Support motion of a finite bar with an external viscous damper

Hao-Chen Kao[1] Jia-Wei Lee[2] Ying-Te Lee[3] Jeng-Tzong Chen[4]

ABSTRACT

In this paper, we extended the previous experience to solve the vibration problem of a finite bar with an external viscous damper on one side and the support motion on the other side. Two analytical methods, the mode superposition method in conjunction with the quasi-static decomposition method and the method of characteristics using the diamond rule, were employed to solve this problem. The non-conservative system with an external viscous damper is solved straightforward by using the method of diamond rule to avoid the complex-valued eigen-system. Both advantages and disadvantages of two methods were discussed. The effect of an external viscous damper on the vibration response is also addressed.

Keywords：support motion, damper, mode superposition method, quasi-static decomposition, diamond rule

1. INTRODUCTION

Support motion is very important in physics and mechanics, because there are various engineering problems which can be modeled by using this model. Many researchers have solved this problem by using various methods, *e.g.*, the mode superposition technique [1], the method of separation variables [2, 3, 4], the method of quasi-static decomposition [3, 5, 6], the method of the diamond rule [3, 7] or the so-called method of characteristics, the image method [6], the finite element method (FEM) [8], the boundary element method (BEM) [9], and the meshless method [10], etc..

The Rayleigh-damped Bernoulli-Euler beam and the string subjected to multi-support excitation have been studied by using many methods including Stokes transformation and Cesaro sum [3, 5, 6]. D'Alembert's solution can provide an exact solution for an infinite string. Method of characteristics (Diamond rule) can be found in the textbook of Farlow [11]. It is widely employed to solve various kinds of problems, *e.g.*, water hammer [12]. The diamond rule on D'Alembert's solution was proposed by John [13] in 1975 and was mainly used to solve the wave problem.

The diamond rule has been employed to solve the one-dimensional vibration problem of an infinite or a semi-infinite string attached by a mass, a spring, or a damper [7], a finite string [3] and a finite bar with an external spring subjected to a support motion [5]. Besides, the animation was also given in [7].

Although the mode superposition method in conjunction with the quasi-static decomposition is a popular approach for solving the support-motion problem, it becomes tedious when the vibration system contains a damper. Three reasons can be explained. One is that the quasi-static solution is not straight forward to be obtained. Another is that the orthogonal relation of complex modes is not easily found. The other is that a complex eigen-system is required. The present solution free of mode superposition is possible since we can employ the method of characteristics in conjunction with the diamond rule for the real response in the time domain.

In this paper, we extend the vibration problem of a finite bar with an external spring [4] to a finite bar with an external viscous damper. For the non-conservative system with an external viscous damper, the effect of an external viscous damper on the vibration response is addressed in more detail.

[1] Graduate student in Department of Harbor and River Engineering, National Taiwan Ocean University (corresponding author: jerrykao6380@gmail.com)
[2] Associate professor in Department of Civil Engineering, Tamkang University
[3] Assistant professor in Department of Harbor and River Engineering, National Taiwan Ocean University
[4] Distinguished chair professor in Department of Harbor and River Engineering, National Taiwan Ocean University

2. PROBLEM STATEMENTS AND METHODS OF SOLUTION

Here, we consider a finite bar with an external viscous damper as shown in Figure 1. The governing equation for the vibration problem of a finite bar is shown below:

$$c^2\frac{\partial^2 u(x,t)}{\partial x^2}=\frac{\partial^2 u(x,t)}{\partial t^2},\ 0<x<L,\ t>0, \qquad (1)$$

where $c=\sqrt{E/\rho}$ and $u(x,t)$ denote the wave speed and displacement in the x direction, respectively. The symbols E, ρ and L denote Young's modulus, the density and the length of bar, respectively. The initial displacement and velocity conditions are

$$u(x,t)\big|_{t=0}=\phi(x)=0, \qquad (2)$$

$$\left.\frac{\partial u(x,t)}{\partial t}\right|_{t=0}=\varphi(x)=0, \qquad (3)$$

where $\phi(x)$ and $\varphi(x)$ are initial displacement and velocity functions, respectively.
The boundary condition at the left hand side (x = 0) can be expressed by the specified support motion as follows:

$$u(0,t)=a(t). \qquad (4)$$

The boundary condition at the right hand side is given from the effect of the viscous damper in Figure 1 as follows:

$$AE\left.\frac{\partial u(x,t)}{\partial x}\right|_{x=L}=-c_d\left.\frac{\partial u(x,t)}{\partial t}\right|_{x=L}, \qquad (5)$$

where c_d denotes the damping coefficient, and A is the area of cross section.

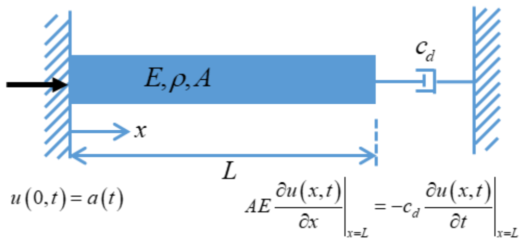


Figure 1 Sketch of a finite bar with an external viscous damper subjected to a support motion

2.1 Method 1: Mode superposition approach in conjunction with the quasi-static decomposition method

The solution can be decomposed into two parts:

$$u(x,t)=U(x,t)+\sum_{n=-\infty}^{\infty}q_n(t)u_n(x), \qquad (6)$$

where $U(x,t)$ denotes the quasi-static solution, and the natural modes $u_n(x)$ weighted by the generalized coordinate, $q_n(t)$ is the generalized coordinate of dynamic contribution due to the inertia effect. The quasi-static part $U(x,t)$, satisfies the governing equation

$$AE\frac{\partial^2 U(x,t)}{\partial x^2}=0,\ 0<x<L, \qquad (7)$$

and is subject to time-dependent boundary conditions at the two sides:

$$U(0,t)=a(t), \qquad (8)$$

$$AE\left.\frac{\partial U(x,t)}{\partial x}\right|_{x=L}=-c_d\left.\frac{\partial U(x,t)}{\partial t}\right|_{x=L}. \qquad (9)$$

By solving Eq. (7) subject to boundary conditions, we have the quasi-static solution,

$$U(x,t)=\alpha(t)x+\beta(t), \qquad (10)$$

where

$$\alpha(t)=e^{-\frac{AE}{c_dL}t}\left[\int_0^t -e^{\frac{AE}{c_dL}\tau}\frac{\dot a(\tau)}{L}d\tau+C\right], \qquad (11)$$

after using the integration factor, where the undetermined constant can be determined by

$$\alpha(0)=a(0)\ \&\ \beta(t)=a(t). \qquad (12)$$

The nth complex-valued function $u_n(x)$ with the complex eigenvalue λ_n is

$$u_n(x)=e^{\lambda_n x}-e^{-\lambda_n x},\ n=0,\ \pm1,\ \pm2,\cdots, \qquad (13)$$

The corresponding complex-valued eigen-values are given below:

$$\lambda_n=-\frac{1}{2L}\ln\left(\frac{c_dc+AE}{c_dc-AE}\right)-\frac{n\pi}{L}\mathrm{i},\ n=0,\ \pm1,\ \pm2,\cdots, \qquad (14)$$

and the corresponding complex frequency is

$$\omega_n=c\lambda_n,\ n=0,\ \pm1,\ \pm2\cdots. \qquad (15)$$

Since the quasi-static part in Eq. (6) satisfies the boundary condition in Eq. (5), we set each basis of the solution to also satisfy the boundary condition in Eq. (5). By taking the following equation

$$AE\left.\frac{\partial(q_n(t)u_n(x))}{\partial x}\right|_{x=L}=-c_d\left.\frac{\partial(q_n(t)u_n(x))}{\partial t}\right|_{x=L}, \qquad (16)$$

we have

$$\dot q_n(t)=\omega_n q_n(t). \qquad (17)$$

Differentiating Eq. (6) with respect to t, we have

$$\frac{\partial u(x,t)}{\partial t}=\dot U(x,t)+\sum_{n=-\infty}^{\infty}\dot q_n(t)u_n(x). \qquad (18)$$

By adopting Eq. (17), Eq. (18) can be written as

$$\frac{\partial u(x,t)}{\partial t}=\dot U(x,t)+\sum_{n=-\infty}^{\infty}\omega_n q_n(t)u_n(x). \qquad (19)$$

Subtracting Eq. (19) from Eq. (18), we have

$$\sum_{n=-\infty}^{\infty}\left[\dot q_n(t)-\omega_n q_n(t)\right]\left(e^{\lambda_n x}-e^{-\lambda_n x}\right)=0. \qquad (20)$$

Then, differentiating Eq. (20) with respect to x and multiplying by the wave speed c, we obtain

$$\sum_{n=-\infty}^{\infty}\left[\dot{q}_n(t)-\omega_n q_n(t)\right]\omega_n\left(e^{\lambda_n x}+e^{-\lambda_n x}\right)=0. \qquad (21)$$

Substituting Eq. (6) into Eq. (1), we have

$$\sum_{n=-\infty}^{\infty}\left[\dot{q}_n(t)-\omega_n q_n(t)\right]\omega_n\left(e^{\lambda_n x}-e^{-\lambda_n x}\right)=-\ddot{U}(x,t). \qquad (22)$$

By adding Eq. (20) to Eq. (22) together and subtracting Eq. (20) from Eq. (22), we obtain

$$\sum_{n=-\infty}^{\infty}\left[\dot{q}_n(t)-\omega_n q_n(t)\right]2\omega_n e^{\lambda_n x}=-\ddot{U}(x,t),\ x\in[0,L] \qquad (23)$$

and

$$\sum_{n=-\infty}^{\infty}\left[\dot{q}_n(t)-\omega_n q_n(t)\right]2\omega_n e^{-\lambda_n x}=\ddot{U}(x,t),\ x\in[0,L] \qquad (24)$$

, respectively. We substitute – x for x in Eq. (24) and the interval is changed from [0, L] to [-L, 0], yields

$$\sum_{n=-\infty}^{\infty}\left[\dot{q}_n(t)-\omega_n q_n(t)\right]2\omega_n e^{-\lambda_n x}=\ddot{U}(-x,t),\ x\in[-L,0] \qquad (25)$$

Substituting the complex-valued eigenvalue into Eqs. (23) and (25) and rearranging them into a single equation, we have

$$\sum_{n=-\infty}^{\infty}\left[\dot{q}_n(t)-\omega_n q(t)\right]2\omega_n e^{-\left(\frac{n\pi}{L}\mathrm{i}\right)x}=\begin{cases}-e^{\frac{1}{2L}\ln\left(\frac{c_d c+AE}{c_d c-AE}\right)x}\ddot{U}(x,t),\ x\in[0,L]\\ e^{\frac{1}{2L}\ln\left(\frac{c_d c+AE}{c_d c-AE}\right)x}\ddot{U}(-x,t),\ x\in[-L,0]\end{cases} \qquad (26)$$

Multiplying the exponential term $e^{(m\pi\mathrm{i}/L)x}$ (where m is an integer) on both sides of Eq. (26) and integrating from - L to L, the left side of Eq. (26) can be expressed as

$$\int_{-L}^{L}\left[\dot{q}_n(t)-\omega_n q(t)\right]2\omega_n e^{-\left(\frac{n\pi}{L}\mathrm{i}\right)x}e^{\left(\frac{m\pi}{L}\mathrm{i}\right)x}dx=\begin{cases}\left[\ddot{q}_n(t)-\omega_n^2 q(t)\right]4\omega_n L,\ n=m\\ 0\qquad\qquad ,\ n\neq m\end{cases} \qquad (27)$$

For the right side of Eq. (26), it can be combined by using the reflection property of integrals as given below:

$$\int_0^L -e^{-\lambda_n x}\ddot{U}(x,t)dx+\int_{-L}^{0}e^{-\lambda_n x}\ddot{U}(-x,t)dx=\int_0^L u_n(x)\ddot{U}(x,t)dx \qquad (28)$$

Combining Eqs. (27) and (28), the first-order ordinary differential equation for the generalized coordinates $q_n(t)$ can be obtained as

$$\dot{q}_n(t)-\omega_n q(t)=\frac{1}{4\omega_n L}\int_0^L u_n(x)\ddot{U}(x,t)dx \qquad (29)$$

The initial condition of generalized coordinates can be determined from the initial conditions of the total solution. Using Eq. (6) to satisfy the initial displacement condition in Eq. (2) and differentiating it with respect to x, we have

$$\frac{\partial u(x,0)}{\partial x}=\frac{\partial\phi(x)}{\partial x}=0=U'(x,0)+\sum_{n=-\infty}^{\infty}q_n(0)\lambda_n\left(e^{\lambda_n x}+e^{-\lambda_n x}\right), \qquad (30)$$

Multiplying Equation (30) by the wave speed c, we have

$$\sum_{n=-\infty}^{\infty}\omega_n q_n(0)\left(e^{\lambda_n x}+e^{-\lambda_n x}\right)=-cU'(x,0). \qquad (31)$$

Similarly, using Eq. (6) to satisfy the initial velocity condition in Eq. (3), we have

$$\frac{\partial u(x,0)}{\partial t}=\varphi(x)=0=\dot{U}(x,0)+\sum_{n=-\infty}^{\infty}\omega_n q_n(0)\left(e^{\lambda_n x}-e^{-\lambda_n x}\right). \qquad (32)$$

By adding Eq. (32) to Eq. (31) together and subtracting Eq. (32) from Eq. (31), we obtain

$$\sum_{n=-\infty}^{\infty}\omega_n q_n(0)2e^{\lambda_n x}=-cU'(x,0)-\dot{U}(x,0),\ x\in[0,L] \qquad (33)$$

and

$$\sum_{n=-\infty}^{\infty}\omega_n q_n(0)2e^{-\lambda_n x}=-cU'(x,0)+\dot{U}(x,0),\ x\in[0,L] \qquad (34)$$

, respectively. We substitute – x for x in Eq. (34) and the interval is changed from [0, L] to [-L, 0], yields

$$\sum_{n=-\infty}^{\infty}\omega_n q_n(0)2e^{\lambda_n x}=-cU'(-x,0)+\dot{U}(-x,0),\ x\in[-L,0] \qquad (35)$$

Similarly, combining Eqs. (33) and (35) into a single equation and rearranging it, we have

$$\sum_{n=-\infty}^{\infty}2\omega_n q_n(0)e^{-\left(\frac{n\pi}{L}\mathrm{i}\right)x}=\begin{cases}e^{\frac{1}{2L}\ln\left(\frac{c_d c+AE}{c_d c-AE}\right)x}\left[-cU'(x,0)-\dot{U}(x,0)\right],\ x\in[0,L]\\ e^{\frac{1}{2L}\ln\left(\frac{c_d c+AE}{c_d c-AE}\right)x}\left[-cU'(-x,0)+\dot{U}(-x,0)\right],\ x\in[-L,0]\end{cases}. \qquad (36)$$

Multiplying the exponential term $e^{(m\pi\mathrm{i}/L)x}$ (where m is an integer) on both sides of Eq. (36) and integrating from - L to L, the left side of Eq. (36) can be expressed as

$$\int_{-L}^{L}\omega_n q_n(0)2e^{-\left(\frac{n\pi}{L}\mathrm{i}\right)x}e^{\left(\frac{m\pi}{L}\mathrm{i}\right)x}dx=\begin{cases}q_n(0)4\omega_n L,\ n=m\\ 0\qquad\quad ,\ n\neq m\end{cases}. \qquad (37)$$

Using of the reflection property of integrals in the right side of Eq. (36), we have

$$\begin{aligned}&\int_0^L e^{-\lambda_n x}\left[-cU'(x,0)-\dot{U}(x,0)\right]dx+\int_{-L}^{0}e^{-\lambda_n x}\left[-cU'(-x,0)+\dot{U}(-x,0)\right]dx\\&=-\int_0^L cU'(x,0)\left(e^{\lambda_n x}+e^{-\lambda_n x}\right)dx+\int_0^L\dot{U}(x,0)u_n(x)dx.\end{aligned} \qquad (38)$$

Combining Eqs. (37) and (38), the initial condition of generalized coordinates $q_n(t)$ can be obtained as

$$q_n(0)=\frac{-1}{4\lambda_n L}\int_0^L U'(x,0)\left(e^{\lambda_n x}+e^{-\lambda_n x}\right)dx+\frac{1}{4\omega_n L}\int_0^L\dot{U}(x,0)u_n(x)dx. \qquad (39)$$

Therefore, we can solve $q_n(t)$ by considering Eqs. (29) and (39) and have

$$\begin{aligned}q_n(t)=\frac{e^{ct\lambda_n}}{2cL\lambda_n^3}\Big\{&\int_0^t e^{-c\lambda_n\tau}\left(\lambda_n\left(-1+\cosh(\lambda_n L)a''(\tau)\right)+\left(\lambda_n L\cosh(\lambda_n L)-\sinh(\lambda_n L)\right)\alpha''(\tau)\right)d\tau\\&-\lambda_n a'(0)+\sinh(\lambda_n L)\left(c\lambda_n\alpha(0)-\alpha'(0)\right)+\lambda_n\cosh(\lambda_n L)\left(a'(0)+L\alpha'(0)\right)\Big\}\end{aligned} \qquad (40)$$

Finally, the series solution for the displacement, $u(x,t)$, is shown below:

$$u(x,t)=U(x,t)+\sum_{n=-\infty}^{\infty}q_n(t)\left(e^{\lambda_n x}-e^{-\lambda_n x}\right), \qquad (41)$$

2.2 Method 2: Method of characteristics in conjunction with the diamond rule

By employing the method of characteristic line, we can assume the general solution of 1D wave equation in Eq. (1) as

$$u(x,t)=P(x+ct)+Q(x-ct), \qquad (42)$$

where $P(x+ct)$ and $Q(x-ct)$ are specified functions to match initial conditions in Eqs. (2) and (3). The functions $P(x+ct)$ and $Q(x-ct)$ represent a left-going-traveling wave and a right-going-traveling wave, respectively. By satisfying Eqs. (2) and (3) for Eq. (42), the D'Alembert's solution for a certain region is expressed as

$$u(x,t)=\frac{1}{2}\left[\phi(x+ct)+\phi(x-ct)\right]+\frac{1}{2c}\int_{x-ct}^{x+ct}\varphi(\tau)d\tau, \qquad (43)$$

where $\phi(x)$ and $\varphi(x)$ are functions of initial displacement and velocity, respectively. Two groups of characteristic lines from Eq. (43) are included in the solution of the wave equation. Moreover, the two groups of parallel characteristic lines can form a parallelogram in the space-time plane as shown in Figure 2. Based on the D'Alembert's solution, we have the equation of the diamond rule [4, 6], as shown below:

$$u_A + u_B = u_C + u_D, \tag{44}$$

where u_A , u_B , u_C and u_D denote the displacement at the four points *A*, *B*, *C* and *D*, respectively. Several parallel characteristic lines separate the domain into many regions of the space-time plane as shown in Figure 3. The diagrams of calculating the displacement by using the diamond rule in the regions I, II, III, IV, V and VI are given in Figure 4. The displacements in the former six regions are given below:

$$u_{\mathrm{I}}(x,t) = 0 \,,\ (x,t) \in \mathrm{I}, \tag{45}$$

$$u_{\mathrm{II}}(x,t) = a\left(\frac{ct-x}{c}\right),\ (x,t) \in \mathrm{II}, \tag{46}$$

$$u_{\mathrm{III}}(x,t) = r_1\left(\frac{x+ct-L}{c}\right),\ (x,t) \in \mathrm{III}, \tag{47}$$

$$u_{\mathrm{IV}}(x,t) = a\left(\frac{ct-x}{c}\right),\ (x,t) \in \mathrm{IV}, \tag{48}$$

$$u_{\mathrm{V}}(x,t) = a\left(\frac{ct-x}{c}\right),\ (x,t) \in \mathrm{V}, \tag{49}$$

$$u'_{\mathrm{VI}}(x,t) = -\frac{1}{c}a'\left(\frac{ct-x}{c}\right) - \frac{1}{c}a'\left(\frac{x+ct-2L}{c}\right) + \frac{1}{c}r_2'\left(\frac{x+ct-L}{c}\right),\ (x,t) \in \mathrm{VI}\ . \tag{50}$$

where the simple form of Eqs. (48)-(50) is due to zero $r_1(t)$ in silent regions of I and III and zero initial displacement $\phi(x)$ and velocity $\varphi(x)$. Following the same procedure, the marching scheme of the time-space plane region and solution can be done. Then, $r_1(t)$ and $r_2(t)$ denote the displacements of $u(L,t),\ 0 \le t \le L/c$ and $u(L,t),\ L/c \le t \le 2L/c$, respectively, which can be obtained from the condition of force equilibrium at $x = L$,

$$AE\left.\frac{\partial u_{\mathrm{III}}(x,t)}{\partial x}\right|_{x=L} = -c_d \left.\frac{\partial u_{\mathrm{III}}(x,t)}{\partial t}\right|_{x=L}, \tag{51}$$

$$AE\left.\frac{\partial u_{\mathrm{VI}}(x,t)}{\partial x}\right|_{x=L} = -c_d \left.\frac{\partial u_{\mathrm{VI}}(x,t)}{\partial t}\right|_{x=L}, \tag{52}$$

Thus, we can determine $r_1(t)$ by using Eq. (47) to satisfy Eq. (51). The displacement at $x = L$, $u_{\mathrm{III}}(L,0)$ and $u_{\mathrm{I}}(L,0)$, must satisfy the displacement continuity. Then, we have

$$r_1(t) = 0 \,,\ 0 \le t \le L/c. \tag{53}$$

Similarly, the response of $r_2(t)$ can be obtained by using Eq. (50) to satisfy Eq. (52). By solving the corresponding first-order ODE for $r_2(t)$ at the end of damper as shown below:

$$r_2'(t) = \frac{2AE}{AE + c_d c} a'\left(\frac{ct-L}{c}\right),\ L/c \le t \le 2L/c, \tag{54}$$

we can obtain

$$r_2(t) = \frac{2AE}{AE + c_d c} a\left(\frac{ct-L}{c}\right) + C\ ,\ L/c \le t \le 2L/c, \tag{55}$$

where the undetermined constant C can be determined by satisfying the displacement continuity of solution in the region IV and VI at $(x,t) = (L, L/c)$ as shown in Figure 4.

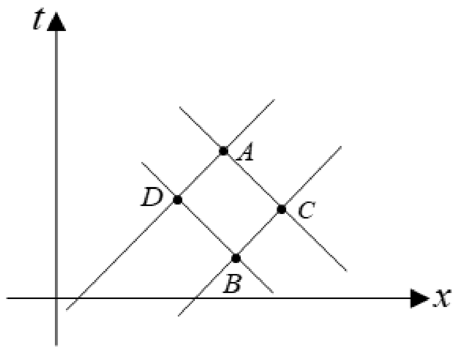


Figure 2 The diamond rule of $u_A + u_B = u_C + u_D$

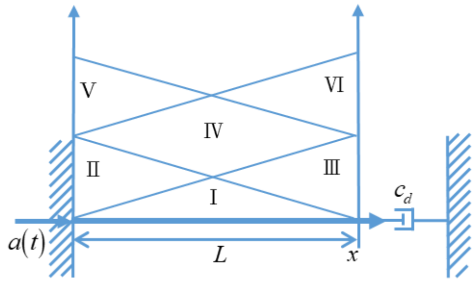


Figure 3 Space-time regions separated by using the characteristic line.

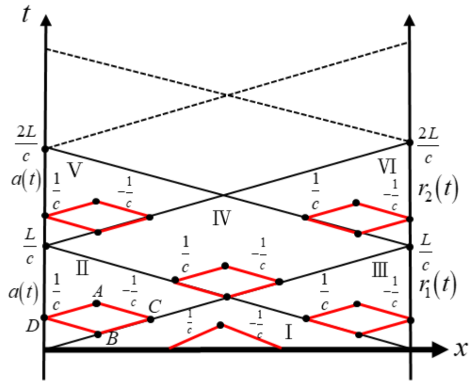


Figure 4 Space-time regions, I, II, III, IV, V and VI and the diamond rule.

3. AN ILLUSTRATIVE EXAMPLE

A finite bar with a damper subjected to a support motion is considered. The model parameters are given as shown below: c=1 m/s, $AE = 1\,N$, $L = 7\,m$ and $c_d = 5\,N \cdot s/m$. By setting the support motion,

$$a(t) = \sin(t), \tag{56}$$

the solutions of two approaches can be obtained as shown in the following subsection.

3.1 Mode superposition method

After substituting model parameters

c, A, E, L, c_d and Eq. (56) into Eq. (41), the vibration response of displacement can be obtained by taking the Eq. (41).

$$u(x,t)=U(x,t)+\sum_{n=-\infty}^{\infty} q_n(t)\left(e^{\lambda_n x}-e^{-\lambda_n x}\right). \quad (57)$$

3.2 Method of characteristics in conjunction with the diamond rule

By substituting model parameters c, A, E, L, c_d and Eq. (56) into Eqs. (45)-(50), we have

$$u_{\mathrm{I}}(x,t)=0,\ (x,t)\in \mathrm{I}, \quad (58)$$

$$u_{\mathrm{II}}(x,t)=\sin(t-x),\ (x,t)\in \mathrm{II}, \quad (59)$$

$$u_{\mathrm{III}}(x,t)=0,\ (x,t)\in \mathrm{III}, \quad (60)$$

$$u_{\mathrm{IV}}(x,t)=\sin(t-x),\ (x,t)\in \mathrm{IV}, \quad (61)$$

$$u_{\mathrm{V}}(x,t)=\sin(t-x),\ (x,t)\in \mathrm{V}, \quad (62)$$

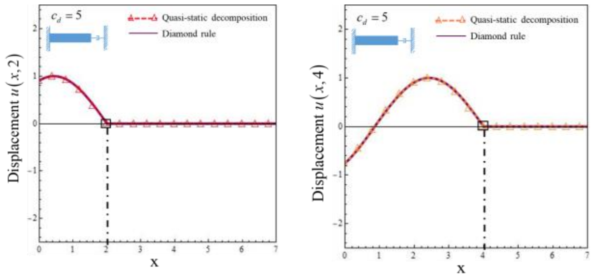
$$u_{\mathrm{VI}}(x,t)=\sin(t-x)-\sin(x+t-14)+r_2(x+t-7),\ (x,t)\in \mathrm{VI}, \quad (63)$$

where

$$r_2(t)=\frac{1}{3}\sin(t-7),\ 7\le t\le 14. \quad (64)$$

The displacement profiles with the silent area for t = 2 and 4 sec by using the mode superposition method and the diamond rule are shown in Figure 5 (a)-(b), respectively. It is interesting to find that the mode superposition method also yields the silent response. In Figure 6, shadow regions, I and III, denote the dead zone. It matches the silent response begins at x = 2 and 4 m to the end of bar (x = 7 m), for the time when t = 2 and 4 sec as shown in Figure 5. It is found that the slope is discontinuous at x = 2 and 4 m when t = 2 and 4 sec, respectively. These discontinuities occur at the locations of (2,2) and (4,4) in the x-t plane as shown in Figure 6. In Figure 6, the shadow region denotes the dead zone. As theoretically predicted, the discontinuity of the slope really occurs at the position of (2,2) and (4,4), on the characteristic line.

Regarding non-silent area, the displacement profiles at t = 8 and 10 sec are shown in Figure 7 (a)-(b), respectively. It is also found that the slope is discontinuous at x = 6 and 4 m when t = 8 and 10 sec, respectively. These slope discontinuities occur at the locations of (6,8) and (4,10) in the x-t plane as shown in Figure 8. This finding matches well from the mathematical requirement that the discontinuity must occur at the position on the characteristic line [14].



(a) t = 2 sec (b) t = 4 sec

Figure 5 Displacement profiles with the silent area by using the quasi-static decomposition and the diamond rule

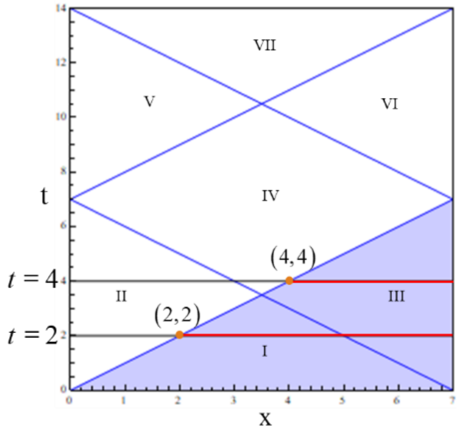
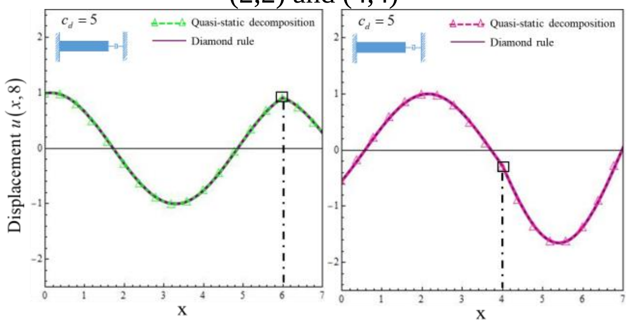


Figure 6 The locations of slope discontinuities at (2,2) and (4,4)



(a) t = 8 sec (b) t = 10 sec

Figure 7 Displacement profiles by using the quasi-static decomposition and the diamond rule

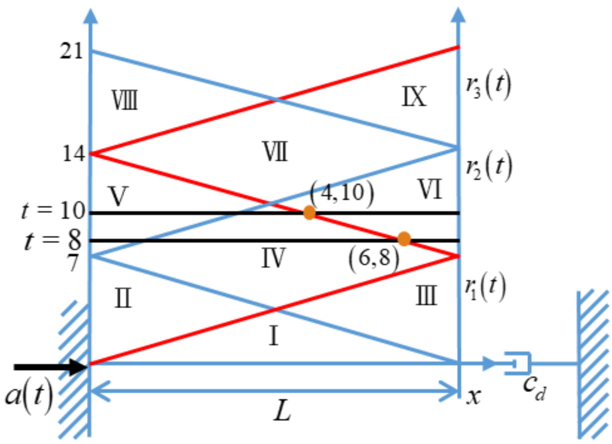


Figure 8 The locations of slope discontinuities (6,8) and (4,10)

4. CONCLUSION

In this paper, we have analytically solved the direct problem of the longitudinal vibration analysis of a finite bar with an external viscous damper on one side and the support motion on the other clamped side by using two methods. The slope discontinuity occurs at the position on the characteristic line as mathematically predicted. The effect of an external viscous damper on the vibration response is also addressed. The method of characteristics line in conjunction with the diamond rule was successfully employed to solve the problem containing the viscous damped boundary in the time domain. Good agreement of results by using the mode superposition method with complex-values eigenvalues and the method of characteristics in conjunction with the diamond rule. Therefore, it is easier to solve the problem of non-conservative system with a damper by using the diamond rule, since it is free to consider the complex-valued eigen-system to solve the problem. Table 5 summaries the comparison of advantages and disadvantages of the two approaches.

Table 5 Comparison of the two approaches for the vibration problem of a finite rod

Method / Item analysis	Mode superposition method in conjunction with the quasi-static decomposition	Method of characteristics in conjunction with the diamond rule
Solution form	Series solution (continuous)	Exact solution (continuous)
Advantage	Without dividing the space-time region to represent the corresponding displacement response	1. Without the truncation error of finite term of series sum 2. It can analytically capture the dead zone 3. General approach for either conservative or non-conservative system 4. Suitable for support excitation of short duration, *e.g.*, earthquake input
Disadvantage	1. Error due to truncation series in the real computation 2. Convergence test is required 3. Complex-valued eigenvalues and eigenequation are required for a damped system	Previous stage error propagates to the later response

REFERENCES

1. G. Oliveto, A. Santini and E. Tripodi (1997), “Complex modal analysis of a flexural vibrating beam with viscous end conditions,” J. Sound Vib., **200**, 327-345.
2. A. J. Hull (1994), “A closed form solution of a longitudinal bar with a viscous boundary condition,” *J. Sound. Vib.*, **169**, 19-28.
3. R. Singh, W. M. Lyons and G. Prater (1989), “complex eigenvalue for longitudinal vibration bars with a viscously damped boundary,” *J. Sound. Vib.*, **133**(2), 364-367.
4. J. T. Chen, Y. S. Jeng (1996), “Dual series representation and its applications a string subjected to support motions,” *Adv. Eng. Softw.*, **27**, 227-238.
5. J. T. Chen, H. C. Kao, Y. T. Lee and J. W. Lee (2022), “Support motion of a finite bar with an external spring,” *J. Low Freq. Noise Vib. Act. Control*, **in press**.
6. J. T. Chen, H.-K. Hong, C. S. Yeh and S. W. Chyuan (1996), “Integral representations and regularization for a divergent series solution of a beam subjected to support motion,” *Earthqu. Eng. Struct. Dynamics*, **25**, 909-925.
7. J. T. Chen, K. S. Chou and S. K. Kao (2009), “One-dimensional wave animation using Mathematica,” *Comput. Appl. Eng. Educ.*, **17**, 323-339.
8. L. Zhao and Q. Chen (2000), “Neumann dynamic stochastic finite element method of vibration for structures with stochastic parameters to random excitation,” *Comput. Struct.*, **77**, 651-657.
9. E. L. Albuquerquea, P. Sollero and P. Fedelinski (2003), “Free vibration analysis of anisotropic material structures using the boundary element method,” *Engng. Anal. Bound. Elem.*, **27**, 977-985.
10. H. Li, Q.X. Wang and K.Y. Lam (2004), “Development of a novel meshless Local Kriging (LoKriging) method for structural dynamic analysis,” *Comput. Method Appl. Mech. Engrg.*, 193, 2599-2619.
11. S. J. Farlow (1937), *Partial Differential Equations for Scientists and Engineers*, John Wiley and Sons, Canada.
12. D. H. Wilkinson and E. M (1980). Curtis, *Water hammer in a thin walled pipe*, UK: Int. Conf. on Pressure Surges, Canterbury; Proc. 3rd ed., 221–240.
13. F. John (1975), *Partial Differential Equation*, Springer-Verlag, New York; 2th ed..
14. G. F. Carrier, C. E. Pearson (1976), *Partial differential equations: theory and technique*, Academic Press, New York; 1st ed