x - \zeta_d|$$

Rigid body term



$$\begin{cases} \otimes\zeta \Rightarrow \ln R_\zeta - \sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{a}{R_\zeta}\right)^m \cos m(\psi-\phi), & \rho\in B_1 \\ \odot\zeta_1 \Rightarrow \ln a - \sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{R_1}{a}\right)^m \cos m(\psi_1-\phi), & \rho\in B_1 \end{cases}$$

$$\because R_1 = \frac{a^2}{R_\zeta} \quad \therefore \ln|x-s_\zeta| - \ln|x-s_1| = \ln R_\zeta - \ln a$$

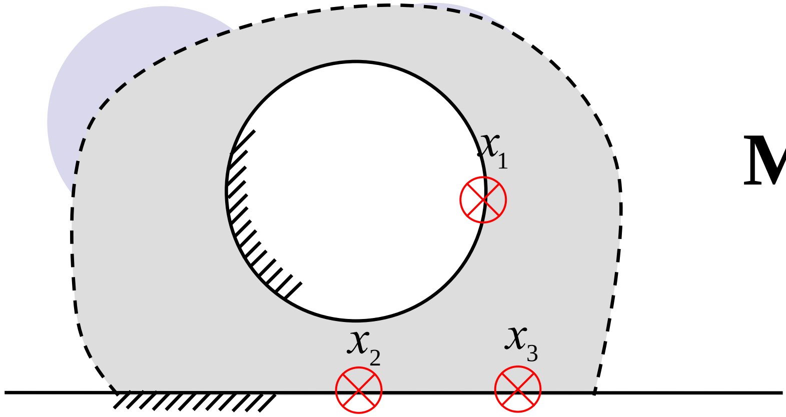
$$\begin{cases} \odot\zeta_2 \Rightarrow \ln R_2 - \sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{a}{R_2}\right)^m \cos m(\psi_2-\phi), & \rho\in B_1 \\ \otimes\zeta_3 \Rightarrow \ln a - \sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{R_3}{a}\right)^m \cos m(\psi_2-\phi), & \rho\in B_1 \end{cases}$$

$$\because R_3 = \frac{a^2}{R_2} \quad \therefore -\ln|x-s_2| + \ln|x-s_3| = \ln a - \ln R_2$$

$$\vdots$$

$$\therefore \ln R_\zeta - \ln a + \ln a + \ln R_2 + ... \Rightarrow e(N)$$

$$G(x,\zeta) = \frac{1}{2\pi}\left\{\ln|x-\zeta| - \lim_{N\to\infty}\left[\sum_{i=1}^{N}\left(\ln|x-\zeta_{4i-3}| + \ln|x-\zeta_{4i-2}| - \ln|x-\zeta_{4i-1}| - \ln|x-\zeta_{4i}|\right) + c(N)\ln|x-\zeta_c| + d(N)\ln|x-\zeta_d| + e(N)\right]\right\}$$

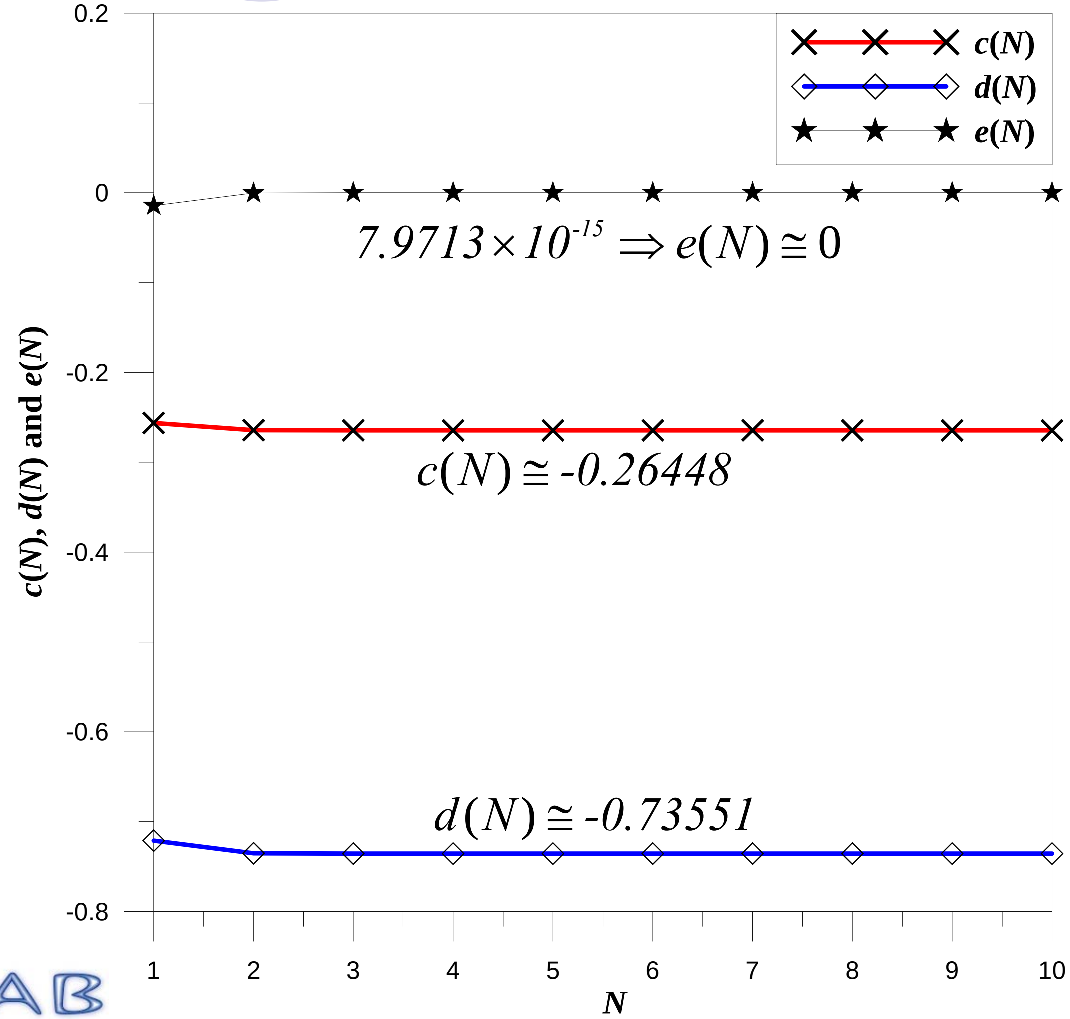


Matching BCs to determine three coefficient

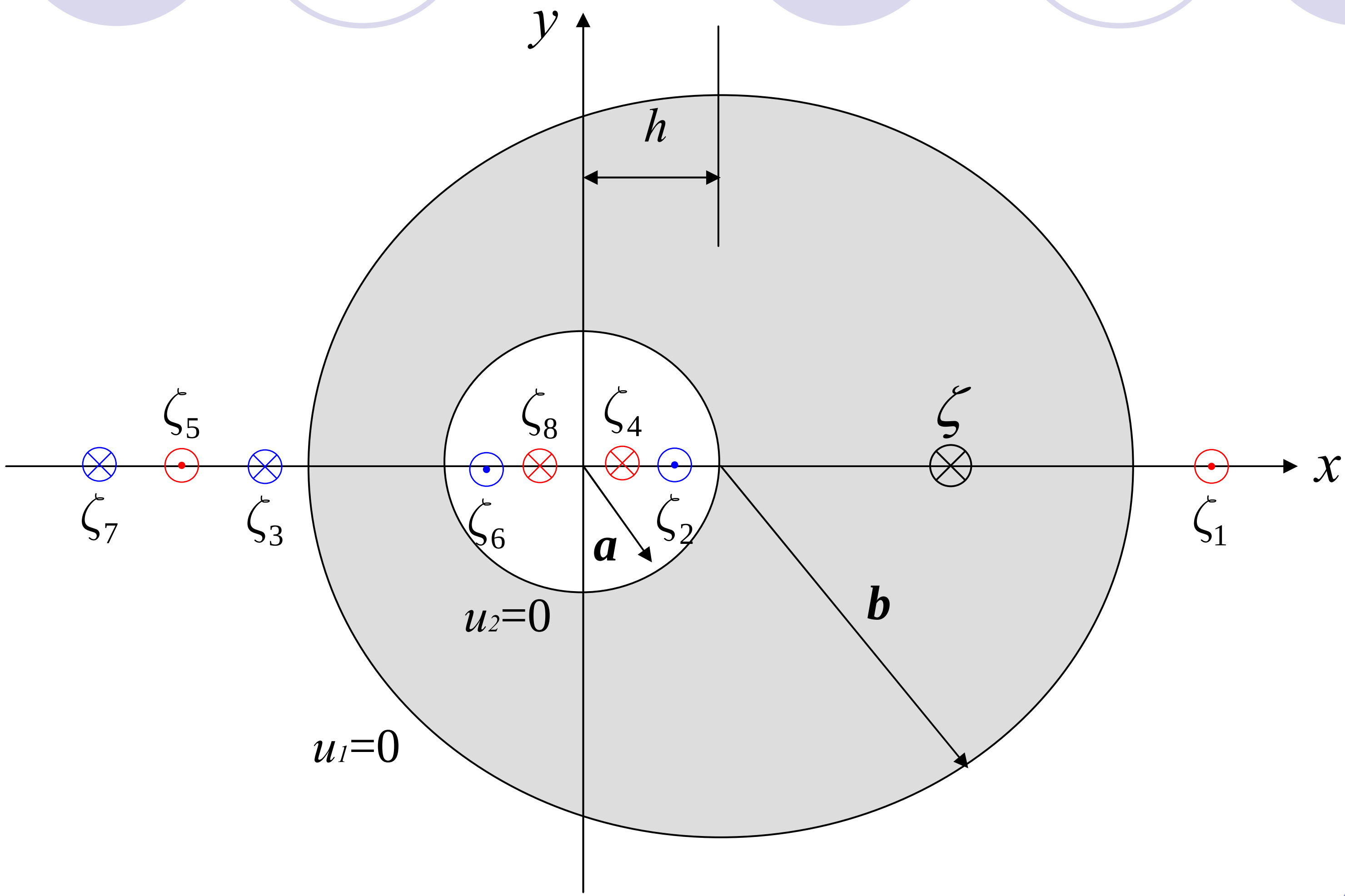
$$\frac{1}{2\pi}\begin{cases} \ln|x_1-\zeta| - \lim_{N\to\infty}\left[\sum_{i=1}^{N}\left(\ln|x_1-\zeta_{4i-3}|+\ln|x_1-\zeta_{4i-2}|-\ln|x_1-\zeta_{4i-1}|-\ln|x_1-\zeta_{4i}|\right)+c(N)\ln|x_1-\zeta_c|+d(N)\ln|x_1-\zeta_d|+e(N)\right]=0 \\ \ln|x_2-\zeta| - \lim_{N\to\infty}\left[\sum_{i=1}^{N}\left(\ln|x_2-\zeta_{4i-3}|+\ln|x_2-\zeta_{4i-2}|-\ln|x_2-\zeta_{4i-1}|-\ln|x_2-\zeta_{4i}|\right)+c(N)\ln|x_2-\zeta_c|+d(N)\ln|x_2-\zeta_d|+e(N)\right]=0 \\ \ln|x_3-\zeta| - \lim_{N\to\infty}\left[\sum_{i=1}^{N}\left(\ln|x_3-\zeta_{4i-3}|+\ln|x_3-\zeta_{4i-2}|-\ln|x_3-\zeta_{4i-1}|-\ln|x_3-\zeta_{4i}|\right)+c(N)\ln|x_3-\zeta_c|+d(N)\ln|x_3-\zeta_d|+e(N)\right]=0 \end{cases}$$

$$\begin{Bmatrix} \ln|x_1-\zeta|-\sum_{i=1}^{N}\left(\ln|x_1-\zeta_{4i-3}|+\ln|x_1-\zeta_{4i-2}|-\ln|x_1-\zeta_{4i-1}|-\ln|x_1-\zeta_{4i}|\right) \\ \ln|x_2-\zeta|-\sum_{i=1}^{N}\left(\ln|x_2-\zeta_{4i-3}|+\ln|x_2-\zeta_{4i-2}|-\ln|x_2-\zeta_{4i-1}|-\ln|x_2-\zeta_{4i}|\right) \\ \ln|x_3-\zeta|-\sum_{i=1}^{N}\left(\ln|x_3-\zeta_{4i-3}|+\ln|x_3-\zeta_{4i-2}|-\ln|x_3-\zeta_{4i-1}|-\ln|x_3-\zeta_{4i}|\right) \end{Bmatrix} + \begin{bmatrix} \ln|x_1-\zeta_c| & \ln|x_1-\zeta_d| & 1 \\ \ln|x_2-\zeta_c| & \ln|x_2-\zeta_d| & 1 \\ \ln|x_3-\zeta_c| & \ln|x_3-\zeta_d| & 1 \end{bmatrix} \begin{Bmatrix} c(N) \\ d(N) \\ e(N) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

Numerical approach to determine $c(N)$, $d(N)$ and $e(N)$



Semi-analytical solution: case 4



MFS-Image group

$\otimes \zeta(R_\zeta + h, \theta)$

$$U(\zeta,x) = \ln b - \sum_{m=1}^{\infty} \frac{1}{m}\left(\frac{R_\zeta}{b}\right)^m \cos\left(m(\theta_\zeta - \phi_1)\right)$$

$$U(\zeta,x) = \ln R_\zeta - \sum_{m=1}^{\infty} \frac{1}{m}\left(\frac{a}{R_\zeta + h}\right)^m \cos\left(m(\theta_\zeta - \phi_2)\right)$$

$\odot \zeta_1(R_1, \theta_1)$

$$U(\zeta_1,x) = \ln R_1 - \sum_{m=1}^{\infty} \frac{1}{m}\left(\frac{b}{R_1}\right)^m \cos\left(m(\theta_1 - \phi)\right)$$

$$U(\zeta_1,x) = \ln R_1 - \sum_{m=1}^{\infty} \frac{1}{m}\left(\frac{a}{R_1}\right)^m \cos\left(m(\theta_1 - \phi)\right)$$

$\odot \zeta_2(R_2, \theta_2)$

$$U(\zeta_2,x) = \ln b - \sum_{m=1}^{\infty} \frac{1}{m}\left(\frac{R_2}{b}\right)^m \cos\left(m(\theta_2 - \phi_2)\right)$$

$$U(\zeta_2,x) = \ln a - \sum_{m=1}^{\infty} \frac{1}{m}\left(\frac{R_2}{a}\right)^m \cos\left(m(\theta_2 - \phi_2)\right)$$

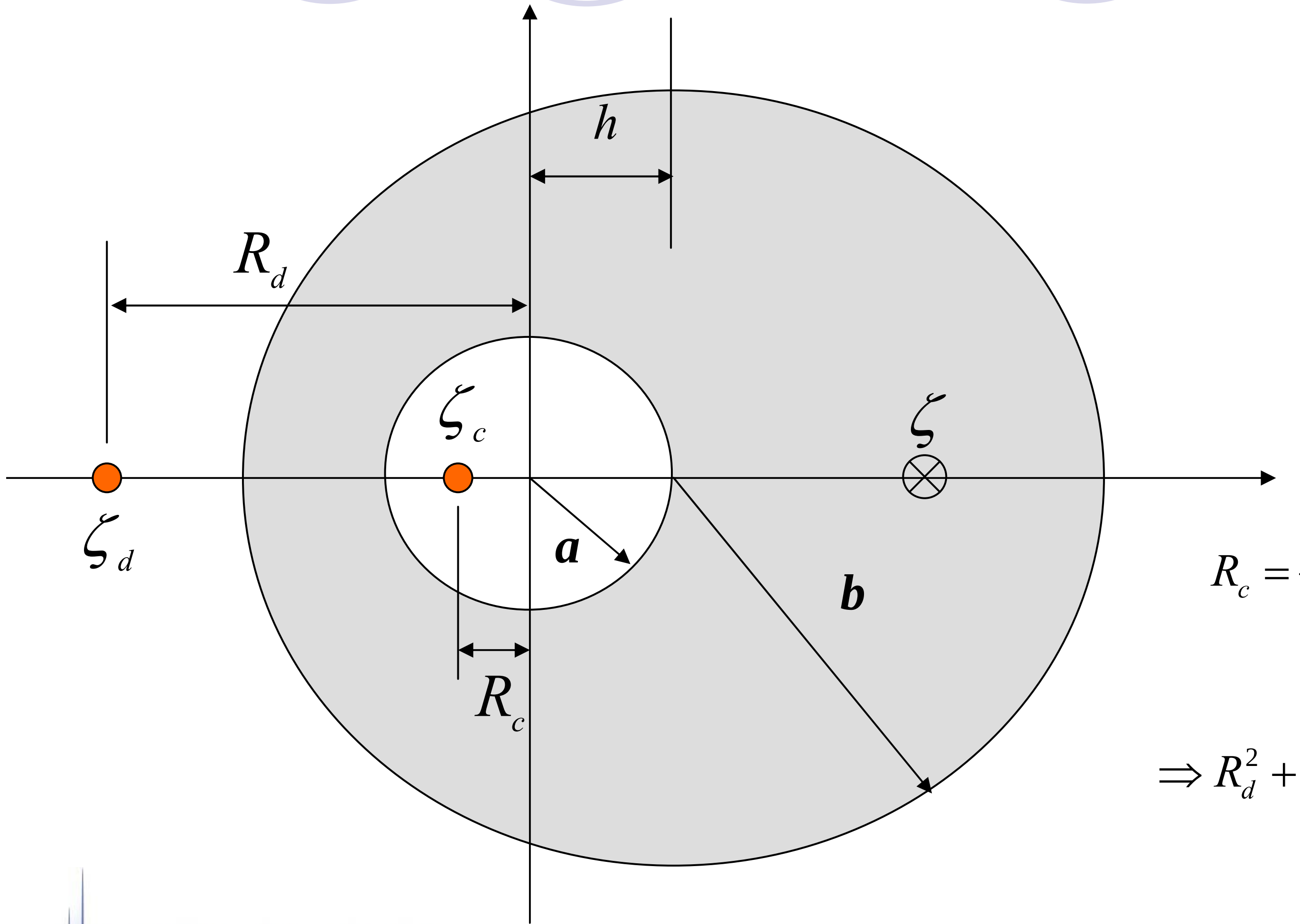
$$R_1 = \frac{b^2}{R_\zeta - h} + h \quad R_5 = \frac{b^2}{h - R_4} - h \quad R_9 = \frac{b^2}{R_8 + e} - e \quad R_{13} = \frac{b^2}{R_{12} + h} - h$$

$$R_2 = \frac{a^2}{R_\zeta} \quad R_6 = \frac{a^2}{R_3} \quad R_{10} = \frac{a^2}{R_7} \quad R_{14} = \frac{a^2}{R_{11}}$$

$$R_3 = \frac{b^2}{h - R_2} - h \quad R_7 = \frac{b^2}{R_6 + h} - h \quad R_{11} = \frac{b^2}{R_{10} + e} - e \quad R_{15} = \frac{b^2}{R_{14} + h} - h \cdots$$

$$R_4 = \frac{a^2}{R_1} \quad R_8 = \frac{a^2}{R_5} \quad R_{12} = \frac{a^2}{R_9} \quad R_{16} = \frac{a^2}{R_{13}}$$

Analytical derivation of location for the two frozen points



$$R_c = \frac{a^2}{R_d},\quad R_d = \frac{b^2}{R_c + h} - h$$

$$\Rightarrow R_d^2 + \frac{(a^2 - b^2 + h^2)R_d}{h} + a^2 = 0$$

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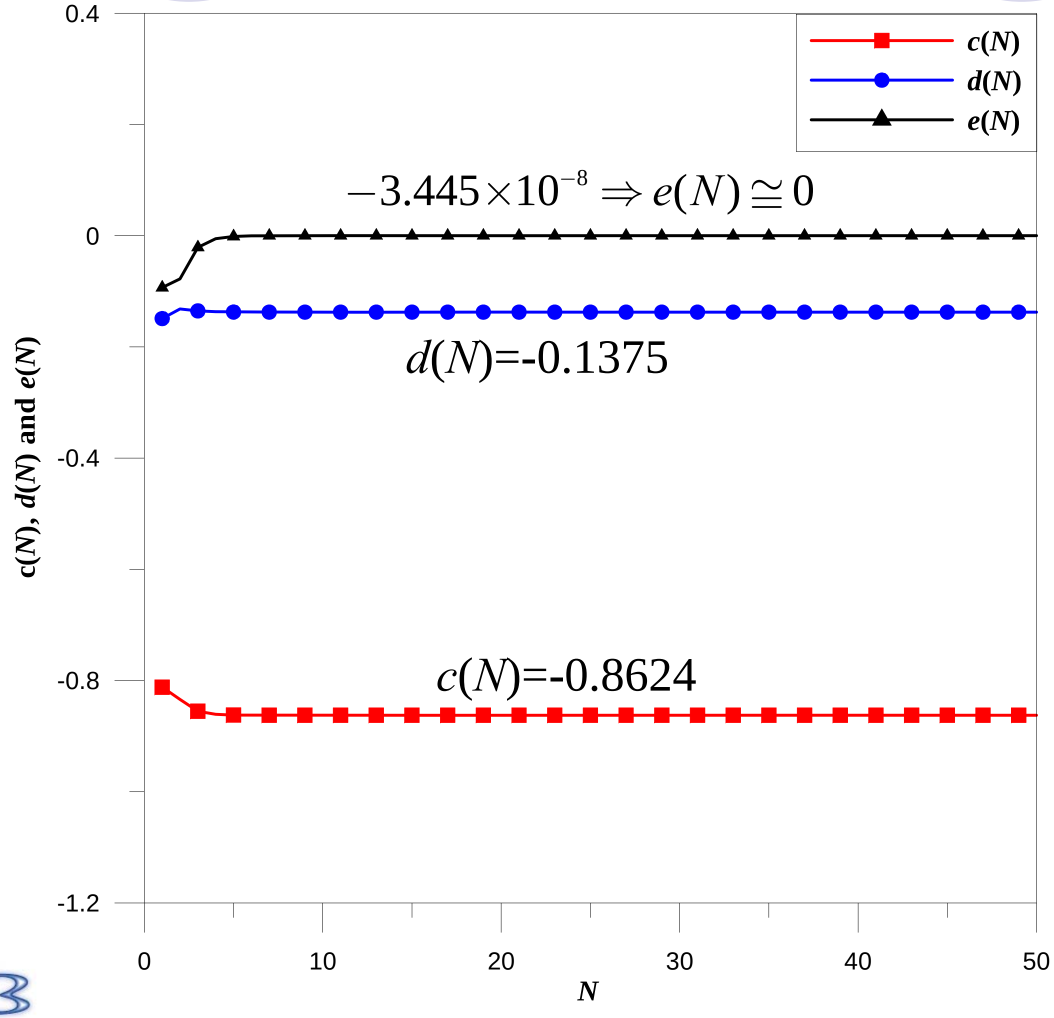
Numerical approach to determine $c(N)$, $d(N)$ and $e(N)$



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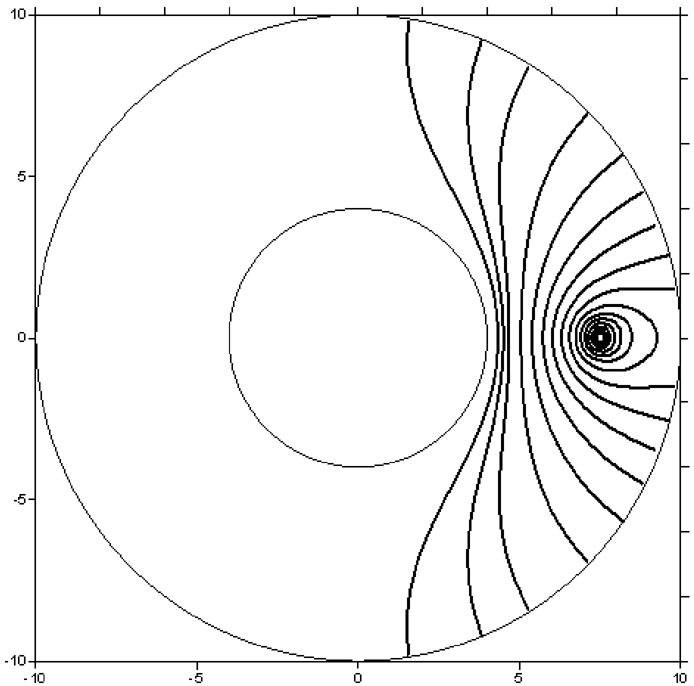
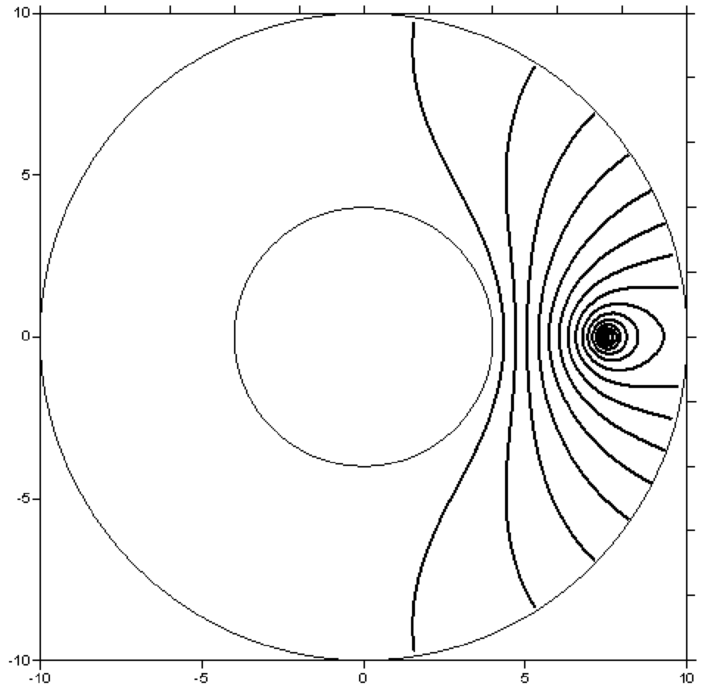
Numerical results: case 1

Fixed-free boundary for the annular case



m=20 (a) Trefftz method



N=20 (b) Image method

Contour plot for the analytical solution (m=N).

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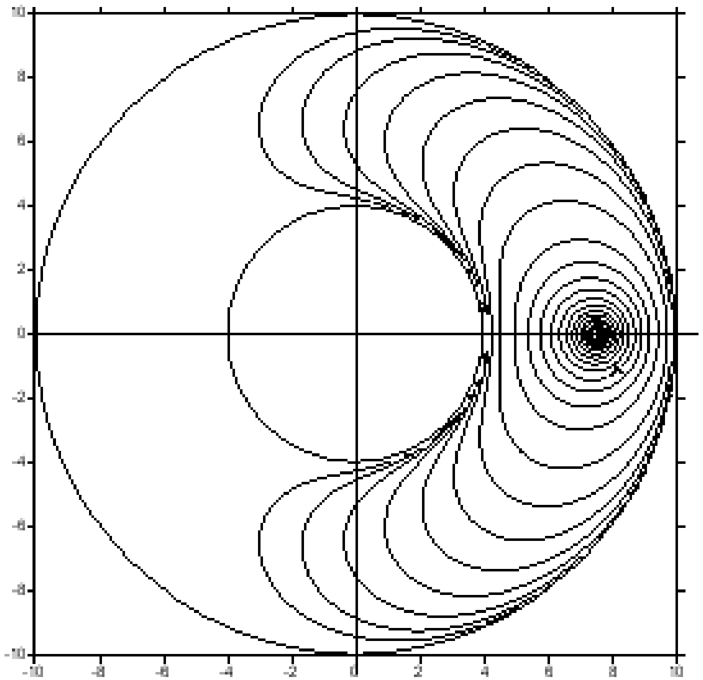
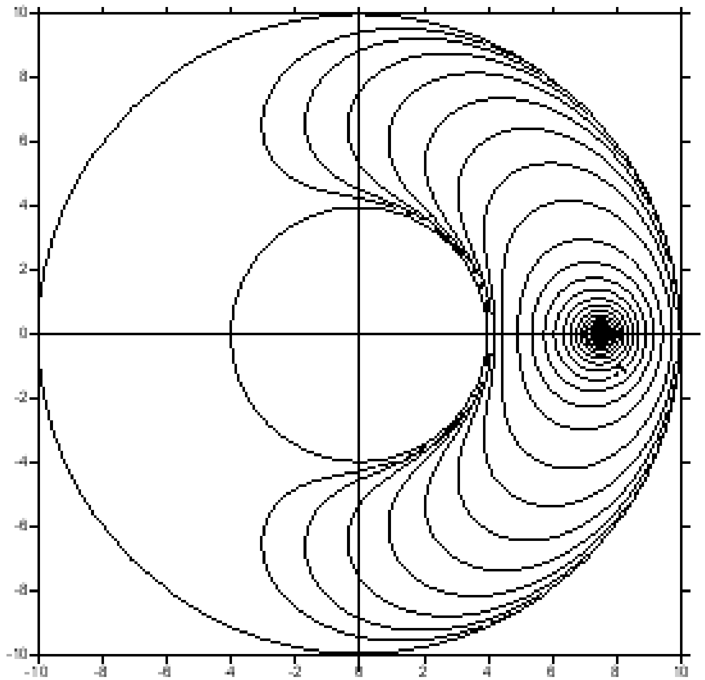
Numerical results: case 2

Fixed-fixed boundary for the annular case



m=20 (a) Trefftz method



N=20 (b) Image method

Contour plot for the analytical solution (m=N).

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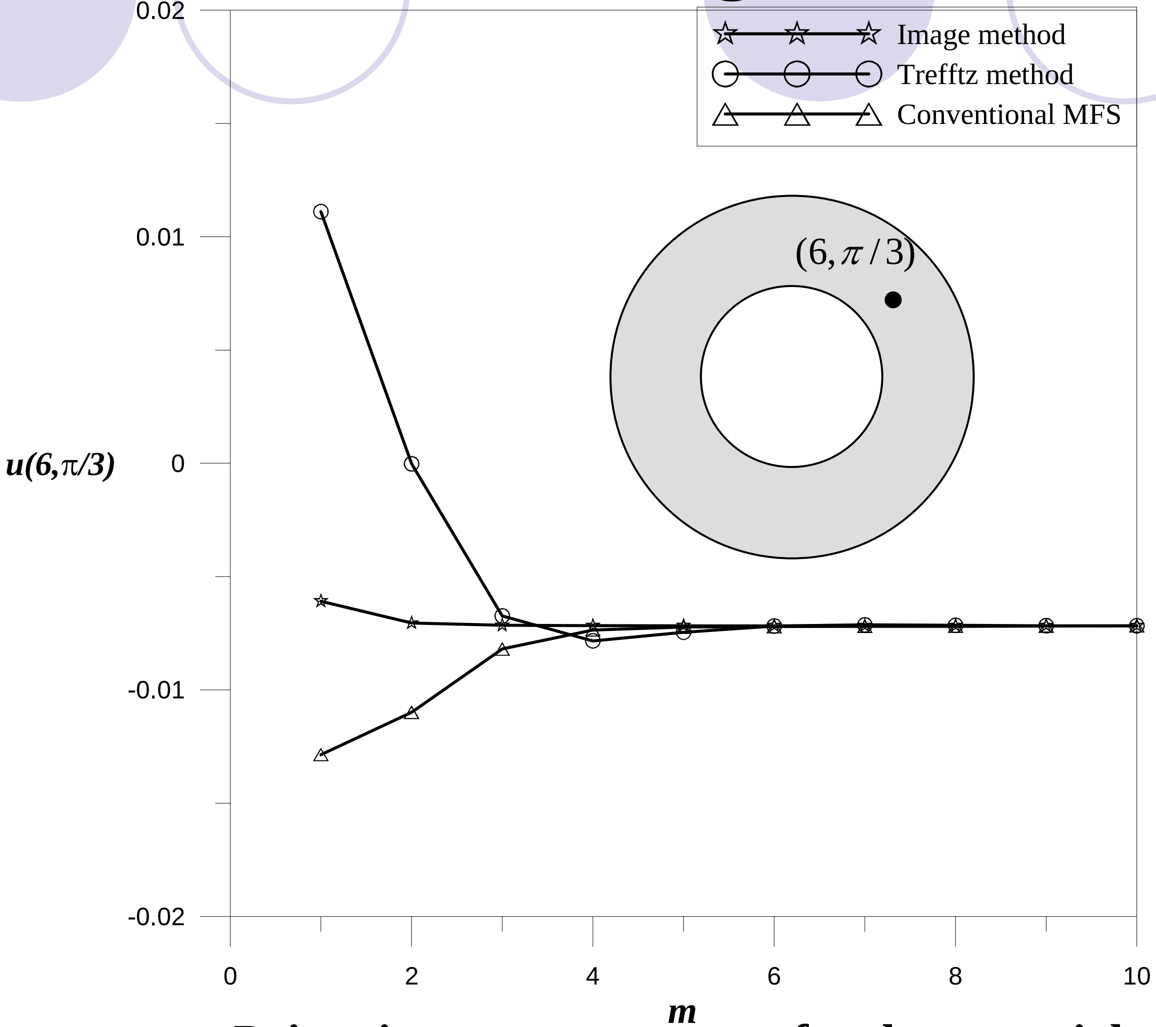
Numerical results: convergence



Pointwise convergence test for the potential by using various approaches.

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Numerical results: convergence rate

Best

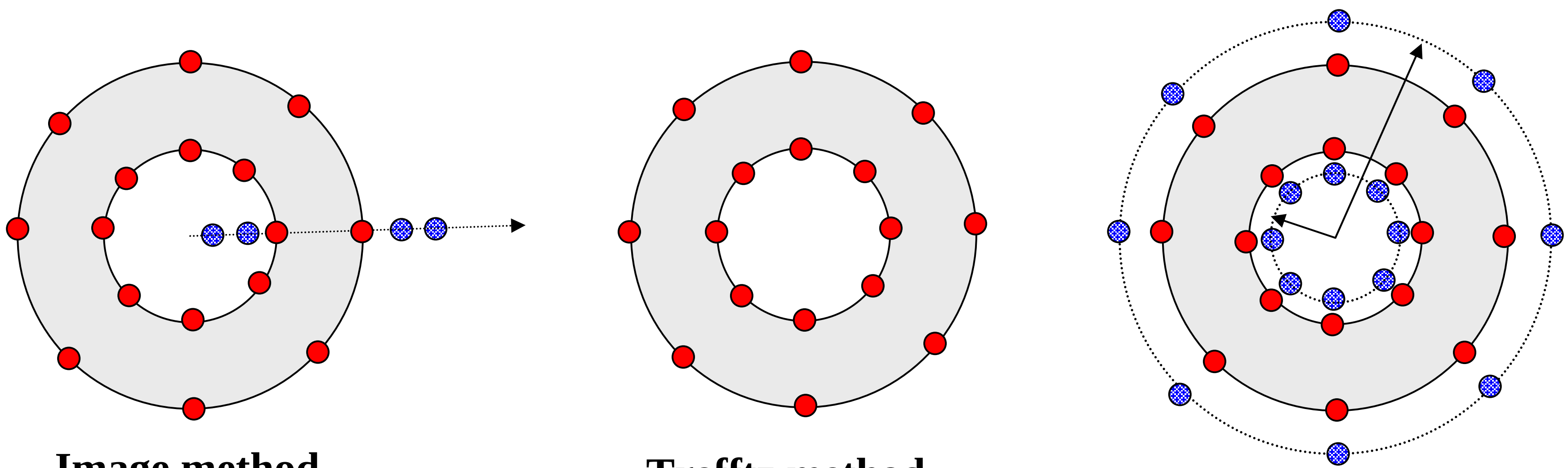


Image method **Trefftz method** **Conventional MFS**

Numerical results: case 3

Dirichlet boundary for the half-plane case

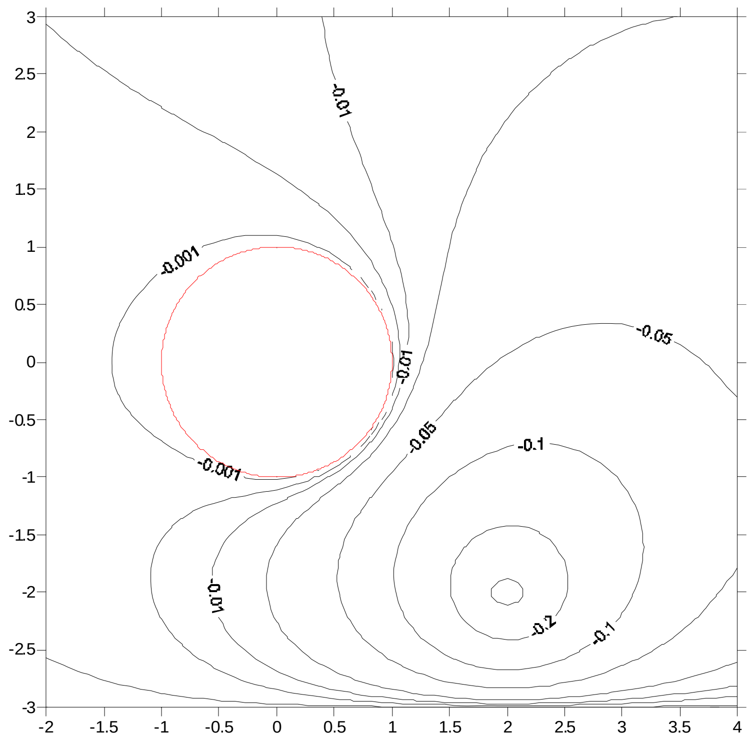
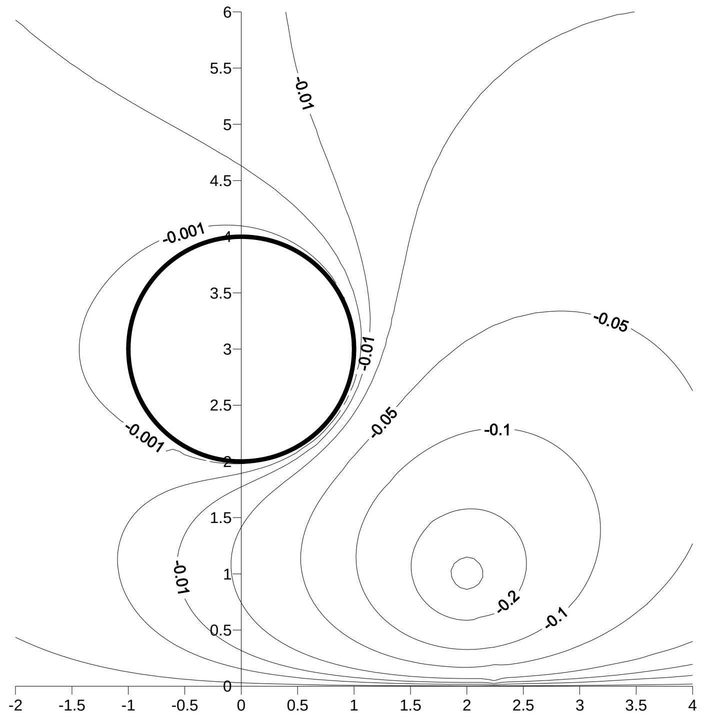


Image method
40+2 points



Null-field BIE approach (addition theorem and superposition technique) (M=50)

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Numerical results: case 4

Dirichlet boundary for the eccentric case

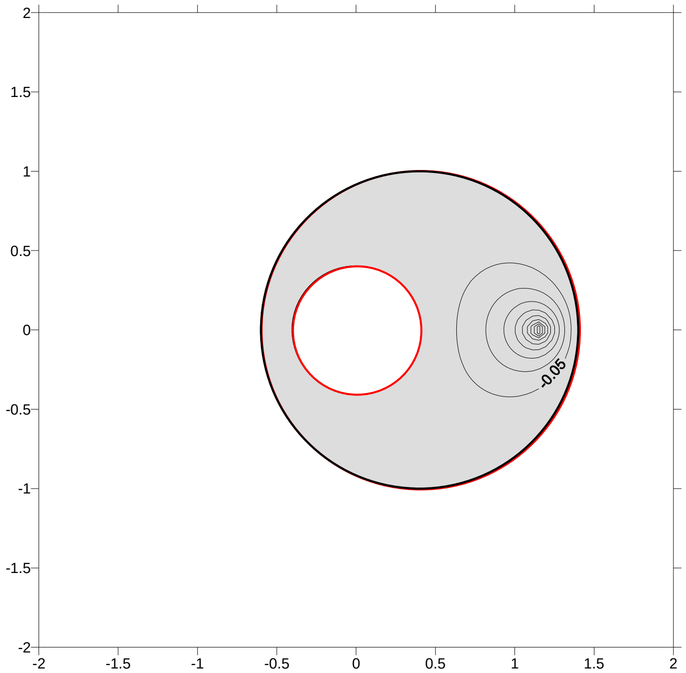
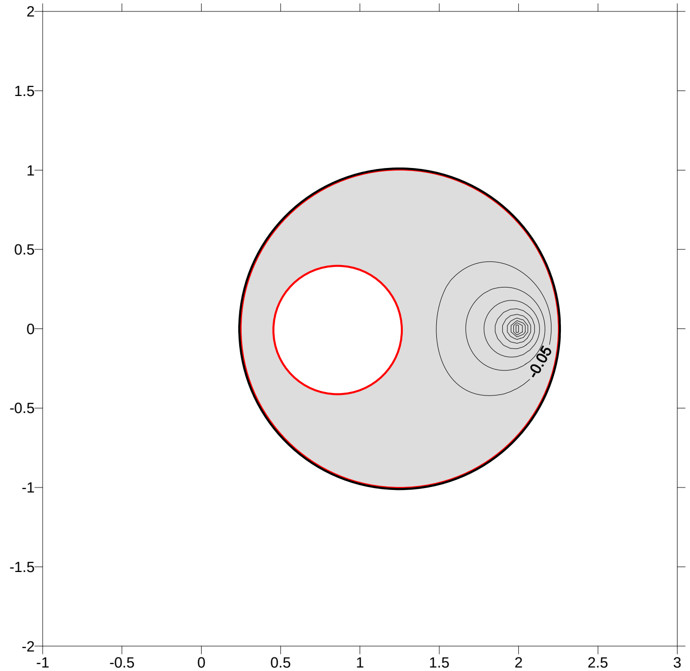


image method



analytical solution

(bi-polar coordinate)

Numerical results: case 5

Dirichlet boundary for the eccentric case

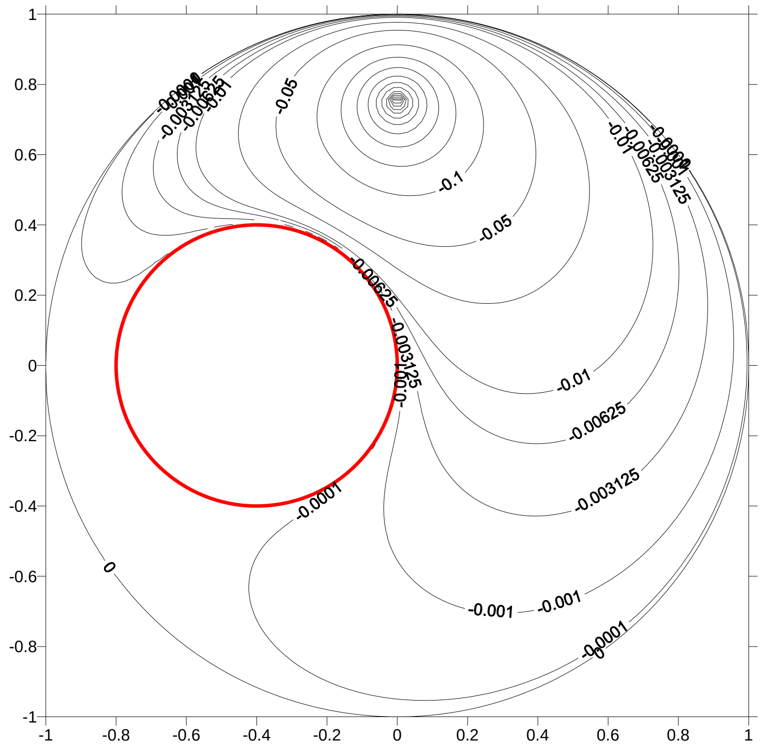
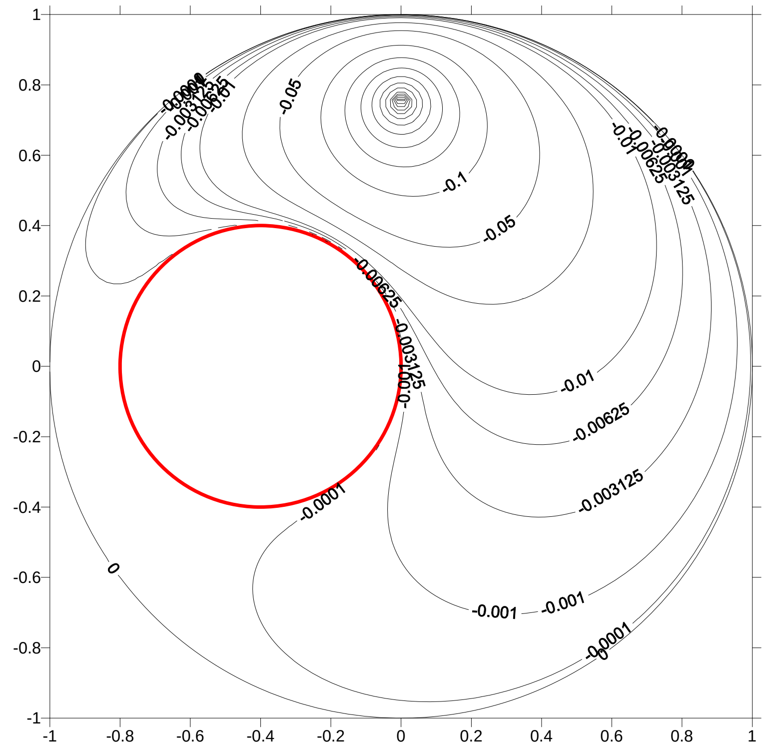


Image method (50+2 point)



Null-field BIE approach (addition theorem and superposition technique) (M=50)

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Conclusions

- The analytical solutions derived by the Trefftz method and MFS were proved to be mathematically equivalent for the annular Green's functions.
- We can find final two frozen image points (one at origin and one at infinity). Their singularity strength can be determined numerically and analytically in a consistent manner.
- The image method can be seen as a special case for method of fundamental solution with optimal locations of sources.

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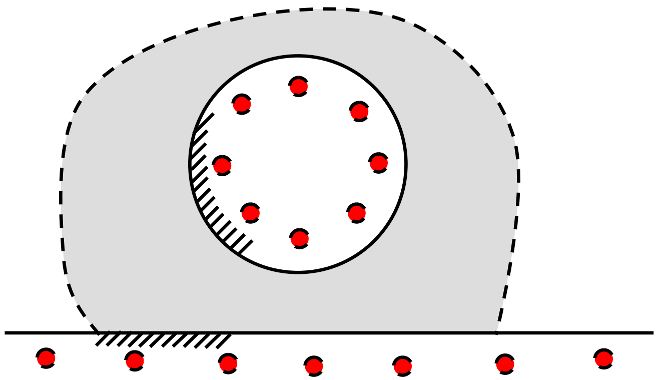
Image method versus MFS

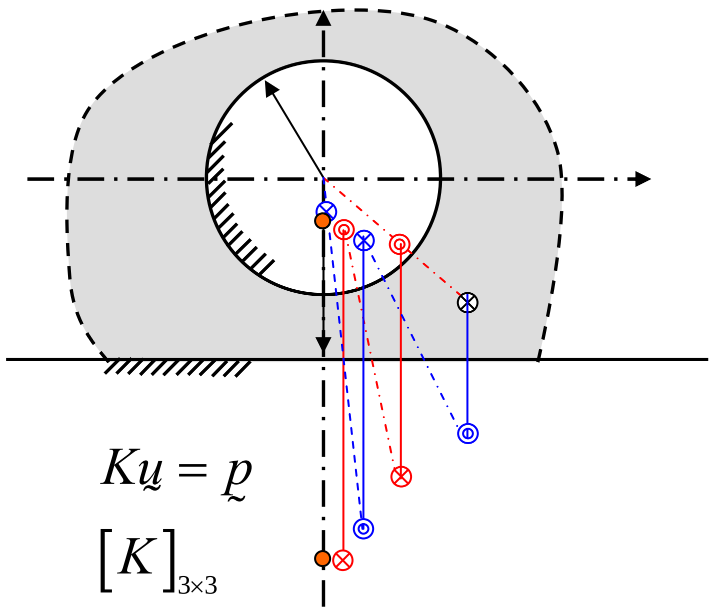


All the strength need to be determined.

$$K\underset{\sim}{u} = \underset{\sim}{p}$$

$$[K]_{N \times N}$$

$$N \to \quad large$$



$$K\underset{\sim}{u} = \underset{\sim}{p}$$

$$[K]_{3\times 3}$$

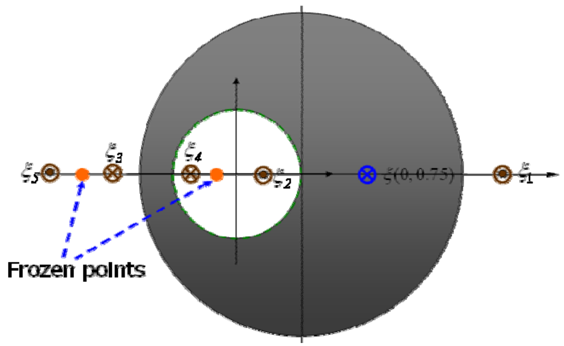
$$optimal$$

Only three coefficients are required to be determined.

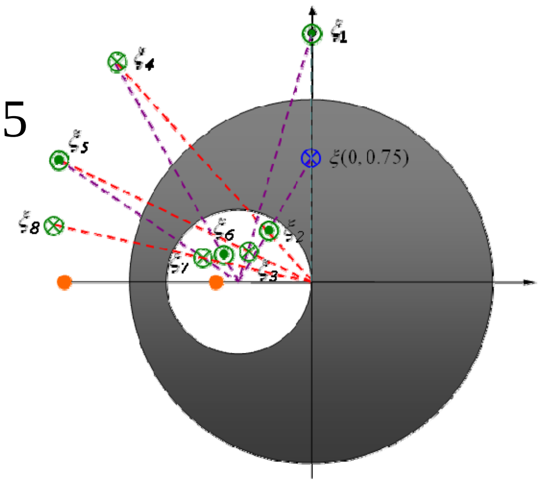
Optimal location of MFS

- Depends on loading

Case 4

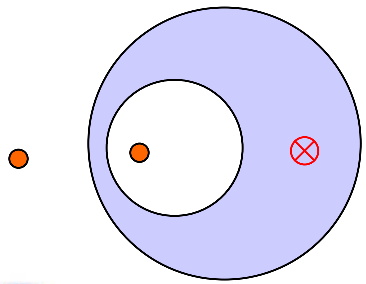


Case 5

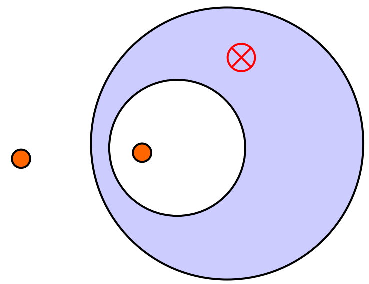


- Depends on geometry

Case 4



Case 5



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- 馬哲儒與虞兆中校長的一則故事
- 莒光日案例
- 帶學生做研究的辛酸與喜樂

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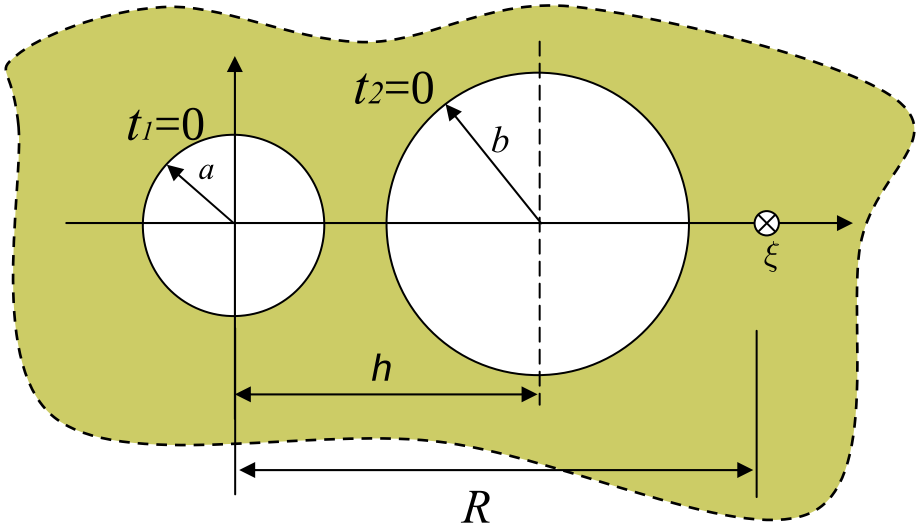
- J.T. Chen, Y.T. Lee, S.R. Yu and S.C. Shieh, Equivalence between Trefftz method and method of fundamental solution for the annular Green's function using the addition theorem and image concept, *EABE*, (2009), in press.
- J.T. Chen, Y.T. Lee and S.C. Shieh, Image method and method of fundamental solution with optimal location of sources, (2009), submitted
- J. T. Chen, K. H. Chou and S. K. Kao, Derivation of Green's function using addition theorem, *MRC*, (2009), in press.
- J.T. Chen, M.H. Tsai and C.S. Liu, Conformal mapping and bipolar coordinate for eccentric Laplace problems, *CAEE* , (2009), Accepted.

Homework (小試身手-海大能，台大一定能!)

Please use the image method to solve the 2-D Laplace problem.



a=0.4

b=1.0

h=2.1

R=3.85

有做出來的同學歡迎寄到我的信箱或MSN

(jtchen@mail.ntou.edu.tw & jt-chen@hotmail.com)

我會義務幫你們核對答案，並為楊老師義務幫你們改作業

NTOU/MSV Lab’s web- http://msvlab.hre.ntou.edu.tw/

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- COE, Japan
- IST, Portugal

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- 終身特聘教授
- 中山科學院 王政盛博士
- 高海大造船系 陳義麟博士
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