On the spurious eigenvalues for a concentric sphere in BIEM

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ABSTRACT

Researchers have paid attention on spurious eigenvalues for multiply-connected domain (2D) eigenproblems by using BEM/BIEM. This paper employs the null-field integral equation method to study the occurring mechanism of spurious eigenvalues for 3D problems with an inner hole. By expanding the fundamental solution into degenerate kernels and expressing the boundary density in terms of spherical harmonics, all boundary integrals can be analytically determined. It is noted that our null-field integral formulation can locate the collocation point on the real boundary thanks to the degenerate kernel. In addition, the spurious eigenvalues are parasitized in the formulations, e.g. singular and hypersingular formulations in the dual BIEM while true eigensolutions are dependent on the boundary condition such as the Dirichlet or Neumann problem. By using the updating terms and updating document of singular value decomposition (SVD) technique, true and spurious eigenvalues can be extracted out, respectively. Besides, true and spurious boundary eigenvectors are obtained in the right and left unitary vectors in the SVD structure of the influence matrices. This finding agrees with that of 2D cases.

Keywords: null-field integral equation, degenerate kernel, eigenproblem, spurious eigenvalue, singular value decomposition.

1. INTRODUCTION

The application of eigenanalysis is gradually increasing for vibration and acoustics. The demand for eigenanalysis calls for an efficient and reliable method of computation for eigenvalues and eigenmodes. Over the past three decades, several boundary element formulations have been employed to solve the eigenproblems [1], e.g., determinant searching method, internal cell method, dual reciprocity method, particular integral method and multiple reciprocity method. In this paper, we will focus on the determinant searching method with emphasis on spurious eigenvalues in using BIEM for 3D problems with an inner hole. Spurious and fictitious solutions stem from non-uniqueness solution problems which appear in different aspects in computational mechanics. First of all, hourglass modes in the finite element method (FEM) using the reduced integration occur due to rank deficiency [2]. Also, loss of divergence-free constraint for the incompressible elasticity results in spurious modes. On the other hand, while solving the differential equation by the finite difference method (FDM), the spurious eigenvalue also appears due to discretization [3-5]. In the real-part BEM [6] or the MRM formulation [7-12], spurious eigensolutions occur in solving eigenproblems. Even though the complex-valued kernel is adopted, the spurious eigensolution also occurs for the multiply-connected problem [13-14] as well as the appearance of fictitious frequency for the exterior acoustics [15]. Spurious eigenvalues in the MFS for 3D problems were also studied by Tsai *et al* [16]. In this paper, a simple case of 3D concentric sphere will be demonstrated to see how spurious eigensolutions occur and how they are suppressed by using SVD.

In the recent years, the SVD technique has been applied to solve problems of fictitious-frequency [15] and continuum mechanics [17]. Two ideas, namely updating term and updating document [15], were successfully applied to extract the true and spurious solutions, respectively. In this paper, the three-dimensional eigenproblem of a concentric sphere is studied in both numerical and analytical ways. Owing to the introduction of degenerate kernel, the collocation point can be located exactly on the real boundary. Besides, true and spurious equations can be found by using the null-field integral equation in conjunction with degenerate kernels and spherical harmonics. Surface distributions of the inner and outer boundaries can be expanded in terms of spherical harmonics. Since a spurious eigenvalue is related to mathematics and has no physical meaning, the remedies, SVD updating term and SVD updating document, are used to extract or filter out true and spurious eigenvalues, respectively. Finally, an example with various boundary conditions is utilized to validate the present approach by using singular and hypersingular formulations.

2. ON THE OCCURRING MECHANISM OF SPURIOUS EIGENVALUES IN

BEM/BIEM

2.1 Problem statements

The governing equation for the eigenproblem of a concentric sphere is the Helmholtz equation as follows:

$$(\nabla^2 + k^2)u(x) = 0, \quad x \in D, \tag{1}$$

where ∇^2, k and D are the Laplacian operator, the wave number and the domain of interest, respectively. The concentric sphere is depicted in Fig. 1. The inner and outer radii are a and b, respectively.

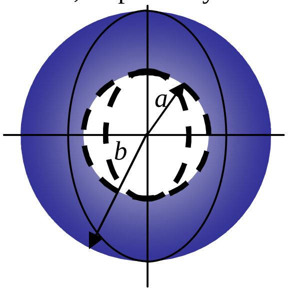

Fig. 1 A concentric sphere

2.2 Dual null-field integral formulation — the conventional version

The dual boundary integral formulation [5] for the domain point is shown below:

$$4\pi u(x) = \int_B T(s,x)u(s)dB(s) - \int_B U(s,x)\frac{\partial u(s)}{\partial n_s}dB(s), \quad x \in D, \tag{2}$$

$$4\pi \frac{\partial u(x)}{\partial n_x} = \int_B M(s,x)u(s)dB(s) - \int_B L(s,x)\frac{\partial u(s)}{\partial n_s}dB(s), \quad x \in D, \tag{3}$$

where x and s are the field and source points, respectively, B is the boundary, n_x and n_s denote the outward normal vector at the field point and the source point, respectively, and the kernel function $U(s,x)$ is the fundamental solution which satisfies

$$(\nabla^2 + k^2)U(s,x) = 4\pi\delta(x-s). \tag{4}$$

where δ is the Dirac-delta function. The other kernel functions can be obtained as

$$T(s,x) = \frac{\partial U(s,x)}{\partial n_s}, \tag{5}$$

$$L(s,x) = \frac{\partial U(s,x)}{\partial n_x}, \tag{6}$$

$$M(s,x) = \frac{\partial^2 U(s,x)}{\partial n_s \partial n_x}. \tag{7}$$

If the collocation point x is on the boundary, the dual boundary integral equations for the boundary point can be obtained as follows:

$$2\pi u(x) = C.P.V.\int_B T(s,x)u(s)dB(s) - R.P.V.\int_B U(s,x)\frac{\partial u(s)}{\partial n_s}dB(s), \quad x \in B, \tag{8}$$

$$2\pi \frac{\partial u(x)}{\partial n_x} = H.P.V.\int_B M(s,x)u(s)dB(s) - C.P.V.\int_B L(s,x)\frac{\partial u(s)}{\partial n_s}dB(s), \quad x \in B, \tag{9}$$

where $R.P.V.$ $C.P.V.$ and $H.P.V.$ are the Riemann principal value, the Cauchy principal value and the Hadamard (or called Mangler) principal value, respectively. By collocating x outside the domain, we obtain the null-field integral equation as shown below:

$$0 = \int_B T(s,x)u(s)dB(s) - \int_B U(s,x)\frac{\partial u(s)}{\partial n_s}dB(s), \quad x \in D^c, \tag{10}$$

$$0 = \int_B M(s,x)u(s)dB(s) - \int_B L(s,x)\frac{\partial u(s)}{\partial n_s}dB(s), \quad x \in D^c, \tag{11}$$

where D^c denotes the complementary domain.

2.3 Dual null-field integral formulation — the present version

By introducing the degenerate kernels, the collocation points can be located on the real boundary free of facing singularity. Therefore, the representations of integral equations including the boundary point can be written as

$$4\pi u(x) = \int_B T(s,x)u(s)dB(s) - \int_B U(s,x)\frac{\partial u(s)}{\partial n_s}dB(s), \quad x \in D \cup B, \tag{12}$$

$$4\pi \frac{\partial u(x)}{\partial n_x} = \int_B M(s,x)u(s)dB(s) - \int_B L(s,x)\frac{\partial u(s)}{\partial n_s}dB(s), \quad x \in D \cup B, \tag{13}$$

and

$$0 = \int_B T(s,x)u(s)dB(s) - \int_B U(s,x)\frac{\partial u(s)}{\partial n_s}dB(s), \quad x \in D^c \cup B, \tag{14}$$

$$0 = \int_B M(s,x)u(s)dB(s) - \int_B L(s,x)\frac{\partial u(s)}{\partial n_s}dB(s), \quad x \in D^c \cup B, \tag{15}$$

once the kernel is expressed in terms of an appropriate degenerate form. It is found that the collocation point is categorized to three positions, domain (Eqs.(2)-(3)), boundary (Eqs.(8)-(9)) and complementary domain (Eqs.(10)-(11)) in the conventional formulation. After using the degenerate kernel for the null-field BIEM, both Eqs.(12)-(13) and Eqs.(14)-(15) can contain the boundary point.

2.4 Expansions of the fundamental solution and boundary density

The fundamental solution as previously mentioned is

$$U(s,x) = -\frac{e^{-ikr}}{r} \tag{16}$$

where $r \equiv |s-x|$ is the distance between the source point and the field point and i is the imaginary number with $i^2=-1$. To fully utilize the property of spherical geometry, the mathematical tools, degenerate (separable or of finite rank) kernel and spherical harmonics, are utilized for the analytical calculation of boundary integrals.

2.4.1 Degenerate (separable) kernel for fundamental solutions

In the spherical coordinate, the field point (x) and source point (s) can be expressed as $x=(\rho,\phi,\theta)$ and $s=(\bar{\rho},\bar{\phi},\bar{\theta})$ in the spherical coordinate**,** respectively. By employing the addition theorem for separating the source point and field point, the kernel functions, $U(s,x)$, $T(s,x)$, $L(s,x)$ and $M(s,x)$, are expanded in terms of degenerate kernel as shown below:

$$U(s,x)=\begin{cases} U^i=ik\sum_{n=0}^{\infty}(2n+1)\sum_{m=0}^{n}\varepsilon_m\dfrac{(n-m)!}{(n+m)!}\cos\left[m\left(\phi-\bar{\phi}\right)\right] \\ \quad P_n^m(\cos\theta)P_n^m\left(\cos\bar{\theta}\right)j_n(k\rho)h_n^{(2)}(k\bar{\rho}),\ \bar{\rho}\geq\rho, \\ U^e=ik\sum_{n=0}^{\infty}(2n+1)\sum_{m=0}^{n}\varepsilon_m\dfrac{(n-m)!}{(n+m)!}\cos\left[m\left(\phi-\bar{\phi}\right)\right] \\ \quad P_n^m(\cos\theta)P_n^m\left(\cos\bar{\theta}\right)j_n(k\bar{\rho})h_n^{(2)}(k\rho),\ \rho>\bar{\rho}. \end{cases}\tag{17}$$

$$T(s,x)=\begin{cases} T^i=ik^2\sum_{n=0}^{\infty}(2n+1)\sum_{m=0}^{n}\varepsilon_m\dfrac{(n-m)!}{(n+m)!}\cos\left[m\left(\phi-\bar{\phi}\right)\right] \\ \quad P_n^m(\cos\theta)P_n^m\left(\cos\bar{\theta}\right)j_n(k\rho)h_n'^{(2)}(k\bar{\rho}),\ \bar{\rho}>\rho, \\ T^e=ik^2\sum_{n=0}^{\infty}(2n+1)\sum_{m=0}^{n}\varepsilon_m\dfrac{(n-m)!}{(n+m)!}\cos\left[m\left(\phi-\bar{\phi}\right)\right] \\ \quad P_n^m(\cos\theta)P_n^m\left(\cos\bar{\theta}\right)j_n'(k\bar{\rho})h_n^{(2)}(k\rho),\ \rho>\bar{\rho}. \end{cases}\tag{18}$$

$$L(s,x)=\begin{cases} L^i=ik^2\sum_{n=0}^{\infty}(2n+1)\sum_{m=0}^{n}\varepsilon_m\dfrac{(n-m)!}{(n+m)!}\cos\left[m\left(\phi-\bar{\phi}\right)\right] \\ \quad P_n^m(\cos\theta)P_n^m\left(\cos\bar{\theta}\right)j_n'(k\rho)h_n^{(2)}(k\bar{\rho}),\ \bar{\rho}>\rho, \\ L^e=ik^2\sum_{n=0}^{\infty}(2n+1)\sum_{m=0}^{n}\varepsilon_m\dfrac{(n-m)!}{(n+m)!}\cos\left[m\left(\phi-\bar{\phi}\right)\right] \\ \quad P_n^m(\cos\theta)P_n^m\left(\cos\bar{\theta}\right)j_n(k\bar{\rho})h_n'^{(2)}(k\rho),\ \rho>\bar{\rho}. \end{cases}\tag{19}$$

$$M(s,x)=\begin{cases} M^i=ik^3\sum_{n=0}^{\infty}(2n+1)\sum_{m=0}^{n}\varepsilon_m\dfrac{(n-m)!}{(n+m)!}\cos\left[m\left(\phi-\bar{\phi}\right)\right] \\ \quad P_n^m(\cos\theta)P_n^m\left(\cos\bar{\theta}\right)j_n'(k\rho)h_n'^{(2)}(k\bar{\rho}),\ \bar{\rho}\geq\rho, \\ M^e=ik^3\sum_{n=0}^{\infty}(2n+1)\sum_{m=0}^{n}\varepsilon_m\dfrac{(n-m)!}{(n+m)!}\cos\left[m\left(\phi-\bar{\phi}\right)\right] \\ \quad P_n^m(\cos\theta)P_n^m\left(\cos\bar{\theta}\right)j_n'(k\bar{\rho})h_n'^{(2)}(k\rho),\ \rho>\bar{\rho}. \end{cases}\tag{20}$$

where the superscripts *"i"* and *"e"* denote the interior and exterior regions, j_n and $h_n^{(2)}$ are the n^{th} order spherical Bessel function of the first kind and the n^{th} order spherical Hankel function of the second kind, respectively, P_n^m is the associated Lengendre polynomial and ε_m is the Neumann factor,

$$\varepsilon_m=\begin{cases}1, & m=0,\\ 2, & m=1,2,\cdots,\infty.\end{cases}\tag{21}$$

It is noted that U and M kernels in Eqs.(17) and (20) contain the equal sign of $\rho=\bar{\rho}$ while T and L kernels do not include the equal sign due to discontinuity.

2.4.2 Spherical harmonics expansion for boundary densities

We apply the spherical harmonics expansion to approximate the boundary density and its normal derivative as expressed by

$$u(s)=\sum_{v=0}^{\infty}\sum_{w=0}^{v}A_{vw}P_v^w\left(\cos\bar{\theta}\right)\cos\left(w\bar{\phi}\right),\quad s\in B,\tag{22}$$

$$t(s)=\frac{\partial u(s)}{\partial n_s}=\sum_{v=0}^{\infty}\sum_{w=0}^{v}B_{vw}P_v^w\left(\cos\bar{\theta}\right)\cos\left(w\bar{\phi}\right),\quad s\in B,\tag{23}$$

where A_{vw} and B_{vw} are the unknown coefficients.

3. PROOF OF THE EXISTENCE OF SPURIOUS EIGENSOLUTIONS FOR A CONCENTRIC SPHERE

In order to fully utilize the geometry of sphere boundary, the potential u and its normal derivative t can be approximated by employing the spherical harmonic. Therefore, the following expressions can be obtained

$$u_1(s)=\sum_{v=0}^{\infty}\sum_{w=0}^{v}A_{vw}^1P_v^w\left(\cos\bar{\theta}\right)\cos\left(w\bar{\phi}\right),\quad s\in B_1,,\tag{24}$$

$$u_2(s)=\sum_{v=0}^{\infty}\sum_{w=0}^{v}A_{vw}^2P_v^w\left(\cos\bar{\theta}\right)\cos\left(w\bar{\phi}\right),\quad s\in B_2,\tag{25}$$

$$t_1(s)=\sum_{v=0}^{\infty}\sum_{w=0}^{v}B_{vw}^1P_v^w\left(\cos\bar{\theta}\right)\cos\left(w\bar{\phi}\right),\quad s\in B_1,\tag{26}$$

$$t_2(s)=\sum_{v=0}^{\infty}\sum_{w=0}^{v}B_{vw}^2P_v^w\left(\cos\bar{\theta}\right)\cos\left(w\bar{\phi}\right),\quad s\in B_2,\tag{27}$$

where A_{vw}^i and B_{vw}^i are the spherical coefficients on B_i ($i=1,2$). When the field point is located on the inner boundary B_1, substitution of Eqs. (24)-(27) into the null-field integral equations yields

$$\begin{aligned}0=&\int_0^{2\pi}\int_0^{\pi}\sum_{n=0}^{\infty}\sum_{m=0}^{n}\sum_{v=0}^{\infty}\sum_{w=0}^{v}ik^2\varepsilon_mA_{vw}^1(2n+1)\frac{(n-m)!}{(n+m)!}j_n'(k\rho)h_n^{(2)}(kR_1)P_n^m(\cos(\theta))\\&\cos\left(m(\phi-\bar{\phi})\right)\cos\left(w\bar{\phi}\right)\left(P_n^m\left(\cos(\bar{\theta})\right)P_v^w\left(\cos(\bar{\theta})\right)\sin(\bar{\theta})\right)R_1^2d\bar{\theta}d\bar{\phi}\\&-\int_0^{2\pi}\int_0^{\pi}\sum_{n=0}^{\infty}\sum_{m=0}^{n}\sum_{v=0}^{\infty}\sum_{w=0}^{v}ik\varepsilon_mB_{vw}^1(2n+1)\frac{(n-m)!}{(n+m)!}j_n(k\rho)h_n^{1(2)}(kR_1)P_n^m(\cos(\theta))\\&\cos\left(m(\phi-\bar{\phi})\right)\cos\left(w\bar{\phi}\right)\left(P_n^m\left(\cos(\bar{\theta})\right)P_v^w\left(\cos(\bar{\theta})\right)\sin(\bar{\theta})\right)R_1^2d\bar{\theta}d\bar{\phi}\\&+\int_0^{2\pi}\int_0^{\pi}\sum_{n=0}^{\infty}\sum_{m=0}^{n}\sum_{v=0}^{\infty}\sum_{w=0}^{v}ik^2\varepsilon_mA_{vw}^2(2n+1)\frac{(n-m)!}{(n+m)!}j_n'(k\rho)h_n^{(2)}(kR_2)P_n^m(\cos(\theta))\\&\cos\left(m(\phi-\bar{\phi})\right)\cos\left(w\bar{\phi}\right)\left(P_n^m\left(\cos(\bar{\theta})\right)P_v^w\left(\cos(\bar{\theta})\right)\sin(\bar{\theta})\right)R_2^2d\bar{\theta}d\bar{\phi}\\&-\int_0^{2\pi}\int_0^{\pi}\sum_{n=0}^{\infty}\sum_{m=0}^{n}\sum_{v=0}^{\infty}\sum_{w=0}^{v}ik\varepsilon_mB_{vw}^2(2n+1)\frac{(n-m)!}{(n+m)!}j_n(k\rho)h_n^{(2)}(kR_2)P_n^m(\cos(\theta))\\&\cos\left(m(\phi-\bar{\phi})\right)\cos\left(w\bar{\phi}\right)\left(P_n^m\left(\cos(\bar{\theta})\right)P_v^w\left(\cos(\bar{\theta})\right)\sin(\bar{\theta})\right)R_2^2d\bar{\theta}d\bar{\phi}.\end{aligned}\tag{28}$$

When the field point is located on the outer boundary B_2, we have

$$\begin{aligned}0=&\int_0^{2\pi}\int_0^{\pi}\sum_{n=0}^{\infty}\sum_{m=0}^{n}\sum_{v=0}^{\infty}\sum_{w=0}^{v}ik^2\varepsilon_mA_{vw}^1(2n+1)\frac{(n-m)!}{(n+m)!}j_n'(kR_1)h_n^{(2)}(k\rho)P_n^m(\cos(\theta))\\&\cos\left(m(\phi-\bar{\phi})\right)\cos\left(w\bar{\phi}\right)\left(P_n^m\left(\cos(\bar{\theta})\right)P_v^w\left(\cos(\bar{\theta})\right)\sin(\bar{\theta})\right)R_1^2d\bar{\theta}d\bar{\phi}\\&-\int_0^{2\pi}\int_0^{\pi}\sum_{n=0}^{\infty}\sum_{m=0}^{n}\sum_{v=0}^{\infty}\sum_{w=0}^{v}ik\varepsilon_mB_{vw}^1(2n+1)\frac{(n-m)!}{(n+m)!}j_n(kR_1)h_n^{1(2)}(k\rho)P_n^m(\cos(\theta))\\&\cos\left(m(\phi-\bar{\phi})\right)\cos\left(w\bar{\phi}\right)\left(P_n^m\left(\cos(\bar{\theta})\right)P_v^w\left(\cos(\bar{\theta})\right)\sin(\bar{\theta})\right)R_1^2d\bar{\theta}d\bar{\phi}\\&+\int_0^{2\pi}\int_0^{\pi}\sum_{n=0}^{\infty}\sum_{m=0}^{n}\sum_{v=0}^{\infty}\sum_{w=0}^{v}ik^2\varepsilon_mA_{vw}^2(2n+1)\frac{(n-m)!}{(n+m)!}j_n'(kR_2)h_n^{(2)}(k\rho)P_n^m(\cos(\theta))\\&\cos\left(m(\phi-\bar{\phi})\right)\cos\left(w\bar{\phi}\right)\left(P_n^m\left(\cos(\bar{\theta})\right)P_v^w\left(\cos(\bar{\theta})\right)\sin(\bar{\theta})\right)R_2^2d\bar{\theta}d\bar{\phi}\\&-\int_0^{2\pi}\int_0^{\pi}\sum_{n=0}^{\infty}\sum_{m=0}^{n}\sum_{v=0}^{\infty}\sum_{w=0}^{v}ik\varepsilon_mB_{vw}^2(2n+1)\frac{(n-m)!}{(n+m)!}j_n(kR_2)h_n^{(2)}(k\rho)P_n^m(\cos(\theta))\\&\cos\left(m(\phi-\bar{\phi})\right)\cos\left(w\bar{\phi}\right)\left(P_n^m\left(\cos(\bar{\theta})\right)P_v^w\left(\cos(\bar{\theta})\right)\sin(\bar{\theta})\right)R_2^2d\bar{\theta}d\bar{\phi}.\end{aligned}\tag{29}$$

For the Dirichlet problem, Eqs. (28) and (29) can be reduced to

$$0=\sum_{n=0}^{\infty}\sum_{m=0}^{n} a^2 k B_{nm}^1 j_n(ka) h_n^{(2)}(ka) P_n^m(\cos(\theta))\cos(m\phi) + \sum_{n=0}^{\infty}\sum_{m=0}^{n} b^2 k B_{nm}^2 j_n(ka) h_n^{(2)}(kb) P_n^m(\cos(\theta))\cos(m\phi), \quad (30)$$

$$0=\sum_{n=0}^{\infty}\sum_{m=0}^{n} a^2 k B_{nm}^1 j_n(ka) h_n^{(2)}(kb) P_n^m(\cos(\theta))\cos(m\phi) + \sum_{n=0}^{\infty}\sum_{m=0}^{n} b^2 k B_{nm}^2 j_n(kb) h_n^{(2)}(kb) P_n^m(\cos(\theta))\cos(m\phi). \quad (31)$$

In order to prove that the spurious eigensolutions of a concentric sphere satisfy the BIE by collocating the inner and outer boundary points, we first derive the true eigensolutions of a sphere subject to the Dirichlet boundary condition.

Now, we consider the sphere with a radius a in the continuous system. By using the null-field integral equation and collocating the point on the boundary, we obtain the true eigenequation

$$j_n(ka)=0, \quad (32)$$

and the corresponding true eigenmode is

$$\{B_{vw}\}, \quad (33)$$

where $\sum\sum|B_{vw}|\neq 0$. By collocating the point in the complementary domain ($x^c \in D^c$) as shown in Fig. 2, the null-field equation yields

$$0=\int_{B_1} U^e(s,x^c)t(s)dB(s), \quad x^c \in D^c. \quad (34)$$

We can obtain the null-field response for x^c as shown below

$$B_{nm}^1 j_n(ka)h_n^{(2)}(ka^+)P_n^m(\cos(\theta))\cos(m\phi)=0, \quad (35)$$

where n and $m \in \mathbb{N}$, since k satisfies Eq. (32).

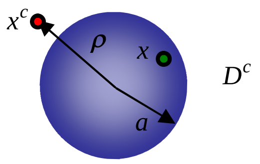

Fig. 2 Collocation point on the sphere boundary from the null-field point ($\rho = a^+$)

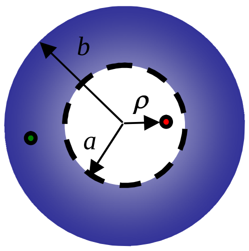

Fig. 3 Collocation point of the concentric sphere ($\rho = a^-$)

Secondly, we consider the spherical case with the fixed-fixed boundary condition as shown in Fig. 3. By selecting a nontrivial inner boundary mode for the boundary mode and trivial outer boundary mode, we have $j_n(ka)=0$ and

$$\begin{Bmatrix} B_{vw}^1 \\ B_{vw}^2 \end{Bmatrix} = \begin{Bmatrix} B_{vw} \\ 0 \end{Bmatrix} \quad (36)$$

This indicates that spurious eigenevalues of $j_n(ka)=0$ and the nontrivial boundary mode of Eq. (36) satisfy Eqs. (30) and (31) due to $U^i(s,a^-)=U^e(x,a^+)$. Therefore, spurious eigenvalues in conjunction with the trivial outer boundary mode happen to be the true eigenvalue of the domain bounded by the inner boundary. Similarly, the concentric sphere subjects to the Neumann boundary condition by using the hypersingular formulation results in the trivial outer boundary mode.

4. SVD TECHNIQUE FOR EXTRACTING OUT TRUE AND SPURIOUS EIGENVALUES BY USING UPDATING TERMS AND UPDATING DOCUMENTS

4.1 Method to extract the true eigensolutions

SVD technique is an important tool in the linear algebra. The matrix $[\mathbf{A}]$ with a dimension M by N can be decomposed into a product of the unitary matrix $[\Phi]$ (M by M), the diagonal matrix $[\Sigma]$ (M by N) with positive or zero elements, and the unitary matrix $[\Psi]$ (N by N)

$$[\mathbf{A}]_{M\times N}=[\Phi]_{M\times M}[\Sigma]_{M\times N}[\Psi]_{N\times N}^H, \quad (37)$$

where the superscript *"H"* is the Hermitian, $[\Phi]$ and $[\Psi]$ are both unitary that their column vectors which satisfy

$$\underset{\sim}{\phi}_i \cdot \underset{\sim}{\phi}_j^H=\delta_{ij}, \quad (38)$$

$$\underset{\sim}{\psi}_i \cdot \underset{\sim}{\psi}_j^H=\delta_{ij}, \quad (39)$$

in which $[\Phi]^H[\Phi]=[\mathbf{I}]_{M\times M}$ and $[\Psi]^H[\Psi]=[\mathbf{I}]_{N\times N}$. For the eigenproblem, we can obtain a nontrivial solution for the homogeneous system from a column vector $\{\psi_i\}$ of $[\Psi]$ when the singular value (σ_i) is zero. For the direct BEM, we have

Singular formulation (UT method)

$$\left[\mathbf{T}^e\right]\{u\}=\left[\mathbf{U}^e\right]\{t\}=\{0\}, \quad (40)$$

Hypersingular formulation (LM method)

$$\left[\mathbf{M}^e\right]\{u\}=\left[\mathbf{L}^e\right]\{t\}=\{0\}, \quad (41)$$

where $\{u\}$ and $\{t\}$ are the boundary excitations.

For the Dirichlet problem, Eq. (40) and (41) can be combined to have

$$\begin{bmatrix} \mathbf{U}^e \\ \mathbf{L}^e \end{bmatrix}\{t\}=\{0\}. \quad (42)$$

By using the SVD technique, the two submatrices in Eqs. (40) and (41) can be combined to have

$$\left[\mathbf{U}^e\right]=\left[\Phi^{(U)}\right]\left[\Sigma^{(U)}\right]\left[\Psi^{(U)}\right]^H \text{ or } \left[\mathbf{U}^e\right]=\sum_j \sigma_j^{(U)}\left\{\phi_j^{(U)}\right\}\left\{\psi_j^{(U)}\right\}^H, \quad (43)$$

$$\left[\mathbf{L}^e\right]=\left[\mathbf{\Phi}^{(L)}\right]\left[\mathbf{\Sigma}^{(L)}\right]\left[\mathbf{\Psi}^{(L)}\right]^H \quad \text{or} \quad \left[\mathbf{L}^e\right]=\sum_j \sigma_j^{(L)}\left\{\phi_j^{(L)}\right\}\left\{\psi_j^{(L)}\right\}^H. \tag{44}$$

where the superscripts, (U) and (L), denote the corresponding matrices. For the linear algebraic system, $\{t\}$ is a column vector of $\{\psi_i\}$ in the matrix $[\Psi]$ corresponding to the zero singular value ($\sigma_i=0$). By setting $\{t\}$ as a vector of $\{\psi_i\}$, in the right unitary matrix for the true eigenvalue k_t, Eq. (42) reduces to

$$\left[\mathbf{U}^e(k_t)\right]\{\psi_i\}=\{0\}, \tag{45}$$

$$\left[\mathbf{L}^e(k_t)\right]\{\psi_i\}=\{0\}. \tag{46}$$

According to Eqs. (43) - (46), we have

$$\sigma_j^{(U)}\left\{\phi_j^{(U)}\right\}=\{0\}, \tag{47}$$

$$\sigma_j^{(L)}\left\{\phi_j^{(L)}\right\}=\{0\}. \tag{48}$$

We can easily extract out the true eigenvalues, $\sigma_j^{(U)}=\sigma_j^{(L)}=\{0\}$, since there exists the same eigensolusion ($\{t\}=\{\psi_i\}$) for the Dirichlet problem using Eqs. (42) or (45) and (46). In a similar way, Eqs. (40) and (41) can be combined to have

$$\begin{bmatrix}\mathbf{T}^e(k_t)\\ \mathbf{M}^e(k_t)\end{bmatrix}\{u\}=\{0\}, \tag{49}$$

for the Neumann problem. We can easily extract out the true eigenvalues for the Neumann problem with respect to the j^{th} zero singular values of $\sigma_j^{(T)}=\sigma_j^{(M)}=\{0\}$.

4.2 Method to filter out the spurious eugensolutions

By employing the LM formulation in the direct BEM, we have

$$\left[\mathbf{M}^e\right]\{u\}=\left[\mathbf{L}^e\right]\{t\}=\{p\}. \tag{50}$$

Since the spurious eigenvalue k_s is embedded in both the Dirichlet and Neumann problems, we have

$$\{p\}^H\{\phi_i\}=\{0\}, \tag{51}$$

where $\{\phi_i\}$ satisfies

$$\left[\mathbf{L}^e(k_s)\right]^H\{\phi_i\}=\{0\}, \quad \text{for the Dirichlet problem} \tag{52}$$

$$\left[\mathbf{M}^e(k_s)\right]^H\{\phi_i\}=\{0\}, \quad \text{for the Neumann problem} \tag{53}$$

according to the Fredholm alternative theorem. By substituting Eq. (50) into Eqs. (52) and (53), we have

$$\{u\}^H\left[\mathbf{M}^e(k_s)\right]^H\{\phi_i\}=\{0\}, \text{ for the Dirichlet problem} \tag{54}$$

$$\{t\}^H\left[\mathbf{L}^e(k_s)\right]^H\{\phi_i\}=\{0\}, \text{ for the Neumann problem} \tag{55}$$

Since $\{u\}$ and $\{t\}$ can be arbitrary boundary excitation for the Dirchlet problem and Neumann problem, respectively, this yields

$$\left[\mathbf{M}^e(k_s)\right]^H\{\phi_i\}=\{0\}, \quad \text{for the Dirichelt problem} \tag{56}$$

$$\left[\mathbf{L}^e(k_s)\right]^H\{\phi_i\}=\{0\}, \quad \text{for the Neumann problem} \tag{57}$$

By combining Eqs. (52) and (53) with Eqs. (56) and (57) for the Dirichlet problem, we have

$$\begin{bmatrix}\left[\mathbf{L}^e\right]^H\\ \left[\mathbf{M}^e\right]^H\end{bmatrix}\{\phi_i\}=\{0\} \quad \text{or} \quad \{\phi_i\}^H\left[\left[\mathbf{L}^e\right]\ \left[\mathbf{M}^e\right]\right]=\{0\}. \tag{58}$$

It indicates that two matrices have the same spurious boundary mode $\{\phi_i\}$ corresponding to the ith zero singular values. By using the SVD technique, the two matrices in Eq. (58) ca be decomposed into

$$\left[\mathbf{L}^e\right]^H=\left[\mathbf{\Psi}^{(L)}\right]\left[\mathbf{\Sigma}^{(L)}\right]\left[\mathbf{\Phi}^{(L)}\right]^H \quad \text{or} \quad \left[\mathbf{L}_\lambda^e\right]=\sum_j \sigma_j^{(L)}\left\{\psi_j^{(L)}\right\}\left\{\phi_j^{(L)}\right\}^H, \tag{59}$$

$$\left[\mathbf{M}^e\right]^H=\left[\mathbf{\Psi}^{(M)}\right]\left[\mathbf{\Sigma}^{(M)}\right]\left[\mathbf{\Phi}^{(M)}\right]^H \quad \text{or} \quad \left[\mathbf{M}_\lambda^e\right]=\sum_j \sigma_j^{(M)}\left\{\psi_j^{(M)}\right\}\left\{\phi_j^{(M)}\right\}^H. \tag{60}$$

By substituting Eqs. (59) and (60) into Eqs. (54) and (55), we have

$$\sigma_j^{(L)}\left\{\psi_j^{(L)}\right\}=\{0\}, \tag{61}$$

$$\sigma_j^{(M)}\left\{\psi_j^{(M)}\right\}=\{0\}. \tag{62}$$

We can easily extract out the spurious eigenvalues since there exists the same spurious boundary mode $\{\phi_i\}$ corresponding the i^{th} zero singular value, $\sigma_i^{(L)}=\sigma_i^{(M)}=0$. Similarly, the spurious eigenvalue parasitized in the UT formulation can be obtained by using SVD updating documents.

5. AN ILLUSTRATIVE EXAMPLE AND DISCUSSIONS

Case 1: A concentric sphere subject to the Dirichlet boundary condition ($u_1=u_2=0$) using the semi-analytical approach

A concentric case with radii a and b ($a=0.5$ m and $b=1.0$ m) is shown in Fig. 1. The analytical solution can be obtained by using the null-filed integral formulation, degenerate kernel and spherical harmonics. The true eigenequation for the Dirichlet problem is

$$j_n(ka)y_n(kb)-j_n(kb)y_n(ka)=0. \tag{63}$$

The common drop locations in Figs. 4(a) and 4(b) indicate the true eigenvalues. We employ the SVD updating term $\begin{bmatrix}U\\L\end{bmatrix}$ to extract the true eigenvalues for the Dirichlet problem as shown in Fig. 4(c). It's found that all the spurious eigenvalues are filtered out. The results agree well with the previous solutions.

Case 2: A concentric sphere subject to the Neumann boundary condition ($t_1 = t_2 = 0$) using the semi-analytical approach

Similarly, the true eigenequation for the Neumann problem is

$$j'_n(ka)y'_n(kb) - j'_n(kb)y'_n(ka) = 0. \quad (64)$$

The common drop locations in Figs. 4(d) and 4(e) indicate the true eigenvalues. Extraction of true eigenvalues by using the SVD updating term $\begin{bmatrix} T \\ M \end{bmatrix}$ is shown in Fig. 4(f). The common drop locations in Figs. 5(a) and 5(b) indicate the spurious eigenvalues for the singular formulation. Similarly, the same drop locations in Figs. 5(d) and 5(e) indicate the spurious eigenvalues for the hypersingular formulation. The spurious eigenequations for the singular and hypersigular formulation are

$$j_n(ka) = 0, \quad (65)$$

$$j'_n(ka) = 0. \quad (66)$$

Finally, we employed the SVD updating document to filter out the spurious eigenvalues. The spurious eigenvalues for singular formulation and hypersingular formulation are extracted as shown in Figs. 5(c) and 5(f), respectively.

6. CONCLUSIONS

Spurious eigenvalues for a concentric sphere were studied analytically and numerically. One example was demonstrated to see how the spurious eigenvalues occur in the concentric sphere. The trivial outer boundary densities were examined in case of spurious eigenvalues which is found to be the true eigenvalue for the domain bounded by the inner boundary. The contribution of the work is to show the existence of spurious eigenvalue for a concentric sphere in an analytical manner by using the degenerate kernels and the spherical harmonics.

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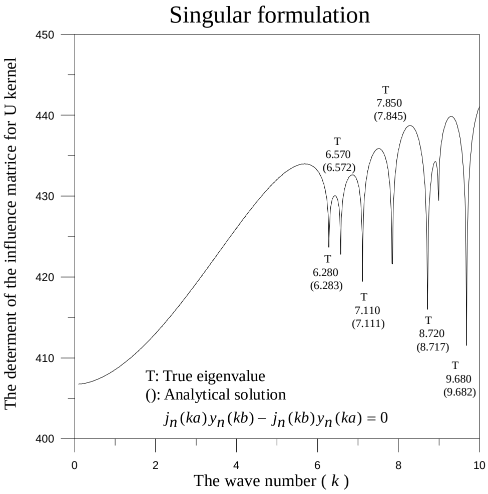
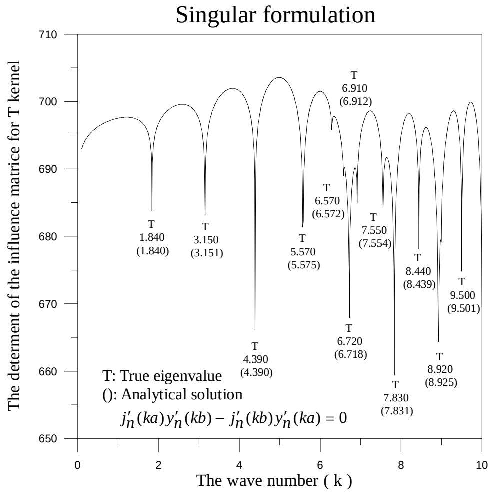
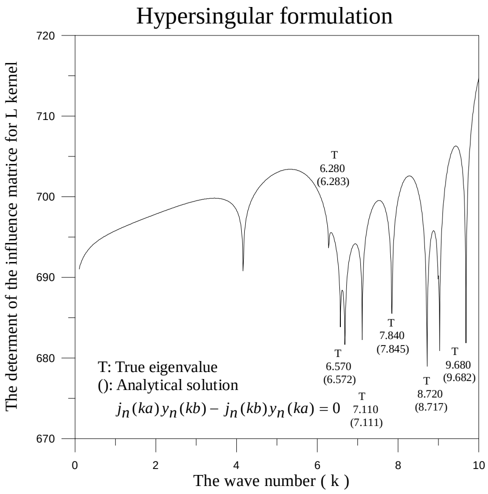


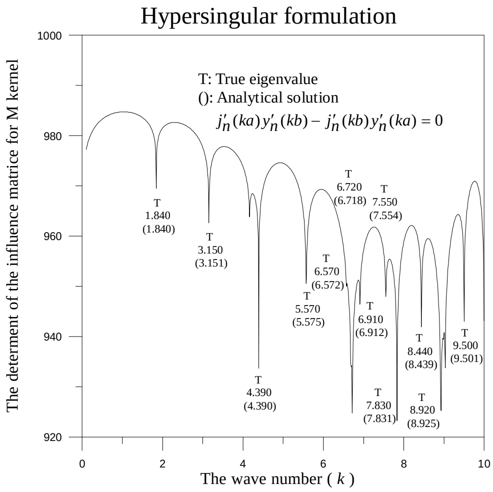
(a) Determinant versus the wave number by using the singular formulation for the Dirichlet condition.



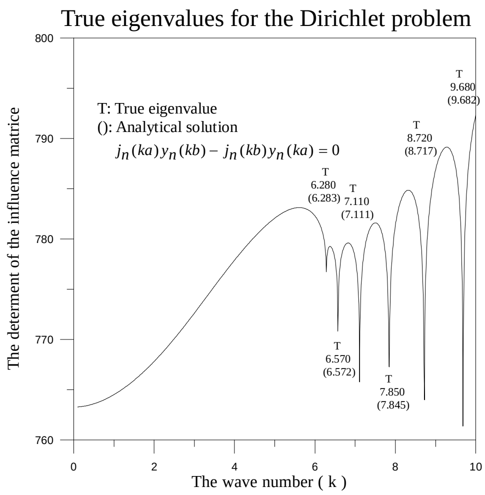
(d) Determinant versus the wave numbers by using the singular formulation for the Neumann condition.



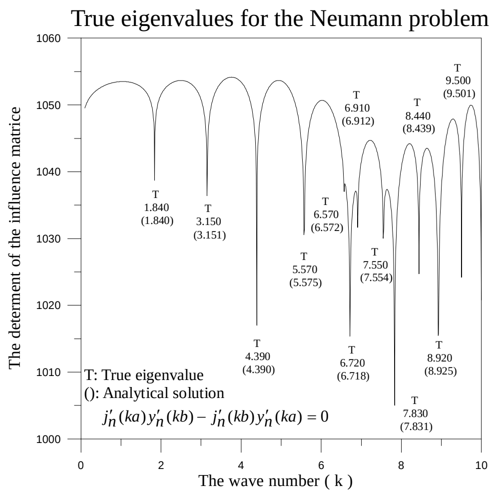
(b) Determinant versus the wave number by using the hypersingular formulation for the Dirichlet condition.



(e) Determinant versus the wave number by using the hypersingular formulation for the Neumann condition.

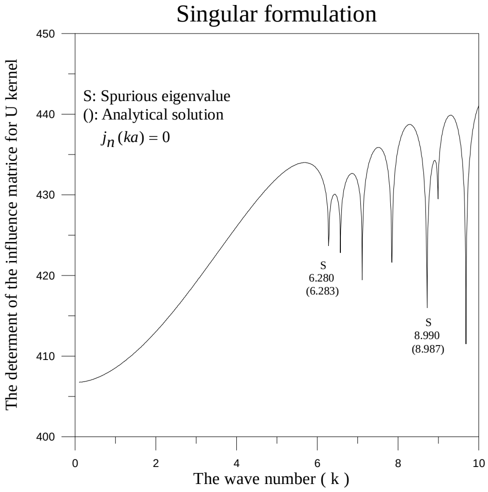


(c) Extraction of true eigenvalues for the Dirichlet problem by using the SVD updating terms.

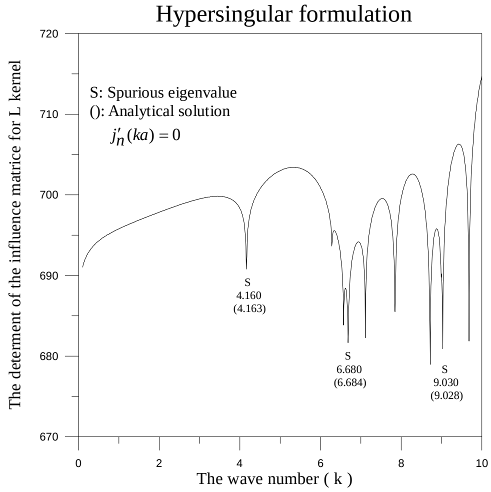


(f) Extraction of true eigenvalues for the Neumann problem by using the SVD updating terms.

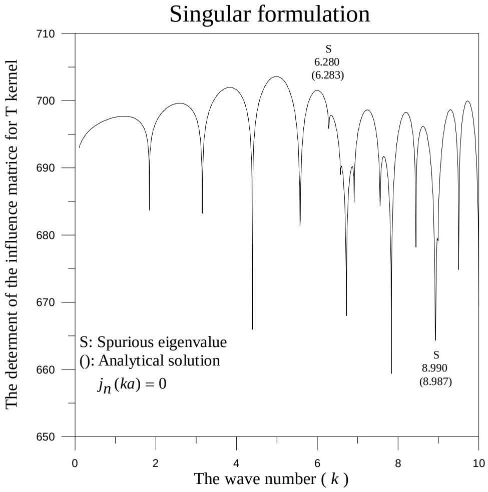
Fig. 4 True eigenvalues for a concentric sphere by using the SVD updating terms ($a = 0.5$ and $b = 1.0$).



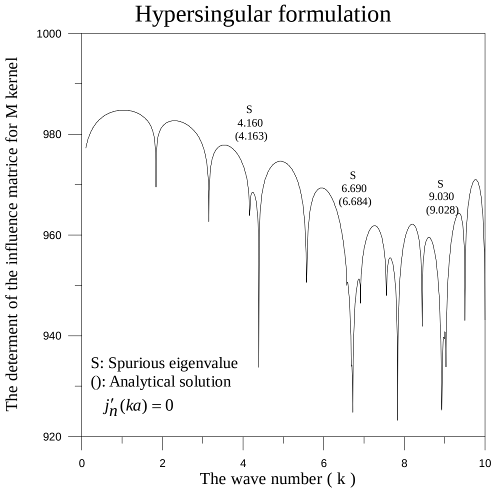
(a) Determinant versus the wave number by using the singular formulation subject to the Dirichlet condition.



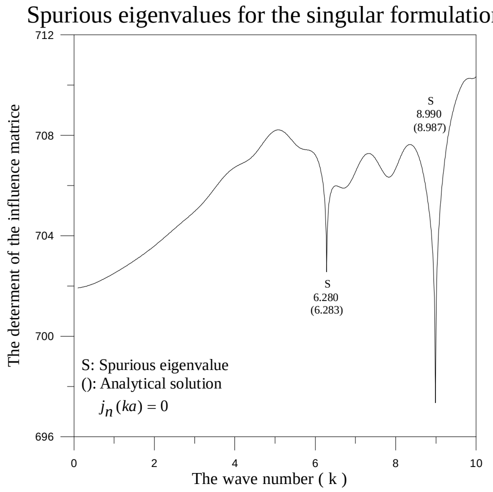
(d) Determinant versus the wave number by using the hypersingular formulation subject to the Dirichlet condition.



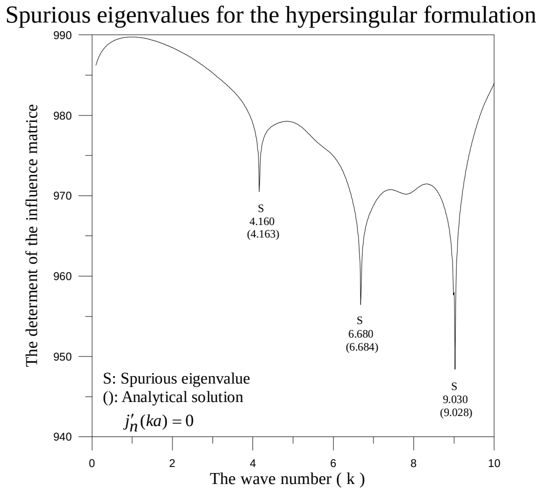
(b) Determinant versus the wave number by using the singular formulation subject to the Neumann condition.



(e) Determinant versus the wave number by using the hypersingular formulation subject to the Neumann condition.



(c) Extraction of the spurious eigenvalues for the singular formulation by using the SVD updating document.



(f) Extraction of the spurious eigenvalues for the hypersingular formulation by using the SVD updating document.

Fig. 5 Extraction of spurious eigenvalues for a concentric sphere by using the SVD updating documents ($a=0.5$ and $b=1.0$).