Regularized meshless method for solving the Cauchy problem

Kuo-Lun Wu[1*], Kue-Hong Chen[2], Jeng-Tzong Chen[1] and Jeng-Hong Kao[1]

[1]**Department of Harbor and River Engineering, National Taiwan Ocean University**
[2]**Department of Information Management, Toko University**
*kevin7442@hotmail.com

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ABSTRACT

In this paper, the Laplace problem with overspecified boundary conditions is investigated by using the regularized meshless method. The solution is represented by a distribution of the kernel functions of double-layer potentials. By using the desingularization technique of adding-back and subtracting terms to regularize the singularity and hypersingularity of the kernel functions, the source points can be located on the real boundary and the diagonal terms of influence matrices are determined. The main difficulty of the coincidence of the source and collocation points then disappears. The accompanied ill-posed problem can be remedied by using Tikhonov regularization technique, linear regularization method and truncated singular value decomposition. The optimal parameters of the Tikhonov technique and linear regularization method and truncated singular value decomposition are derived by adopting L-curve concept. The numerical evidences of the regularized meshless method are given to verify the accuracy of the solutions after comparing with the results of analytical solution. The comparison of Tikhonov regularization technique, linear regularization method and truncated singular value decomposition are also discussed in the example.

Keywords: regularized meshless method, Tikhonov technique, linear regularization method, truncated singular value decomposition, L-curve technique, Cauchy problem.

1. INTRODUCTION

Inverse problems are presently becoming more important in many fields of science and engineering [15,20]. They may be one of the following problems or their combinations. (Ⅰ) lack the determination of the domain, its boundary, or an inner unknown boundary, (Ⅱ) lack inference of the governing equation, (Ⅲ) lack identification of boundary conditions and/or initial conditions (Cauchy problem), (Ⅳ) lack determination of the material properties involved, (Ⅴ) lack determination of the forces or inputs acting in the domain [18]. The Cauchy problem is focused in this paper.

Sometimes, unreasonable results occur in the Cauchy problem subjected to the measured and contaminated errors on the overspecified boundary condition, because of the ill-posed behavior in the linear algebraic system [14,19]. Mathematically speaking, the Cauchy problem is ill-posed since the solution is very sensitive to the given data. Such a divergent problem could be avoided by using regularization methods [15,20]. For examples, truncated singular value decomposition (TSVD) [17], Tikhonov regularization technique [2] and linear regularization method [9] have been applied to treat with the divergent problems. The three methods can obtain the convergent solution more precisely and reasonably. The TSVD, Tikhonov regularization technique and linear regularization method, had been successfully applied to overcome the ill-posed problem of the Laplace equation [4,10]. In this paper, the comparison of three regularization techniques is made to obtain a better method.

For the Cauchy problem, the influence matrix is often ill-posed such that the regularization technique which regularizes the influence matrix is necessary. The TSVD transform the ill-posed matrix into a well-posed one by choosing an appropriate truncated number for *i*. Similarly, the Tikhonov technique and linear regularization method transform into a well-posed one by choosing an appropriate parameter for λ and $\lambda * H$ [3]. The appropriate truncated number (or parameter) can be determined according to a compromise point between regularization errors (due to data smoothing) and perturbation errors (due to noise disturbance) by implementing the L-curve concept [11,16]. The corner of the L-curve determines the optimal value of λ (or *i*) which will be employed to provide the compromise point and will be elaborated on later.

During the last decade, scientific researchers have paid attention to the meshless methods for solving Cauchy problems in which the mesh or element is free [5]. The method of fundamental solutions (MFS) which is a kind of meshless methods has been extensively applied to solve some engineering problems [1,8]. However, the location of source and observation point is vital to the accuracy of the solution by implementing the conventional MFS. But it still accompanies some difficulties at the ill-posed problem. Consequently, a novel meshless method - regularized meshless method (RMM) [6,7,12,13] has been employed to solve the potential problems based on the potential theory as well as the desingularization of subtracting and adding-back technique to regularize the singularity and hypersingularity of the kernel functions. The proposed method distributes the observation and source points on the coincident locations of the real boundary even using the singular kernels (double-layer potentials) instead of non-singular kernels and still maintains the spirit of the MFS. The diagonal terms of the influence matrices can be derived by using the proposed technique.

In this paper, we are going to employ the RMM in conjunction with the TSVD, Tikhonov technique, linear regularization method and L-curve concept to circumvent the ill-posed problem. To obtain the optimal truncated number or parameter, L-curve concept is employed. Finally, the results of the example contaminated with artificial noise on the overspecified boundary condition are given to illustrate the validity of the proposed technique. Good agreements are observed as comparing analytical solutions.

2. Formulation

2.1 Governing equation and over-specified boundary condition

To consider the inverse problem for Laplace equation with overspecified boundary condition as shown in Fig. 1 satisfies:

$$\nabla^2\phi(x)=0, \qquad x\in D \tag{1}$$

subjected to the boundary condition on B_1 as

$$\phi(x)=\overline{\phi},\quad \psi(x)=\overline{\psi},\quad x\in B_1 \tag{2}$$

where ∇^2 is the Laplacian operator, D is the domain of interesting, $\psi(x)=\dfrac{\partial\phi(x)}{\partial n_x}$ in which n_x is the normal vector at x, B is the whole boundary which consists of the known boundary (B_1), and the unknown boundary (B_2).

2.2 Method of solution

2.2.1 Method of fundamental solutions

By employing the radial basis functions (RBFs) concept [6], the representation of the solution for interior problem can be approximated in terms of the strengths α^j of the singularities s^j as

$$\phi(x^i)=\sum_{j=1}^{N+M}A^{(i)}(s^j,x^i)\alpha^j, \tag{3}$$

$$\psi(x^i)=\sum_{j=1}^{N+M}B^{(i)}(s^j,x^i)\alpha^j, \tag{4}$$

where $A^{(i)}(s^j,x^i)$ is RBF in which the superscript (i) denotes the interior domain, $B^{(i)}(s^j,x^i)=\dfrac{\partial A^{(i)}(s^j,x^i)}{\partial n_{x^i}}$, α^j is the j th unknown coefficient (strength of the singularity), s^j is j th source point (singularity), x^i is i th observation point, and N is number of the boundary points on B_1 and M is number of the boundary points on B_2. Boundary condition is satisfied at the boundary points, $\{x^i\}_{i=1}^{N+M}$ so that the coefficients $\{\alpha^j\}_{j=1}^{N+M}$ can be determined.

The chosen RBFs of Eqs. (3) and (4) in this paper are the double-layer potentials in the potential theory as

$$A^{(i)}(s^j,x^i)=\frac{-n_k y_k}{\bar{r}_{ij}^{2}}, \tag{5}$$

$$B^{(i)}(s^j,x^i)=2\frac{y_k y_l n_k \overline{n}_l}{\bar{r}_{ij}^{4}}-\frac{n_l\overline{n}_l}{\bar{r}_{ij}^{2}}, \tag{6}$$

where $\bar{r}_{ij}=\left|s^j-x^i\right|$, n_k is the kth component of the outward normal vector at s^j ; $\overline{n_k}$ is the kth component of the outward normal vector at x^i and $y_k=x_k^i-s_k^j$.

2.2.1 Regularized meshless method

It is noted that the double layer potentials have both singularity and hypersingularity when the source point and the observation point are coincided, which lead to troublesome singular kernels and controversially auxiliary boundary in the conventional MFS. The off-set distance between the off-set (auxiliary) boundary (B') and the real boundary (B) defined by d as shown in Fig. 2 (a) and (b) needs to be chosen deliberately. To overcome the abovementioned drawback, s^j is distributed on the real boundary as shown in Fig. 2 (c) and (d) by using the proposed regularization technique.

When the collocation point x^i approaches to the source point s^j, Eqs. (3) and (4) become singular. Eqs. (3) and (4) for the interior problems need to be regularized by using subtracting and adding-back technique [6] as follows :

$$\begin{aligned}\phi(x^i)&=\sum_{j=1}^{N+M}A^{(i)}(s^j,x^i)\alpha^j-\sum_{j=1}^{N+M}A^{(e)}(s^j,x^i)\alpha^i\\&=\sum_{j=1}^{i-1}A^{(i)}(s^j,x^i)\alpha^j+\sum_{j=i+1}^{N+M}A^{(i)}(s^j,x^i)\alpha^j\\&\quad+\left[\sum_{m=1}^{N+M}A^{(i)}(s^m,x^i)-A^{(i)}(s^i,x^i)\right]\alpha^i,\ x^i\in B\end{aligned} \tag{7}$$

$$\begin{aligned}\psi(x^i)&=\sum_{j=1}^{N+M}B^{(i)}(s^j,x^i)\alpha^j-\sum_{j=1}^{N+M}B^{(e)}(s^j,x^i)\alpha^i\\&=\sum_{j=1}^{i-1}B^{(i)}(s^j,x^i)\alpha^j+\sum_{j=i+1}^{N+M}B^{(i)}(s^j,x^i)\alpha^j\\&\quad-\left[\sum_{m=1}^{N+M}B^{(i)}(s^m,x^i)-B^{(i)}(s^i,x^i)\right]\alpha^i,\ x^i\in B\end{aligned} \tag{8}$$

in which

$$\sum_{j=1}^{N+M}A^{(e)}(s^j,x^i)=0,\ x^i\in B \tag{9}$$

$$\sum_{j=1}^{N+M}B^{(e)}(s^j,x^i)=0,\ x^i\in B \tag{10}$$

The detailed derivations of Eqs. (9) and (10) are given in reference [6]. The superscript (e) of $A^{(e)}(s^j,x^i)$

and $B^{(e)}(s^j, x^i)$ denotes the exterior domain, the term of $\sum_{m=1}^{N+M} A^{(i)}(s^m, x^i)$ and $\sum_{m=1}^{N+M} B^{(i)}(s^m, x^i)$ are the adding-back terms and the terms of $A^{(i)}(s^i, x^i)$ and $B^{(i)}(s^i, x^i)$ are the subtracting terms in two brackets for the special treatment technique.

2.2.3 Derivation of diagonal coefficients of influence matrices

We can obtain the following linear algebraic system after collocating $N+M$ observation points, $\{x^i\}_{i=1}^{N+M}$, to the real boundary in Eq. (7) as :

$$\begin{Bmatrix} \{\bar{\phi}_1\}_{N\times 1} \\ \{\phi_2\}_{M\times 1} \end{Bmatrix} = \begin{bmatrix} [A_1]_{N\times(N+M)} \\ [A_2]_{M\times(N+M)} \end{bmatrix} \{\alpha\}_{(N+M)\times 1} \tag{11}$$

where

$$\{\bar{\phi}_1\}_{N\times 1} = \begin{Bmatrix} \bar{\phi}_1 \\ \bar{\phi}_2 \\ \vdots \\ \bar{\phi}_N \end{Bmatrix}, \quad \{\phi_2\}_{M\times 1} = \begin{Bmatrix} \phi_{N+1} \\ \phi_{N+2} \\ \vdots \\ \phi_{N+M} \end{Bmatrix} \tag{12}$$

$$[A_1]_{N\times(N+M)} = \begin{bmatrix} \sum_{m=1}^{N+M} a_{1,m} - a_{1,1} & a_{1,2} & a_{1,3} & \cdots & a_{1,N} & \cdots & a_{1,N+M} \\ a_{2,1} & \sum_{m=1}^{N+M} a_{2,m} - a_{2,2} & a_{2,3} & \cdots & a_{2,N} & \cdots & a_{2,N+M} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{N,1} & a_{N,2} & a_{N,3} & \cdots & \sum_{m=1}^{N+M} a_{N,m} - a_{N,N} & \cdots & a_{N,N+M} \end{bmatrix},$$

$$[A_2]_{M\times(N+M)} = \begin{bmatrix} a_{N+1,1} & a_{N+1,2} & a_{N+1,3} & \cdots & \sum_{m=1}^{N+M} a_{N+1,m} - a_{N+1,N+1} & \cdots & a_{N+1,N+M} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{N+M,1} & a_{N+M,2} & a_{N+M,3} & \cdots & a_{N+M,N+1} & \cdots & \sum_{m=1}^{N+M} a_{N+M,m} - a_{N+M,N+M} \end{bmatrix} \tag{13}$$

$$\{\alpha\}_{(N+M)\times 1} = \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_N \\ \alpha_{N+1} \\ \vdots \\ \alpha_{N+M} \end{Bmatrix}, \tag{14}$$

in which $a_{ij} = A^{(i)}(s^j, x^i)$ and $i, j = 1, 2, \ldots, N+M$.

In a similar way, Eq. (8) yields

$$\begin{Bmatrix} \{\bar{\psi}_1\}_{N\times 1} \\ \{\psi_2\}_{M\times 1} \end{Bmatrix} = \begin{bmatrix} [B_1]_{N\times(N+M)} \\ [B_2]_{M\times(N+M)} \end{bmatrix} \{\alpha\}_{(N+M)\times 1} \tag{15}$$

where

$$\{\bar{\psi}_1\}_{N\times 1} = \begin{Bmatrix} \bar{\psi}_1 \\ \bar{\psi}_2 \\ \vdots \\ \bar{\psi}_N \end{Bmatrix}, \quad \{\psi_2\}_{M\times 1} = \begin{Bmatrix} \psi_{N+1} \\ \psi_{N+2} \\ \vdots \\ \psi_{N+M} \end{Bmatrix} \tag{16}$$

$$[B_1]_{N\times(N+M)} = \begin{bmatrix} -\left[\sum_{m=1}^{N+M} b_{1,m} - b_{1,1}\right] & b_{1,2} & b_{1,3} & \cdots & b_{1,N} & \cdots & b_{1,N+M} \\ b_{2,1} & -\left[\sum_{m=1}^{N+M} b_{2,m} - b_{2,2}\right] & b_{2,3} & \cdots & b_{2,N} & \cdots & b_{2,N+M} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ b_{N,1} & b_{N,2} & b_{N,3} & \cdots & -\left[\sum_{m=1}^{N+M} b_{N,m} - b_{N,N}\right] & \cdots & b_{N,N+M} \end{bmatrix},$$

$$[B_2]_{M\times(N+M)} = \begin{bmatrix} b_{N+1,1} & b_{N+1,2} & b_{N+1,3} & \cdots & -\left[\sum_{m=1}^{N+M} b_{N+1,m} - b_{N+1,N+1}\right] & \cdots & b_{N+1,N+M} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ b_{N+M,1} & b_{N+M,2} & b_{N+M,3} & \cdots & b_{N+M,N+1} & \cdots & -\left[\sum_{m=1}^{N+M} b_{N+M,m} - b_{N+M,N+M}\right] \end{bmatrix} \tag{17}$$

$$\{\alpha\}_{(N+M)\times 1} = \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_N \\ \alpha_{N+1} \\ \vdots \\ \alpha_{N+M} \end{Bmatrix}, \tag{18}$$

in which $b_{ij} = B^{(i)}(s^j, x^i)$ and $i, j = 1, 2, \ldots, N+M$.

Rearrange the influence matrices of Eqs. (11) and (15) together into the linearly algebraic solver system as

$$\begin{Bmatrix} \{\bar{\phi}_1\}_{N\times 1} \\ \{\bar{\psi}_1\}_{N\times 1} \end{Bmatrix} = \begin{bmatrix} [A_1]_{N\times(N+M)} \\ [B_1]_{N\times(N+M)} \end{bmatrix} \{\alpha\}_{(N+M)\times 1} \tag{19}$$

The linear equations in Eq. (19) can be generally written as

$$[A]\{x\} = \{b\} \tag{20}$$

where $[A] = \begin{bmatrix} [A_1]_{N\times(N+M)} \\ [B_1]_{N\times(N+M)} \end{bmatrix}$, $\{x\} = \{\alpha\}_{(N+M)\times 1}$ and $\{b\} = \begin{Bmatrix} \{\bar{\phi}_1\}_{N\times 1} \\ \{\bar{\psi}_1\}_{N\times 1} \end{Bmatrix}$.

For the Cauchy problem of the Laplace equation, the influence matrix $[A]$ is often ill-posed such that the regularization technique in section 2.2.4 which regularizes the influence matrix is necessary.

2.2.4 Regularization techniques for Cauchy problem

2.2.4.1 Truncated singular value decomposition

In the singular value decomposition (SVD), the matrix $[A]$ is decomposed into

$$[A] = [U]\ [\Sigma]\ [V]^T \tag{21}$$

where $[U] = [u_1, u_2, \cdots, u_m]$ and $[V] = [v_1, v_2, \cdots, v_m]$ are column orthonormal matrices, with column vectors called left and right singular vectors, respectively, T denotes the matrix transposition, and $[\Sigma] = diag(\sigma_1, \sigma_2, \cdots, \sigma_m)$ is a diagonal matrix with nonnegative diagonal elements in nonincreasing order, which are the singular values of $[A]$.

A convenient measure of the conditioning of the matrix $[A]$ is the condition number *Cond* defined as

$$Cond = \frac{\sigma_1}{\sigma_m}, \tag{22}$$

where σ_1 is the maximum singular value and σ_m is the minimum singular value i.e. the ratio between the largest singular value and the smallest singular value. By means of the SVD, the solution a^0 can be written as

$$a^0 = \sum_{i=1}^{k} \frac{u_i^T b}{\sigma_i} v_i, \tag{23}$$

where k is the rank of $[A]$, u_i is the element of the left singular vector and v_i is the element of the right singular vector. For an ill-conditioned matrix equation, there are small singular values, therefore the solution is dominated by contributions from small singular values when noise is present in the data. One simple remedy to the difficulty is to leave out contributions from small singular values, i.e. taking a^p as an approximate solution, where a^p is defined as:

$$a^p = \sum_{i=1}^{p} \frac{u_i^T b}{\sigma_i} v_i \,, \tag{24}$$

where $p \le k$ is the regularization parameter, which determines when one starts to leave out small singular values. Note that if $p = k$, the approximate solution is exactly the least squares solution. This method is known as TSVD in the inverse problem community [4].

2.2.4.2 Tikhonov technique

Tikhonov proposed a method to transform this ill-posed problem into a well-posed one. Instead of solving $[A]\{x\} = \{b\}$ directly, the procedures of Tikhonov technique are written as follows:

(Ⅰ). Minimize $\|x\|^2$ subject to $\|Ax-b\|^2 \le \varepsilon$ (25)

where ε is the prescribed error tolerance.

(Ⅱ). The proposed problem in Eq. (25) is equivalent to [10]

minimize $\|Ax-b\|^2$ subject to $\|x\|^2 \le \varepsilon^*$, (26)

and the Euler-Lagrange equation obtained from reference [10] can be written as

$$(A^T A + \lambda I)x = A^T b \tag{27}$$

Where λ is the regularization parameter (Lagrange parameter).

2.2.4.3 Linear regularization method

The single central idea in inverse theory is the prescription [9] ,

minimize: $P[x] + \lambda Q[x]$ (28)

where $P[x] > 0$ and $Q[x] > 0$ are two positive functions of x.

Then, using equation X^2 [9] , the minimization principle Eq.28 is

minimize: $P[x] + \lambda Q[x] = |A \cdot x - b|^2 + \lambda x \cdot H \cdot x$ (29)

in vector notation,

$$(A^T A + \lambda H)x = A^T b \tag{30}$$

where

$$[H]_{M\times M} = [B^T]_{M\times(M-1)} \cdot [B]_{(M-1)\times M} = \begin{pmatrix} 1 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & & & & \ddots & & & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & -1 & 1 \end{pmatrix} \tag{31}$$

in which

$$[B]_{(M-1)\times M} = \begin{pmatrix} -1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & & & & \ddots & & & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & -1 & 1 \end{pmatrix} \tag{32}$$

and T denotes matrix transposition.

2.2.5 The L-curve and its applications

The L-curve concept is proposed to aid us in selecting the optimal parameter λ (or i, truncated number). Two indices are frequently used, one represents the sensitivity of the influence matrix on the solution and the other represents the degree of distortion to the original system. Usually, the norm error is $\|u - u_e\| = \left\{ \int_a^b |u - u_e|^2 dx \right\}$, where u is the numerical result and u_e is the analytical result, is chosen as the index of sensitivity and λ (or i) is chosen as the index of degree of distortion. A sketch diagram for the TSVD method、Tikhonov technique and Linear regularization method combined with the L-curve concept is illustrated in Fig. 3. One can find that when the regularization parameter, λ or i, is small, $\|u - u_e\|$ tends to very large even though λ (or i) is small. It is shown that the regularization parameter is too small such that not much improvement of ill-posed remedy in the influence matrix is done. On the other hand, when the regularization parameter, λ (or i), is large, λ (or i) tends to be very large even though $\|u - u_e\|$ tends to small value which shows that the regularization parameter is too large such that the original system is distorted too much. Therefore, the compromised results of $\|u - u_e\|$ and λ (or i) lead us to choose the corresponding value in the corner of the L-shape curve as the optimal regularization parameter.

3. Numerical example

To illustrate application of the TSVD, Tikhonov technique, linear regularization method and L-curve for the Laplace equation with overspecified B.C.s. A circle domain, R = 1, is chosen as a representation example. Three kinds of treatments in the problem is considered: TSVD, Tikhonov technique and Linear regularization method all for the inverse problem with noise.

The present model of the inverse problem with noise can be described as shown in Fig. 4. By using random data simulation, we can obtain 1% random errors contaminating the input data, as shown in Fig.5. If regularization techniques are not employed, the results are unreasonable as shown in Fig. 6 .

When the TSVD, Tikhonov technique and linear regularization method are applied in the analysis for the case, we can obtain solutions with many values of λ (or i), as shown in Fig. 7(a), 7(b), 7(c). Therefore, we can find the relationship between the norm error and the value of λ (or i); i.e., the L-curve, as shown in Fig. 8(a), 8(b), 8(c), can be constructed. As expected from the mathematical point of view, a corner is present in the L-curve. If the corner of the L-curve is chosen as an optimal point, the appropriate value is 104 for TSVD, 0.000042 for Tikhonov technique and 0.21 for linear regularization method, respectively. Therefore, the deconvolution results will be regularized to approximate the analytical solution, as shown in Fig. 9(a), 9(b), 9(c). We can find that the appropriate solutions obtained by using the regularization techniques look more reasonable in comparison with the analytical solution than do the results obtained without using regularization. Although some differences still occur among TSVD, Tikhonov technique and linear regularization method. Therefore, we can find the differences among the L-curves of three treatments, as shown in Fig. 10. The results of three treatments are compared with the analytical solution, as shown in Fig. 11. Then we can figure out the norm error of the L-curve by linear regularization method is much lower than others, also the result after regularized by Linear regularization method is agree the analytical solution better than others.

4. CONCLUSION

In this paper, we used the RMM to solve the Laplace equation with overspecified boundary condition. Only the boundary nodes on the real boundary are required. The major difficulty of the coincidence of the source and collocation points in the conventional MFS is then circumvented. Besides, the regularization techniques using the TSVD, Tikhonov technique and linear regularization method, together with the L-curve, plays a role in determining the optimal parameter λ (or *i*) which can maintain the system characteristic and can make the system insensitive to contaminating noise. Furthermore, the numerical results obtained by using the linear regularization method for the case are in very close agreements with the analytical solutions and is superior to other regularization techniques.

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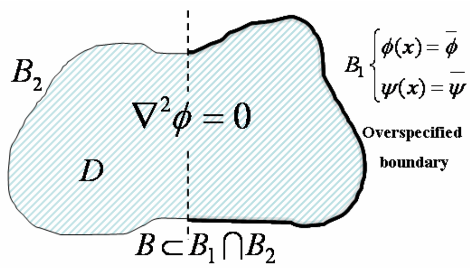

Figure 1 Sketch diagram of the inverse problem with overspecified boundary condition.

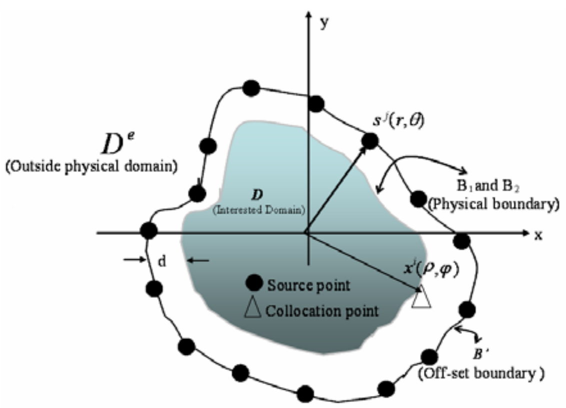

Figure 2 (a).

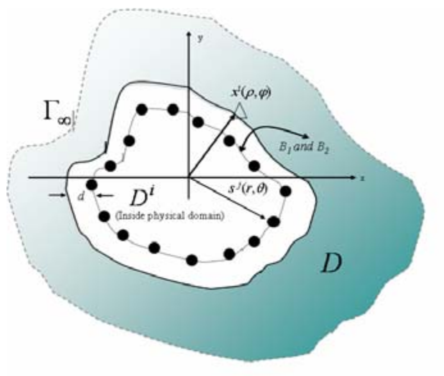

Figure 2 (b).

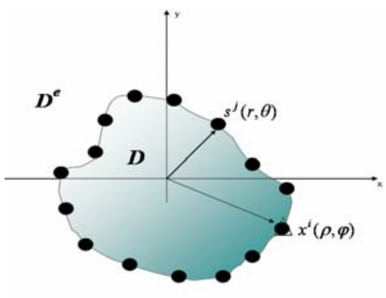

Figure 2 (c).

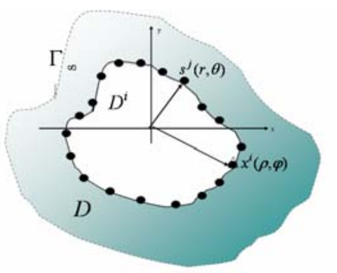

Figure 2 (d).

Figure 2 The source point and observation point distributions and definitions of r,θ,ρ,φ by using the conventional MFS and the regularized meshless method for the interior and exterior problems: (a) interior problem (MFS), (b) exterior problem (MFS), (c) interior problem (proposed method), (d) exterior problem (proposed method).

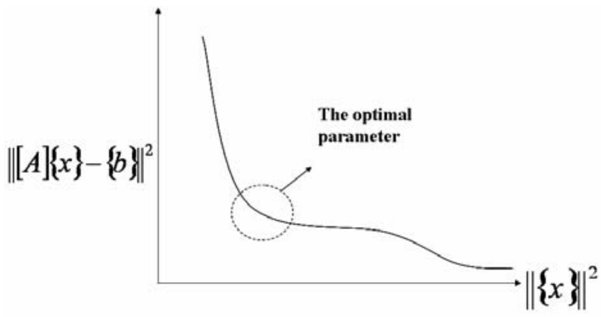

Figure 3 L-curve concept.

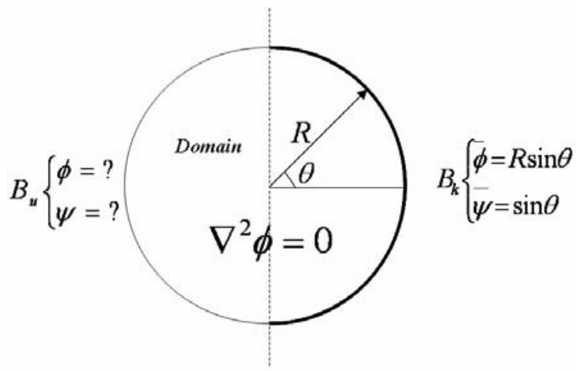

Figure 4 Problem sketch.

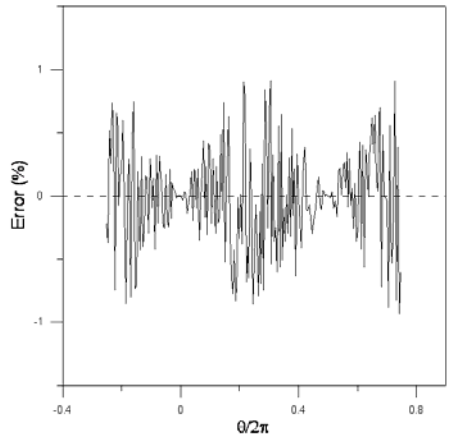

Figure 5 The random error.

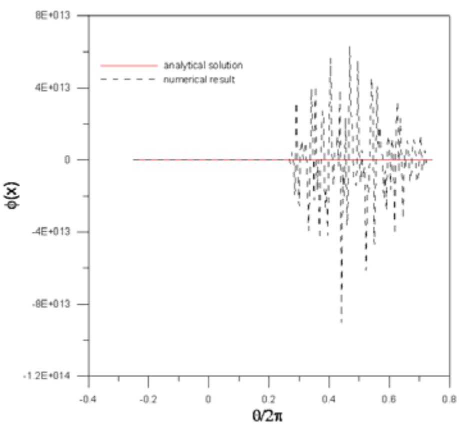

Figure 6 The boundary potential without regularization techniques.

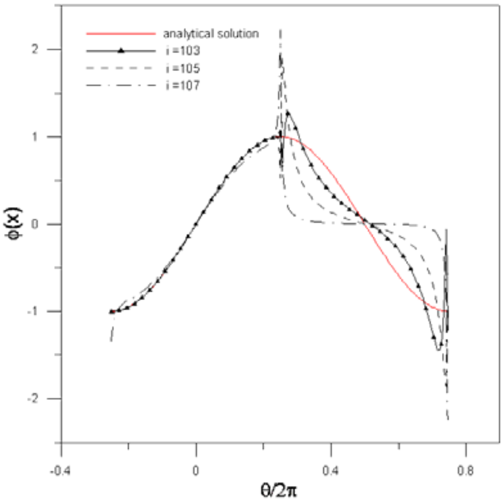

Figure 7(a).

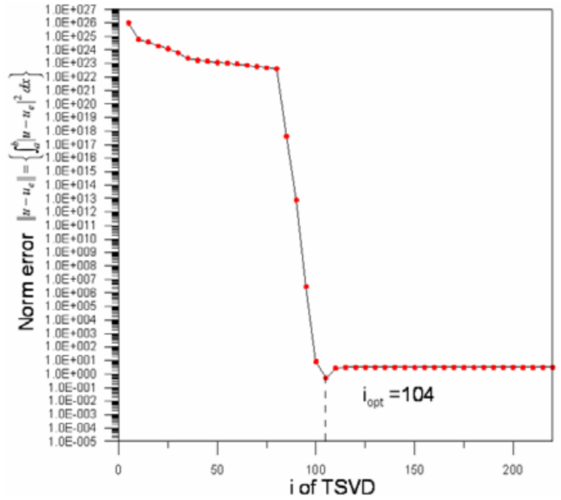

Figure 8(a).

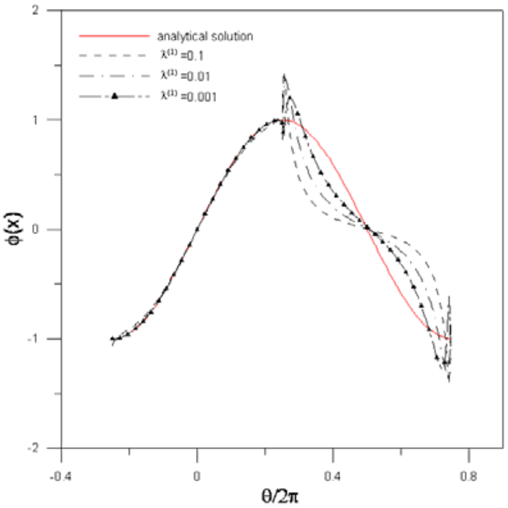

Figure 7(b).

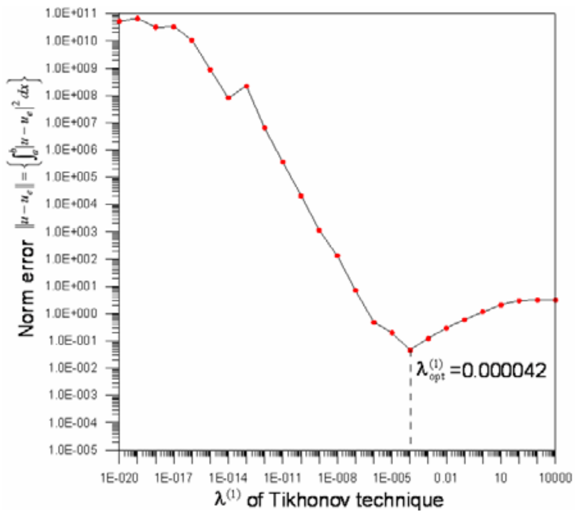

Figure 8(b).

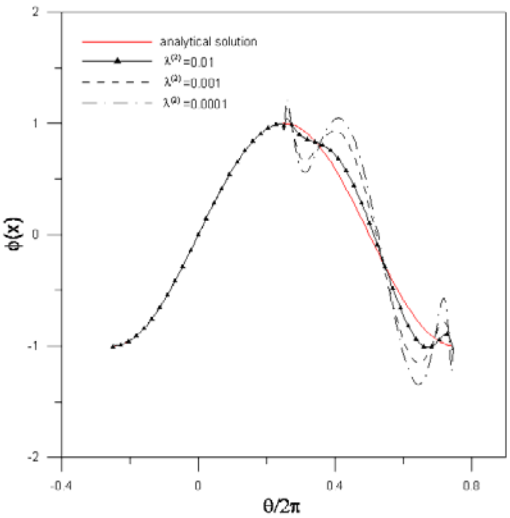

Figure 7(c).

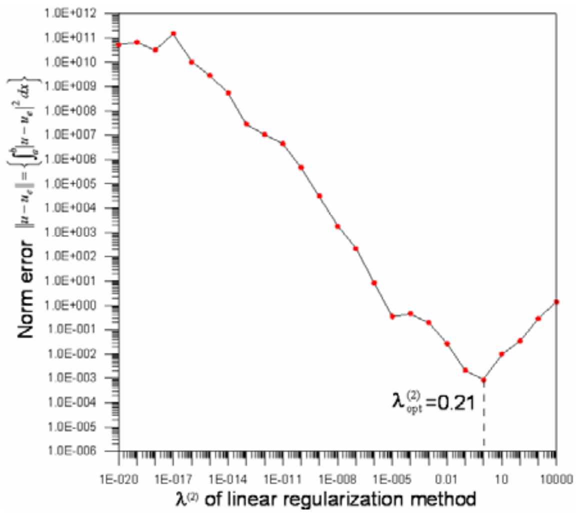

Figure 8(c).

Figure 7 The boundary potential with different values of λ (or *i*) by using (a) TSVD, (b) Tikhonov technique, (c) linear regularization method.

Figure 8 L-curve by (a) TSVD, (b) Tikhonov technique, (c) linear regularization method.

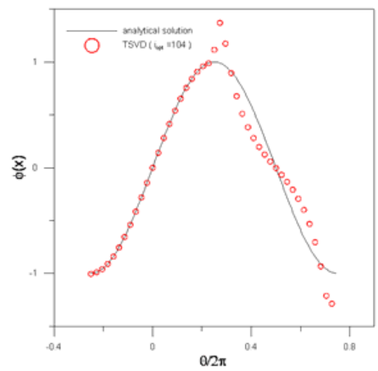

Figure 9(a).

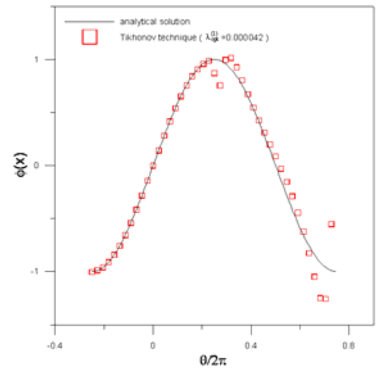

Figure 9(b).

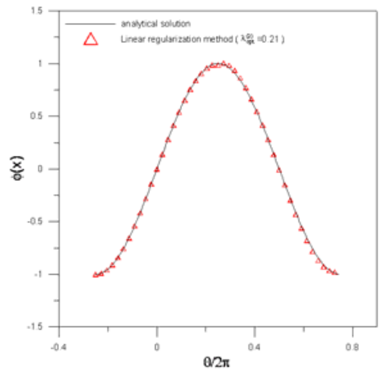

Figure 9(c).

Figure 9 The boundary potential with the optimal value of λ (or *i*) by (a) TSVD, (b) Tikhonov technique, (c) linear regularization method.

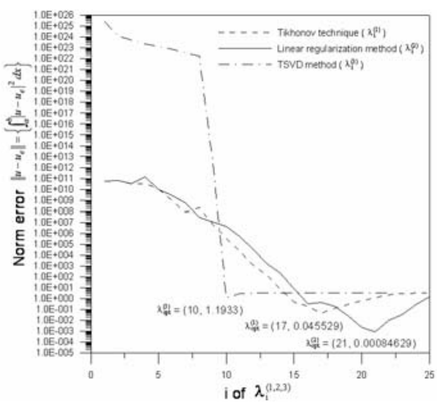

Figure 10 L-curve by TSVD, Tikhonov technique and Linear regularization method.

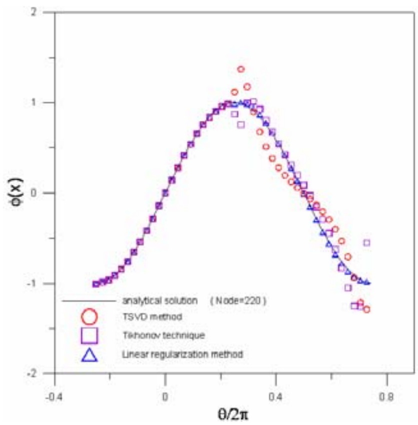

Figure 11 The boundary potential by TSVD, Tikhonov technique and Linear regularization method with optimal values.

正規化無網格法求解柯西問題

吳國綸[1]　陳桂鴻[2]　陳正宗[1]　高政宏[1]

[1]國立台灣海洋大學河海工程學系
[2]稻江科技暨管理學院資訊管理學系

摘要

本文是利用正規化無網格法求解過定邊界之拉普拉斯問題，使用雙層勢能來表示整個場解，且使用一加一減技巧來正規化處理奇異及超奇異核函數。使用提出的數值方法有別於傳統基本解法須將源點佈在虛假邊界上，可將奇異源放在真實的邊界上，並可獲得線性代數方程。配合邊界條件，即可輕易的決定出線性代數系統的未知係數。然而伴隨著的病態問題可藉由截取式奇異值分解法、Tikhonov 技術及線性正規化法來克服，在最佳化參數方面，則可用 L 曲線的觀念來得到。所得之數值結果在與解析解作比較後可獲得滿意的結果，並對其三種克服病態問題之方法加以比較討論。

關鍵詞：正規化無網格法、Tikhonov 技術、線性正規化法、截取式奇異值分解法、L 曲線技術、柯西問題。