REGULARIZED MESHLESS METHOD FOR SOLVING LAPLACE EQUATION WITH MULTIPLE HOLES

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ABSTRACT

In this paper, a regularized meshless method (RMM) is developed to solve the two-dimension Laplace problem with multiply-connected domain. The solution is represented by using the double layer potential. The source points can be located on the real boundary by using the proposed regularized technique to regularize the singularity and hypersingularity of the kernel functions. The difficulty of the coincidence of the source and collocation points in traditional method of fundamental solutions is avoided and thereby the diagonal terms of influence matrices are easily determined. The numerical results demonstrate the accuracy of the solutions after comparing with those of exact solution and BEM for the Dirichlet, mixed-type and arbitrary-shape problems with multiple holes. Good agreements are observed.

1. INTRODUCTION

In recent years, science and engineering communities have paid much attention to the meshless method in which the element is free. Because of neither domain nor boundary meshing required for the meshless method, it is very attractive for engineers in model creation. Therefore, the meshless method becomes promising in solving engineering problems.

The method of fundamental solutions (MFS) is one of the meshless methods and belongs to a boundary method of boundary value problems, which can be viewed as a discrete type of indirect boundary element method. The MFS was attributed to Kupradze in 1964 [10], and had been applied to potential [9], Helmholtz [5], diffusion [4], biharmonic [11] and elasticity problems [3]. In the MFS, the solution is approximated by a set of fundamental solutions of the governing equations which are expressed in terms of sources located outside the physical domain. The unknown coefficients in the linear combination of the fundamental solutions are determined by matching the boundary condition. The method is relatively easy to implement. It is adaptive in the sense that it can take into account sharp changes in the solution and in the geometry of the domain and can easily incorporate complex boundary conditions [11]. A survey of the MFS and related method over the last thirty years can be found in Ref. [9]. However, the MFS is still not a popular method because of the debatable artificial boundary (off-set boundary) distance for source location in numerical implementation especially for a complicated geometry. The diagonal coefficients of influence matrices are divergent in conventional case when the off-set boundary approaches the real boundary. In spite of its gain of singularity free, the influence matrices become ill-posed when the off-set boundary is far away from the real boundary. It results in an ill-posed problem since the condition number for the influence matrix becomes very large.

Recently, Young et al. [13] developed a modified MFS, namely regularized meshless method (RMM), to overcome the drawback of MFS for solving the Laplace equation. The method eliminates the well-known drawback of equivocal artificial boundary. The subtracting and adding-back technique [13] can regularize the singularity and hypersingularity of the kernel functions. This method can simultaneously distribute the observation and source points on the real boundary even using the singular kernels instead of non-singular kernels [8]. The diagonal terms of the influence matrices can be extracted out by using the proposed technique. However, the problem solved in [13] is limited for simply-connected problems. For the Laplace problem with multiply-connected domain, the solutions can be obtained by using the finite difference method (FDM) [12] and the boundary element method (BEM) [1,6]. The conventional MFS has also been employed to solve the Laplace problem with multiple circular holes [7].

Following the sources of [13] for simply-connected problems, we extend to the multiply-connected problems by using the RMM in this paper. A general-purpose program is developed to solve the multiply-connected Laplace problems. The results will be compared with those of the BEM and analytical solutions. Furthermore, the sensitivity and convergent test will be studied through several examples to show the validity of our method.

2. FORMULATION

2.1 Governing equation and boundary conditions

Consider a boundary value problem with a potential $u(x)$, which satisfies the Laplace equation as follows:

$$\nabla^2 u(x) = 0, \quad x \in D, \tag{1}$$

subject to boundary conditions,

$$u(x) = \bar{u}, \quad x \in B_p^{\bar{u}}, \quad p = 1, \ 2, \ 3, \cdots, \ m \tag{2}$$

$$t(x) = \bar{t}, \quad x \in B_q^{\bar{t}}, \quad q = 1, \ 2, \ 3, \cdots, \ m \tag{3}$$

where ∇^2 is Laplacian operator, D is the domain of the problem, $t(x) = \frac{\partial u(x)}{\partial n_x}$, m is the total number of boundaries including m-1 numbers of inner boundaries and one outer boundary (the mth boundary), $B_p^{\bar{u}}$ is the essential boundary (Dirichlet boundary) of the pth boundary in which the potential is prescribed by $\bar{u}$ and $B_q^{\bar{t}}$ is the natural boundary (Neumann boundary) of the qth boundary in which the flux is prescribed by $\bar{t}$. Both $B_p^{\bar{u}}$ and $B_q^{\bar{t}}$ construct the whole boundary of the domain D as shown in Figure 1.

2.2 Conventional method of fundamental solutions

By employing the RBF technique [2], the representation of the solution for multiply-connected problem as shown in Figure 1 can be approximated in terms of the α_j strengths of the singularities at s_j as

$$\begin{aligned} u(x_i) &= \sum_{j=1}^{N} T(s_j, x_i)\alpha_j \\ &= \sum_{j=1}^{N_1} T(s_j, x_i)\alpha_j + \sum_{j=N_1+1}^{N_1+N_2} T(s_j, x_i)\alpha_j + \cdots \\ &\quad + \sum_{j=N_1+N_2+\cdots+N_{m-1}+1}^{N} T(s_j, x_i)\alpha_j \end{aligned} \tag{4}$$

$$\begin{aligned} t(x_i) &= \sum_{j=1}^{N} M(s_j, x_i)\alpha_j \\ &= \sum_{j=1}^{N_1} M(s_j, x_i)\alpha_j + \sum_{j=N_1+1}^{N_1+N_2} M(s_j, x_i)\alpha_j + \cdots \\ &\quad + \sum_{j=N_1+N_2+\cdots+N_{m-1}+1}^{N} M(s_j, x_i)\alpha_j \end{aligned} \tag{5}$$

where x_i and s_j represent ith observation point and jth source point, respectively, α_j are the jth unknown coefficients (strength of the singularity), $N_1, N_2, \cdots, N_{m-1}$ are the numbers of source points on $m-1$ numbers of inner boundaries, respectively, N_m is the number of source points on the outer boundary, while N is the total numbers of source points $(N = N_1 + N_2 + \cdots + N_m)$ and $M(s_j, x_i) = \frac{\partial T(s_j, x_i)}{\partial n_{x_i}}$.

The coefficients $\{\alpha_j\}_{j=1}^{N}$ are determined so that BCs are satisfied at the boundary points. The distributions of source points and observation points are shown in Figure 2 (a) for the MFS. The chosen bases are the double layer potentials [3,4,5] as

$$T(s_j, x_i) = \frac{((x_i - s_j), n_j)}{r_{ij}^2}, \tag{6}$$

$$M(s_j, x_i) = \frac{2((x_i - s_j), n_j)((x_i - s_j), \bar{n}_i)}{r_{ij}^4} - \frac{(n_j, \bar{n}_i)}{r_{ij}^2}, \tag{7}$$

where (,) is the inner product of two vectors, r_{ij} is $|s_j - x_i|$, n_j is the normal vector at s_j and $\bar{n}_i$ is the normal vector at x_i.

It is noted that the double layer potentials have both singularity and hypersingularity when source and filed points coincide, which lead to difficulty in the conventional MFS. The off-set distance between the off-set (auxiliary) boundary (B') and the real boundary (B), defined by d, shown in Figure 2 (a) needs to be chosen deliberately. To overcome the abovementioned shortcoming, s_j is distributed on the real boundary as shown in Figure 2 (b), by using the proposed regularized technique as written in section 2.3. The rationale for choosing double layer potential instead of the single layer potential as used in the RMM for the form of RBFs is to take the advantage of the regularization of the subtracting and adding-back technique, so that no off-set distance is needed when evaluating the diagonal coefficients of influence matrices which will be explained in Section 2.4. The single layer potential can not be chosen because the following Eqs. (9), (12), (15) and (18) in Section 2.3 are not satisfied. If the single layer potential is used, the regularization of subtracting and adding-back technique fails.

2.3 Regularized meshless method

When the collocation point x_i approaches the source point s_j, the potentials in Eqs. (4) and (5) become singular. Eqs. (4) and (5) for the multiply-connected problems need to be regularized by using the regularization of subtracting and adding-back technique [13] as follows:

$$\begin{aligned} u(x_i^I) = &\sum_{j=1}^{N_1} T(s_j^I, x_i^I)\alpha_j + \cdots + \sum_{j=N_1+\cdots+N_{p-1}+1}^{N_1+\cdots+N_p} T(s_j^I, x_i^I)\alpha_j + \cdots \\ &+ \sum_{j=N_1+\cdots+N_{m-2}+1}^{N_1+\cdots+N_{m-1}} T(s_j^I, x_i^I)\alpha_j \\ &+ \sum_{j=N_1+\cdots+N_{m-1}+1}^{N} T(s_j^O, x_i^I)\alpha_j \\ &- \sum_{j=N_1+\cdots+N_{p-1}+1}^{N_1+\cdots+N_p} T(s_j^I, x_i^I)\alpha_i, \\ &x_i^I \in B_p,\ p=1,\ 2,\ 3,\cdots,\ m-1 \end{aligned} \tag{8}$$

where x_i^I is located on the inner boundary ($p=1,\ 2,\ 3,\cdots,\ m-1$) and the superscript I and O denote the inward and outward normal vectors, respectively, and

$$\begin{aligned} &\sum_{j=N_1+\cdots+N_{p-1}+1}^{N_1+\cdots+N_p} T(s_j^I, x_i^I) = 0, \\ &x_i^I \in B_p,\ p=1,\ 2,\ 3,\cdots,\ m-1 \end{aligned} \tag{9}$$

Therefore, we can obtain

$$\begin{aligned} u(x_i^I) = &\sum_{j=1}^{N_1} T(s_j^I, x_i^I)\alpha_j + \cdots + \sum_{j=N_1+\cdots+N_{p-1}+1}^{i-1} T(s_j^I, x_i^I)\alpha_j \\ &+ \sum_{j=i+1}^{N_1+\cdots+N_p} T(s_j^I, x_i^I)\alpha_j + \cdots \\ &+ \sum_{j=N_1+\cdots+N_{m-2}+1}^{N_1+\cdots+N_{m-1}} T(s_j^I, x_i^I)\alpha_j \\ &+ \sum_{j=N_1+\cdots+N_{m-1}+1}^{N} T(s_j^O, x_i^I)\alpha_j \\ &-\left[\sum_{j=N_1+\cdots+N_{p-1}+1}^{N_1+\cdots+N_p} T(s_j^I, x_i^I) - T(s_i^I, x_i^I)\right]\alpha_i, \\ &x_i^I \in B_p,\ p=1,\ 2,\ 3,\cdots,\ m-1 \end{aligned} \tag{10}$$

When the observation point x_i^O locates on the outer boundary (*p=m*), Eq. (8) becomes

$$\begin{aligned} u(x_i^O) = &\sum_{j=1}^{N_1} T(s_j^I, x_i^O)\alpha_j + \sum_{j=N_1+1}^{N_1+N_2} T(s_j^I, x_i^O)\alpha_j + \cdots \\ &+ \sum_{j=N_1+\cdots+N_{m-2}+1}^{N_1+\cdots+N_{m-1}} T(s_j^I, x_i^O)\alpha_j + \sum_{j=N_1+\cdots+N_{m-1}+1}^{N} T(s_j^O, x_i^O)\alpha_j \\ &- \sum_{j=N_1+\cdots+N_{m-1}+1}^{N} T(s_j^I, x_i^I)\alpha_i,\ x_i^{O\ and\ I} \in B_p,\ p=m \end{aligned} \tag{11}$$

where

$$\sum_{j=N_1+\cdots+N_{m-1}+1}^{N} T(s_j^I, x_i^I)\alpha_i = 0,\ x_i^I \in B_p,\ p=m \tag{12}$$

Hence, we obtain

$$\begin{aligned} u(x_i^O) = &\sum_{j=1}^{N_1} T(s_j^I, x_i^O)\alpha_j + \sum_{j=N_1+1}^{N_1+N_2} T(s_j^I, x_i^O)\alpha_j + \cdots \\ &+ \sum_{j=N_1+\cdots+N_{m-2}+1}^{N_1+\cdots+N_{m-1}} T(s_j^I, x_i^O)\alpha_j + \sum_{j=N_1+\cdots+N_{m-1}+1}^{i-1} T(s_j^O, x_i^O)\alpha_j \\ &+ \sum_{j=i+1}^{N} T(s_j^O, x_i^O)\alpha_j \\ &-\left[\sum_{j=N_1+\cdots+N_{m-1}+1}^{N} T(s_j^I, x_i^I) - T(s_i^O, x_i^O)\right]\alpha_i, \\ &x_i^{I\ and\ O} \in B_p,\ p=m \end{aligned} \tag{13}$$

Similarly, the boundary flux is obtained as

$$\begin{aligned} t(x_i^I) = &\sum_{j=1}^{N_1} M(s_j^I, x_i^I)\alpha_j + \cdots + \sum_{j=N_1+\cdots+N_{p-1}+1}^{N_1+\cdots+N_p} M(s_j^I, x_i^I)\alpha_j + \cdots \\ &+ \sum_{j=N_1+\cdots+N_{m-2}+1}^{N_1+\cdots+N_{m-1}} M(s_j^I, x_i^I)\alpha_j \\ &+ \sum_{j=N_1+\cdots+N_{m-1}+1}^{N} M(s_j^O, x_i^I)\alpha_j \\ &- \sum_{j=N_1+\cdots+N_{p-1}+1}^{N_1+\cdots+N_p} M(s_j^I, x_i^I)\alpha_i, \\ &x_i^I \in B_p,\ p=1,\ 2,\ 3,\cdots,\ m-1 \end{aligned} \tag{14}$$

where

$$\begin{aligned} &\sum_{j=N_1+\cdots+N_{p-1}+1}^{N_1+\cdots+N_p} M(s_j^I, x_i^I) = 0, \\ &x_i^I \in B_p,\ p=1,\ 2,\ 3,\cdots,\ m-1 \end{aligned} \tag{15}$$

Therefore, we can obtain

$$\begin{aligned} t(x_i^I) = &\sum_{j=1}^{N_1} M(s_j^I, x_i^I)\alpha_j + \cdots + \sum_{j=N_1+\cdots+N_{p-1}+1}^{i-1} M(s_j^I, x_i^I)\alpha_j \\ &+ \sum_{j=i+1}^{N_1+\cdots+N_p} M(s_j^I, x_i^I)\alpha_j + \cdots \\ &+ \sum_{j=N_1+\cdots+N_{m-2}+1}^{N_1+\cdots+N_{m-1}} M(s_j^I, x_i^I)\alpha_j \\ &+ \sum_{j=N_1+\cdots+N_{m-1}+1}^{N} M(s_j^O, x_i^I)\alpha_j \\ &-\left[\sum_{j=N_1+\cdots+N_{p-1}+1}^{N_1+\cdots+N_p} M(s_j^I, x_i^I) - M(s_i^I, x_i^I)\right]\alpha_i, \\ &x_i^I \in B_p,\ p=1,\ 2,\ 3,\cdots,\ m-1 \end{aligned} \tag{16}$$

When the observation point locates on the outer boundary (*p=m)*, Eq. (14) yields

$$
\begin{aligned}
t(x_i^O) &= \sum_{j=1}^{N_1} M(s_j^I, x_i^O)\alpha_j + \sum_{j=N_1+1}^{N_1+N_2} M(s_j^I, x_i^O)\alpha_j + \cdots \\
&+ \sum_{j=N_1+\cdots+N_{m-2}+1}^{N_1+\cdots+N_{m-1}} M(s_j^I, x_i^O)\alpha_j + \sum_{j=N_1+\cdots+N_{m-1}+1}^{N} M(s_j^O, x_i^O)\alpha_j \\
&- \sum_{j=N_1+\cdots+N_{m-1}+1}^{N} M(s_j^I, x_i^I)\alpha_i, \quad x_i^{O \text{ and } I} \in B_p, p = m
\end{aligned} \tag{17}
$$

where

$$
\sum_{j=N_1+\cdots+N_{m-1}+1}^{N} M(s_j^I, x_i^I) = 0, \quad x_i^I \in B_p, \quad p = m \tag{18}
$$

Hence, we obtain

$$
\begin{aligned}
t(x_i^O) &= \sum_{j=1}^{N_1} M(s_j^I, x_i^O)\alpha_j + \sum_{j=N_1+1}^{N_1+N_2} M(s_j^I, x_i^O)\alpha_j + \cdots \\
&+ \sum_{j=N_1+\cdots+N_{m-2}+1}^{N_1+\cdots+N_{m-1}} M(s_j^I, x_i^O)\alpha_j + \sum_{j=N_1+\cdots+N_{m-1}+1}^{i-1} M(s_j^O, x_i^O)\alpha_j \\
&+ \sum_{j=i+1}^{N} M(s_j^O, x_i^O)\alpha_j \\
&- \left[\sum_{j=N_1+\cdots+N_{m-1}+1}^{N} M(s_j^I, x_i^I) - M(s_i^O, x_i^O) \right]\alpha_i, \\
& x_i^{O \text{ and } I} \in B_p, \quad p = m
\end{aligned} \tag{19}
$$

The detailed derivations of Eqs. (9), (12), (15) and (18) are given in the reference [13]. According to the dependence of the normal vectors for inner and outer boundaries [13], their relationships are

$$
\begin{cases} T(s_j^I, x_i^I) = -T(s_j^O, x_i^O), & i \neq j \\ T(s_j^I, x_i^I) = T(s_j^O, x_i^O), & i = j \end{cases} \tag{20}
$$

$$
\begin{cases} M(s_j^I, x_i^I) = M(s_j^O, x_i^O), & i \neq j \\ M(s_j^I, x_i^I) = M(s_j^O, x_i^O), & i = j \end{cases} \tag{21}
$$

where the left hand side and right hand side of the equal sign in Eqs.(20) and (21) denote the kernels for observation and source point with the inward and outward normal vectors, respectively.

By using the proposed technique, the singular terms in Eqs. (4) and (5) have been transformed into regular terms ($-\left[\sum_{j=N_1+N_2+\cdots+N_{P-1}+1}^{N_1+N_2+\cdots+N_p} T(s_j^I, x_i^I) - T(s_i^{I \text{ or } O}, x_i^{I \text{ or } O}) \right]$ and $-\left[\sum_{j=N_1+\cdots+N_{p-1}+1}^{N_1+\cdots+N_p} M(s_j^I, x_i^I) - M(s_i^{I \text{ or } O}, x_i^{I \text{ or } O}) \right]$), in Eqs. (10), (13), (16) and (19), respectively, where $p = 1, \; 2, \; 3, \cdots, \; m$. The terms of $\sum_{j=N_1+\cdots+N_{P-1}+1}^{N_1+\cdots+N_p} T(s_j^I, x_i^I)$ and $\sum_{j=N_1+\cdots+N_{p-1}+1}^{N_1+\cdots+N_p} M(s_j^I, x_i^I)$ are the adding-back terms and the terms of $T(s_i^{I \text{ or } O}, x_i^{I \text{ or } O})$ and $M(s_i^{I \text{ or } O}, x_i^{I \text{ or } O})$ are the subtracting terms in the two brackets for reqularization. After using the abovementional method of regularization of subtracting and adding-back technique [13], we are able to remove the singularity and hypersingularity of the kernel functions.

2.4 Derivation of influence matrices for arbitrary domain problems

By collocating N observation points to match with the BCs from Eqs. (10) and (13) for the Dirichlet problem, and the linear algebraic equation is obtained

$$
\begin{Bmatrix} \bar{u}_1 \\ \vdots \\ \bar{u}_N \end{Bmatrix} = \begin{bmatrix} [T_{11}]_{N_1 \times N_1} & \cdots & [T_{1m}]_{N_1 \times N_m} \\ \vdots & \ddots & \vdots \\ [T_{m1}]_{N_m \times N_1} & \cdots & [T_{mm}]_{N_m \times N_m} \end{bmatrix}_{N \times N} \begin{Bmatrix} \alpha_1 \\ \vdots \\ \alpha_N \end{Bmatrix} \tag{22}
$$

where

$$
\begin{aligned}
[T_{11}] &= \begin{bmatrix} T_{11}^{11} & \cdots & T(s_{N_1}^I, x_1^I) \\ \vdots & \ddots & \vdots \\ T(s_1^I, x_{N_1}^I) & \cdots & T_{11}^{N_1 N_1} \end{bmatrix}_{N_1 \times N_1} \\
T_{11}^{11} &= -\left[\sum_{j=1}^{N_1} T(s_j^I, x_1^I) - T(s_1^I, x_1^I) \right] \\
T_{11}^{N_1 N_1} &= -\left[\sum_{j=1}^{N_1} T(s_j^I, x_{N_1}^I) - T(s_{N_1}^I, x_{N_1}^I) \right]
\end{aligned} \tag{23}
$$

$$
[T_{1m}] = \begin{bmatrix} T(s_{N_1+\cdots+N_{m-1}+1}^O, x_1^I) & \cdots & T(s_N^O, x_1^I) \\ \vdots & \ddots & \vdots \\ T(s_{N_1+\cdots+N_{m-1}+1}^O, x_{N_1}^I) & \cdots & T(s_N^O, x_{N_1}^I) \end{bmatrix}_{N_1 \times N_m} \tag{24}
$$

$$
\begin{aligned}
[T_{m1}] &= \begin{bmatrix} T_{m1}^{11} & \cdots & T_{m1}^{1N_1} \\ \vdots & \ddots & \vdots \\ T(s_1^I, x_N^O) & \cdots & T(s_{N_1}^I, x_N^O) \end{bmatrix}_{N_m \times N_1} \\
T_{m1}^{11} &= T(s_1^I, x_{N_1+\cdots+N_{m-1}+1}^O) \\
T_{m1}^{1N_1} &= T(s_{N_1}^I, x_{N_1+\cdots+N_{m-1}+1}^O)
\end{aligned} \tag{25}
$$

$$
\begin{aligned}
[T_{mm}] &= \begin{bmatrix} T_{mm}^{11} & \cdots & T(s_N^O, x_{N_1+\cdots+N_{m-1}+1}^O) \\ \vdots & \ddots & \vdots \\ T(s_{N_1+\cdots+N_{m-1}+1}^O, x_N^O) & \cdots & T_{mm}^{N_m N_m} \end{bmatrix}_{N_m \times N_m} \\
T_{mm}^{11} &= -[\sum_{j=N_1+\cdots+N_{m-1}+1}^{N} T(s_j^I, x_{N_1+\cdots+N_{m-1}+1}^I) \\
&\quad - T(s_{N_1+\cdots+N_{m-1}+1}^O, x_{N_1+\cdots+N_{m-1}+1}^O)] \\
T_{mm}^{N_m N_m} &= -\left[\sum_{j=N_1+\cdots+N_{m-1}+1}^{N} T(s_j^I, x_N^I) - T(s_N^O, x_N^O) \right]
\end{aligned} \tag{26}
$$

For the Neumann problem, Eqs. (16) and (19) yield

$$
\begin{Bmatrix} \bar{t}_1 \\ \vdots \\ \bar{t}_N \end{Bmatrix} = \begin{bmatrix} [M_{11}]_{N_1 \times N_1} & \cdots & [M_{1m}]_{N_1 \times N_m} \\ \vdots & \ddots & \vdots \\ [M_{m1}]_{N_m \times N_1} & \cdots & [M_{mm}]_{N_m \times N_m} \end{bmatrix}_{N \times N} \begin{Bmatrix} \alpha_1 \\ \vdots \\ \alpha_N \end{Bmatrix} \tag{27}
$$

in which

$$
\begin{aligned}
[M_{11}] &= \begin{bmatrix} M_{11}^{11} & \cdots & M(s_{N_1}^I, x_1^I) \\ \vdots & \ddots & \vdots \\ M(s_1^I, x_{N_1}^I) & \cdots & M_{11}^{N_1 N_1} \end{bmatrix}_{N_1 \times N_1} \\
M_{11}^{11} &= -\left[\sum_{j=1}^{N_1} M(s_j^I, x_1^I) - M(s_1^I, x_1^I) \right] \\
M_{11}^{N_1 N_1} &= -\left[\sum_{j=1}^{N_1} M(s_j^I, x_{N_1}^I) - M(s_{N_1}^I, x_{N_1}^I) \right]
\end{aligned} \tag{28}
$$

$$
[M_{1m}] = \begin{bmatrix} M(s_{N_1+\cdots+N_{m-1}+1}^O, x_1^I) & \cdots & M(s_N^O, x_1^I) \\ \vdots & \ddots & \vdots \\ M(s_{N_1+\cdots+N_{m-1}+1}^O, x_{N_1}^I) & \cdots & M(s_N^O, x_{N_1}^I) \end{bmatrix}_{N_1 \times N_m} \tag{29}
$$

$$[M_{m1}]=\begin{bmatrix} M_{m1}^{11} & \cdots & M_{m1}^{1N_1} \\ \vdots & \ddots & \vdots \\ M(s_1^I, x_N^O) & \cdots & M(s_{N_1}^I, x_N^O) \end{bmatrix}_{N_m \times N_1} \quad (30)$$

$$M_{m1}^{11} = M(s_1^I, x_{N_1+\cdots+N_{m-1}+1}^O)$$

$$M_{m1}^{1N_1} = M(s_{N_1}^I, x_{N_1+\cdots+N_{m-1}+1}^O)$$

$$[M_{mm}]=\begin{bmatrix} M_{mm}^{11} & \cdots & M(s_N^O, x_{N_1+\cdots+N_{m-1}+1}^O) \\ \vdots & \ddots & \vdots \\ M(s_{N_1+\cdots+N_{m-1}+1}^O, x_N^O) & \cdots & M_{mm}^{N_m N_m} \end{bmatrix}_{N_m \times N_m}$$

$$M_{mm}^{11} = -[\sum_{j=N_1+\cdots+N_{m-1}+1}^{N} M(s_j^I, x_{N_1+\cdots+N_{m-1}+1}^I) - M(s_{N_1+\cdots+N_{m-1}+1}^O, x_{N_1+\cdots+N_{m-1}+1}^O)] \quad (31)$$

$$M_{mm}^{N_m N_m} = -\left[\sum_{j=N_1+\cdots+N_{m-1}+1}^{N} M(s_j^I, x_N^I) - M(s_N^O, x_N^O)\right]$$

For the mixed-type problem, a linear combination of Eqs. (22) and (27) is required to satisfy the mixed-type BCs. After the unknown density ($\{\alpha_j\}_{j=1}^N$) are obtained by solving the linear algebraic equations, the field solution can be solved by using Eqs. (4) and (5).

3. NUMERICAL EXAMPLES

Case 1: Dirichlet problem

The multiply-connected Dirichlet problem is shown in Figure 3, and an analytical solution is

$$u(r,\theta) = \frac{1}{r}\cos(\theta) \, , \quad (32)$$

The exact solution is plotted in Figure 4 . The field solutions by using the RMM (360 points) and the BEM (360 elements) are shown in Figure 5 (a) and Figure 5 (b).

Case 2: Mixed-type problem

The mixed-type problem for multiply-connected domain is shown in Figure 6, and an analytical solution is available as follows:

$$u = r^3 \cos(3\theta) \, , \quad (33)$$

The exact solution is plotted in Figure 7. The defined norm error is

$$\int_0^{2\pi} \left| u_{exact}(r = 0.5, \theta) - u(r = 0.5, \theta) \right|^2 d\theta \quad (34)$$

The norm error of the RMM versus the total number N of source points is shown in Figure 8 and the convergent result is found after distributing 200 points. The field solutions by using the RMM (400 points) and the BEM (800 elements) are shown in Figures 9 (a) and (b), respectively. After comparing Figure 9 (a) with Figure 9 (b) and Figure 7, the RMM result agrees with the exact solution and the BEM result.

Case 3: Arbitrary-shape problem

The arbitrary-shape problem for continuous boundary conditions are given in Figure 10. An analytical solution is available as follows:

$$u = e^x \cos(y) \quad (35)$$

The field potential in Eq. (35) is shown in Figure 11 (a). The field solutions by using the RMM (400 points) is shown in Figures 11 (b). The norm error is defined as

$$\int_0^{2\pi} \left| u_{exact}(r = 0.9, \theta) - u(r = 0.9, \theta) \right|^2 d\theta \quad (36)$$

The norm error versus the total number N of source points is shown in Figure 12 and the convergent result is found after distributing over 200 points.

4. CONCLUSIONS

In this study, we used the RMM to solve the Laplace problems with multiply-connected domain subject to the Dirichle、mixed-type and arbitrary-shape. Only the boundary nodes on the real boundary are required. The major difficulty of the coincidence of the source and collocation points in the conventional MFS is then circumvented. Furthermore, the controversy of the off-set boundary outside the physical domain by using the conventional MFS no longer exists. Although it results in the singularity and hypersingularity due to the use of double layer potential, the finite values of the diagonal terms for the influence matrices have been extracted out by employing the regularization technique. The numerical results were obtained well by applying the developed program to three examples after compared with those of analytical solutions and BEM.

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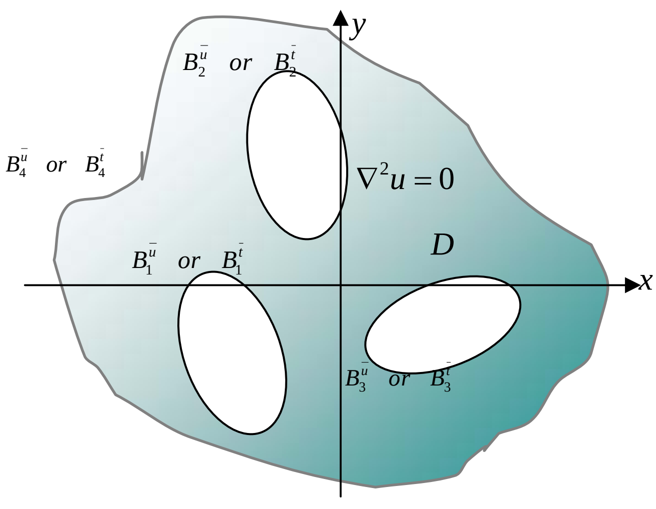

Figure 1 Laplace problem with holes

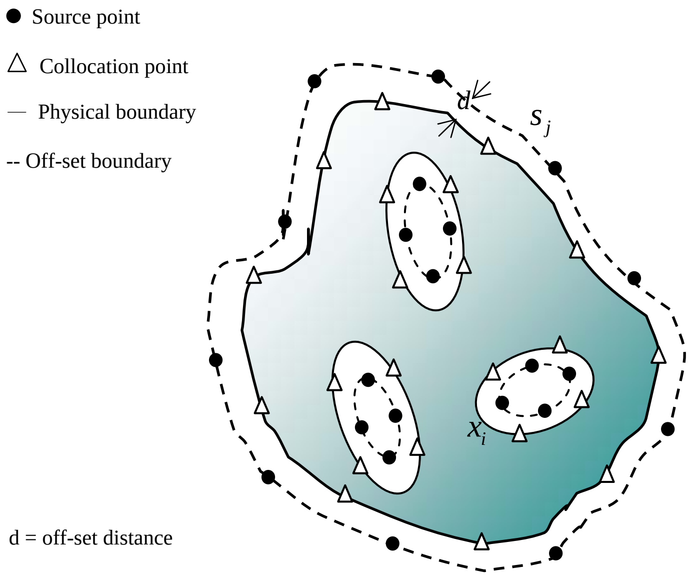

Figure 2 (a) Conventional MFS

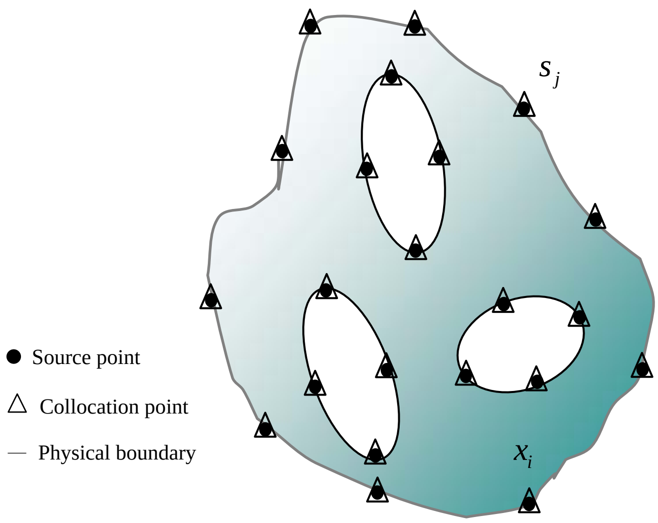

Figure 2 (b) RMM

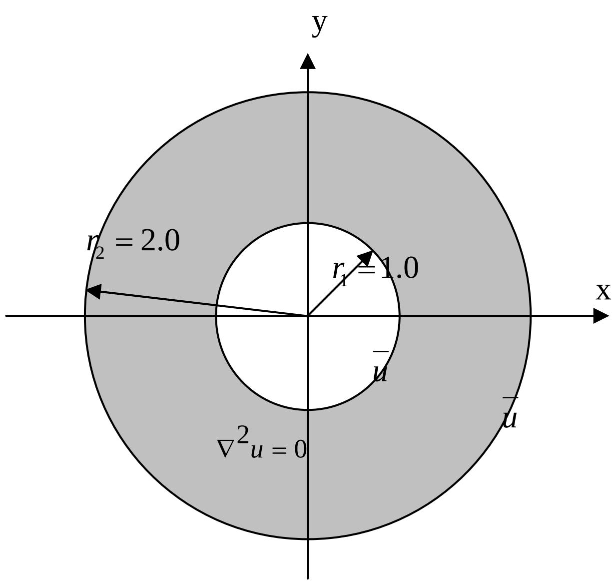

Figure 3 Problem sketch

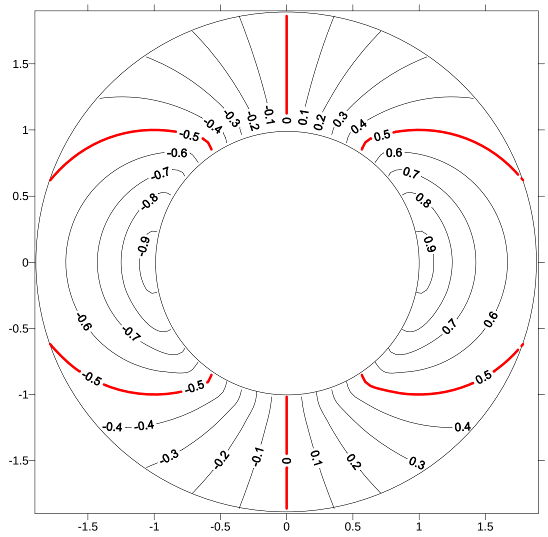

Figure 4 Exact solution for the case 1

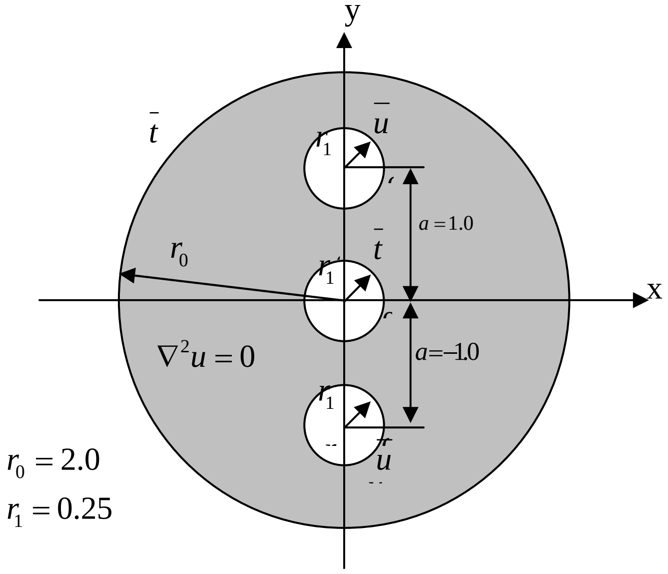

Figure 6 Problem sketch

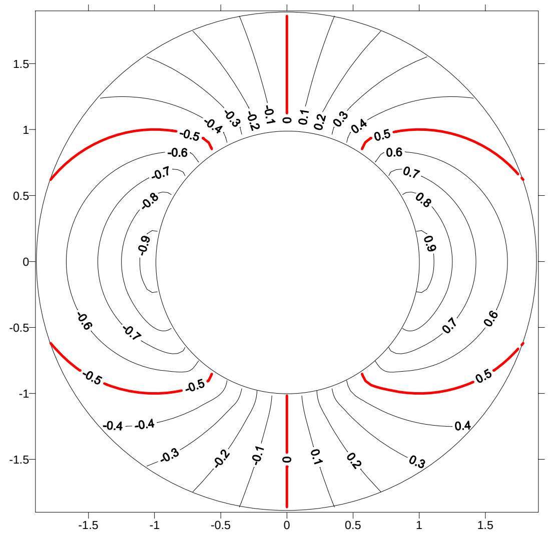

Figure 5 (a) RMM for the case 1

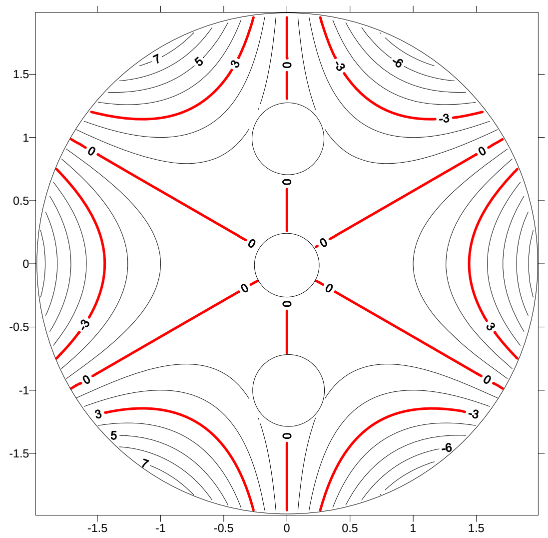

Figure 7 Exact solution for the case 2

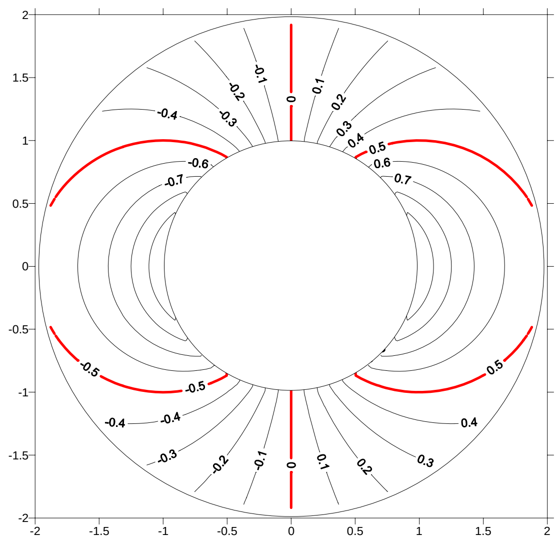

Figure 5 (b) BEM for the case 1

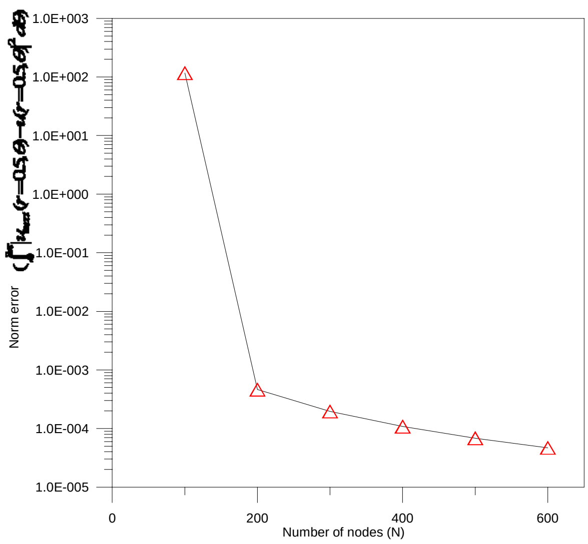

Figure 8 The norm error along radius $r = 0.5$ versus total number of nodes

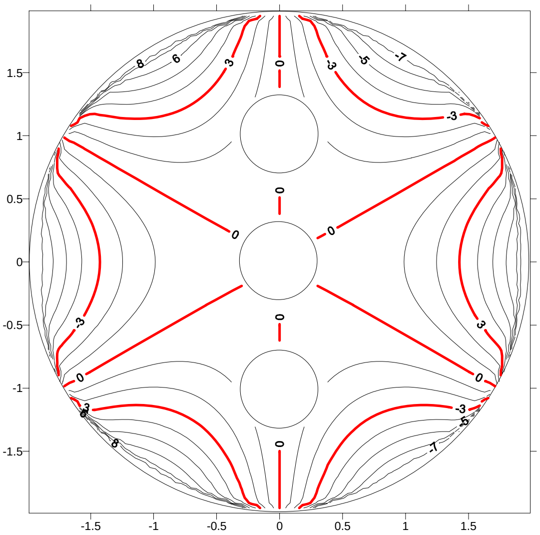

Figure 9 (a) RMM for the case 2

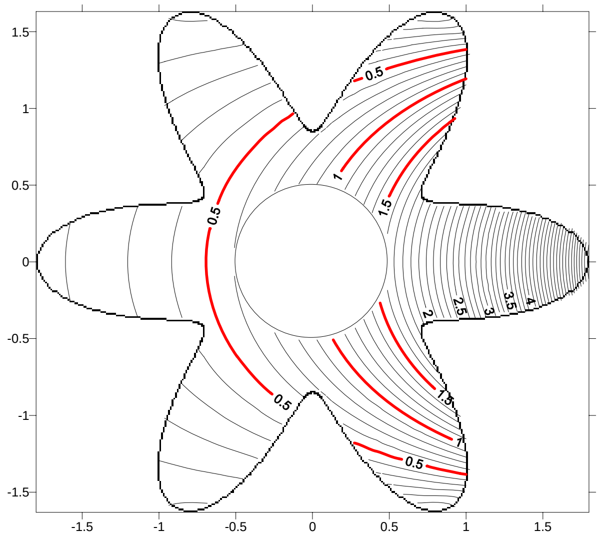

Figure 11 (a) Exact solution for the case 3

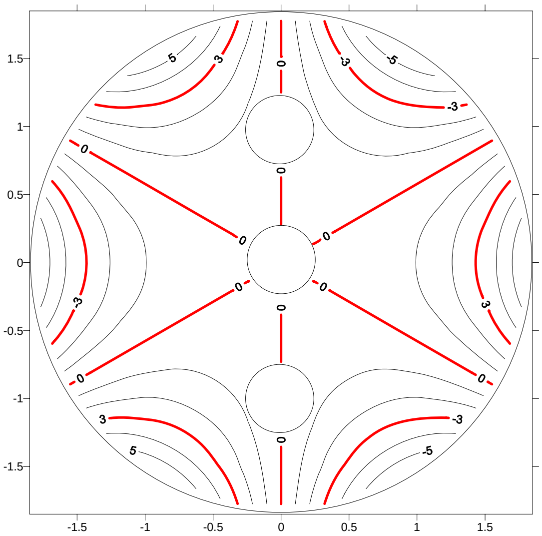

Figure 9 (b) BEM for the case 2

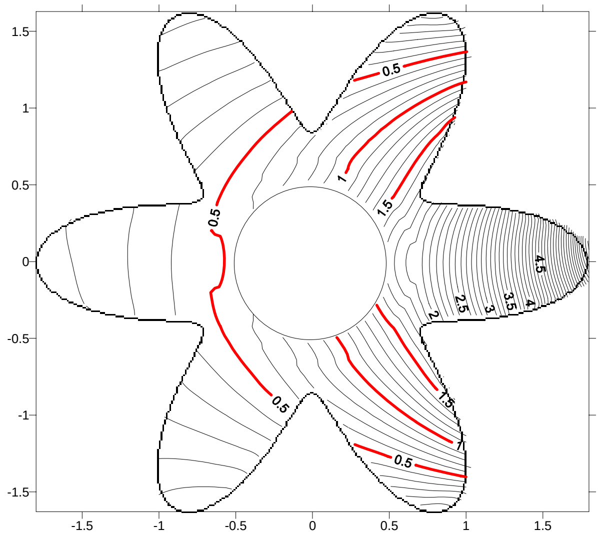

Figure 11 (b) RMM for the case 3

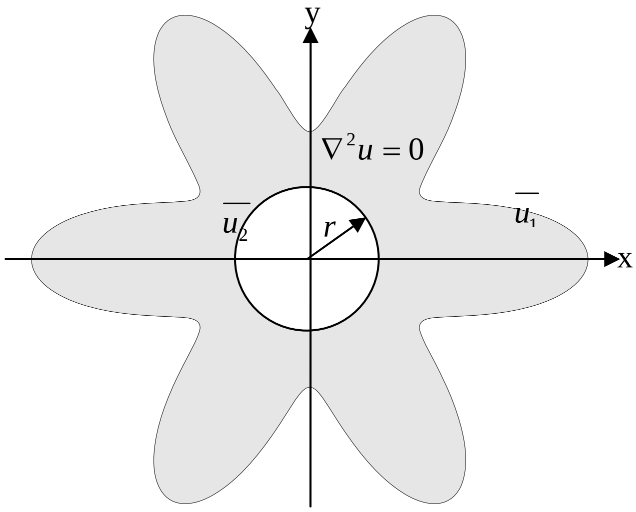

Figure 10 Problem sketch

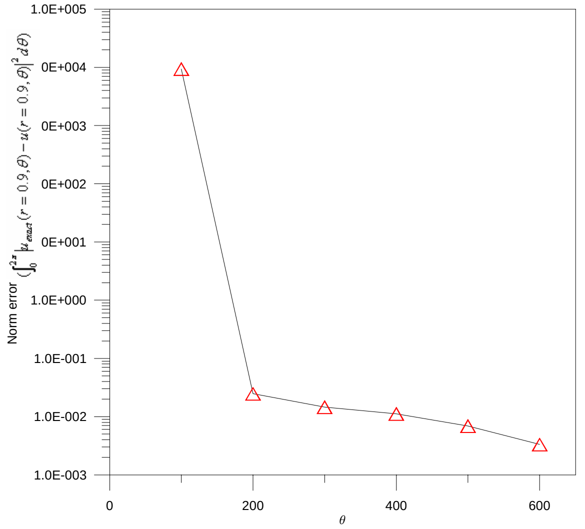

Figure 12. The norm error along the radius $r = 0.9$ versus the number of nodes for the case 3.