國立中山大學應用數學系
學術演講

On the equivalence of
the Trefftz method and method of fundamental solutions
for Laplace and biharmonic equaitons

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日　期：94年04月14日

時　間：PM14:10~15:00

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Outlines

1. Laplace and biharmonic problems
2. Trefftz method and MFS
3. Connection between the two methods
4. Numerical examples
5. Conclusions

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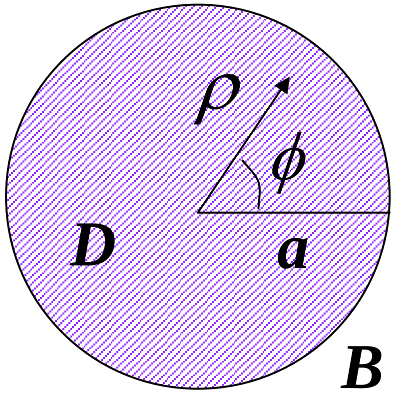
Laplace problem

Two-dimensional Laplace problem with a circular domain

G.E.: $\nabla^2 u(x) = 0, \quad x \in D$

B.C.: $u(x) = \bar{u}, \quad x \in B$



where

∇^2 **denotes the Laplacian operator**

$u(x)$ **is the potential function**

ρ **is the radius of the field point**

ϕ **is the angle along the field point**

Biharmonic problem

Two-dimensional biharmonic problem with a circular domain

G.E.: $\nabla^4 u(x) = \frac{w(x)}{D},\ x \in \Omega$ → Splitting method → $\nabla^4 u^*(x) = 0,\ x \in \Omega$

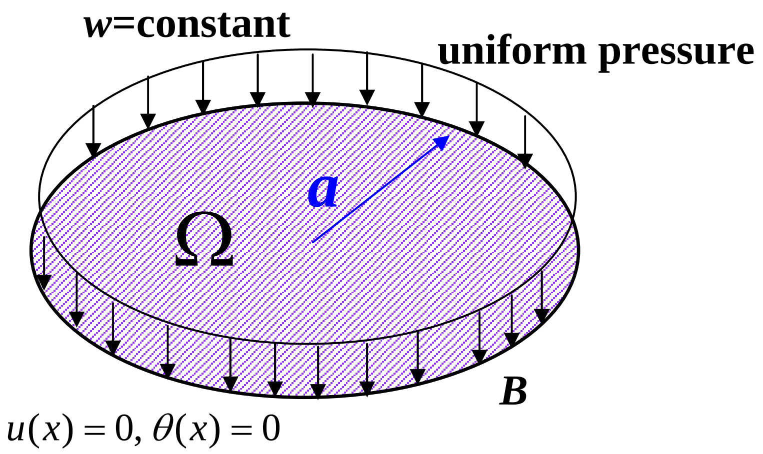
B.C.: $u(x) = 0,\ \theta(x) = 0,\ x \in B$ → $u^*(x) = \bar{u}^*(x),\ \theta^*(x) = \bar{\theta}^*(x),\ x \in B$

$u(x)$: deflection of the circular plate

D : flexure rigidity

$w(x)$: uniform distributed load

Ω : domain of interest



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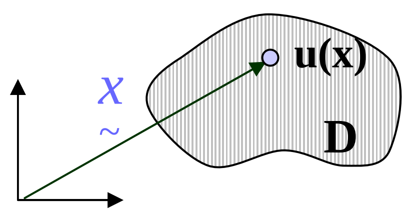
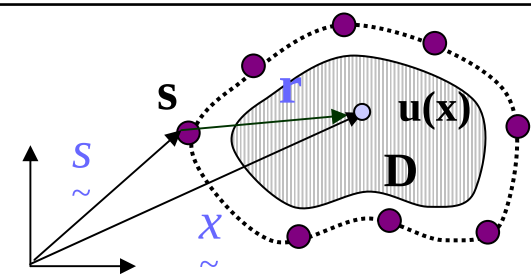
Trefftz method and MFS

Method	Trefftz method	MFS
Definition	$u(x) = \sum_{j=1}^{N_T} c_j u_j(x)$	$u(x) = \sum_{j=1}^{N_M} w_j U(x, s_j)$
		
Base	$u_j(x)$ (T-complete function)	$U(x, s) = \psi(r)$, $r = \|x-s\|$
G.E.	$\mathcal{L} u(x)=0, \quad x \in D$	$\mathcal{L} U(x,s)=0, \quad x \in D$ (singularity at s)
Match B.C.	*Determine* c_j	*Determine* w_j

N_T is the number of complete functions

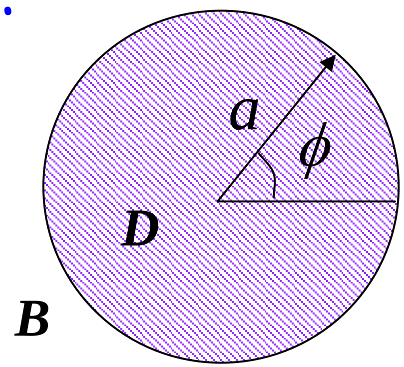
N_M is the number of source points in the MFS

Laplace problem

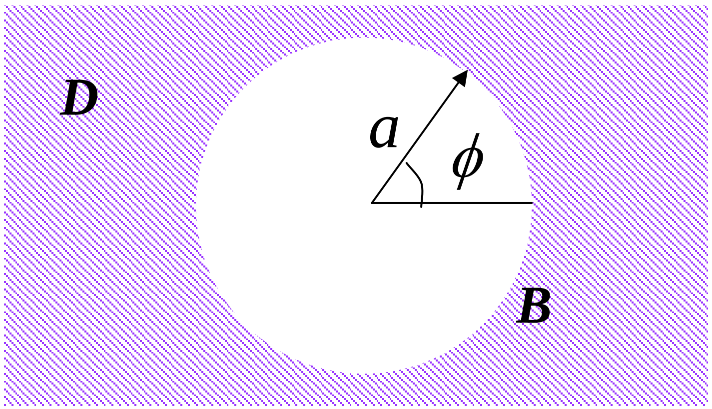
Two-dimensional Laplace problem with a circular domain

G.E.: $\nabla^2 u(x) = 0, \quad x \in D$ **B.C.:** $u(x) = \bar{u}, \quad x \in B$

Interior :



Exterior :



Boundary Condition: $u(a,\phi) = \bar{a}_0 + \sum_{n=1}^{N} \bar{a}_n \cos(n\phi) + \sum_{n=1}^{N} \bar{b}_n \sin(n\phi)$

Trefftz method

Representation of the field solution :

$$u(x) = \sum_{j=1}^{2N_T+1} w_j u_j(x)$$

where

$2N_T + 1$ **is the number of complete functions**

w_j **is the unknown coefficient**

u_j **is the T-complete function which satisfies the Laplace equation**

T-complete set function

$$1$$

$$\nabla^2 1 = 0$$

$$\rho^n \cos(n\phi)$$

$$\nabla^2 \rho^n \cos(n\phi) = 0$$

$$\rho^n \sin(n\phi)$$

$$\nabla^2 \rho^n \sin(n\phi) = 0$$

Laplace Equation

where

$$\nabla^2 = \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial \phi^2}$$

Derivation of unknown coefficients (Trefftz)

T-complete set functions : **Interior:** $1,\ \rho^n\cos(n\phi),\ \rho^n\sin(n\phi)$

Exterior: $\ln\rho,\ \rho^{-n}\cos(n\phi),\ \rho^{-n}\sin(n\phi)$

Field solution:

Interior :

$$u^I(a,\phi) = a_0 + \sum_{n=1}^{N_T} a_n a^n \cos(n\phi) + \sum_{n=1}^{N_T} b_n a^n \sin(n\phi)$$

Exterior :

$$u^E(a,\phi) = a_0 \ln a + \sum_{n=1}^{N_T} a_n a^{-n} \cos(n\phi) + \sum_{n=1}^{N_T} b_n a^{-n} \sin(n\phi)$$

By matching the boundary condition at $\rho = a$

Interior problem:

$$a_0 = \bar{a}_0,$$

$$a_n = \frac{\bar{a}_n}{a^n}, \qquad n = 1,2,\ldots N_T$$

$$b_n = \frac{\bar{b}_n}{a^n} \qquad n = 1,2,\ldots N_T$$

Exterior problem:

$$a_0 = \frac{1}{\ln a}\bar{a}_0,$$

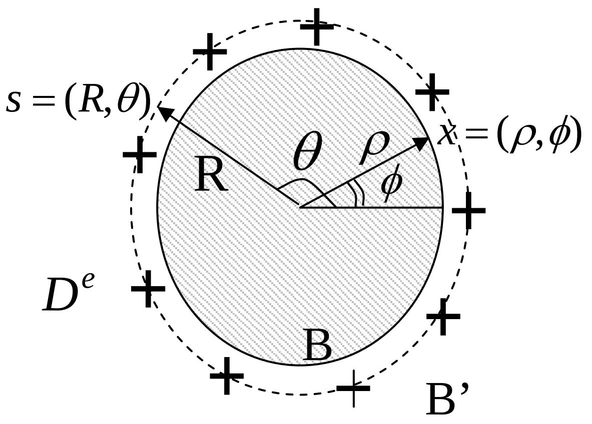
$$a_n = a^n \bar{a}_n, \qquad n = 1,2,\ldots N_T$$

$$b_n = a^n \bar{b}_n \qquad n = 1,2,\ldots N_T$$

Method of Fundamental Solutions (MFS)

Field solution :

$$u(x) = \sum_{j=1}^{N_M} c_j U(x, s_j), \quad s_j \in D^e$$



where

N_M is the number of source points in the MFS

c_j is the unknown coefficient

$U(x, s_j)$ is the fundamental solution

D^e is the complementary domain

S is the source point

X is the collocation point

Fundamental Solution

$$\nabla_x^2 U(x,s) = 2\pi\delta(x-s)$$

$$\Downarrow$$

$$U(x,s) = \ln(r) \qquad r = |\underline{x} - \underline{s}|$$

where

$$\nabla_x^2 = \frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}$$

Derivation of unknown coefficients (MFS)

Degenerate kernel :
$$U(R,\theta,\rho,\phi)=\ln r=\begin{cases} U^i(R,\theta;\rho,\phi)=\ln(R)-\sum_{n=1}^{\infty}\frac{1}{n}(\frac{\rho}{R})^n\cos(n(\theta-\phi)), & R\ge\rho \\ U^e(R,\theta;\rho,\phi)=\ln(\rho)-\sum_{n=1}^{\infty}\frac{1}{n}(\frac{R}{\rho})^n\cos(n(\theta-\phi)), & R<\rho \end{cases}$$

Field solution:

Interior :
$$u^I(a,\phi)=\sum_{j=1}^{N_M}c_j[\ln(R)-\sum_{n=1}^{\infty}\frac{1}{n}(\frac{a}{R})^n\cos(n(\theta_j-\phi))],\quad 0<\phi<2\pi$$

Exterior :
$$u^E(a,\phi)=\sum_{j=1}^{N_M}c_j[\ln(a)-\sum_{n=1}^{\infty}\frac{1}{n}(\frac{R}{a})^n\cos(n(\theta_j-\phi))],\quad 0<\phi<2\pi$$

Interior problem:
$$\bar{a}_0=\sum_{j=1}^{N_M}c_j\ln(R)$$
$$\frac{\bar{a}_n}{\rho^n}=-\sum_{j=1}^{N_M}c_j\frac{1}{n}(\frac{1}{R})^n\cos(n\theta_j),\quad n=1,2,\ldots N_M$$
$$\frac{\bar{b}_n}{\rho^n}=-\sum_{j=1}^{N_M}c_j\frac{1}{n}(\frac{1}{R})^n\sin(n\theta_j),\quad n=1,2,\ldots N_M$$

Exterior problem:
$$\frac{1}{\ln(a)}\bar{a}_0=\sum_{j=1}^{N_M}c_j$$
$$a^n\bar{a}_n=-\sum_{j=1}^{N_M}c_j\frac{1}{n}(R)^n\cos(n\theta_j),\quad n=1,2,\ldots N_M$$
$$a^n\bar{b}_n=-\sum_{j=1}^{N_M}c_j\frac{1}{n}(R)^n\sin(n\theta_j),\quad n=1,2,\ldots N_M$$

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Relationship between the two methods

$$\{\underline{w}\} = [K]\{\underline{c}\}$$

Trefftz MFS

By setting $2N_T + 1 = N_M = 2N + 1$

Trefftz method MFS

$$\underline{w} = \begin{Bmatrix} a_0 \\ a_1 \\ b_1 \\ a_2 \\ b_2 \\ \vdots \\ a_N \\ b_N \end{Bmatrix}$$

Interior:

$$[K^I] = \begin{bmatrix} \ln R & \ln R & \ln R & \cdots & \ln R \\ \frac{-1}{R}\cos(\theta_1) & \frac{-1}{R}\cos(\theta_2) & \frac{-1}{R}\cos(\theta_3) & \cdots & \frac{-1}{R}\cos(\theta_{2N+1}) \\ \frac{-1}{R}\sin(\theta_1) & \frac{-1}{R}\sin(\theta_2) & \frac{-1}{R}\sin(\theta_3) & \cdots & \frac{-1}{R}\sin(\theta_{2N+1}) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{-1}{N}(\frac{1}{R})^N\cos(N\theta_1) & \frac{-1}{N}(\frac{1}{R})^N\cos(N\theta_2) & \frac{-1}{N}(\frac{1}{R})^N\cos(N\theta_3) & \cdots & \frac{-1}{N}(\frac{1}{R})^N\cos(N\theta_{2N+1}) \\ \frac{-1}{N}(\frac{1}{R})^N\sin(N\theta_1) & \frac{-1}{N}(\frac{1}{R})^N\sin(N\theta_2) & \frac{-1}{N}(\frac{1}{R})^N\sin(N\theta_3) & \cdots & \frac{-1}{N}(\frac{1}{R})^N\sin(N\theta_{2N+1}) \end{bmatrix}$$

Exterior:

$$[K^E] = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ -R\cos(\theta_1) & -R\cos(\theta_2) & -R\cos(\theta_3) & \cdots & -R\cos(\theta_{2N+1}) \\ -R\sin(\theta_1) & -R\sin(\theta_2) & -R\sin(\theta_3) & \cdots & -R\sin(\theta_{2N+1}) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{-1}{N}(R)^N\cos(N\theta_1) & \frac{-1}{N}(R)^N\cos(N\theta_2) & \frac{-1}{N}(R)^N\cos(N\theta_3) & \cdots & \frac{-1}{N}(R)^N\cos(N\theta_{2N+1}) \\ \frac{-1}{N}(R)^N\sin(N\theta_1) & \frac{-1}{N}(R)^N\sin(N\theta_2) & \frac{-1}{N}(R)^N\sin(N\theta_3) & \cdots & \frac{-1}{N}(R)^N\sin(N\theta_{2N+1}) \end{bmatrix}$$

$$\underline{c} = \begin{Bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ \vdots \\ c_{2N} \\ c_{2N+1} \end{Bmatrix}$$

Matrix ($K = T_R \, T_\theta$)

$$
[T_\theta] = \begin{bmatrix}
1 & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & 1 \\
\cos(\theta_1) & \cos(\theta_2) & \cdots & \cdots & \cdots & \cdots & \cdots & \cos(\theta_{2N+1}) \\
\sin(\theta_1) & \sin(\theta_2) & \cdots & \cdots & \cdots & \cdots & \cdots & \sin(\theta_{2N+1}) \\
\cos(2\theta_1) & \cos(2\theta_2) & \cdots & \cdots & \cdots & \cdots & \cdots & \cos(2\theta_{2N+1}) \\
\sin(2\theta_1) & \sin(2\theta_2) & \cdots & \cdots & \cdots & \cdots & \cdots & \sin(2\theta_{2N+1}) \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\cos(N\theta_1) & \cos(N\theta_2) & \cdots & \cdots & \cdots & \cdots & \cdots & \cos(N\theta_{2N+1}) \\
\sin(N\theta_1) & \sin(N\theta_2) & \cdots & \cdots & \cdots & \cdots & \cdots & \sin(N\theta_{2N+1})
\end{bmatrix}_{(2N+1)\times(2N+1)}
$$

$$
\Rightarrow \quad [T_\theta][T_\theta]^T = \begin{bmatrix}
2N+1 & 0 & \cdots & \cdots & 0 \\
0 & \frac{2N+1}{2} & \cdots & \cdots & 0 \\
0 & 0 & \frac{2N+1}{2} & \cdots & \vdots \\
\vdots & \vdots & \cdots & \ddots & 0 \\
0 & 0 & \cdots & 0 & \frac{2N+1}{2}
\end{bmatrix}_{(2N+1)\times(2N+1)}
$$

$$
\Rightarrow \quad \det[T_\theta] = \frac{(2N+1)^{N+\frac{1}{2}}}{2^N} \neq 0, \quad N \in \textit{natural number}
$$

Matrix ($K = T_R T_\theta$)

$$[T_R]^I = \begin{bmatrix} \ln(R) & 0 & 0 & 0 & \cdots & \cdots & 0 & 0 \\ 0 & \frac{-1}{R} & 0 & 0 & \cdots & \cdots & 0 & 0 \\ 0 & 0 & \frac{-1}{R} & 0 & \cdots & \cdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \frac{-1}{2}(\frac{1}{R})^2 & \cdots & \cdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \frac{-1}{2}(\frac{1}{R})^2 & \cdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \cdots & \ddots & 0 & 0 \\ 0 & 0 & 0 & \cdots & \cdots & \cdots & \frac{-1}{N}(\frac{1}{R})^N & 0 \\ 0 & 0 & 0 & \cdots & \cdots & \cdots & 0 & \frac{-1}{N}(\frac{1}{R})^N \end{bmatrix}_{(2N+1)\times(2N+1)}$$

$R >> a$, $N \to \infty$ ⇔ **ill-posed problem**

$\ln R = 0$ $(R = 1)$ ⇔ **Degenerate scale problem**

$\frac{\ln(a)}{\ln(a)}$ ⇐

$$[T_R]^E = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & \cdots & 0 & 0 \\ 0 & -R & 0 & 0 & \cdots & \cdots & 0 & 0 \\ 0 & 0 & -R & 0 & \cdots & \cdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \frac{-1}{2}(R)^2 & \cdots & \cdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \frac{-1}{2}(R)^2 & \cdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \cdots & \ddots & 0 & 0 \\ 0 & 0 & 0 & \cdots & \cdots & \cdots & \frac{-1}{N}(R)^N & 0 \\ 0 & 0 & 0 & \cdots & \cdots & \cdots & 0 & \frac{-1}{N}(R)^N \end{bmatrix}_{(2N+1)\times(2N+1)}$$

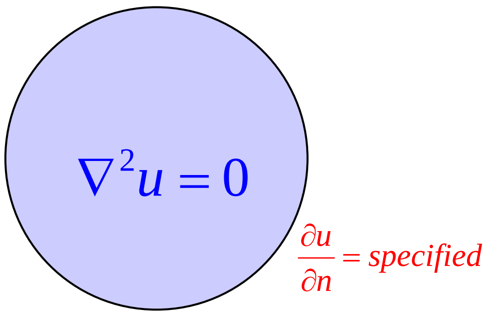
$R << a$, $N \to \infty$ ⇔ **ill-posed problem**

$(a = 1)$ ⇔ **Nonuniqueness**

The nonuniqueness problem

The nonuniqueness problem

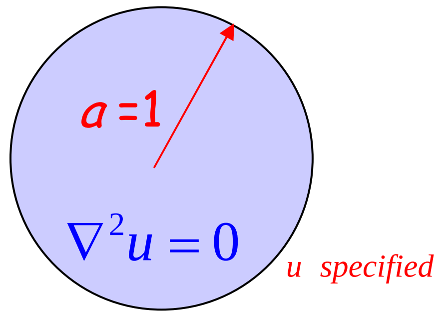
- Rigid body for Neumann problem
- Degenerate scale in plane BVP
- Hypersingular BIE for multiply-connected problems



$[U]\{t\} = [T]\{u\}$

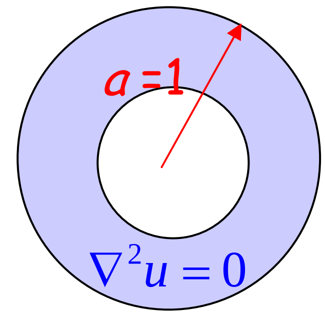
$[L]\{t\} = [M]\{u\}$

Mathematically and physically realizable



$[U]\{t\} = [T]\{u\}$

Mathematically realizable



$[U]\{t\} = [T]\{u\}$ Degenerate scale

$[L]\{t\} = [M]\{u\}$ Nonuniqueness

Mathematically realizable

Papers of degenerate scale (Taiwan)

1. J. T. Chen, J. H. Lin, S. R. Kuo and Y. P. Chiu, 2001, Analytical study and numerical experiments for degenerate scale problems in boundary element method using degenerate kernels and circulants, *Engineering Analysis with Boundary Elements*, Vol.25, No.9, pp.819-828. (SCI and EI)
2. J. T. Chen, S. R. Kuo and J. H. Lin, 2002, Analytical study and numerical experiments for degenerate scale problems in the boundary element method for two-dimensional elasticity, *Int. J. Numer. Meth. Engng.*, Vol.54, No.12, pp.1669-1681. (SCI and EI)
3. J. T. Chen, C. F. Lee, I. L. Chen and J. H. Lin, 2002 An alternative method for degenerate scale problem in boundary element methods for the two-dimensional Laplace equation, *Engineering Analysis with Boundary Elements*, Vol.26, No.7, pp.559-569. (SCI and EI)
4. J. T. Chen, S. R. Lin and K. H. Chen, 2005, Degenerate scale for Laplace equation using the dual BEM, *Int. J. Numer. Meth. Engng*, Vol.62, No.2, pp.233-261. (SCI and EI).

Papers of degenerate scale (China)

1. 胡海昌, 調和函數邊界積分方程的充要條件, 1989, *固體力學學報*, Vol.2, No.2, pp.99-104
2. 胡海昌, 平面調和函數的充要的邊界積分方程, 1992, *中國科學學報*, Vol.4, pp.398-404
3. W. J. He, H. J. Ding and H. C. Hu, Nonuniqueness of the conventional boundary integral formulation and its elimination for two-dimensional mixed potential problems, Computers and Structures, Vol.60, No.6, pp.1029-1035, 1996.

Trefftz method and MFS for biharmonic equation

Analytical solution:

$$u^*(\rho,\phi)=a_0+\sum_{m=1}^{N_T}a_m\rho^m\cos(m\phi)+\sum_{m=1}^{N_T}b_m\rho^m\sin(m\phi)+c_0(\rho^2)+\sum_{m=1}^{N_T}c_m\rho^{m+2}\cos(m\phi)+\sum_{m=1}^{N_T}d_m\rho^{m+2}\sin(m\phi)$$

Trefftz method:

Field solution : $u(x)=\sum_{j=1}^{N_T}g_j u_j^*(x)$ $\quad \theta(x)=\dfrac{\partial u(x)}{\partial n_x}=\sum_{j=1}^{N_T}h_j\theta_j^*(x)$

T-complete functions: $1,\ \rho^m\cos(m\phi),\ \rho^m\sin(m\phi),\ \rho^2,\ \rho^{m+2}\cos(m\phi),\ \rho^{m+2}\sin(m\phi)$

MFS:

Field solution : $u(x)=\sum_{j=1}^{N_M}v_jU(x,s_j),$

$$\frac{\partial u(x)}{\partial n_x}=\theta(x)=\sum_{j=1}^{N_M}v_j\frac{\partial U(x,s_j)}{\partial n_x}=\sum_{j=1}^{N_M}v_j\Theta(x,s_j)$$

Relationship between the Trefftz method and MFS

$$
\begin{Bmatrix} a_0 \\ a_1 \\ b_1 \\ \vdots \\ a_N \\ b_N \\ \hdashline c_0 \\ c_1 \\ d_1 \\ \vdots \\ c_N \\ d_N \end{Bmatrix}_{(4N+2)\times 1}
=
\left[\begin{array}{cccc}
R^2 \ln R & R^2 \ln R & \cdots\ \cdots & R^2 \ln R \\
-R(1+\ln R)\cos(\theta_1) & -R(1+\ln R)\cos(\theta_2) & \cdots\ \cdots & -R(1+\ln R)\cos(\theta_{N_M}) \\
-R(1+\ln R)\sin(\theta_1) & -R(1+\ln R)\sin(\theta_2) & \cdots\ \cdots & -R(1+\ln R)\sin(\theta_{N_M}) \\
\vdots & \vdots & \ddots\ \ddots & \vdots \\
\frac{1}{N(N-1)}(\frac{1}{R})^{N-2}\cos(N\theta_1) & \frac{1}{N(N-1)}(\frac{1}{R})^{N-2}\cos(N\theta_2) & \cdots\ \cdots & \frac{1}{N(N-1)}(\frac{1}{R})^{N-2}\cos(N\theta_{N_M}) \\
\frac{1}{N(N-1)}(\frac{1}{R})^{N-2}\sin(N\theta_1) & \frac{1}{N(N-1)}(\frac{1}{R})^{N-2}\sin(N\theta_2) & \cdots\ \cdots & \frac{1}{N(N-1)}(\frac{1}{R})^{N-2}\sin(N\theta_{N_M}) \\
\hdashline
1+\ln R & 1+\ln R & \cdots\ \cdots & 1+\ln R \\
\frac{-1}{2R}\cos(\theta_1) & \frac{-1}{2R}\cos(\theta_2) & \cdots\ \cdots & \frac{-1}{2R}\cos(\theta_{N_M}) \\
\frac{-1}{2R}\sin(\theta_1) & \frac{-1}{2R}\sin(\theta_2) & \cdots\ \cdots & \frac{-1}{2R}\sin(\theta_{N_M}) \\
\vdots & \vdots & \ddots\ \ddots & \vdots \\
\frac{1}{N(N+1)}\frac{-1}{R^N}\cos(N\theta_1) & \frac{1}{N(N+1)}\frac{-1}{R^N}\cos(N\theta_2) & \cdots\ \cdots & \frac{1}{N(N+1)}\frac{-1}{R^N}\cos(N\theta_{N_M}) \\
\frac{1}{N(N+1)}\frac{-1}{R^N}\sin(N\theta_1) & \frac{1}{N(N+1)}\frac{-1}{R^N}\sin(N\theta_2) & \cdots\ \cdots & \frac{1}{N(N+1)}\frac{-1}{R^N}\sin(N\theta_{N_M})
\end{array}\right]_{(4N+2)\times(4N+2)}
\begin{Bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ v_{4N+1} \\ v_{4N+2} \end{Bmatrix}_{(4N+2)\times 1}
$$

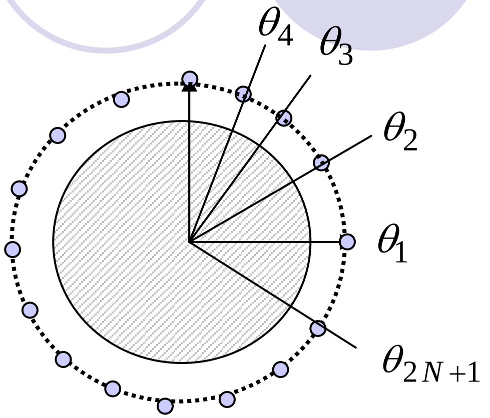
Coefficients of the Trefftz method

Mapping matrix [K]

Coefficients of the MFS

Decomposition of the K matrix

$$[K]=[T_R][T_\theta]$$



$$[T_R]=\begin{bmatrix} R^2\ln R &&&&&&&&&&& \\ & -R(1+2\ln R) &&&&&&&&&& \\ && -R(1+2\ln R) &&&&&&&&& \\ &&& \ddots &&&&&&&& \\ &&&& \frac{1}{R^{N-2}N(N-1)} &&&&&&& \\ &&&&& \frac{1}{R^{N-2}N(N-1)} &&&&&& \\ &&&&&& 1+\ln R &&&&& \\ &&&&&&& \frac{-1}{2R} &&&& \\ &&&&&&&& \frac{-1}{2R} &&& \\ &&&&&&&&& \ddots && \\ &&&&&&&&&& \frac{1}{R^N N(N+1)} & \\ &&&&&&&&&&& \frac{1}{R^N N(N+1)} \end{bmatrix}_{(4N+2)\times(4N+2)}$$

$$[T_\theta]=\begin{bmatrix} 1 & 1 & \cdots & \cdots & 1 & 1 \\ \cos\theta_1 & \cos\theta_2 & \cdots & \cdots & \cos\theta_{4N+1} & \cos\theta_{4N+2} \\ \sin\theta_1 & \sin\theta_2 & \cdots & \cdots & \sin\theta_{4N+1} & \sin\theta_{4N+2} \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \cos N\theta_1 & \cos N\theta_2 & \cdots & \cdots & \cos N\theta_{4N+1} & \cos N\theta_{4N+2} \\ \sin N\theta_1 & \sin N\theta_2 & \cdots & \cdots & \sin N\theta_{4N+1} & \sin N\theta_{4N+2} \\ \hline 1 & 1 & \cdots & \cdots & 1 & 1 \\ \cos\theta_1 & \cos\theta_2 & \cdots & \cdots & \cos\theta_{4N+1} & \cos\theta_{4N+2} \\ \sin\theta_1 & \sin\theta_2 & \cdots & \cdots & \sin\theta_{4N+1} & \sin\theta_{4N+2} \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \cos N\theta_1 & \cos N\theta_2 & \cdots & \cdots & \cos N\theta_{4N+1} & \cos N\theta_{4N+2} \\ \sin N\theta_1 & \sin N\theta_2 & \cdots & \cdots & \sin N\theta_{4N+1} & \sin N\theta_{4N+2} \end{bmatrix}_{(4N+2)\times(4N+2)}$$

$$\Rightarrow \det[T_\theta]=2(2N+1)^{2N+1}\neq 0$$

Diagonal matrix T_R

$$[T_R]=\begin{bmatrix} R^2\ln R & & & & & & & & & & \\ & -R(1+2\ln R) & & & & & & & & & \\ & & -R(1+2\ln R) & & & & & & & & \\ & & & \ddots & & & & & & & \\ & & & & \frac{1}{R^{N-2}N(N-1)} & & & & & & \\ & & & & & \frac{1}{R^{N-2}N(N-1)} & & & & & \\ & & & & & & 1+\ln R & & & & \\ & & & & & & & \frac{-1}{2R} & & & \\ & & & & & & & & \frac{-1}{2R} & & \\ & & & & & & & & & \ddots & \\ & & & & & & & & & & \frac{1}{R^N N(N+1)} \\ & & & & & & & & & & & \frac{1}{R^N N(N+1)} \end{bmatrix}_{(4N+2)\times(4N+2)}$$

Existence of the degenerate scales

$\ln R$

$1+2\ln R$

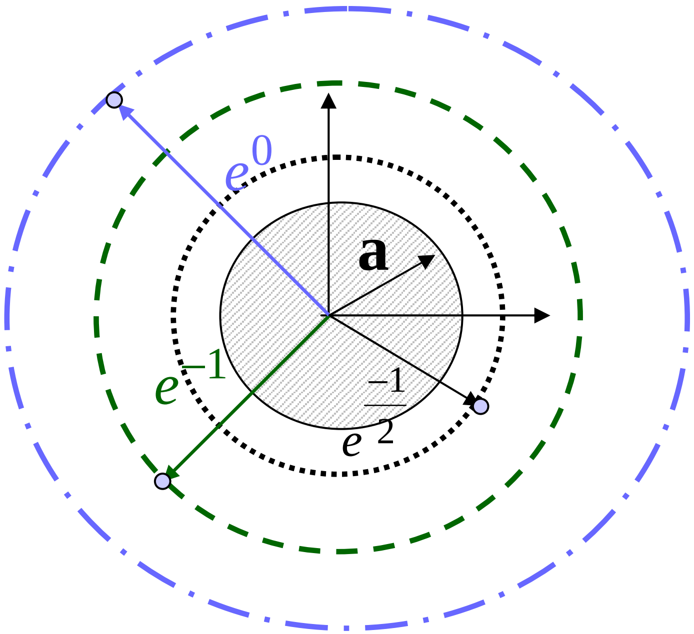
$1+\ln R$

$\neq 0$: $\{g_j\}_{(4N+2)\times 1}=[K]_{(4N+2)\times(4N+2)}\{v_j\}_{(4N+2)\times 1}$ **O. K.!**

$=0$: **Degenerate scale problem** ⇨ $R=e^0,\ e^{\frac{-1}{2}},\ e^{-1}$ ⇨ **Nonuniqueness (in numerical aspect)**

The occurrence of the degenerate scales using the MFS

Special size:



Degenerate scale problem

⇩

Mathematics: rank-deficiency problem

(nonuniqueness problem)

⇩

Numerical failure

○ **: position of the source points**

On the complete set of the Trefftz method and the MFS using the degenerate kernel

T-complete functions of the Trefftz method:

$$1,\ \rho^m\cos(m\phi),\ \rho^m\sin(m\phi),\ \rho^2,\ \rho^{m+2}\cos(m\phi),\ \rho^{m+2}\sin(m\phi)$$

Degenerate kernel of the MFS:

m=0 m=0 m=1 m=1

$$u^*(\rho,\phi)=\rho^2(1+\ln R)+1\cdot R^2\ln R-(1+2\ln R)R\rho\cos(\theta-\phi)-\frac{1}{2}\frac{\rho^3}{R}\cos(\theta-\phi)$$

$$-\sum_{m=2}^{\infty}\frac{\rho^{m+2}}{m(m+1)R^m}\cos(m(\theta-\phi))+\sum_{m=2}^{\infty}\frac{1}{m(m-1)}\frac{\rho^m}{R^{m-2}}\cos(m(\theta-\phi))$$

m=2, 3….. m=2, 3…..

Comparison between the Trefftz method and MFS

	Trefftz method	MFS
Objectivity (Frame of difference)	Bad	Good
Degenerate scale	Disappear	Appear
Ill-posed behavior	Appear	Appear

Outlines

1. Laplace and biharmonic problems
2. Trefftz method and MFS
3. Connection between the two methods
4. Numerical examples
5. Conclusions

The efficiency between the Trefftz method and the MFS

We propose an example for exact solution:

$$u(r,\theta) = r^{50}\cos(50\theta),$$

Trefftz method :

$N_T = 50$

N=101 terms

MFS :

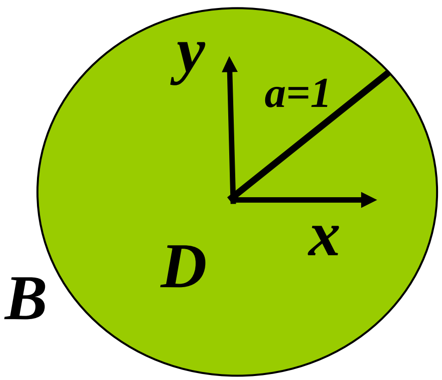
$N_M < 50$

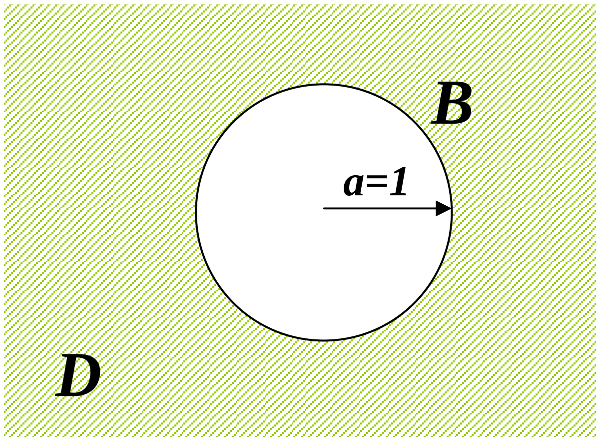
N < 101 terms

Numerical Example 1

$$G.E.: \nabla^2 u(x) = 0$$

$$B.C.: u(x) = \cos(3\theta)$$



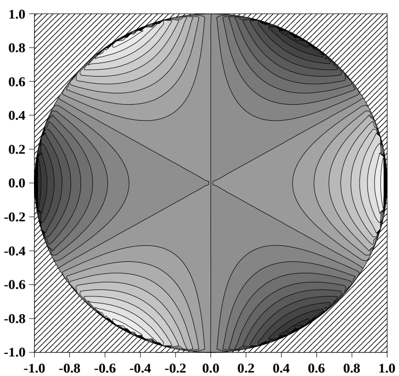
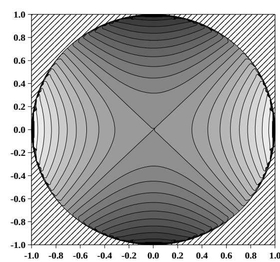
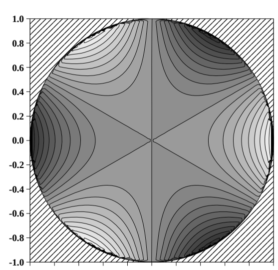
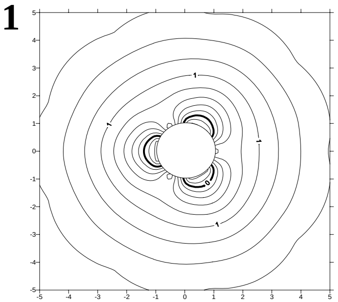
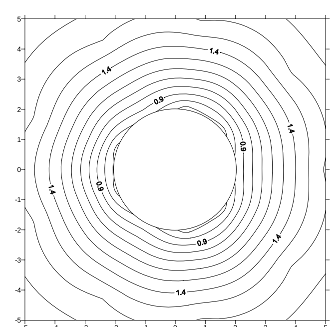
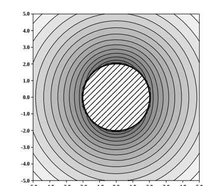
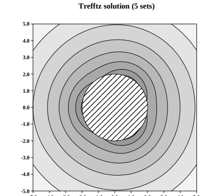


Exact solution: $u(r,\theta) = r^3 \cos(3\theta)$ $\quad u(r,\theta) = c \ln r + \frac{1}{r^3}\cos(3\theta)$

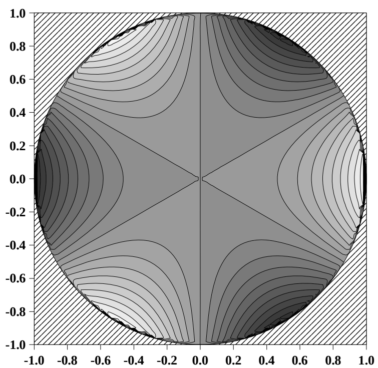
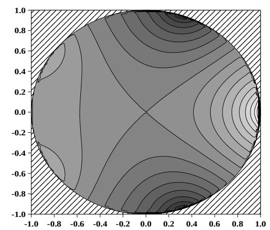
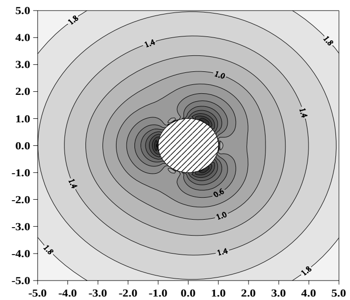
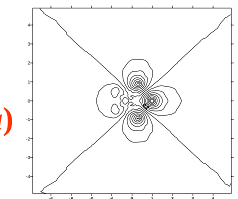
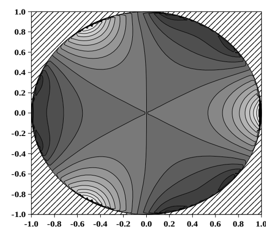
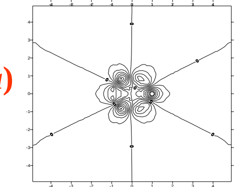
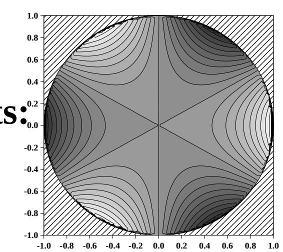
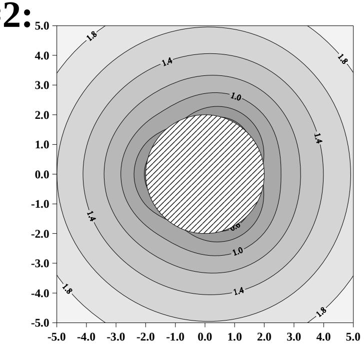
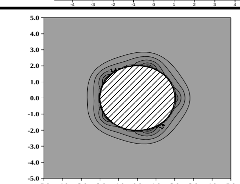
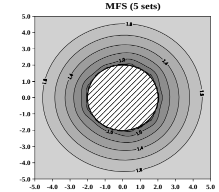
1. Trefftz method for simply-connected problem

2. MFS for simply-connected problem

Trefftz method

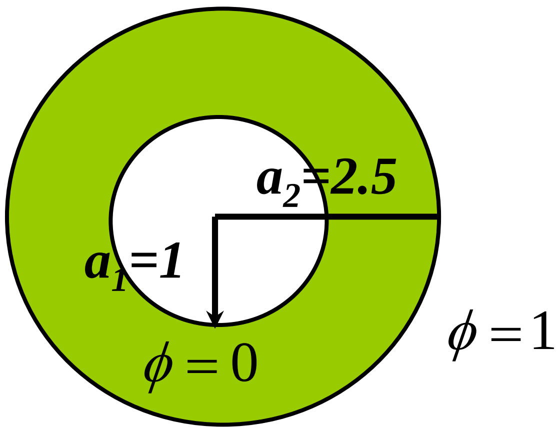
Trefftz method for the simply-connected problem			
Interior problem		**Exterior problem**	
Exact solution	**Numerical solution**	**Exact solution**	**Numerical solution**
 Exact solution	**5 Points:** B.C. aliasing base deficiency  Trefftz method (5 sets) **9 Points:**  Trefftz method (9 sets)	***a*=1** 	**5 Points:** B.C. aliasing Failure ($\ln\rho$) **9 Points:** Failure ($\ln\rho$)
		***a*=2** 	**5 Points:** B.C. aliasing  Trefftz solution (5 sets) **9 Points:**  Trefftz solution (9 sets)

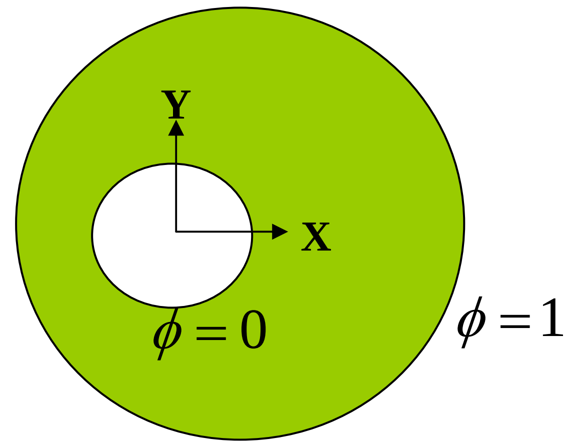
MFS

MFS for simply-connected problem			
Interior problem		Exterior problem	
Exact solution	Numerical solution	Exact solution	Numerical solution
 Exact solution	5 Points: B.C. aliasing  MFS (5 sets)	*a*=1:  Exact solution (r=1)	5 Points: B.C. aliasing Failure (ln *a*) 
	9 Points:  MFS (9 sets)		9 Points: Failure (ln *a*) 
	55 Points:  MFS (55 sets)	*a*=2:  Exact solution (r=2)	5 Points: B.C. aliasing  MFS (5 sets)
			9 Points:  MFS (9 sets)

Numerical Example 2

$$G.E.: \nabla^2 \phi = 0$$





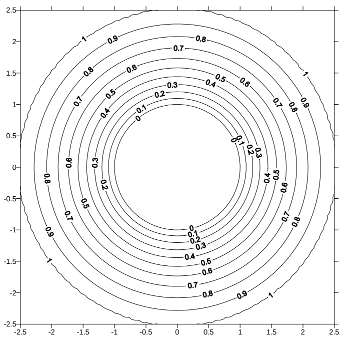
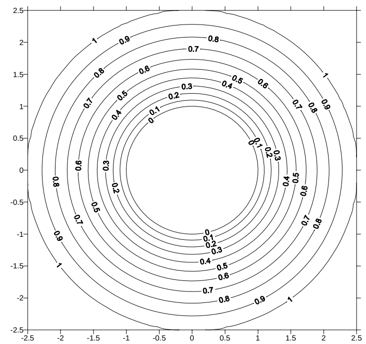
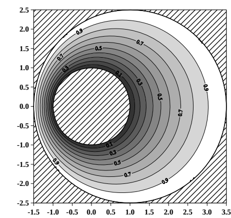
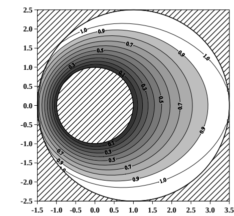
Exact solution:

$$u(\rho,\phi) = \frac{\ln \rho}{\ln 2.5}$$

$$u(\rho,\phi) = \frac{1}{2\ln 2}\left\{\frac{16\rho^2 + 1 + 8\rho\cos\phi}{\rho^2 + 16 + 8\rho\cos\phi}\right\}$$

1. Trefftz method for multiply-connected problem
2. MFS for multiply-connected problem

Trefftz

<table>
<tr><th colspan="4">Trefftz method for multiply-connected problem</th></tr>
<tr><th colspan="2">Concentric circle</th><th colspan="2">Eccentric circle</th></tr>
<tr><th>Exact solution</th><th>Numerical solution</th><th>Exact solution</th><th>Numerical solution</th></tr>
<tr>
<td rowspan="3">26 Points

$$u(\rho,\phi)=\frac{\ln\rho}{\ln 2.5}$$</td>
<td rowspan="3">26 Points
</td>
<td>
</td>
<td>6 Points

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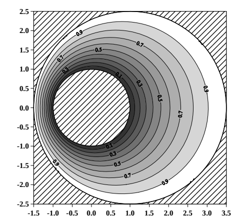
<td>
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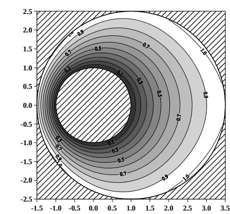
<td>14 Points

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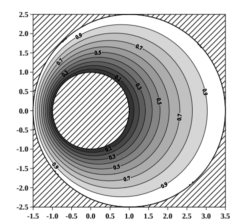
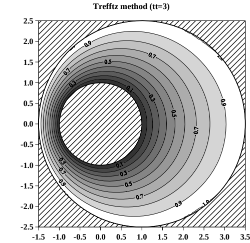
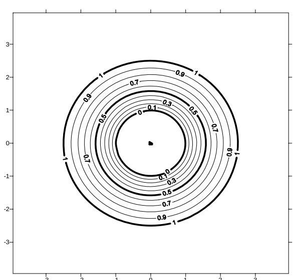
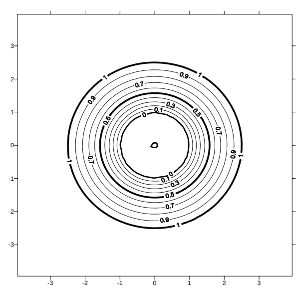
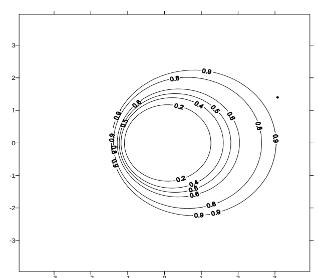
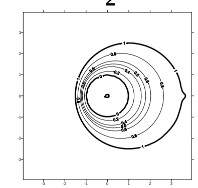
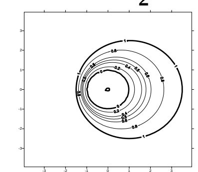
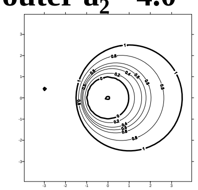
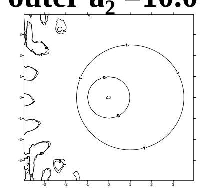
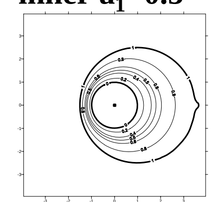
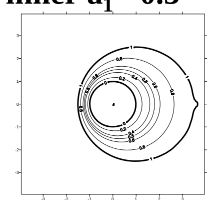
<tr>
<td>
</td>
<td>26 Points

</td>
</tr>
</table>

MFS

MFS for the multiply-connected problem

Concentric circle		Eccentric circle	
Exact solution	Numerical solution	Exact solution	Numerical solution
Inner circle: 20 points outer circle: 60points $u(\rho,\phi)=\frac{\ln\rho}{\ln 2.5}$ 	Inner circle: $a_1=0.9$ outer circle :$a_2=2.6$ Inner circle: 20 points outer circle: 60points 	Inner circle: 20 points outer circle: 60points $u(\rho,\phi)=\frac{1}{2\ln 2}\times\{\frac{16\rho^2+1+8\rho\cos\alpha}{\rho^2+16+8\rho\cos\alpha}\}$ 	Inner: 20points; outer: 60points; inner $a_1=0.9$ outer $a_2=2.6$  outer $a_2=3.0$  outer $a_2=4.0$  outer $a_2=10.0$  Inner: 20points; outer: 60points; outer $a_2$2.6 inner $a_1=0.5$  inner $a_1=0.3$ 

Outlines

1. Laplace and biharmonic problems
2. Trefftz method and MFS
3. Connection between the two methods
4. Numerical examples
5. Conclusions

Conclusions

1. The proof of the mathematical equivalence between the Trefftz method and MFS for Laplace equation was derived successfully.
2. The T-complete set functions in the Trefftz method for interior and exterior problems are imbedded in the degenerate kernels of the fundamental solutions
3. The sources of degenerate scale and ill-posed behavior in the MFS are easily found in the present formulation.
4. It is found that MFS can approach the exact solution more efficiently than the Trefftz method under the same number of degrees of freedom.
5. The content of this talk will be published in Computers and Mathematics with Applications.

歡迎參觀

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E-mail: jtchen@mail.ntou.edu.tw

The End

Thanks for your kind attention

Degenerate kernel

Degenerate kernel :

$$U(R,\theta,\rho,\phi)=\begin{cases} U^i(R,\theta,\rho,\phi)=\ln(R)-\sum_{m=1}^{\infty}\frac{1}{m}(\frac{\rho}{R})^m\cos(m(\theta-\phi)), & R>\rho \\ U^e(R,\theta,\rho,\phi)=\ln(\rho)-\sum_{m=1}^{\infty}\frac{1}{m}(\frac{R}{\rho})^m\cos(m(\theta-\phi)), & R<\rho \end{cases}$$

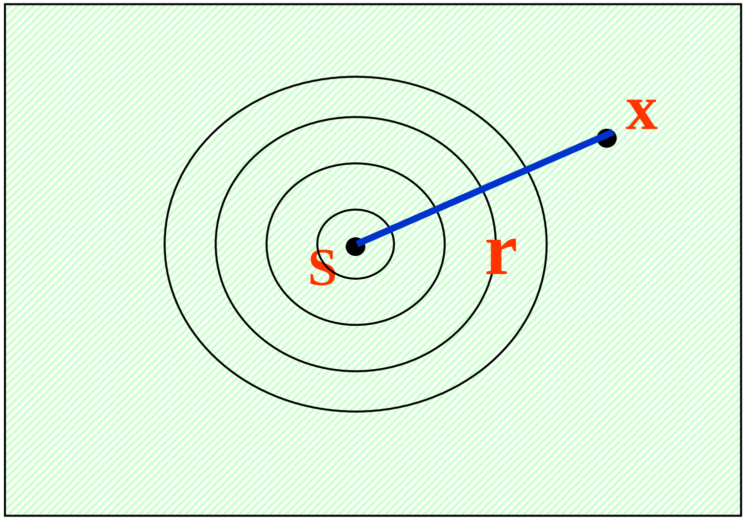
Symmetry property for kernel :

$$U(x,s_j)=U(s_j,x) \Longrightarrow u(x)=\sum_{j=1}^{N_M}c_jU(s_j,x),\ \ s_j\in D^e.$$

Degenerate kernel (step1)

Step 1

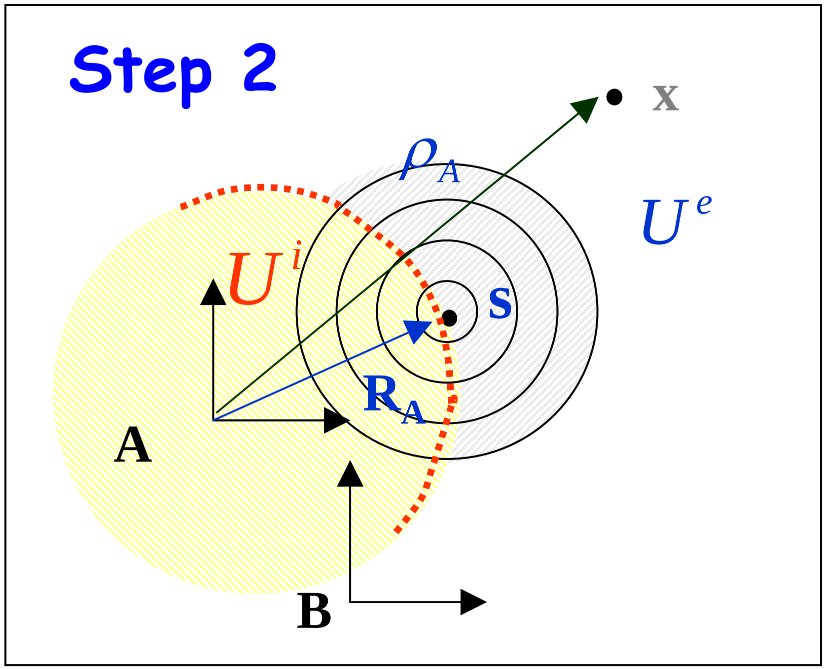
$$U(s,x) = \ln(r) = \ln\left|\underline{s} - \underline{x}\right|$$

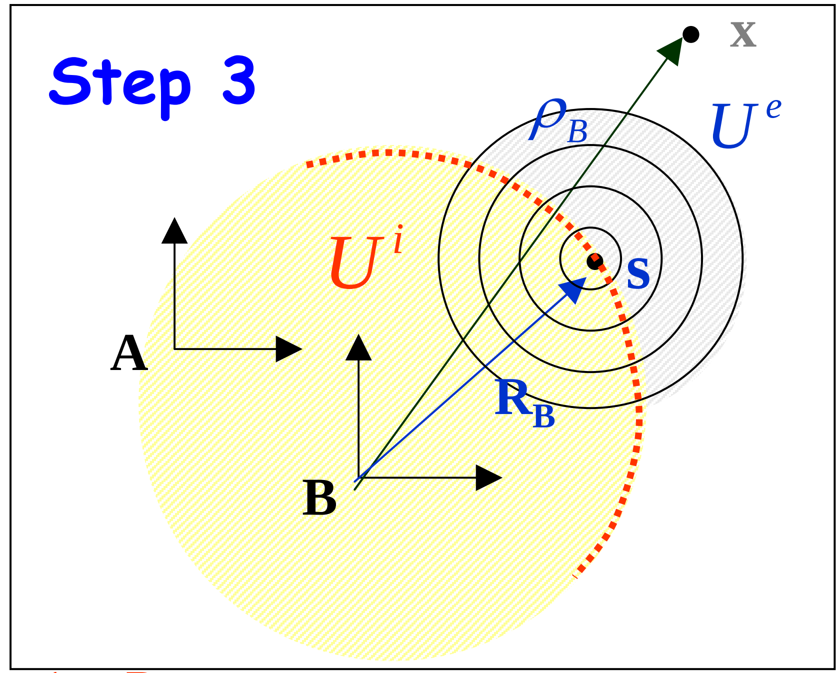


x: variable

s: fixed

Degenerate kernel (Step 2, Step 3)





$$U^i(R,\theta,\rho,\phi) = \ln(\rho) - \sum_{m=1}^{\infty} \frac{1}{m}\left(\frac{R}{\rho}\right)^m \cos(m(\theta-\phi)), \quad R > \rho$$

$$U^e(R,\theta,\rho,\phi) = \ln(R) - \sum_{m=1}^{\infty} \frac{1}{m}\left(\frac{\rho}{R}\right)^m \cos(m(\theta-\phi)), \quad R < \rho$$

Derivation of degenerate kernel

Use the Complex Variable method to derive the degenerate kernel:

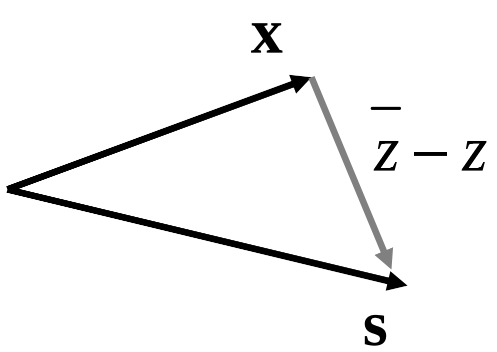
Motivation: $z = \ln r + i\theta$

$$\underset{\sim}{x} = (\rho, \phi) \rightarrow \bar{z}, \qquad \underset{\sim}{s} = (R, \theta) \rightarrow z$$

$$\Rightarrow \quad \begin{aligned} \bar{z} &= \ln \rho + i\phi \quad (x) \\ z &= \ln R + i\theta \quad (s) \end{aligned}$$

Derivation of degenerate kernel

Not important



$$\ln(\bar{z} - z) = \ln r + i\theta$$

$$\Rightarrow \quad \ln r = \mathrm{Re}[\ln(\bar{z} - z)]$$

$$\Rightarrow \quad \ln(\bar{z} - z) = \ln \bar{z}\left(1 - \frac{z}{\bar{z}}\right)$$

Due to: $\left|\frac{z}{\bar{z}}\right| < 1$

$$\Rightarrow \quad \ln|1 - x| = -\sum_{m=1}^{\infty} \frac{1}{m}(x)^m$$

$$\Rightarrow \quad \mathrm{Re}[\ln z + \sum\left(\frac{-1}{m}\right)\left(\frac{z}{\bar{z}}\right)^m] = \ln \rho + \sum\left(\frac{-1}{m}\right)\left(\frac{\rho}{R}\right)\cos(m(\theta - \phi))$$