APCOM'07 in conjunction with EPMESC XI, *December 3-6, 2007, Kyoto, JAPAN*

Dual BEM since 1986

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Abstract In this talk, the development of dual BEM is reviewed since its appearance in 1986 by Hong and Chen. Roles of hypersigularity are also examined in the computational mechanics. A novel method using the SVD (singular value decomposition) for problems with degenerate boundaries was proposed without employing hypersingular formulation and subdomain approach in 2003. Some traps of BEM in engineering applications, degenerate scale, spurious eigenvalues and fictitious frequencies, are investigated. We provide a perspective on the nonuniqueness and its treatment, including degenerate boundary, degenerate scale, spurious eigensolution, fictitious frequency and corner problems as shown in Table 1, in the boundary integral formulation and boundary element method (BEM). All the nonuniqueness problems originate from the rank deficiency in the influence matrix. Both the Fredholm alternative theorem and SVD technique are employed to study the nonuniqueness problems. Based on the circulant properties and degenerate kernels as analytical tools for circular and annular cases, mathematical analysis can be done. Updating terms and updating documents of the SVD technique are utilized. The roles of right and left unitary vectors of SVD in BEM and their relations to true, spurious and fictitious modes are examined by using the Fredholm alternative theorem. Two methods, CHIEF and CHEEF techniques, for dealing with the nonuniqueness problems in BEM are proposed. Null-fields of nonuniqueness problems are shown in Table 2. The related works since 1984 are shown in Table 3. The three-years project on this topic supported by NSC Wu Ta-You Award is also shown in Table 4. Only key references of NTOU/MSV group are contained here.

Key words: dual BEM, dual BIEM, rank deficiency, degenerate problem, spurious eigenvalue and fictitious eigenfrequency

1. INTRODUCTION

The boundary integral equation method (BIEM) and the boundary element method (BEM) have received much attention since Rizzo proposed a numerical treatment of the boundary integral equation for elastostatics. Most of the efforts have been focused on the singular boundary integral equation for primary fields, e.g. potential or displacement. For most problems, the formulation of a singular boundary integral equation for the primary field provides sufficient conditions to ensure a unique solution. In some cases, e.g., those with Hermite polynomial elements, degenerate boundaries, corners, the construction of a symmetric matrix, the improvement of condition numbers, the construction of an image system, the tangent flux or hoop stress calculation on the boundary, an error indicator in the adaptive BEM, fictitious (irregular) frequencies in exterior acoustics, spurious eigenvalues, degenerate scale, the free surface seepage flow and the Tikhonov formulation for inverse problems, it is found that the integral representation for a primary field (Cauchy integral formula) can not provide sufficient constraints. In other words, the influence matrices are rank deficient. It is well known now that the hypersingular equation plays an important role in the aforementioned problems. Mathematicians and engineers have paid attention to the hypersingular equation from the viewpoint of mathematics and engineering, respectively. A review article by Chen and Hong on hypersingularity can be found. The hypersingular formulation provides the theoretical base of BIE for degenerate boundary problems. Natural boundary element method by Feng and Yu also focused on the hypersingular formulation. Totally speaking, four degenerate problems in BEM, degenerate scale, degenerate boundary, spurious eigenvalues and fictitious frequency, are encountered. In the following, we will first examine the roles of hypersingularity and then review the rank-deficiency problems in a unified manner.

2. ROLES OF HYPERSINGULARITY IN BEM/BIEM

In dual integral equations, hypersingularity is present in the integral equation with the hypersingular kernel. In this section, we discuss why hypersingularity is important in treating certain problems and summarize the roles that it plays in the boundary element methods. One important role of the hypersingular equation is that it can provide additional constraints to ensure a unique solution in the former six items of sections 2.1-2.6. Based on the foundation of potential theory of single, double-layer potentials and their derivatives, the dual frame provide the well-posed mathematical model.

2.1 Higher order element

In order to improve accuracy for a coarse mesh, Watson chose the Hermite cubic element. An immediate gain is the interelement continuity of the first derivative of the primary variable. Another gain is that the smoothness of the density function for the primary variable can improve the condition of existence for the hypersingular integral. Since the number of unknown data doubles at the same time, taking the gradient of the displacement integral equation, which introduces hypersingularity, is necessary.

2.2 Degenerate boundary

In a degenerate boundary problem, the spatial coincidence of the two sides of the degenerate boundary leads to the result that the singular integral equation on one side is indistinguishable from that on the other side where the displacements on the two sides may be different. Although the hypersingular integral equations are different between the two collocation points on the two sides by their corresponding normal vectors, they are dependent since the two normal vectors differ only by a minus sign. To obtain additional independent equations, both singular and hypersingular equations, collocated on the degenerate boundary, are necessary. They were first given the name of the dual (boundary) integral equations for elasticity by Hong and Chen and were later implemented into the BEPO2D program for potential flow. Cruse formulated this degenerate boundary problem in terms of the density functions by using the displacement discontinuity and traction summation on the two sides of the degenerate boundary. Cruse noticed that this formulation introduced double unknowns and then additional equations were required. He went on further to survey the traction BIE. Although Watson proposed another type of additional equation, the kernels that he derived were different from the kernels in the dual boundary integral equations, and the properties of his kernels had not been investigated thoroughly. Chen in 1986 established the unified dual formulation, which incorporates the displacement and traction boundary integral equations under the supervision of Prof. Hong. Besides, Gray independently found the formulation of dual integral equations. Feng and Yu also proposed a similar formulation of natural BEM. A review paper in fracture mechanics using dual BEMs was written by Aliabadi. In mathematical physics, a degenerate boundary is often present, e.g., a cutoff wall in potential flow, a crack problem in elasticity, a thin airfoil in aerodynamics, a baffle in heat conduction, a screen barrier in acoustics, a thin breakwater and a magnetic wave across an antenna. All these problems have been successfully solved by NTOU/MSV group. The dual formulation is closely related to the theory of pseudo-differential operators. The four kernels in the dual formulation obey the property of Calderon projector.

2.3 Degenerate scale

For a plane elasticity, plate and potential problems, the BIE approach and BEM have been shown to yield a nonunique solution when the geometry size is equal to degenerate scale. Not only the simply connected domain but also the multiply connected problems have the transfinite boundaries. By employing the hypersingular formulation, no degenerate scale occurs since a zero eigenvalue is not imbedded in the influence matrix. Other alternatives to avoid the degenerate scale are the addition of rigid body term and CHEEF approach. The degenerate scale (critical value) of the circular and annular cases can be obtained in the continuous and discrete systems by using the degenerate kernel, the Fourier series and the circulant. The degenerate scale depends on the outer boundary of the problems of doubly-connected domain. Mathematically speaking, degenerate scale stems from Γ contour and log capacity.

2.4 Spurious eigenvalues in the dual multiple reciprocity method (MRM) and the real-part BEM for simply-connected eigen problems as well as the complex-valued BEM for multiply-connected eigen problems

The conventional MRM result is spurious eigenvalues. To deal with this problem the dual MRM provides an ideal framework to solve the eigenproblems by using the real-valued computation. To distinguish whether the eigenvalue is true or not, Chen and Wong applied a hypersingular MRM formulation to obtain sufficient constraints for the eigenequations. The dual formulation for the MRM has also been successfully extended to solve the acoustic modes for a two-dimensional cavity with an incomplete partition. The singular value decomposition (SVD) technique can also be used to filter out spurious eigenvalues for an overdeterminate system in the dual MRM. Another advantage by using the SVD for the overdetermined system in the dual MRM was its ability to determine the multiplities of the eigenvalues. Moreover, a series-type complex-valued dual BEM called the complete MRM was derived by Yeih et al. Five methods, the complete MRM, the complex-valued dual BEM, the real-part dual BEM, the imaginary-part dual BEM, and the conventional dual MRM, were summarized in the WCCM 4 keynote lecture by Chen. Regarding to the complex-valued BEM, it is found that spurious eigenvalues also appear for the multiply connected eigen problems. Both the annular membrane and plate as well as eccentric cases were theoretically and numerically demonstrated to show the appearance of spurious eigenvalues which exactly match the true eigenvalue of the domain bounded by the inner boundary. Not only BEM but also method of fundamental solutions (MFS) result in spurious eigenvalues.

2.5 Fictitious frequencies (irregular values)

It is worth noting that the dual integral equations in acoustic applications received much attention earlier than did the Laplace problem since the exterior problem using the singular integral equation with the weakly singular and strongly singular kernels resulted in fictitious eigenfrequencies. Although Schenck found a unique solution by using the CHIEF method, which is generally preferred by the engineering community, this method has limitations. For example, it can not be used to solve the exterior problem with a degenerate boundary (such as a noise barrier) since an interior point is not available. Burton and Miller were first to propose the combined use of dual integral representation for the acoustic problem with all wave numbers. Terai applied dual integral equations to the acoustic problem with the degenerate boundary of a screen. Wu and Wan also applied dual integral equations to the acoustic radiation and scattering problems for thin bodies. For the indirect method, Brakhage and Werner employed the mixed-potential approach. Several researchers have dealt with hypersingularity to overcome the fictitious-frequency problems. The available methods have been summarized from the viewpoint of the dual integral representation. From this viewpoint, the fictitious frequencies depend on the kernel of the integral representation of the solution and on the location where the singularity is distributed. In other words, the boundary condition can not change the position of fictitious frequencies once the integral representation is chosen. To demonstrate that these statements are true, Chen has given three examples for one-, two- and three-dimensional problems by using the indirect method and the direct method. In the three examples, the degenerate kernels in the frequency domain have been employed to represent the potentials in the interior and exterior domains. The analytical results can be obtained and the mechanism of fictitious frequencies can be easily understood upon considering the difference of the stiffness matrix between the exterior and interior problems. Therefore, some misleading comments by Shaw and Rizzo have been corrected. Besides, the mathematical structures of the four matrices, U, T, L and M can be unified by using the SVD. It is found that fictitious modes are imbedded in the left unitary vectors of the influence matrix after using the SVD while true modes are given in the right unitary vectors.

2.6 Corner problem

The corner problem with the Dirichlet boundary condition is another problem in which the number of equations is not sufficient for the conventional BEM. The double-node technique was utilized to tackle this problem. Researchers have tried to find better and additional constraints. Again, the hypersingular integral formulation plays a role in providing independent constraints for the boundary unknowns. For the case that the displacement (or potential) is specified at the corner, the traction (or potential flux) unknowns are doubled due to the different normal vectors. Unfortunately, the singular equations alone can not distinguish the normal vectors of the collocation points at the corner. The second equation of the dual integral representation can be collocated to the points before the corner and after the corner with two different independent normal vectors, causing the equations to be independent. The detailed derivations were derived by Chen and Hong. Therefore, a unique solution can be achieved by balancing the number of equations and unknowns after choosing any two of the three independent equations. The three methods can all match the exact solution well. However, it has been reported that using two hypersingular equations for the two points before and after the corner results in lower accuracy than that by using one singular and one hypersingular equation.

2.7 Adaptive boundary element methods

An essential ingredient for all adaptive boundary element methods is a reliable estimate of the local error. The hypersingular integral equation is a complementary equation available for error estimation. By using this concept, the error indicator can successfully track the form of the exact error curve which is the guide of mesh generation. Error estimation and adaptive BEMs were successfully applied to solve for the Laplace, Helmhotlz and modified Helmholtz equations by NTOU/MSV group.

2.8 Calculation of the tangent flux or the hoop stress on and near the boundary

The hypersingular integral equation can be used to directly calculate the boundary stress instead of using the numerical derivative of the obtained displacement field through the Hooke's law. The tangent derivative along the boundary has been formulated in terms of both the boundary potential and the boundary normal flux. For elasticity problems, Huber et al. have shown that the accuracy of the numerical derivative is lower than that of the direct calculation of the boundary stress by using the hypersingular formulation. Since the integral representation of the solution exhibits the jump behavior across the boundary, the stress or flux near the boundary often displays the Gibbs phenomenon. By using the regularized version of dual integral equations, accuracy near the boundary can be ensured. Numerical examples have been provided by NTOU/MSV group. To avoid the boundary-layer effect, null-field formulation in conjunction with degenerate kernel can well capture the discontinuity across the boundary. This approach can also deal with the nearly singular integrals. Successful applications in the Laplace, Helmholtz, biharmonic and biHelmholtz equations were done by the author’s group.

2.9 Symmetric formulation

In the coupled use of FEM and BEM, the symmetry requirement of the stiffness matrix is especially useful. The four kernel functions in the dual integral equations display the elegant structure of potential theory. The symmetry and transpose symmetry properties for the four kernel functions have been found by Hong and Chen. The dual integral representations can be used to assemble the four kernel functions of the dual integral equations into a global symmetric matrix by using the symmetry and transpose symmetry properties of the kernel functions. In order to establish the symmetry for the interpolation function, the quadratic energy form of double integration was needed in the symmetric-Galerkin formulation by Shiau, Chiu, Bonnet, Kane, Parreira, and Sirtori. The numerical experiment has been perormed successfully by Shiau and Chiu. However, all the symmetric formulations in the literature need double boundary integrations are time-consuming. For reduction to a single boundary integration, degenerate kernels can be employed. Construction of symmetric matrices has been investigated in more detail in Italy.

2.10 Improvement of condition numbers

In the dual integral representation, the potentials resulted from integrating the U and M kernels are continuous when the field point moves across the boundary, while those from integrating the T and L kernels show the jump behavior. The jump terms make the [T] and [L] matrices diagonally dominant and preferably lower their condition numbers. For the case of the Dirichlet problem, an inversion of the [U] matrix is needed when the first equation of the dual integral representation is considered. If we adopt the second equation, an inversion of the [L] matrix is preferred since it is more well-conditioned. From the viewpoint of the pseudo-differential operators, the orders of T and L kernels are of zero order, which are numerically stabler than the U kernel of order minus one and the M kernel of order one. This agrees with the above statement that the inversion of a matrix with diagonal dominance is numerically stable. Also, the properties of Caleron projector can be found.

2.11 The Tikhonov formulation for inverse problems with overspecified boundary conditions

In solving an ill-posed inverse problem with overspecified boundary conditions by the Tikhonov formulation, double boundary integrals occur naturally. The inner integrals in the double integrals are hypersingular. To avoid hypersingularity, Yeih, Koya and Mura employed a fictitious BEM to deal with the inverse problem. Fictitious BEM proposed by Yeih et al. is not absolutely necessary since the hypersingular integral can be evaluated by using the regularization techniques.

2.12 Free-surface problems

The free-surface problem can be treated as a moving boundary problem with overspecified boundary conditions. An iterative scheme for the free-surface seepage was proposed by Niwa using the conventional BEM. By employing the hypersingular integral equation, the rate of convergence can be accelerated. The number of iteration using the hypersingular equation is lower than that using the singular formulation.

2.13 Construction of the image system

In the half-plane, half-space, quarter-plane or quarter space problems, special Green's functions subjected to certain boundary conditions are often used as auxiliary systems to establish integral equations which can eliminate integrations on the rectilinear or plane boundaries such as the ground surface. Conventionally, we always locate the source outside the domain to adjust for the satisfaction of boundary conditions. Based on the physical meaning of the dual integral equations in potential theory, an additional degree of freedom for adjusting is available, which is the normal vector of the dipole or dislocation source. Illustrative examples have been given in the book by Chen and Hong.

3. RANK DEFICIENCY IN BEM/BIEM

3.1 Degenerate boundary in boundary value problems

For the problem with a degenerate boundary, the dual integral representation has been proposed for crack problems in elasticity by Hong and Chen, and boundary element researchers have increasingly paid attention to the second equation of the dual representation. The second equation, which is derived for the secondary field, e.g., flux or traction, is very popular now and is termed the hypersingular boundary integral equation. Hong and Chen presented the theoretical bases of the dual integral equations and a general formulation which incorporates the displacement and traction boundary integral equations. Huang and So extended the concept of he Hadamard principal value in the dual integral equations to determine the dynamic stress intensity factors of multiple cracks. Gray also independently found the hypersingular integral representations for the Laplace equation and the Navier equation although he did not coin the formulation “dual”. Martin, Rizzo and Gonsalves called the new kernel in the dual integral equations “hypersingular” while Kaya earlier called the kernel “superstrong singularity”. Since the formulation was derived for the secondary field, by analogy with the term “natural boundary condition”, Feng and Yu called the method ``natural BEM" or ``canonical integral equations". Balas, Sladek and Sladek in their book proposed a unified theory for crack problems by using the displacement boundary integral equation and another integro-differential equation for the traction field. Based on the dual integral representation for the degenerate boundary problems, Hong and Chen developed the dual BEM programs for crack and potential flow problems with a cutoff wall. Besides, Chen and his coworkers extended the dual BEM program for the Laplace equation and the Navier equation to three programs. One is for the Helmholtz equation by the dual MRM. Another is for the Helmholtz equation by the complex-valued formulation.The other is for the modified Helmholtz equation. A general purpose program, BEASY, was developed for crack problems by the Wessex Institute of Technology (WIT) and termed the ``dual boundary element method (DBEM)" . This program has been extended to solve crack growth problems more efficiently by using the benefit of the single-domain approach. Chen and Hong, Mi and Aliabadi extended two-dimensional cases to three-dimensional crack problems. A program implemented by Lutz et al. was also reported. In the mathematical literature, the relationships between the boundary integral operators and various layer potentials are obtainable through the so-called Calderon projector. Four identities to relate the four kernels have been constructed. The order of pseudo-differential operator for the integral equations on the circular case in the dual formulation was discussed by Amini, Chen and Chiu. These mathematical problems were first studied by Hadamard and Mangler. The hypersingular integral equation was derived by Hadamard in solving the cylindrical wave equation by employing the spherical means of descent. The improper integral was then defined by Tuck as the ``Hadamard principal value''. Almost at the same time of Hadamard's work, Mangler derived the same mathematical form in solving a thin airfoil problem. This is the reason why the improper integral of hypersingularity is called the ``Mangler principal value'' in theoretical aerodynamics. This nonintegrable integral of hypersingularity arises naturally in the dual boundary integral representations especially for problems with degenerate boundaries, *e.g.*, crack problems in elasticity, heat flow through a baffle, Darcy flow around a cutoff wall, a cracked bar under torsion, screen impinging in acoustics, antenna in electromagnetic wave, a thin breakwater and aerodynamic problems of a thin airfoil. Applications of the hypersingular integral equation in mechanics were discussed by Martin et al. and by Chen and Hong. Combining the singular integral equation, *e.g.*, Green's identity (scalar field) or Somigliana's identity (vector field), with the hypersingular integral equation, we can construct the dual integral equations according to the continuous and discontinuous properties of the potential as the field point moves across the boundary. From the above point of view, the definition of the dual (boundary) integral equations is quite different from that of the dual integral equations given by Sneddon and Lowangrub and Buecker, which, indeed, come from the same equation but different collocation points in crack problems of elastodynamics. The solution for the conventional dual integral equations was first studied by Beltrami. The dual boundary integral equations for the primary and secondary fields defined and coined by Hong and Chen are generally independent of each other, and only for very special cases are they dependent. To deal with the degenerate boundary problems, the hypersingular formulation is a powerful method in conjunction with the dual BEM. However, regularization for hypersingularity is required. To avoid hypersingularity, one alternative has been proposed by using the multi-domain approach of singular equation in sacrifice of introducing artificial boundary where the continuity and equilibrium conditions on the interface boundary are considered to condense the matrix. We may wonder whether it is possible to solve the degenerate problems by using only the singular equation in the single-domain approach. The SVD technique was successfully considered to achieve the goal in 2003 by Chen et al..

3.2 Degenerate scale for 2-D Laplace, biharmonic and Navier problems

It is well known that rigid body motion test or so called use of simple solution can be employed to examine the singular matrices in BEM for the strongly singular and hypersingular kernels in the problems without degenerate boundaries. Zero eigenvalues associated with rigid body modes are imbedded in the corresponding influence matrices. In such a case, singular matrix occurs physically and mathematically. The nonunique solution for a singular matrix is found to include a rigid body term for the interior Neumann (traction) problem. However, for a certain geometry, the influence matrix of the weakly singular kernel may be singular for the Dirichlet problem or displacement specified problem. In other words, the numerical results may be unstable when the used scale is changed or the considered domain is expanded to a special size. The nonunique solution is not physically realizable but results from the zero eigenvalue of the influence matrix in the BEM. The special geometry dimension which results in a nonunique solution for a potential problem is called a degenerate scale by He and Chen et al.. The term ``scale'' stems from the fact that degenerate mechanism depends on the geometry size used in the BEM implementation. Some mathematicians coined it a critical value (C.V.), logarithmic capacity, Gamma contour and transfinite boundary since it is mathematically realizable. For several specific boundary conditions, some studies for potential problems (Laplace equations), plate problems (biharmonic equations) and plane elasticity problems (Navier equation) have been done. The difficulties due to nonuniqueness of solutions were overcome by the necessary and sufficient boundary integral formulation and boundary contour method. The degenerate scale problems in the BEM have been studied analytically by Kuhn and Constanda and numerical experiments have been performed. Degenerate kernels and circulant matrices were employed to determine the eigenvalues for the influence matrices analytically in a discrete system for circular and annular problems. The singularity pattern distributed along a ring boundary resulting in a null-field can be obtained when the ring boundary is a degenerate scale. An annular region has also been considered for the harmonic equation and the biharmonic equation and the possible degenerate scales were investigated. Hypersingular formulation is an alternative to study the degenerate scale problems for simply-connected problems, since eigenvalues are never zero. Another simple approach is to superimpose a rigid body motion in the fundamental solution so that the zero eigenvalue can be shifted to be nonzero. However, this treatment results in another degenerate scale. By employing the CHIEF concept, a CHEEF approach was developed to obtain the independent constraint. A unified method was proposed to study the problem by using the Fredholm alternative theorem and the SVD updating technique. Both the spurious mode (mathematically realizable) and rigid body mode (physically realizable) can be determined. The left and right unitary matrices in the SVD for BEM are found to be the true and spurious modes. In addition, a direct treatment in the matrix operation instead of adding a rigid body term in the fundamental solution can be derived. The general proof of degenerate scale for eccentric Laplace problems was obtained to show the outer radius of one is the critical value.

3.3 Spurious eigensolutions for interior eigenproblems

For interior problems, eigendata are very important informations in vibrations and acoustics. According to the complex-valued boundary element method, the eigenvalues and eigenmodes can be determined. Nevertheless, complex arithmetic is required. To avoid complex arithmetic, many approaches including the multiple reciprocity method (MRM), the real-part and the imaginary-part BEMs have been proposed. For example, Tai and Shaw employed only real-part kernel in the integral formulation. A simplified method using only the real-part or imaginary-part kernel was also presented by De Mey, Yasko and Hutchinson. Although De Mey found that the zeros for a real-part of the complex determinant may be different from the determinant by using the real-part kernel, the spurious eigensolutions were not discovered analytically. Chen and Wong and Yeih et al. found the spurious eigensolutions analytically in the MRM by using simple examples of rod and beam, respectively. Later, Kamiya et al. and Yeih et al. independently claimed that MRM is no more than the real-part BEM. Kang et al. employed the Nondimensional Dynamic Influence Function method (NDIF) to solve the eigenproblem. Chen et al. commented that the NDIF method is a special case of imaginary-part BEM. Kang and Lee also found the spurious eigensolutions and filtered out the spurious eigenvalues by using the net approach. Later, they extended to solve plate vibration problems. Chen et al. proposed a double-layer potential approach to filter out the spurious eigenmodes. The reason why spurious eigenvalues occur in the real-part BEM is the loss of the constraints, which was investigated by Yeih et al.. The spurious eigensolutions and fictitious frequencies arise from an improper approximation of the null space operator. The fewer number of constraint equations makes the solution space larger. Spurious eigensolutions were also found in the Maxwell equation. The spurious eigensolutions can be filtered out by using many alternatives, e.g., the complex-valued BEM, the domain partition technique, the dual formulation in conjunction with the SVD updating techniques and the CHEEF(Combined Helmholtz Exterior integral Equation Formulation) method. Besides, the spurious eigensolution for the multiply-connected problem was found even though the complex-valued kernel was used. A unified formulation to study the phenomenon was proposed by using the Fredholm alternative theorem and the SVD technique. The SVD updating techniques in conjunction with the dual formulation was employed to sort out the true and spurious eigenvalues. In addition, the relation between the left unitary vector in the SVD and the spurious mode was constructed by NTOU/MSV group.

3.4 Fictitious frequency in exterior acoustics

For exterior acoustics, the solution to the boundary is unique for all wave numbers. This is not the case for the numerical treatment of integral equation formulation, which breaks down at certain frequency known as irregular frequency or fictitious frequency. This problem is completely nonphysical because there are no discrete eigenvalues for the exterior problems. It was found that the singular (UT) equation results in fictitious frequencies which are associated with the interior eigenfrequency of the Dirichlet problems while the hypersingular (LM) equation produces fictitious frequencies which are associated with the interior eigenfrequency of the Neumann problems. The general derivation was provided in a continuous system, and a discrete system was analytically studied by using the properties of circulant for a circular case. Schenck proposed a CHIEF (Combined Helmholtz Interior integral Equation Formulation) method, which is easy to implement and is efficient but still has some drawbacks. Burton and Miller proposed an integral equation that was valid for all wave numbers by forming a linear combination of the singular integral equation and its normal derivative through an imaginary constant. In case of a fictitious frequency, the resulting coefficient matrix for the exterior acoustic problems becomes ill-conditioned. This means that the boundary integral equations are not linearly independent and the resulted matrix is rank deficient. In the fictitious-frequency case, the rank of the coefficient matrix is less than the number of the boundary unknowns. The SVD updating technique can be employed to detect the possible fictitious frequencies and modes by checking whether the first minimum singular value, is zero or not.

By employing the Fredholm alternative theorem and the SVD updating technique, the degenerate mechanism for the four numerical problems, degenerate boundary, degenerate scale, spurious eigenvalues and fictitious frequencies, were studied in a unified manner. This formulation was employed to solve for rank-deficiency problems. Extension to multiple scattering was done by NTOU/MSV group.

4. MATHEMATICAL TOOLS

Mathematical tools including the hypersingular equation, pseudo-differential operator, Calderon projector, degenerate kernel, circulant, Fourier series, Fredholm alternative theorem, singular value decomposition and techniques of updatig term and updating document were required in the unified formulation. For more detail, please visit the web site http://ind.ntou.edu.tw/~msvlab.

5. CONCLUSIONS

In the past twenty years, the dual BEM as well as the hypersingularity have been well developed in both theoretical and applied aspects. More than one thousand papers on this topic have been published. Rank deficiency occurring in the four degenerate problems (degenerate boundary, degenerate scale, spurious eigenvalue and fictitious frequencies) in the BEM were reviewed. Besides, the roles of hypersingularity in BEM/BIEM were examined. A unified formulation to study rank-deficiency problems in the BEM was proposed. Mathematical tools including degenerate kernel, Fourier series, circulant, SVD updating techniques were utilized. Spurious mode and true mode were found to be imbedded in the left and right unitary vectors of the influence matrices, respectively. Fredholm alternative theorem was adopted to obtain the updating documents in the SVD. Numerical examples have been demonstrated to check the validity of the unified formulation by NTOU/MSV Group.

ACKNOWLEDGEMENTS

The author is grateful for the continuous support of National Science Council, Taiwan. Special thanks are for the Life-time Distinguished Professor, Hong H.-K., the author's colleagues and former students for their contribution during this work.

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Table 1 Nonuniqueness and its treatment in BEM

Nonuniqueness in BEM	Treatment
Degenerate boundary	Dual BEM (hypersingularity) Multi-domain BEM Conventional BEM + SVD
Degenerate scale	Addition of rigid body term Hypersingular formulation CHEEF method SVD updating technique Pseudo-inverse
Fictitious frequency	Burton and Miller approach CHIEF method SVD updating technique GSVD
Spurious frequency	Burton and Miller method SVD updating technique Dual BEM Subdomain method GSVD CHEEF method (simply-connected) CHIEF method (multiply-connected)
Corner singularity	Double node technique Hypersingular formulation

Table 2 Null fields in the nonuniqueness problem using BEM and BIE

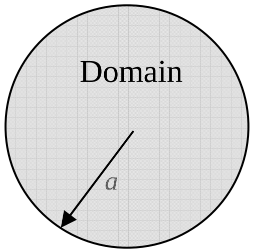
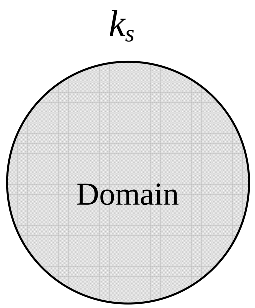
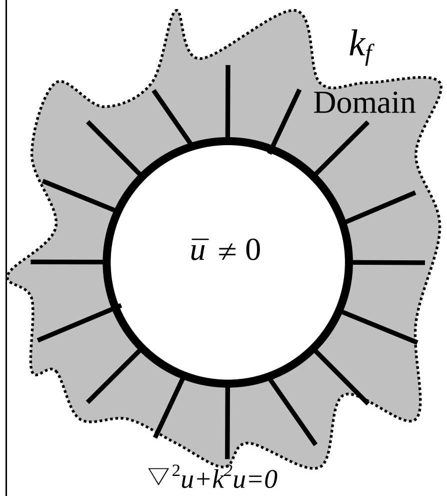
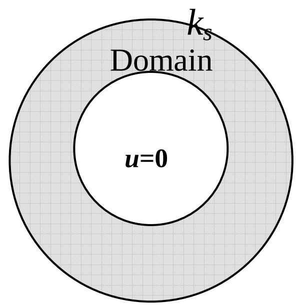
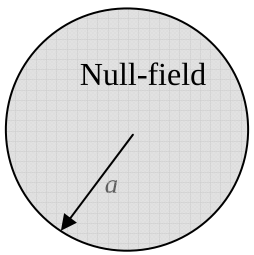
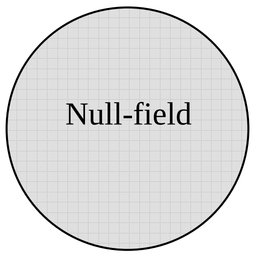
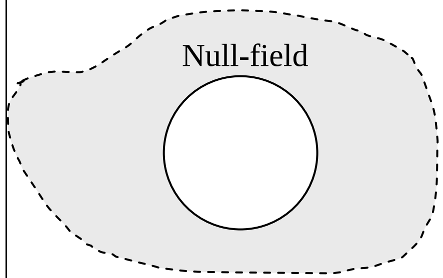
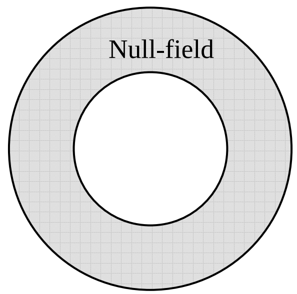
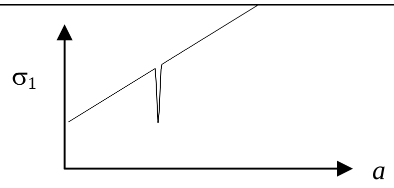
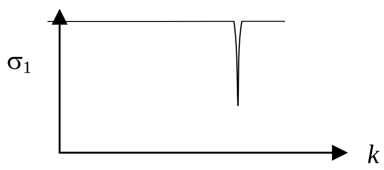
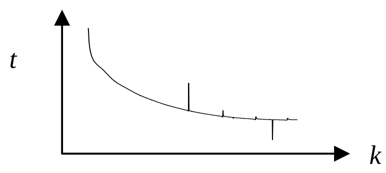
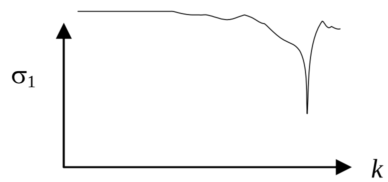
	Degenerate scale (potential problem)	**Spurious eigenvalues** (interior acoustics)	**Fictitious frequencies** (exterior acoustics)	**Spurious eigenvalues** (multiply-connected acoustics and plate)
Figure sketch	 **Special scale** **(*a*=1)**	 **Simply-connected domain** $\nabla^2 u = -k^2 u$		 $\nabla^2 u = -k^2 u$ (*acoustics*) $\nabla^4 u = \lambda^4 u$ (*plate*)
Null-field				
Rank deficiency				

Table 3 Research topics of NTOU / MSV LAB (1984-2007)

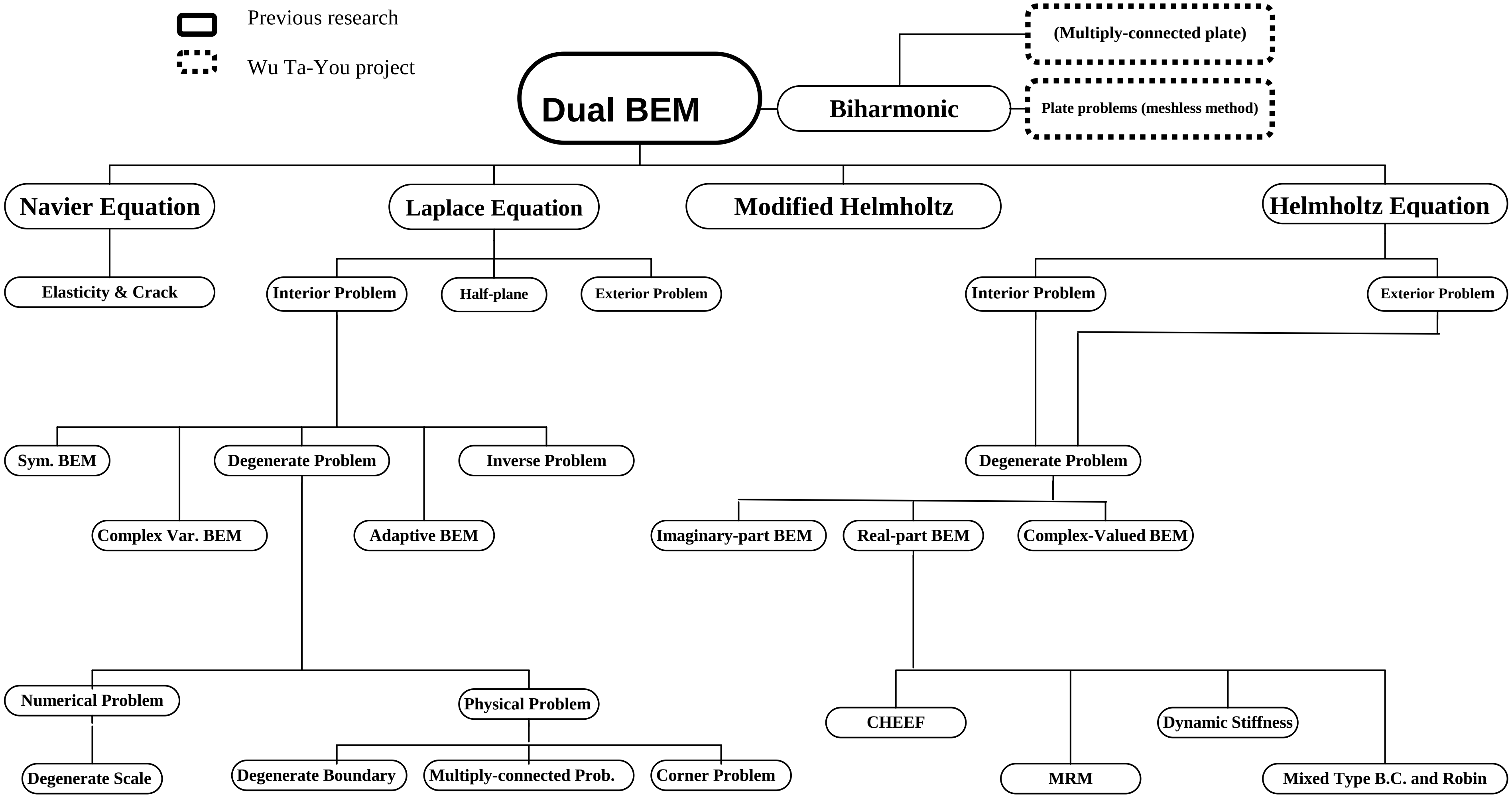
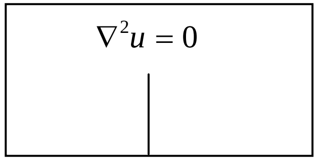
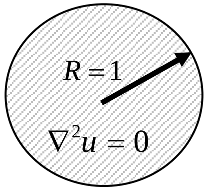
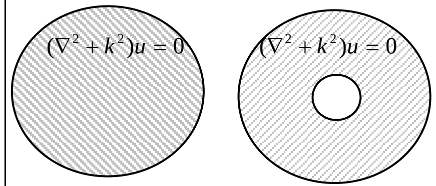
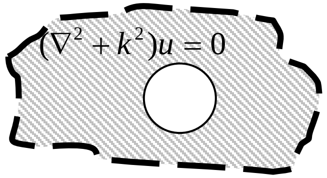
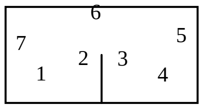
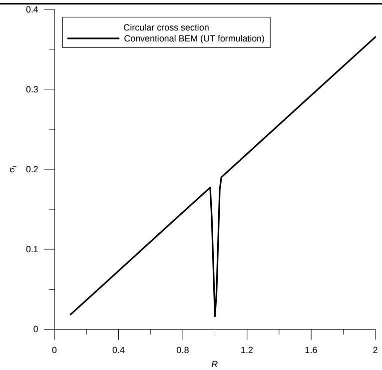
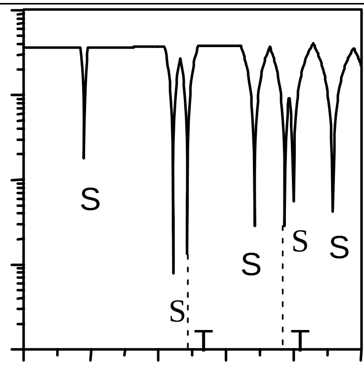
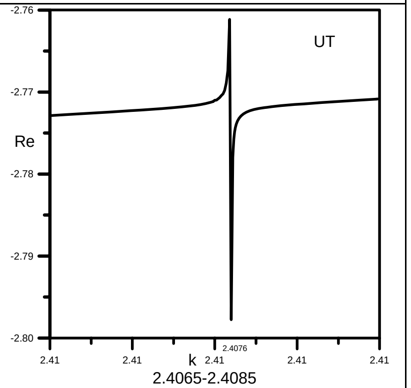


Table 4 The three-years NSC Wu Ta-You Award project

Project	Project 1		Project 2	Project 3
Physical problems	Potential problem with a degenerate boundary	Torsion problem with a degenerate scale	Spurious eigenvalues of interior acoustics (simply-connected and multiply-connected problems)	Fictitious frequency of exterior acoustics
Mathematical problem				
Mathematical formulation & numerical method	$[U]\tilde{t} = [T]\tilde{u}$ $[L]\tilde{t} = [M]\tilde{u}$	$[U]\tilde{t} = [T]\tilde{u}$ $[L]\tilde{t} = [M]\tilde{u}$	$[U]\tilde{t} = [T]\tilde{u}$ $[L]\tilde{t} = [M]\tilde{u}$	$[U]\tilde{t} = [T]\tilde{u}$ $[L]\tilde{t} = [M]\tilde{u}$
Numerical trouble	 $[T] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{17} \\ \times & \times & \times & \cdots & \times \\ \times & \times & \times & \cdots & \times \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ a_{71} & a_{72} & a_{73} & \cdots & a_{77} \end{bmatrix}$			
Treatment	1. Dual BEM (hypersingularity) 2. Multi-domain BEM 3. Conventional BEM + SVD	1. Addition of rigid body term 2. Hypersingular formulation 3. CHEEF method 4. SVD updating technique 5. Pseudo-inverse	1. Burton and Miller method 2. SVD updating technique 3. Dual BEM 4. Subdomain method 5. GSVD 6. CHEEF method (simply-connected) 7. CHIEF method (multiply-connected)	1. Burton and Miller approach 1. CHIEF method 2. SVD updating technique 3. GSVD