Analytical solutions for the Green's functions of biharmonic problems with circular boundaries

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Outlines

- Overview of BEM and motivation
- Unified formulation for the Green's function of null-field approach
  - Boundary integral equations and null-field equations
  - Expansions of boundary densities and kernels
  - Series representation for the Green's function of the annular plate
- Analytical solutions
- Conclusions
- Further studies

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Overview of numerical methods

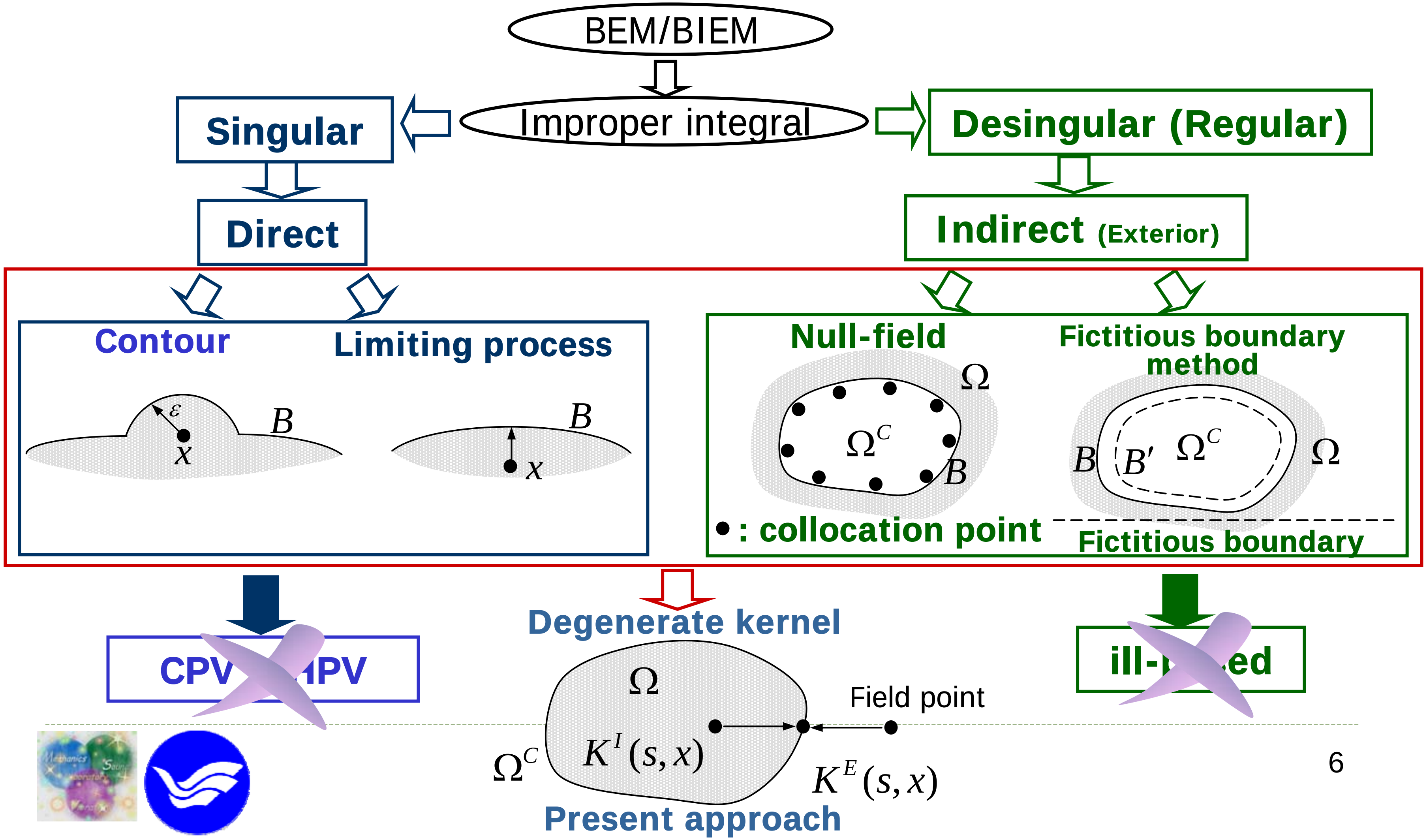
- Numerical methods
  - Domain type
    - FEM
    - FDM
  - Boundary type
    - BEM
    - MFS

Number of Papers of FEM, BEM and FDM

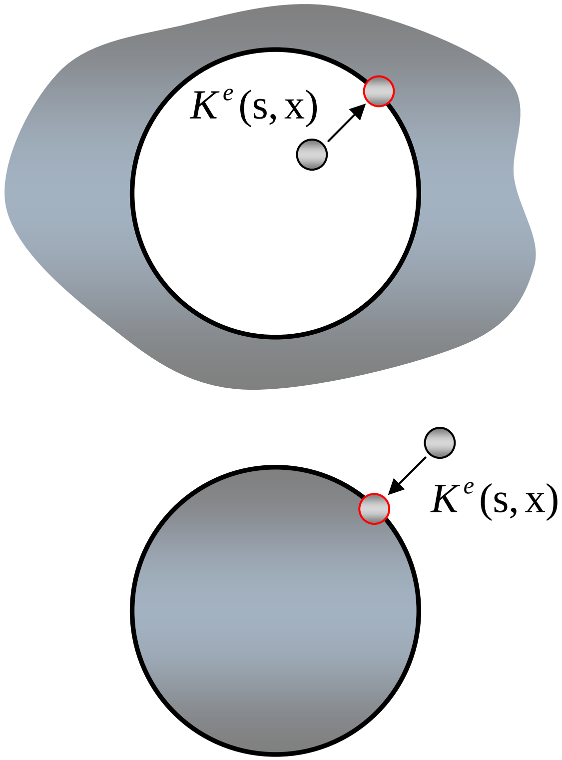
Numerical method	*Search phrase in topic field*	*No. of entries*	*Rank*	*Ratio*
FEM	"Finite element" or "finite elements"	66237	1	66.77%
FDM	"Finite difference" or "Finite differences"	19531	2	19.69%
BEM	"Boundary element" or "Boundary elements" or "boundary integral"	10126	3	10.21%
FVM	"Finite volume method" or "finite volume methods"	1695	4	1.71%
CM	"Collocation method" or "collocation methods"	1615	5	1.63%

Search data: May, 3, 2004
Data from: Prof. Cheng A. H. D.

Motivation


BEM/BIEM
Improper integral
Singular
Desingular (Regular)
Direct
Indirect (Exterior)
Contour
Limiting process
ε
B
x
B
x
Null-field
Ω
Ω^C
B
Fictitious boundary method
B
B′
Ω^C
Ω
● : collocation point
Fictitious boundary
CPV
HPV
Degenerate kernel
Ω
Field point
Ω^C
K^I(s,x)
K^E(s,x)
Present approach
ill-posed

Present approach



$$\varphi(x) = \int_B K(s, x)\psi(s)dB(s)$$

Degenerate kernel

$$\begin{cases} K^i(s, x), |s| \geq |x| \\ K^e(s, x), |x| > |s| \end{cases}$$

→ No principal value

Fundamental solution

$\ln r \qquad r^2 \ln r$

→ RPV, CPV, HPV and FP

Advantages of degenerate kernel

1. No principal value
2. Well-posed
3. Exponential convergence
4. Free of boundary-layer effect
5. Mesh free

Literature review

Engineering problems

Laplace equation

$$\nabla^2 u = 0$$

Shen (2005), Wu (2006)

- Anti-plane shear (EABE, 2006)
- Torsion bar (CMES, 2006)
- Piezoelectricity (EABE, 2006)
- Degenerate scale (MRC, 2007)
- Multiple inclusions(ASME-JAM, 2007)

Helmholtz equation

$$(\nabla^2 + k^2)u = 0$$

Chen (2005), Chen (2006)

- Multiple scattering (CMAME and SDEE, 2007)
- Eigenproblem (JCA, 2007)

Biharmonic equation

$$\nabla^4 u = 0$$

Hsiao (2005)

- Plate problem (ASME-JAM, 2006)

Bihelmholtz equation

$$\nabla^4 u = \lambda u$$

Lee (2007), Lee (2007)

- Plate vibration (JSV, 2007)

Green's function

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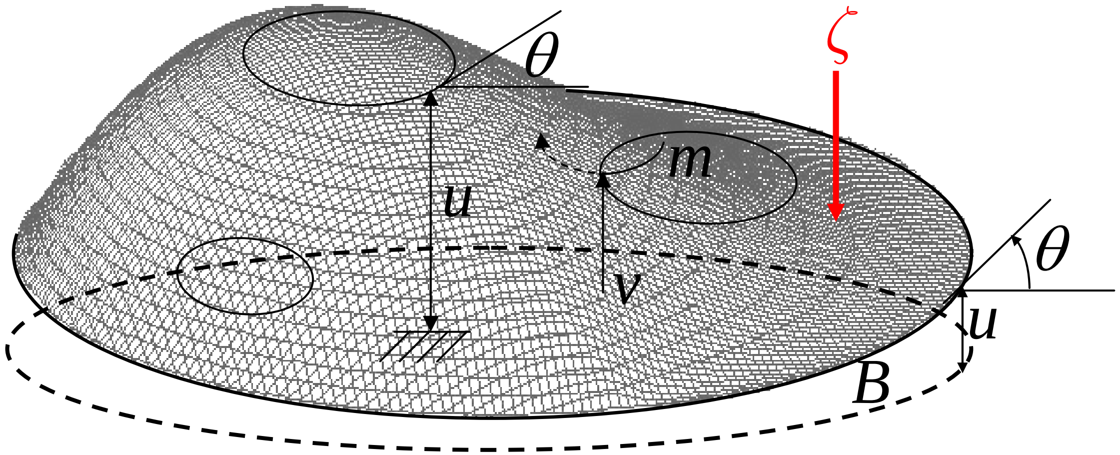
  - Expansions of boundary densities and kernels
  - Series representation for the Green's function of the clamped-free annular plate
  - Discussions on Adewale's results
- Analytical solutions
- Conclusions
- Further studies

Problem statement

Governing equation: $\nabla^4 G(x,\zeta) = \delta(x-\zeta), \quad x \in \Omega$



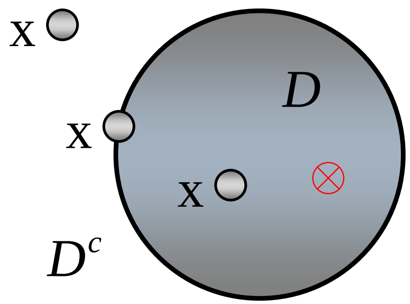
Essential boundary condition: $G(x,\zeta),\ K_{\theta,x}[G(x,\zeta)],\ x \in B$

$G(x,\zeta)$: lateral displacement, $K_{\theta,x}[G(x,\zeta)]$: slope

Natural boundary condition: $K_{m,x}[G(x,\zeta)],\ K_{v,x}[G(x,\zeta)],\ x \in B$

$K_{m,x}[G(x,\zeta)]$: moment, $K_{v,x}[G(x,\zeta)]$: shear force

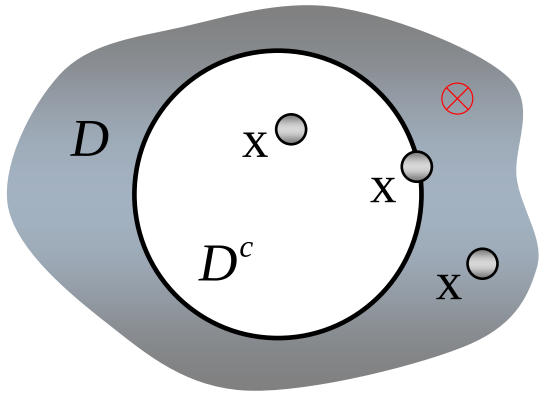
Boundary integral equation and null-field integral equation



$$K_{\theta,x}[\cdot] = \frac{\partial}{\partial n_x}$$

$$K_{m,x}[\cdot] = \nu\nabla_x^2 + (1-\nu)\frac{\partial^2}{\partial^2 n_x}$$

$$K_{v,x}[\cdot] = \frac{\partial \nabla_x^2}{\partial n_x} + (1-\nu)\frac{\partial}{\partial t_x}\left[\frac{\partial}{\partial n_x}\left(\frac{\partial}{\partial t_x}\right)\right]$$



$$\mathbf{8\pi G(x,\zeta)} = -\int_B U(s,x)K_{v,s}[G(s,\zeta)]dB(s) + \int_B \Theta(s,x)K_{m,s}[\mathbf{G(s,\zeta)]dB(s)}$$
$$-\int_B \mathbf{M}(s,x)K_{\theta,s}[G(s,\zeta)]dB(s) + \int_B V(s,x)\mathbf{G(s,\zeta)dB(s)} + \mathbf{U(\zeta,x)}, \quad \boldsymbol{x} \in \boldsymbol{\Omega} \cup B$$

$$4\pi G(x,\zeta) = -R.P.V.\int_B U(s,x)K_{v,s}[G(s,\zeta)]dB(s) + R.P.V.\int_B \Theta(s,x)K_{m,s}[G(s,\zeta)]dB(s)$$
$$-R.P.V.\int_B M(s,x)K_{\theta,s}[G(s,\zeta)]dB(s) + C.P.V.\int_B V(s,x)G(s,\zeta)dB(s) + U(\zeta,x), \quad x \in B$$

Degenerate kernel

$$\mathbf{0} = -\int_B U(s,x)K_{v,s}[G(s,\zeta)]dB(s) + \int_B \Theta(s,x)K_{m,s}[G(s,\zeta)]dB(s)$$
$$-\int_B M(s,x)K_{\theta,s}[G(s,\zeta)]dB(s) + \int_B V(s,x)G(s,\zeta)dB(s) + U(\zeta,x), \quad x \in \Omega^c \cup B$$

Present approach

Boundary integral equation

$$8\pi G(x,\zeta) = -\int_B U(s,x)K_{v,s}[G(s,\zeta)]dB(s) + \int_B \Theta(s,x)K_{m,s}[G(s,\zeta)]dB(s) - \int_B M(s,x)K_{\theta,s}[G(s,\zeta)]dB(s) + \int_B V(s,x)G(s,\zeta)dB(s) + U(\zeta,x), \quad x \in \Omega \cup B$$

$$8\pi K_{\theta,x}[G(x,\zeta)] = -\int_B U_\theta(s,x)K_{v,s}[G(s,\zeta)]dB(s) + \int_B \Theta_\theta(s,x)K_{m,s}[G(s,\zeta)]dB(s) - \int_B M_\theta(s,x)K_{\theta,s}[G(s,\zeta)]dB(s) + \int_B V_\theta(s,x)G(s,\zeta)dB(s) + U_\theta(\zeta,x), \quad x \in \Omega \cup B$$

$$8\pi K_{m,x}[G(x,\zeta)] = -\int_B U_m(s,x)K_{v,s}[G(s,\zeta)]dB(s) + \int_B \Theta_m(s,x)K_{m,s}[G(s,\zeta)]dB(s) - \int_B M_m(s,x)K_{\theta,s}[G(s,\zeta)]dB(s) + \int_B V_m(s,x)G(s,\zeta)dB(s) + U_m(\zeta,x), \quad x \in \Omega \cup B$$

$$8\pi K_{v,x}[G(x,\zeta)] = -\int_B U_v(s,x)K_{v,s}[G(s,\zeta)]dB(s) + \int_B \Theta_v(s,x)K_{m,s}[G(s,\zeta)]dB(s) - \int_B M_v(s,x)K_{\theta,s}[G(s,\zeta)]dB(s) + \int_B V_v(s,x)G(s,\zeta)dB(s) + U_v(\zeta,x), \quad x \in \Omega \cup B$$

Null-field equation

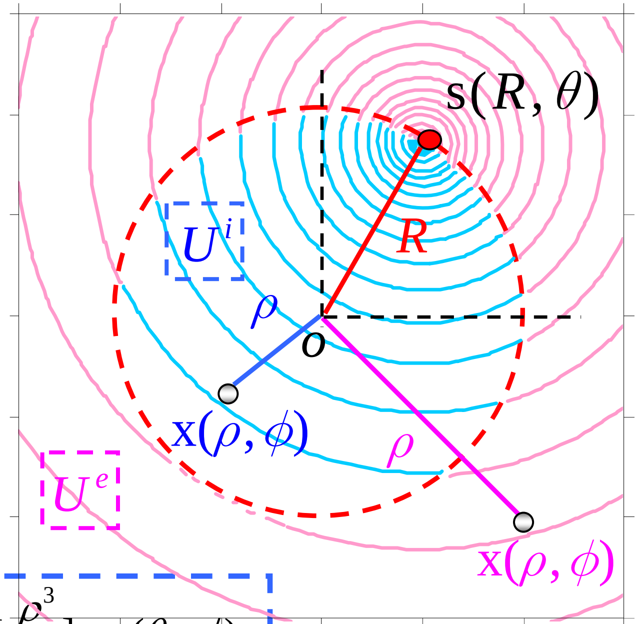
$$0 = -\int_B U(s,x)K_{v,s}[G(s,\zeta)]dB(s) + \int_B \Theta(s,x)K_{m,s}[G(s,\zeta)]dB(s) - \int_B M(s,x)K_{\theta,s}[G(s,\zeta)]dB(s) + \int_B V(s,x)G(s,\zeta)dB(s) + U(\zeta,x), \quad x \in \Omega^c \cup B$$

$$0 = -\int_B U_\theta(s,x)K_{v,s}[G(s,\zeta)]dB(s) + \int_B \Theta_\theta(s,x)K_{m,s}[G(s,\zeta)]dB(s) - \int_B M_\theta(s,x)K_{\theta,s}[G(s,\zeta)]dB(s) + \int_B V_\theta(s,x)G(s,\zeta)dB(s) + U_\theta(\zeta,x), \quad x \in \Omega^c \cup B$$

$$0 = -\int_B U_m(s,x)K_{v,s}[G(s,\zeta)]dB(s) + \int_B \Theta_m(s,x)K_{m,s}[G(s,\zeta)]dB(s) - \int_B M_m(s,x)K_{\theta,s}[G(s,\zeta)]dB(s) + \int_B V_m(s,x)G(s,\zeta)dB(s) + U_m(\zeta,x), \quad x \in \Omega^c \cup B$$

$$0 = -\int_B U_v(s,x)K_{v,s}[G(s,\zeta)]dB(s) + \int_B \Theta_v(s,x)K_{m,s}[G(s,\zeta)]dB(s) - \int_B M_v(s,x)K_{\theta,s}[G(s,\zeta)]dB(s) + \int_B V_v(s,x)G(s,\zeta)dB(s) + U_v(\zeta,x), \quad x \in \Omega^c \cup B$$

Expansions of fundamental solution and boundary density



Degenerate kernel – fundamental solution

$$U(\mathrm{s},\mathrm{x}) = \begin{cases} U^I(\mathrm{s},\mathrm{x}) = \sum_j A_j(\mathrm{s}) B_j(\mathrm{x}), & |\mathrm{s}| \geq |\mathrm{x}| \\ U^E(\mathrm{s},\mathrm{x}) = \sum_j A_j(\mathrm{x}) B_j(\mathrm{s}), & |\mathrm{x}| > |\mathrm{s}| \end{cases}$$

Fourier series expansion – boundary density

$$U(s,x) = r^2 \ln r = \begin{cases} U^I(R,\theta;\rho,\phi) = \rho^2(1+\ln R) + R^2 \ln R - [R\rho(1+2\ln R) + \frac{1}{2}\frac{\rho^3}{R}]\cos(\theta-\phi) \\ \quad -\sum_{m=2}^{\infty}[\frac{1}{m(m+1)}\frac{\rho^{m+2}}{R^m} - \frac{1}{m(m-1)}\frac{\rho^m}{R^{m-2}}]\cos[m(\theta-\phi)], \quad R \geq \rho \\ U^E(R,\theta;\rho,\phi) = R^2(1+\ln\rho) + \rho^2 \ln\rho - [\rho R(1+2\ln\rho) + \frac{1}{2}\frac{R^3}{\rho}]\cos(\theta-\phi) \\ \quad -\sum_{m=2}^{\infty}[\frac{1}{m(m+1)}\frac{R^{m+2}}{\rho^m} - \frac{1}{m(m-1)}\frac{R^m}{\rho^{m-2}}]\cos[m(\theta-\phi)], \quad \rho > R \end{cases}$$

Relation among the kernels and singularities

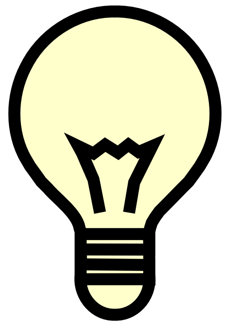
$\varepsilon \equiv |x-s|$

		$K_{\theta,s}(\cdot)$	$K_{m,s}(\cdot)$	$K_{v,s}(\cdot)$
	$U(s,x)$ $\varepsilon^2 \ln\varepsilon$	$\Theta(s,x)$ $\varepsilon\ln\varepsilon$	$M(s,x)$ $\ln\varepsilon$	$V(s,x)$ $1/\varepsilon$
$K_{\theta,x}(\cdot)$	$U_\theta(s,x)$ $\varepsilon\ln\varepsilon$	$\Theta_\theta(s,x)$ $\ln\varepsilon$	$M_\theta(s,x)$ $1/\varepsilon$	$V_\theta(s,x)$ $1/\varepsilon^2$
$K_{m,x}(\cdot)$	$U_m(s,x)$ $\ln\varepsilon$	$\Theta_m(s,x)$ $1/\varepsilon$	$M_m(s,x)$ $1/\varepsilon^2$	$V_m(s,x)$ $1/\varepsilon^3$
$K_{v,x}(\cdot)$	$U_v(s,x)$ $1/\varepsilon$	$\Theta_v(s,x)$ $1/\varepsilon^2$	$M_v(s,x)$ $1/\varepsilon^3$	$V_v(s,x)$ $1/\varepsilon^4$

$U(s,x) = r^2 \ln r$

is the fundamental solution, which satisfies

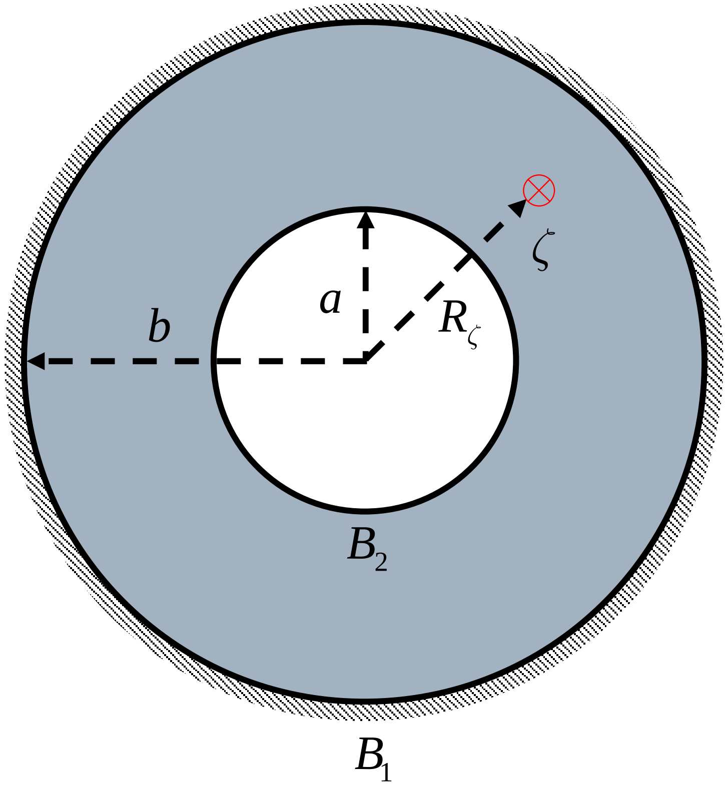
$$\nabla^4 U(s,x) = 8\pi\delta(s-x)$$



Series representation for the Green's function of the clamped-free annular plate

Governing equation: $\nabla^4 G(x,\zeta) = \frac{1}{D}\delta(x-\zeta)$

where D is the flexural rigidity



Fixed boundary condition:

$$G(x,\zeta) = 0,\ K_{\theta,x}[G(x,\zeta)] = 0$$

Free boundary condition:

$$K_{m,x}[G(x,\zeta)] = 0,\ K_{v,x}[G(x,\zeta)] = 0$$

Simply-supported boundary condition:

$$G(x,\zeta) = 0,\ K_{m,x}[G(x,\zeta)] = 0$$

Unknown boundary conditions:

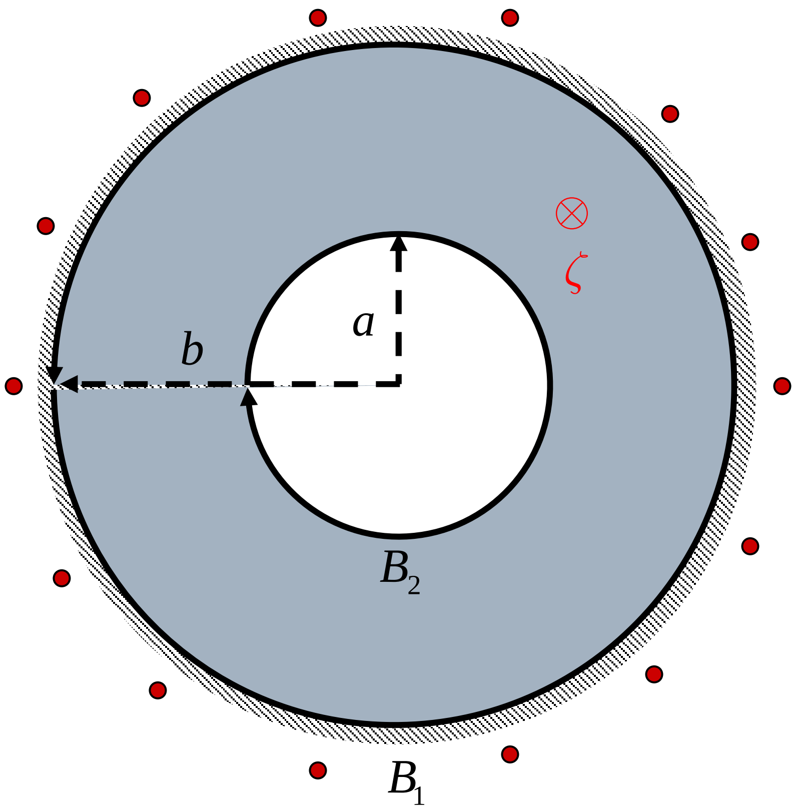
$$K_{v,s}[G(s,\zeta)] = \sum_{n=0}^{M}(a_n \cos n\theta + b_n \sin n\theta)$$

$$K_{m,s}[G(s,\zeta)] = \sum_{n=0}^{M}(\overline{a}_n \cos n\theta + \overline{b}_n \sin n\theta)$$

$$K_{\theta,s}[G(s,\zeta)] = \sum_{n=0}^{M}(p_n \cos n\theta + q_n \sin n\theta)$$

$$G(s,\zeta) = \sum_{n=0}^{M}(\overline{p}_n \cos n\theta + \overline{q}_n \sin n\theta)$$

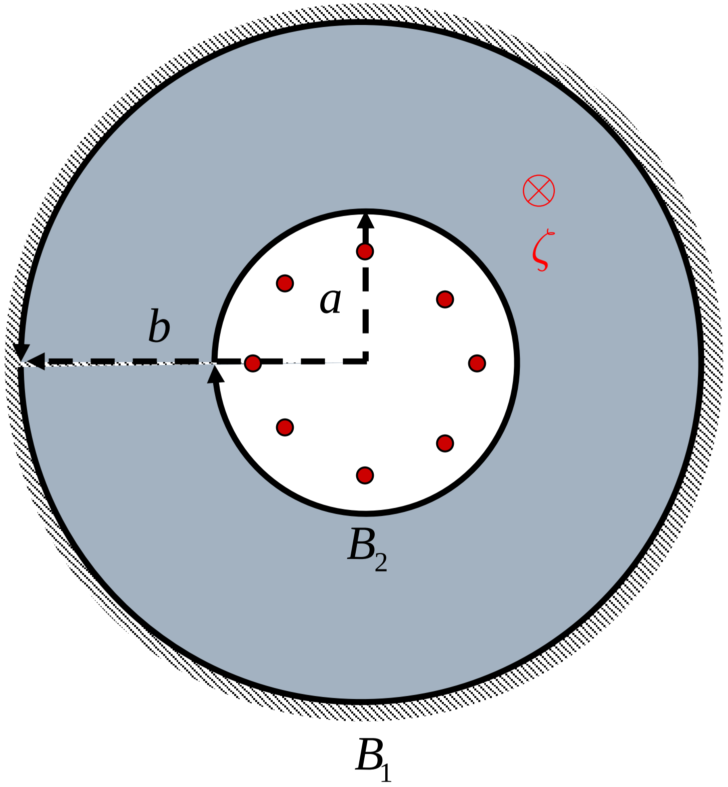
Null-field equations $x \in B$



$$0 = -\int_{B_1} U^e(s,x) K_{v,s}[G(s,\zeta)]dB(s) + \int_{B_1} \Theta^e(s,x) K_{m,s}[G(s,\zeta)]dB(s) - \int_{B_1} M^e(s,x) K_{\theta,s}[G(s,\zeta)]dB(s) + \int_{B_1} V^e(s,x) G(s,\zeta)dB(s) - \int_{B_2} U^e(s,x) K_{v,s}[G(s,\zeta)]dB(s) + \int_{B_2} \Theta^e(s,x) K_{m,s}[G(s,\zeta)]dB(s) - \int_{B_2} M^e(s,x) K_{\theta,s}[G(s,\zeta)]dB(s) + \int_{B_2} V^e(s,x) G(s,\zeta)dB(s) + U^e(\zeta,x), \quad \rho \in b^+$$

$$0 = -\int_{B_1} U^e{}_\theta(s,x) K_{v,s}[G(s,\zeta)]dB(s) + \int_{B_1} \Theta^e{}_\theta(s,x) K_{m,s}[G(s,\zeta)]dB(s) - \int_{B_1} M^e{}_\theta(s,x) K_{\theta,s}[G(s,\zeta)]dB(s) + \int_{B_1} V^e{}_\theta(s,x) G(s,\zeta)dB(s) - \int_{B_2} U^e{}_\theta(s,x) K_{v,s}[G(s,\zeta)]dB(s) + \int_{B_2} \Theta^e{}_\theta(s,x) K_{m,s}[G(s,\zeta)]dB(s) - \int_{B_2} M^e{}_\theta(s,x) K_{\theta,s}[G(s,\zeta)]dB(s) + \int_{B_2} V^e{}_\theta(s,x) G(s,\zeta)dB(s) + U^e{}_\theta(\zeta,x), \quad \rho \in b^+$$

Null-field equations $x \in B$



$$0 = -\int_{B_1} U^i(s,x) K_{v,s}[G(s,\zeta)] dB(s) + \int_{B_1} \Theta^i(s,x) K_{m,s}[G(s,\zeta)] dB(s)$$
$$- \int_{B_1} M^i(s,x) K_{\theta,s}[G(s,\zeta)] dB(s) + \int_{B_1} V^i(s,x) G(s,\zeta) dB(s)$$
$$- \int_{B_2} U^i(s,x) K_{v,s}[G(s,\zeta)] dB(s) + \int_{B_2} \Theta^i(s,x) K_{m,s}[G(s,\zeta)] dB(s)$$
$$- \int_{B_2} M^i(s,x) K_{\theta,s}[G(s,\zeta)] dB(s) + \int_{B_2} V^i(s,x) G(s,\zeta) dB(s)$$
$$+ U^i(\zeta, x), \quad \rho \in a^-$$

$$0 = -\int_{B_1} U^i{}_{\theta}(s,x) K_{v,s}[G(s,\zeta)] dB(s) + \int_{B_1} \Theta^i{}_{\theta}(s,x) K_{m,s}[G(s,\zeta)] dB(s)$$
$$- \int_{B_1} M^i{}_{\theta}(s,x) K_{\theta,s}[G(s,\zeta)] dB(s) + \int_{B_1} V^i{}_{\theta}(s,x) G(s,\zeta) dB(s)$$
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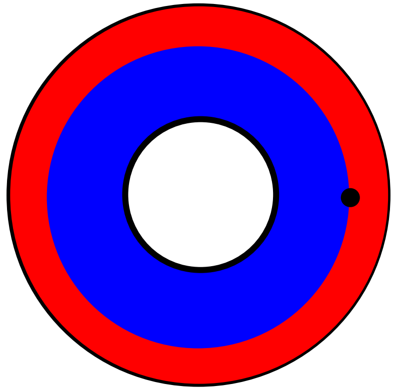
$$+ U^i{}_{\theta}(\zeta, x), \quad \rho \in a^-$$

Field solutions $x \in \Omega$



$$8\pi G(x,\zeta) = -\int_{B_1} U^i(s,x) K_{v,s}[G(s,\zeta)]dB(s) + \int_{B_1} \Theta^i(s,x) K_{m,s}[G(s,\zeta)]dB(s)$$
$$-\int_{B_1} M^i(s,x) K_{\theta,s}[G(s,\zeta)]dB(s) + \int_{B_1} V^i(s,x) G(s,\zeta)dB(s)$$
$$-\int_{B_2} U^e(s,x) K_{v,s}[G(s,\zeta)]dB(s) + \int_{B_2} \Theta^e(s,x) K_{m,s}[G(s,\zeta)]dB(s)$$
$$-\int_{B_2} M^e(s,x) K_{\theta,s}[G(s,\zeta)]dB(s) + \int_{B_2} V^e(s,x) G(s,\zeta)dB(s)$$
$$+U(\zeta,x), \quad a \le x \le b$$

$$8\pi G(x,\zeta) = -\int_{B_1} U^i(s,x) K_{v,s}[G(s,\zeta)]dB(s) + \int_{B_1} \Theta^i(s,x) K_{m,s}[G(s,\zeta)]dB(s)$$
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$$-\int_{B_2} U^e(s,x) K_{v,s}[G(s,\zeta)]dB(s) + \int_{B_2} \Theta^e(s,x) K_{m,s}[G(s,\zeta)]dB(s)$$
$$-\int_{B_2} M^e(s,x) K_{\theta,s}[G(s,\zeta)]dB(s) + \int_{B_2} V^e(s,x) G(s,\zeta)dB(s)$$
$$+U^i(\zeta,x), \quad a < x \le R_\zeta$$

$$8\pi G(x,\zeta) = -\int_{B_1} U^i(s,x) K_{v,s}[G(s,\zeta)]dB(s) + \int_{B_1} \Theta^i(s,x) K_{m,s}[G(s,\zeta)]dB(s)$$
$$-\int_{B_1} M^i(s,x) K_{\theta,s}[G(s,\zeta)]dB(s) + \int_{B_1} V^i(s,x) G(s,\zeta)dB(s)$$
$$-\int_{B_2} U^e(s,x) K_{v,s}[G(s,\zeta)]dB(s) + \int_{B_2} \Theta^e(s,x) K_{m,s}[G(s,\zeta)]dB(s)$$
$$-\int_{B_2} M^e(s,x) K_{\theta,s}[G(s,\zeta)]dB(s) + \int_{B_2} V^e(s,x) G(s,\zeta)dB(s)$$
$$+U^e(\zeta,x), \quad R_\zeta < x \le b$$

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- Conclusions
- Further studies

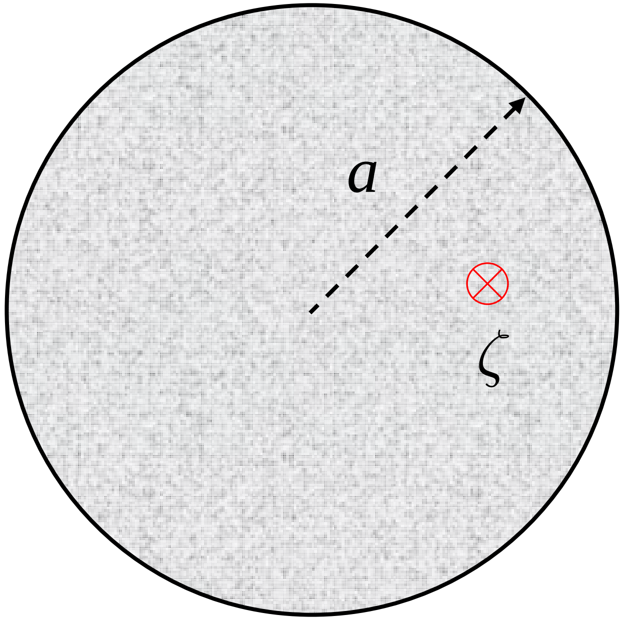
Analytical solutions

- Robin boundary condition (Laplace)
- A circular plate (Biharmonic)
  - The load at the center
  - The eccentric load
- An annular plate (Biharmonic)
  - Fixed-Free (Adewale's results)
  - Fixed-Fixed
  - Free-Simply supported

Analytical solutions

- ***Robin boundary condition (Laplace)***
- A circular plate (Biharmonic)
  - The load at the center
  - The eccentric load
- An annular plate (Biharmonic)
  - Fixed-Free (Adewale's results)
  - Fixed-Fixed
  - Free-Simply supported

Problem statement (Laplace)



Governing equation:

$$\nabla^2 G(x,\zeta) = \delta(x-\zeta), \quad x \in \Omega$$

Boundary condition:

$$\partial G(x,\zeta) \Big/ \partial n_x + \beta G(x,\zeta) = 0, \quad x \in B$$

where β is the impedance coefficient

Analytical solutions

$$G(x,\zeta)=\frac{1}{2\pi}\left\{\frac{1}{\beta}+\ln\frac{|x\bar{\zeta}-1|}{|x-\zeta|}+2\,\mathrm{Re}\left[\omega^{-\beta}\int_0^{\omega}\frac{t^{\beta-1}}{1-t}dt\right]\right\}$$

Closed-form

$$=\frac{1}{2\pi}\left[\frac{1}{a\beta}+\ln\frac{a^3}{|x-\zeta||x\bar{\zeta}-a^2|}-\sum_{m=1}^{\infty}\frac{2a\beta}{m(m+a\beta)}\left(\frac{\rho R_s}{a^2}\right)^m\cos m(\phi-\theta_s)\right]$$

Series-form

where $x=\rho e^{i\phi}$ $\zeta=R_\zeta e^{i\theta_\zeta}$ $\omega=\rho R_\zeta e^{i(\theta_\zeta-\phi)}$

Melnikov's result

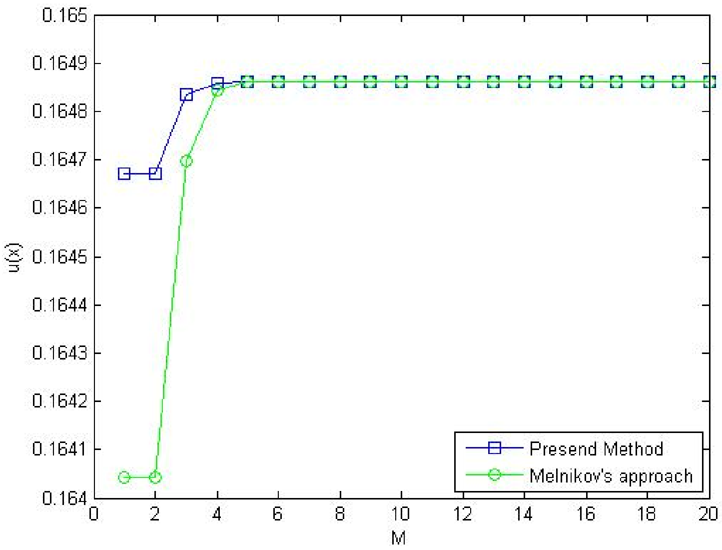
$$G(x,\zeta)=(1+a\beta\ln a)\,p_0+\sum_{m=1}^{\infty}\frac{\rho^m}{a^m}\frac{m-a\beta}{2m}(p_m\cos m\phi+q_m\sin m\phi)+\frac{\ln|x-\zeta|}{2\pi}$$

where $p_0=-\dfrac{1}{2\pi a\beta}$ $p_m=-\dfrac{R_\zeta^m}{\pi(m+a\beta)a^m}\cos m\theta_\zeta$ $q_m=-\dfrac{R_\zeta^m}{\pi(m+a\beta)a^m}\sin m\theta_\zeta$

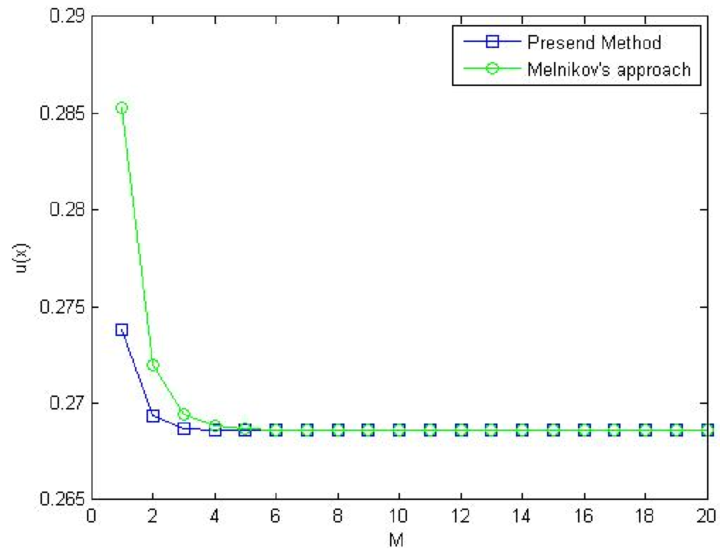
Our solution

Equivalence is proved in the thesis

The convergence rate

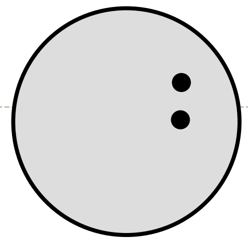


$$(\rho,\phi)=(2,0)$$

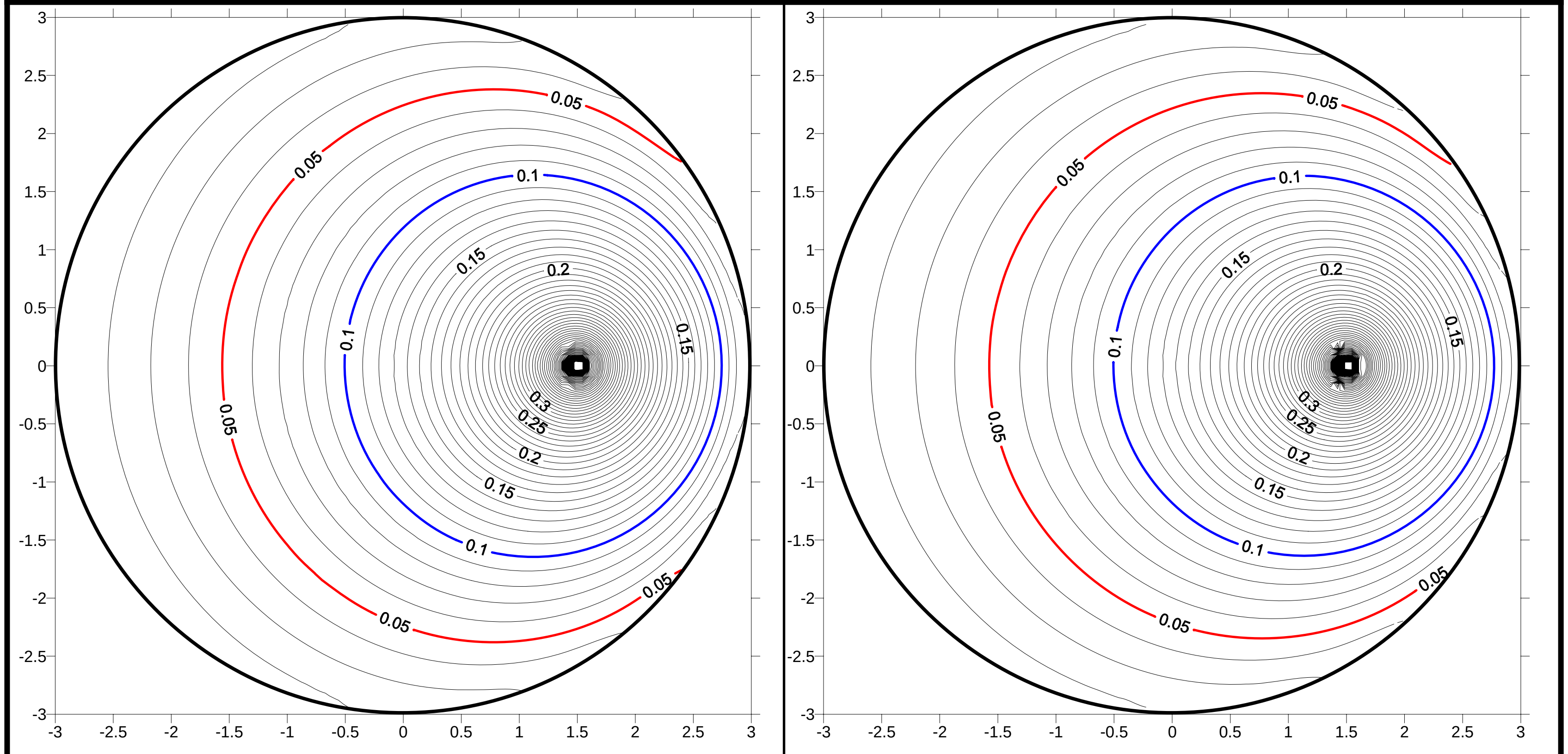


$$(\rho,\phi)=\left(\sqrt{2},\frac{\pi}{4}\right)$$

The present method converges better than Melnikov's



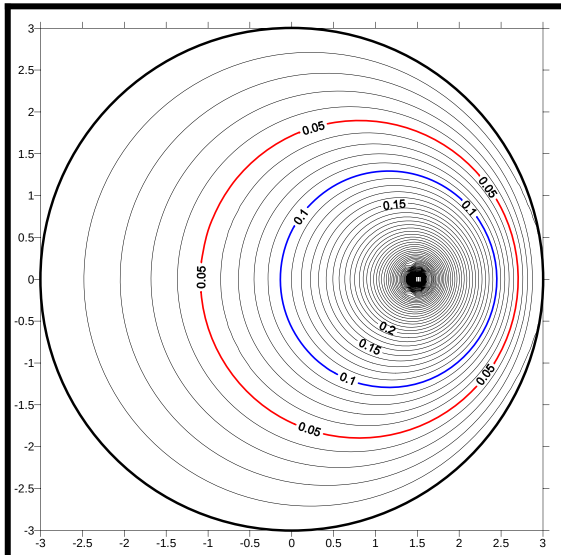
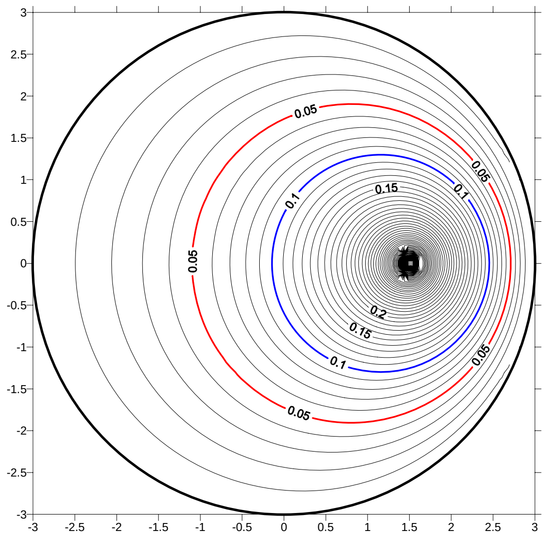
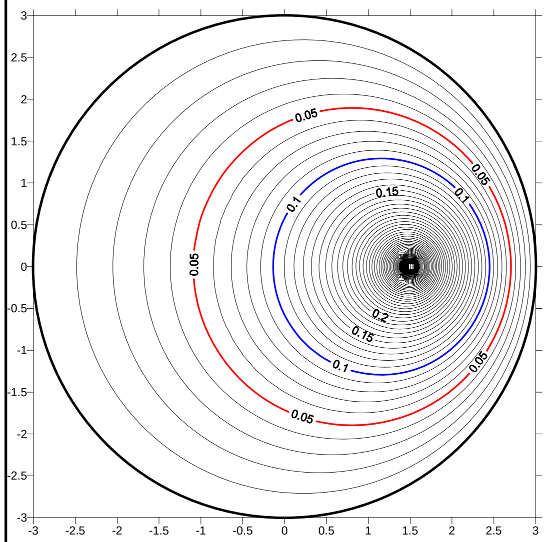
Potential contours



Melnikov's (M=20)	Present method (M=20)

$a = 3 \qquad R_\zeta = 1.5 \qquad \theta_\zeta = 0 \qquad \beta = 2$

The limiting case (fixed) $\beta \to \infty$

		
Present method (M=50)	Melnikov's results (M=50)	Closed-from

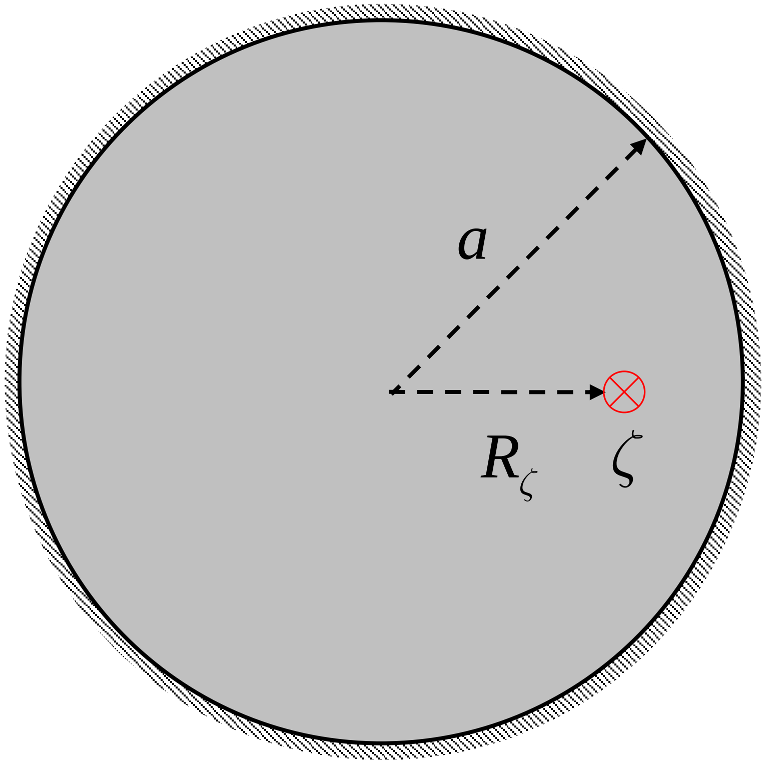
Closed-form solution:

$$G(x,\zeta) = (\ln|x-\zeta| - \ln|x-\zeta'| + \ln a - \ln R_\zeta)/2\pi$$

Analytical solutions

- Robin boundary condition (Laplace)
- ***A circular plate*** (Biharmonic)
  - The load at the center
  - The eccentric load
- An annular plate (Biharmonic)
  - Fixed-Free (Adewale's results)
  - Fixed-Fixed
  - Free-Simply supported

Problem statement (Biharmonic)



Fixed boundary conditions:

$$G(x,\zeta)=0,\ \ K_{\theta,x}[G(x,\zeta)]=0,\ \ x\in B$$

Governing equation:

$$\nabla^4 G(x,\zeta)=\delta(x-\zeta),\ \ x\in\Omega$$

Analytical solutions

$$G(x,\zeta)=\frac{1}{16\pi D}\left(a^2-\rho^2+2\rho^2\ln\frac{\rho}{a}\right)$$

Szilard

$$G(x,\zeta)=\frac{1}{8\pi D}\left[\frac{1}{2a^2}\left(a^2-|z|^2\right)\left(a^2-|\zeta|^2\right)-(z-\zeta)^2\ln\left|\frac{a^2-z\bar{\zeta}}{a|z-\zeta|}\right|\right]$$

Melnikov

$$8\pi G(x,\zeta)=-2\pi a\rho^2(1+\ln a)a_0-2\pi a^3\ln a\,a_0+\left[a\rho(1+2\ln a)+\frac{1}{2}\frac{\rho^3}{a}\right]\pi a(a_1\cos\phi+b_1\sin\phi)$$

Present method

$$+\sum_{n=2}^{\infty}\left[\frac{1}{n(n+1)}\frac{\rho^{n+2}}{a^n}-\frac{1}{n(n-1)}\frac{\rho^n}{a^{n-2}}\right]\pi a(a_n\cos n\phi+b_n\sin n\phi)$$

$$+2\pi\rho^2p_0+2\pi a^2(1+2\ln a)p_0-\left[\rho(3+2\ln a)-\frac{1}{2}\frac{\rho^3}{a^2}\right]\pi a(p_1\cos\phi+q_1\sin\phi)$$

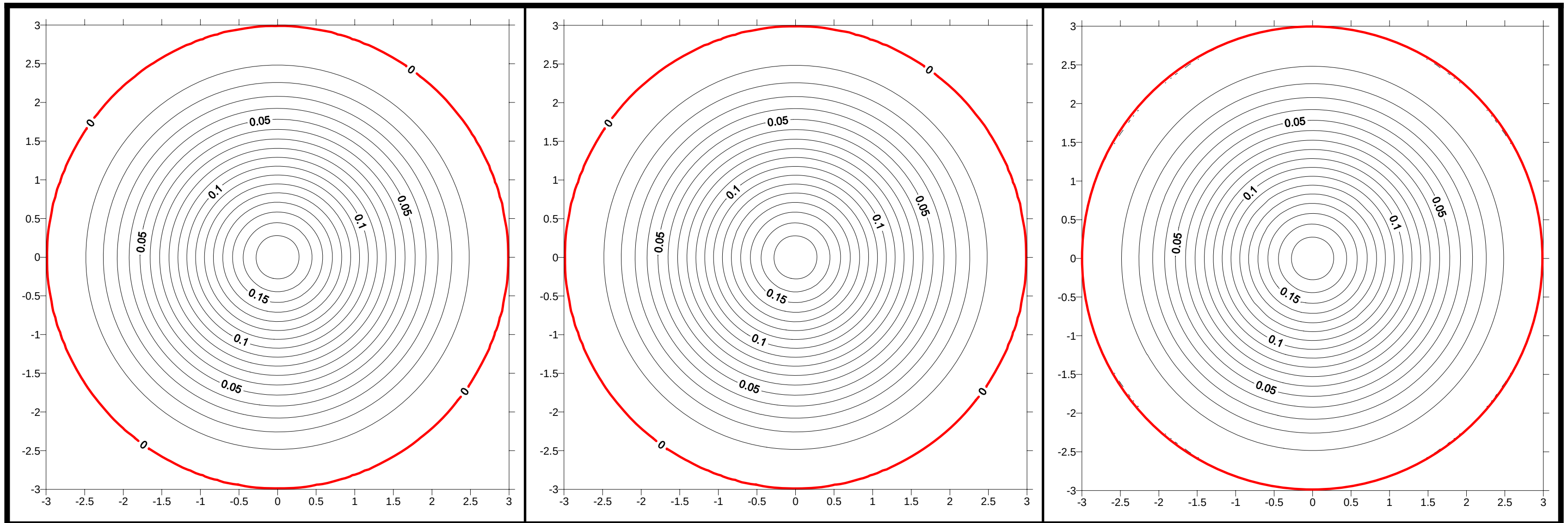
$$+\sum_{n=2}^{\infty}\left[\frac{1}{n+1}\frac{\rho^{n+2}}{a^{n+1}}-\frac{n-2}{n(n-1)}\frac{\rho^n}{a^{n-1}}\right]\pi a(p_n\cos n\phi+q_n\sin n\phi)+(x-\zeta)^2\ln|x-\zeta|,\quad x\in\Omega,$$

$$\begin{Bmatrix}a_0\\p_0\end{Bmatrix}=\begin{Bmatrix}\frac{1}{2\pi a}\\\frac{a^2-R_s^2}{4\pi a^2}\end{Bmatrix}\quad\begin{Bmatrix}a_1\\p_1\end{Bmatrix}=\begin{Bmatrix}\frac{-\cos\theta_sR_s\left(R_s^2-3a^2\right)}{2\pi a^4}\\\frac{\cos\theta_sR_s\left(a^2-R_s^2\right)}{2\pi a^3}\end{Bmatrix}\quad\begin{Bmatrix}b_1\\q_1\end{Bmatrix}=\begin{Bmatrix}\frac{-\sin\theta_sR_s\left(R_s^2-3a^2\right)}{2\pi a^4}\\\frac{\sin\theta_sR_s\left(a^2-R_s^2\right)}{2\pi a^3}\end{Bmatrix}\quad\begin{Bmatrix}a_n\\p_n\end{Bmatrix}=\begin{Bmatrix}\frac{a^{-n-3}\cos n\theta_sR_s^n\left[(n+2)a^2-nR_s^2\right]}{2\pi}\\\frac{a^{-n-2}\cos n\theta_sR_s^n\left[a^2-R_s^2\right]}{2\pi}\end{Bmatrix}$$

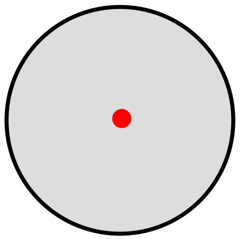
$$\begin{Bmatrix}b_n\\q_n\end{Bmatrix}=\begin{Bmatrix}\frac{a^{-n-3}\sin n\theta_sR_s^n\left[(n+2)a^2-nR_s^2\right]}{2\pi}\\\frac{a^{-n-2}\sin n\theta_sR_s^n\left[a^2-R_s^2\right]}{2\pi}\end{Bmatrix}$$

Equivalence is proved in the thesis

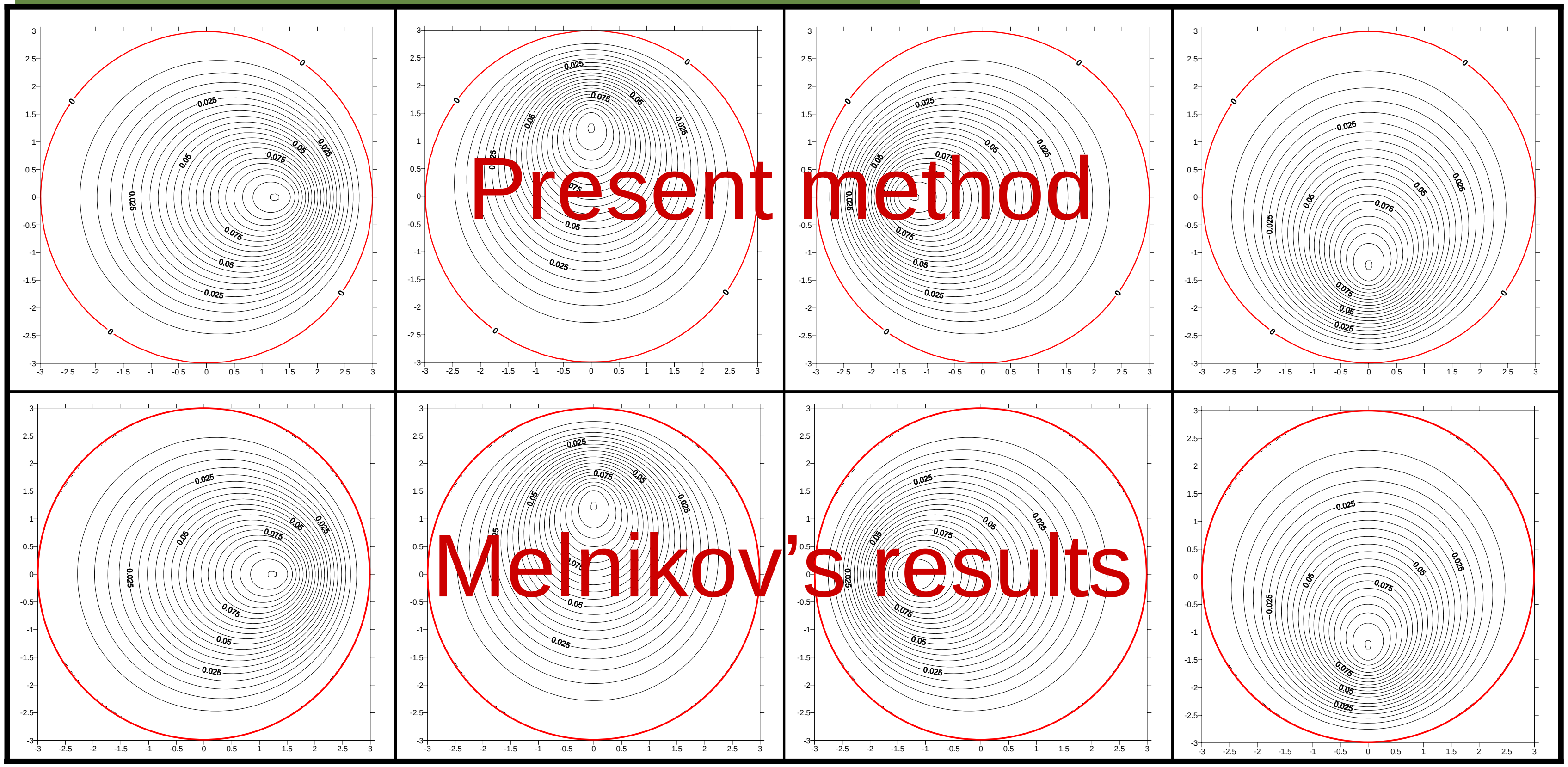
Displacement contours

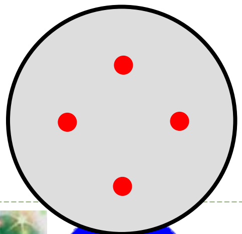


Present method (M=1) | Szilard's approach | Melnikov's approach



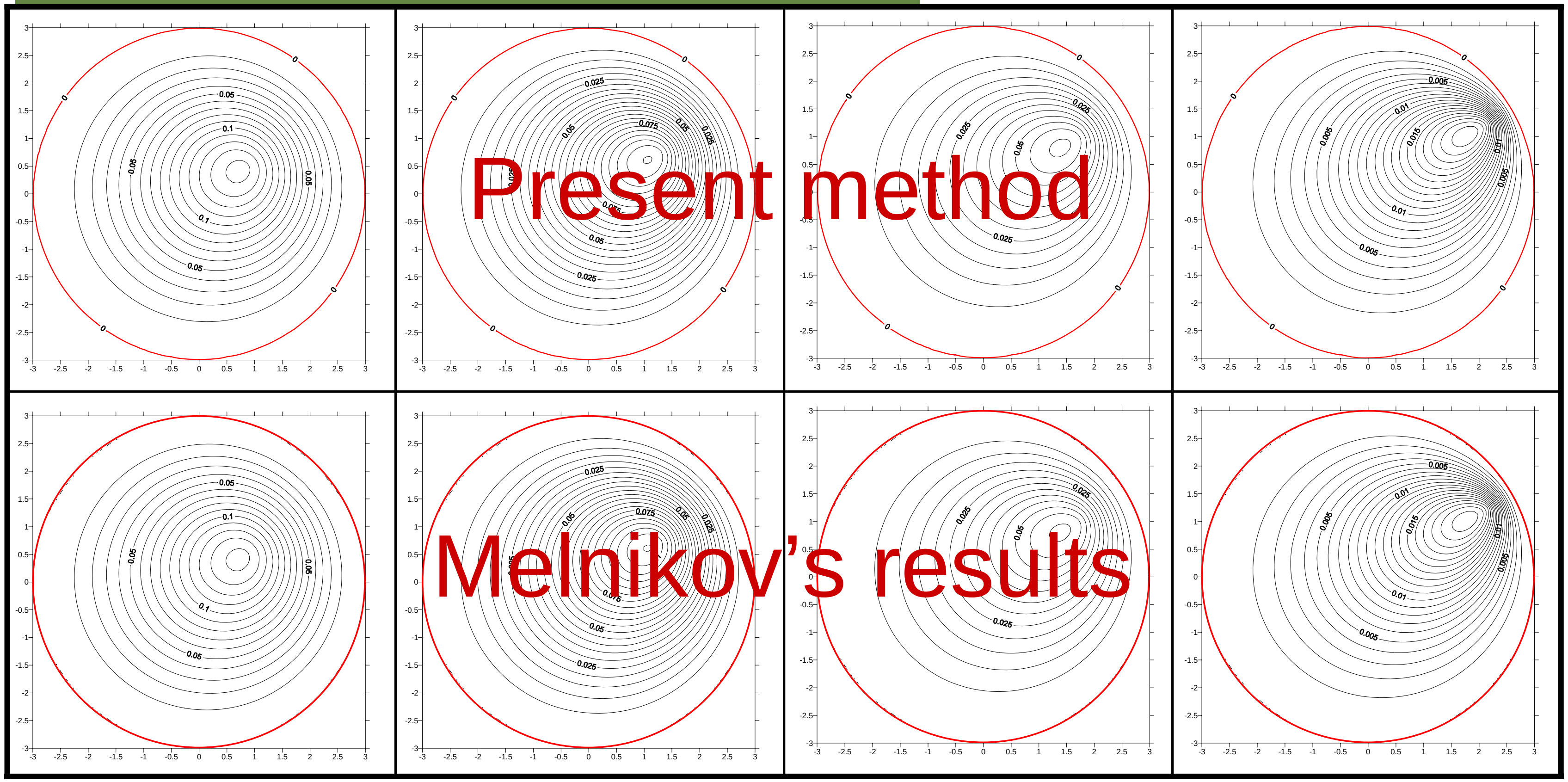
Displacement contours

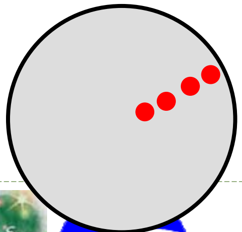




$$(a = 3 \quad , \quad R_s = 1.5 \quad , \quad M = 50)$$

Displacement contours





$$\left(a = 3 \quad , \quad \theta_s = \frac{\pi}{6} \quad , \quad M = 50 \right)$$

Analytical solutions

- Robin boundary condition (Laplace)
- A circular plate (Biharmonic)
  - The load at the center
  - The eccentric load
- ***An annular plate (Biharmonic)***
  - Fixed-Free (Adewale's results)
  - Fixed-Fixed
  - Free-Simply supported

Discussions on Adewale's results

Ajayi O. Adewale, 2006, Isotropic clamped-free thin annular circular plate subjected to a concentrated load, *ASME Journal of Applied Mechanics*, **73**, 658-663.

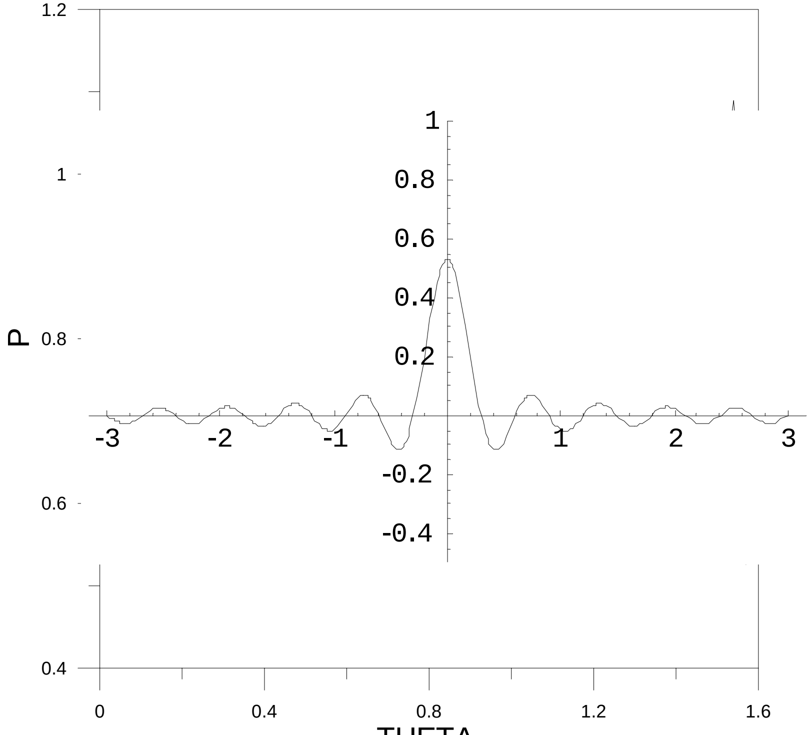
$u=\sum_{m=0}^{\infty} R_m(r)\cos m\theta$ → $\nabla^4 u = 0$ → Trefftz bases → Four boundary conditions; Four continuity conditions

Four continuity conditions → Unknown coefficients → Displacement → Simulation of a concentrated load; The definition of shear force

Simulation of a concentrated load

$$P \approx P\left[\frac{1}{2}+\sum_{k=1}^{\infty}\frac{2\sin\frac{(2k-1)\pi}{2}}{(2k-1)\pi}\cos\left((2k-1)\theta\right)\right], 0 \leq \theta \leq \frac{\pi}{2}$$

$$\int_{-\infty}^{\infty}\delta(x)\,dx = 1$$

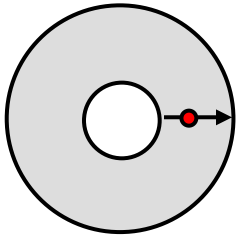


The definition of shear force

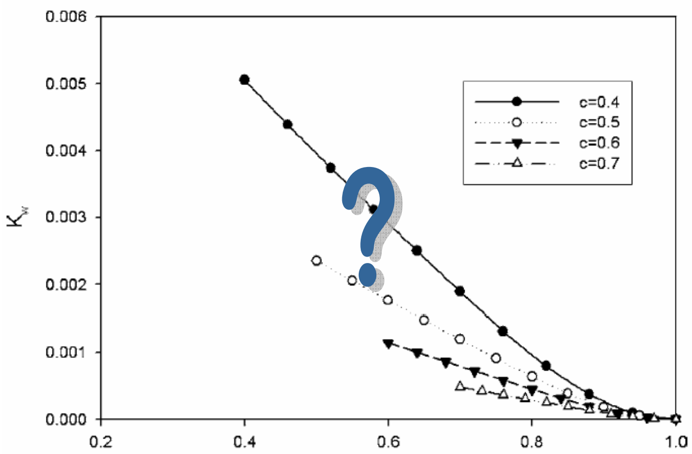
$\left(\frac{\partial^3}{\partial r^3}-\frac{1}{r^2}\frac{\partial}{\partial r}+\frac{1}{r}\frac{\partial^2}{\partial r^2}-\frac{m^2}{r^2}\frac{\partial}{\partial r}\right)R_m(r)\Big\vert_{r=a}=0$?	
$-D\frac{\partial}{\partial r}\left(\nabla^2 u\right)+\frac{1}{r}\frac{\partial}{\partial\theta}\left[-D(1-\nu)\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial u}{\partial\theta}\right)\right]$	Szilard
$-D\left[\frac{\partial}{\partial r}\nabla_r^2 u+\frac{1-\nu}{r}\frac{\partial}{\partial\varphi}\left(\frac{1}{r}\frac{\partial^2 u}{\partial r\partial\varphi}-\frac{1}{r^2}\frac{\partial u}{\partial\varphi}\right)\right]$	Leissa
$\frac{\partial\nabla_x^2 u}{\partial n_x}+(1-\nu)\frac{\partial}{\partial t_x}\left[\frac{\partial}{\partial n_x}\left(\frac{\partial u}{\partial t_x}\right)\right]$	Present method

$$\frac{\partial^3 R_m}{\partial r^3}-\frac{1}{r^2}\frac{\partial R_m}{\partial r}+\frac{1}{r}\frac{\partial^2 R_m}{\partial r^2}+\frac{2m^2}{r^3}R_m-\frac{m^2}{r^2}\frac{\partial R_m}{\partial r}+(1-\nu)\left[\frac{m^2}{r^3}R_m-\frac{m^2}{r^2}\frac{\partial R_m}{\partial r}\right]$$

The results



Adewale's

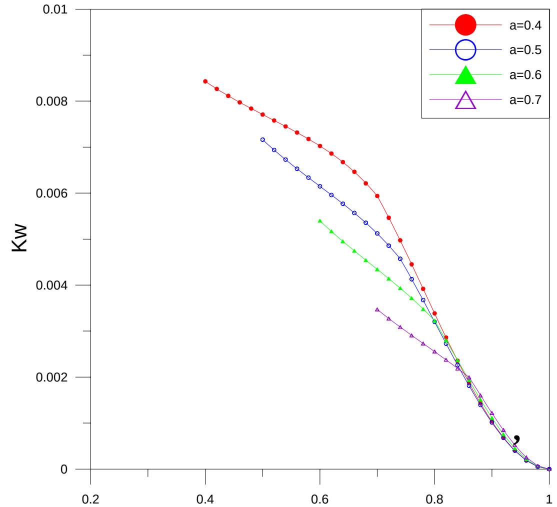


$$a = 0.4, \quad 0.5, \quad 0.6, \quad 0.7 \qquad D = 1$$

$$b = 1.0 \qquad \nu = 0.3$$

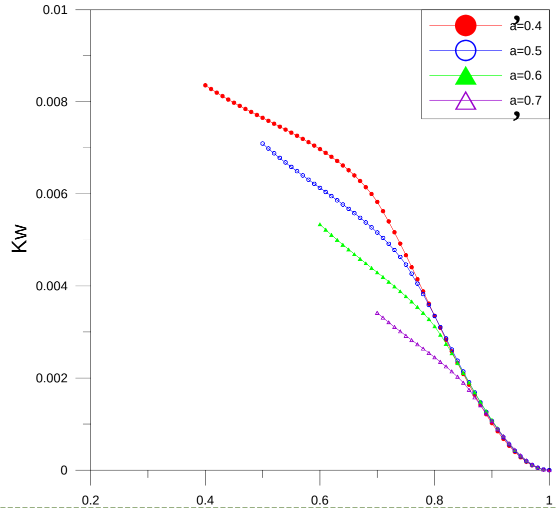
$$R_\zeta = \frac{a+b}{2} \qquad k_w = \frac{wD}{P}$$

ABAQUS

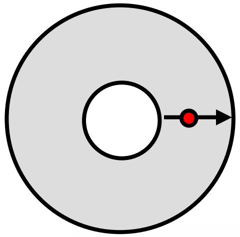


Present method

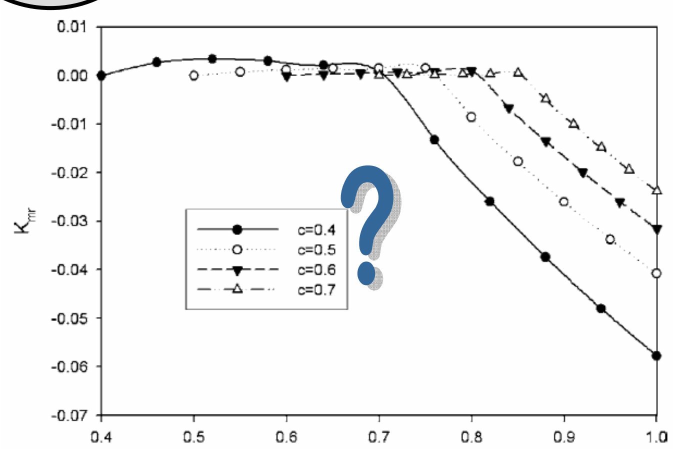
M=50



The results



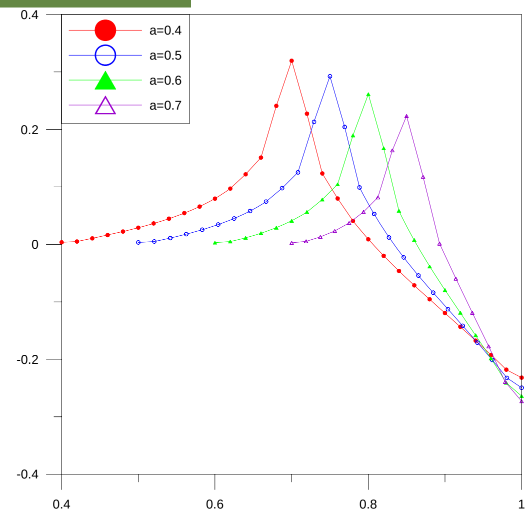
Adewale's



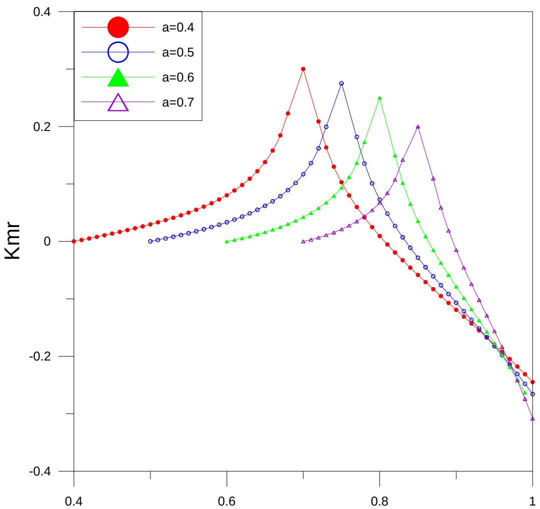
$$a = 0.4, \quad 0.5, \quad 0.6, \quad 0.7 \qquad D = 1$$

$$b = 1.0 \qquad \nu = 0.3$$

$$R_\zeta = \frac{a+b}{2} \qquad k_{mr} = \frac{M_r D}{P}$$



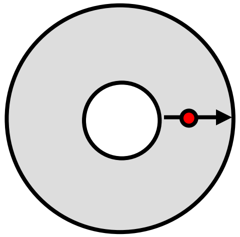
ABAQUS



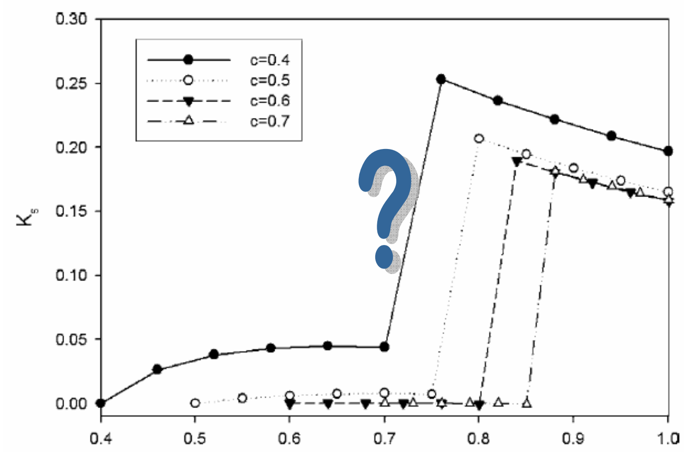
Present method

M=50

The results



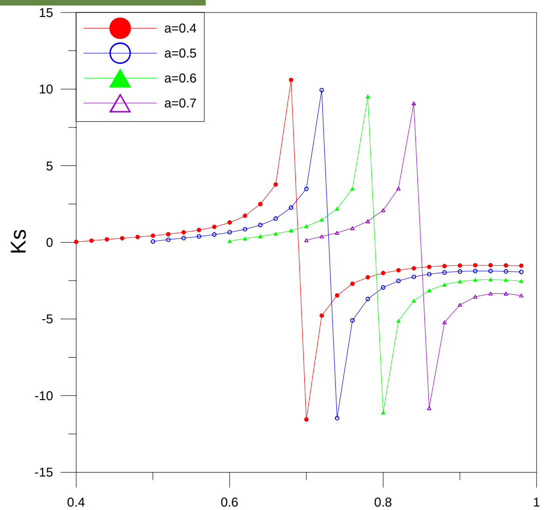
Adewale's



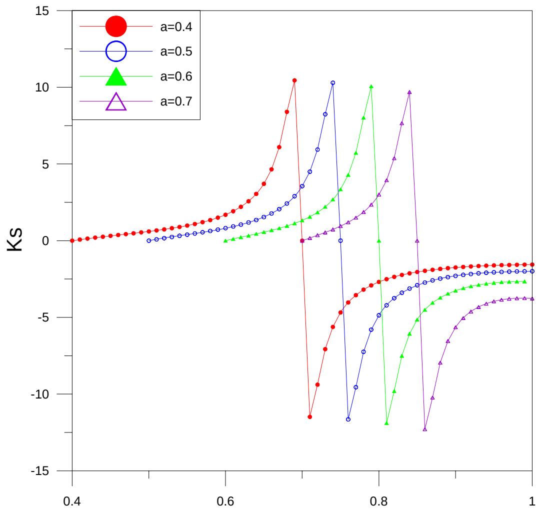
$$a = 0.4, \quad 0.5, \quad 0.6, \quad 0.7 \qquad D = 1$$

$$b = 1.0 \qquad \nu = 0.3$$

$$R_\zeta = \frac{a+b}{2} \qquad k_s = \frac{M_s D}{P}$$



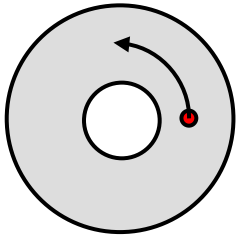
ABAQUS



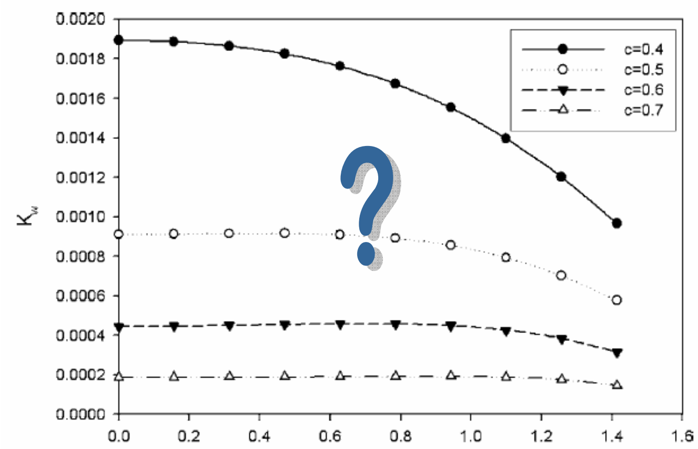
Present method

M=50

The results



Adewale's



$$a = 0.4, \quad 0.5, \quad 0.6, \quad 0.7 \qquad D = 1$$

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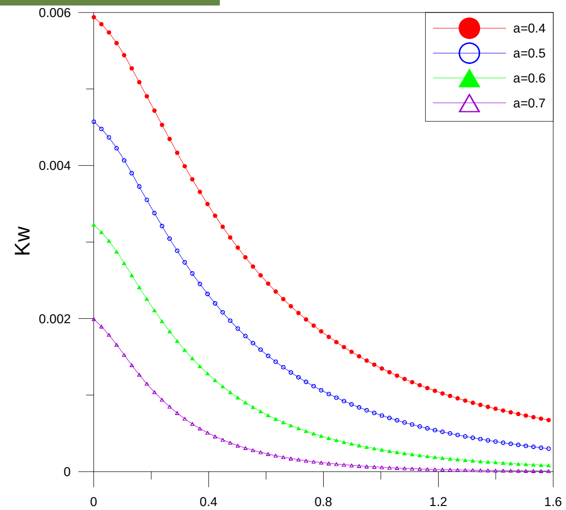
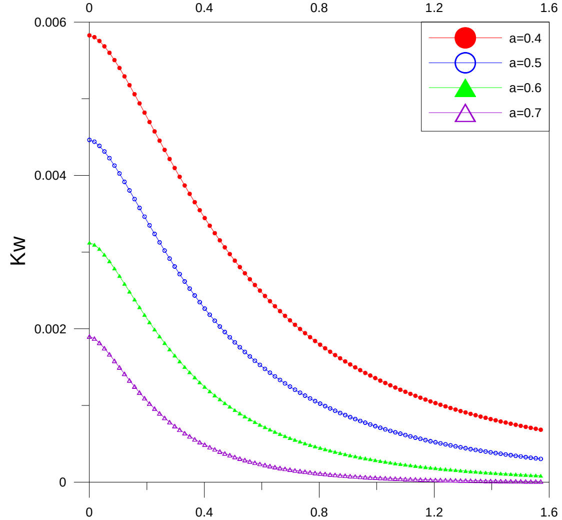
$$R_\zeta = \frac{a+b}{2} \qquad k_w = \frac{wD}{P}$$



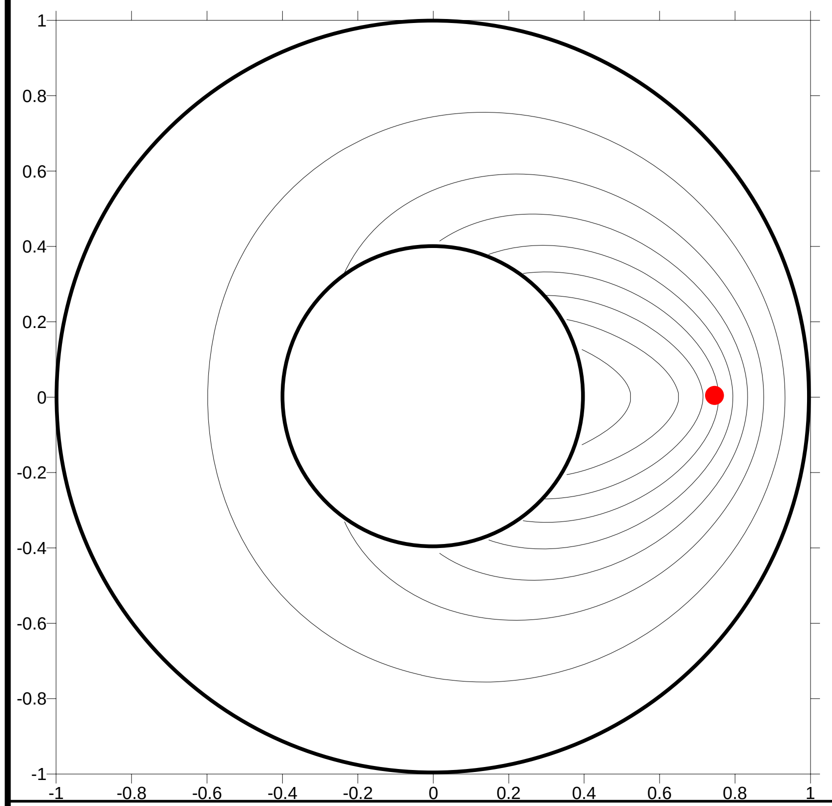
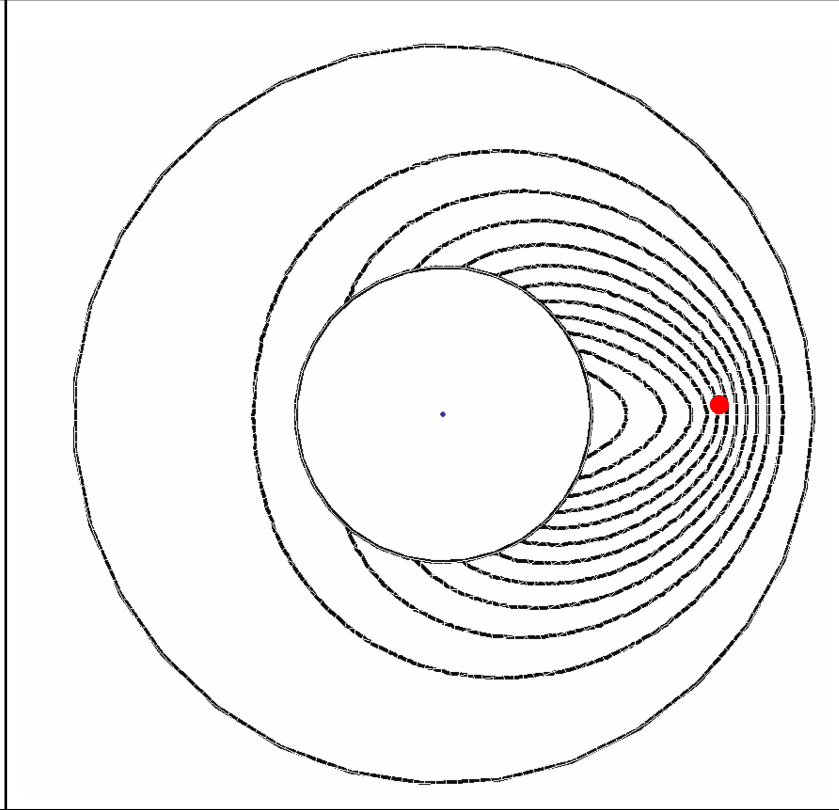
ABAQUS



Present method

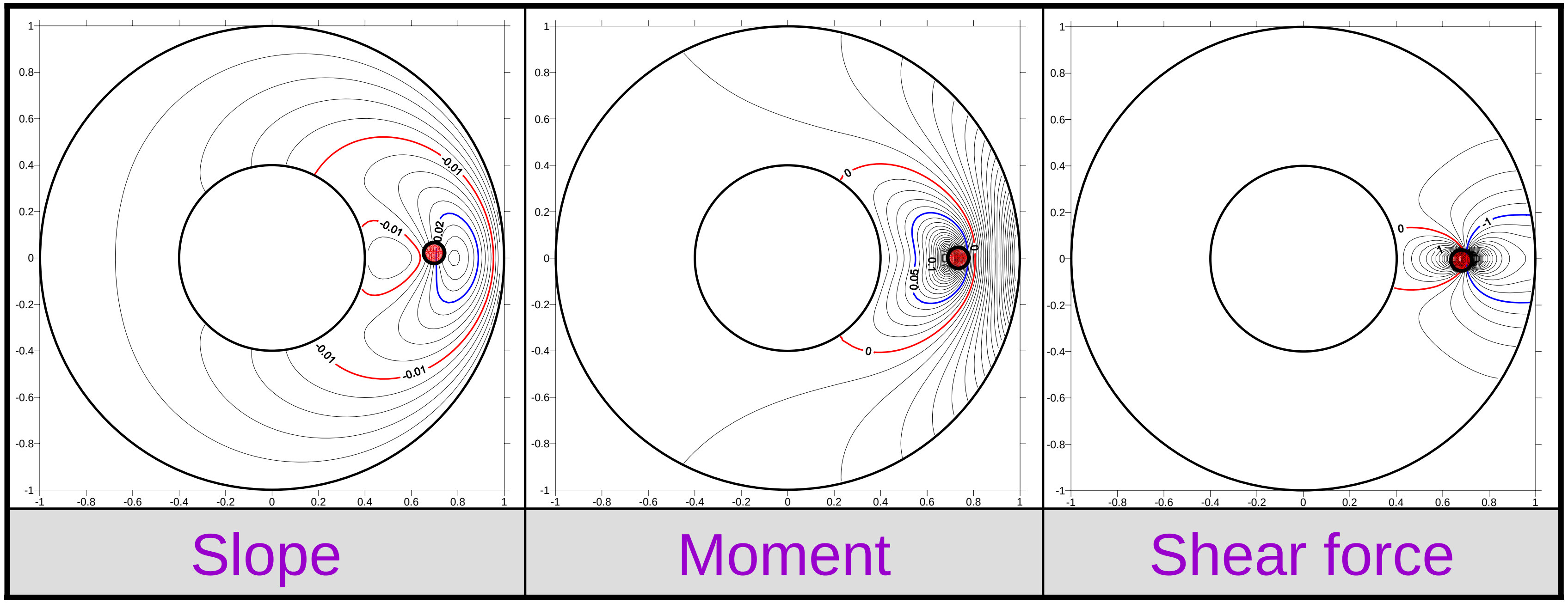
M=50

Displacement contours (fixed-free)

	
Present method (M=50)	ABAQUS

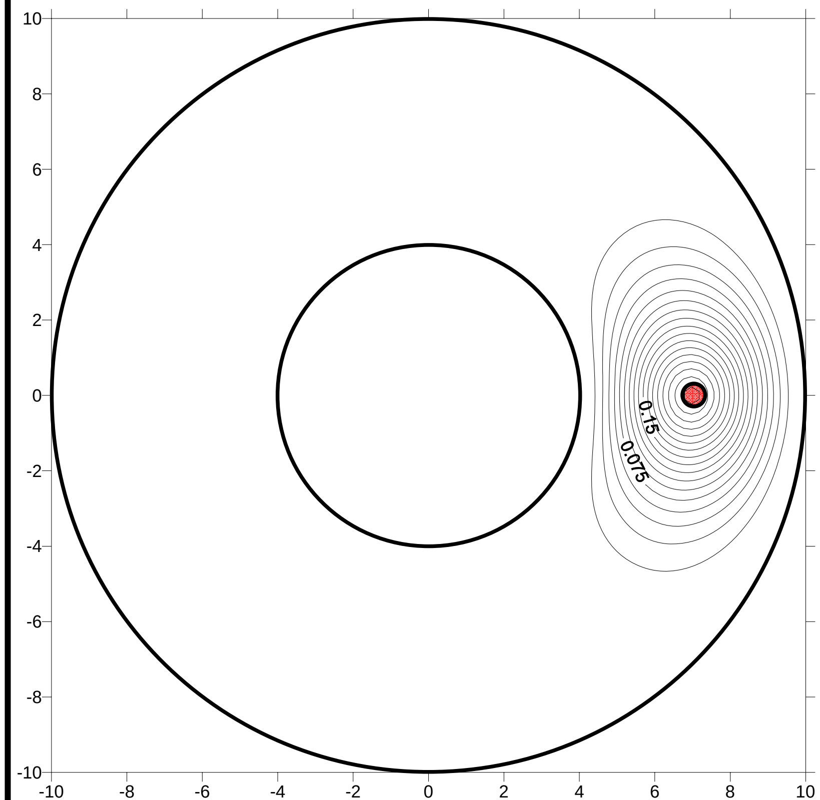
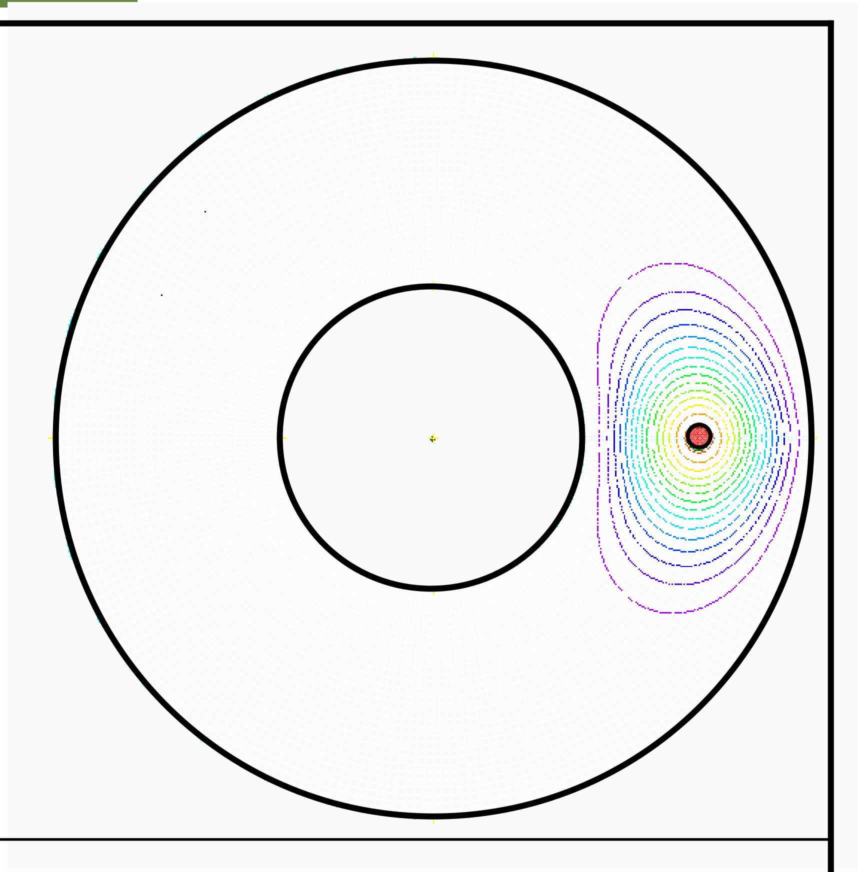
$(a = 0.4 \;,\; b = 1 \;,\; R_\zeta = 0.7 \;,\; \theta_\zeta = 0)$

The slope, moment and shear force contours

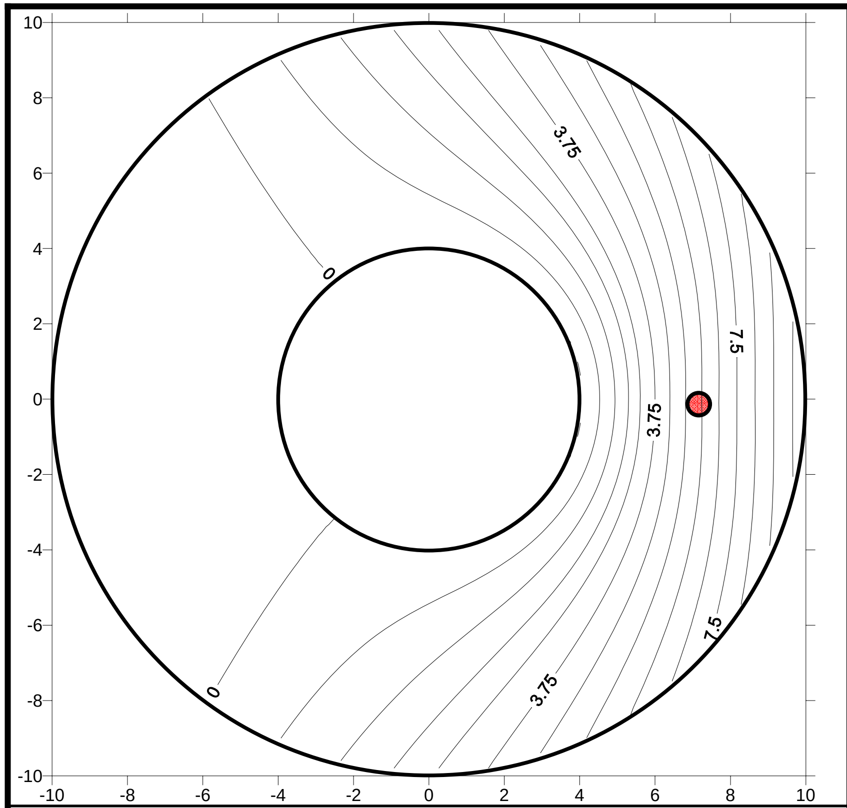


$$(R_\zeta = 0.7,\ \theta_\zeta = 0,\ M = 50)$$

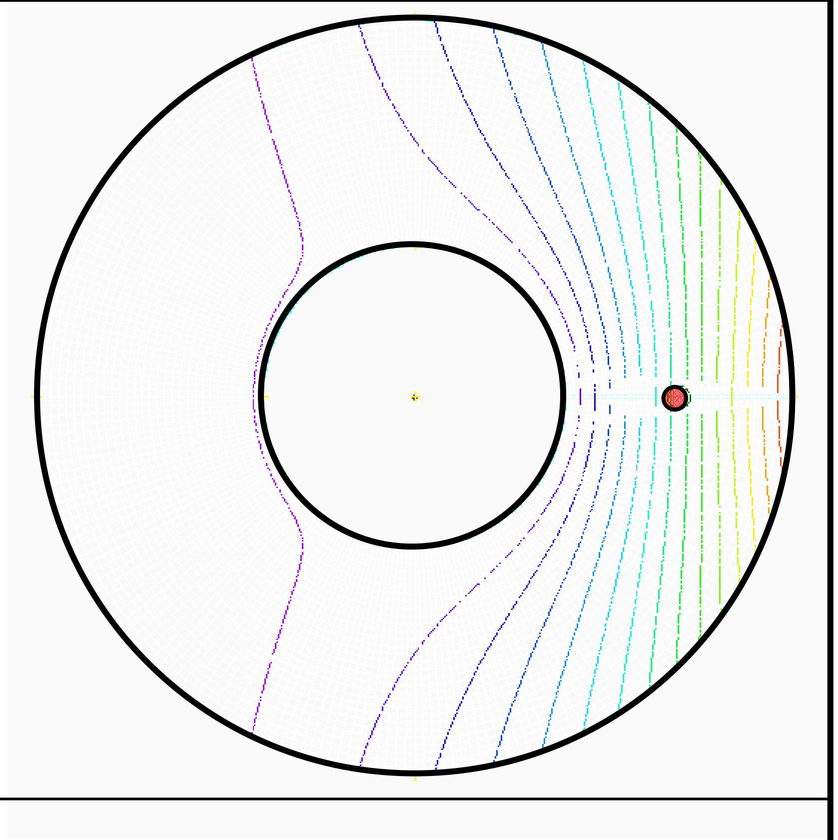
Displacement contours (fixed-fixed)

	
Present method $M = 50$	ABAQUS

Displacement contours (free-simply supported)



Present method (M=50)



ABAQUS

Outlines

- Overview of BEM and motivation
- Unified formulation for the Green's function of null-field approach
  - Boundary integral equations and null-field equations
  - Expansions of boundary densities and kernels
  - Series representation for the Green's function of the annular plate
- Analytical solutions
- ***Conclusions***
- Further studies

Conclusions

- An analytical approach using degenerate kernels and Fourier series for null-field integral equations has been successfully proposed to solve Green's function.
- According to analytical results, only few terms of Fourier series can achieve accurate solutions.
- Good agreement was obtained after compared with previous results, exact solution and ABAQUS data.
- Adewale's results were reexamined.
- Five goals of singularity free, boundary-layer effect free, exponential convergence, well-posed model and mesh free are obtained.

Outlines

- Overview of BEM and motivation
- Unified formulation for the Green's function of null-field approach
  - Boundary integral equations and null-field equations
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  - Series representation for the Green's function of the annular plate
- Analytical solutions
- Conclusions
- ***Further studies***

Further studies

- Degenerate kernel for ellipse, crack or square.
- 2-D problems to 3-D problems.
- Image method with degenerate kernel.
- Plate problem subject to a concentrated load with multiple holes.

The end

Thanks for your kind attention.

Your comments will be highly appreciated.

Welcome to visit the web site of MSVLAB: http://ind.ntou.edu.tw/~msvlab

Simulative test

- What is the Kirchoff plate ?
- How to use the image approach in Robin boundary condition ?
- Why didn't the plate problem involve the Robin boundary condition ?