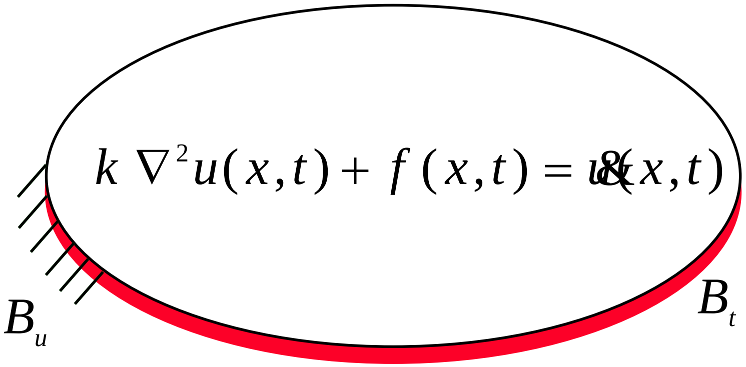
On the Relations of Hypersingular Kernel and Divergent Series in Heat Conduction Problem Using BEM

$$k\,\nabla^2 u(x,t) + f(x,t) = \dot{u}(x,t)$$

$$u(x,t) = \bar{u}(x_{B_u}, t) \qquad \frac{\partial u(x,t)}{\partial n} = \bar{t}(x_{B_t}, t)$$



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Dual Integral Equations and Dual Series Representation

- **Dual Integral Equations:**

$$u(x,t) = \int_0^t \int_B U(s,x;\tau,t)\ \hat{t}(s,\tau) dB(s) d\tau - \int_0^t \int_B T(s,x;\tau,t)\ \hat{u}(s,\tau) dB(s) d\tau$$
$$+ \int_0^t \int_V U(s,x;\tau,t)\ f(s,\tau) dV(s) d\tau$$

$$t(x,t) = \int_0^t \int_B L(s,x;\tau,t)\ \hat{t}(s,\tau) dB(s) d\tau - \int_0^t \int_B M(s,x;\tau,t)\ \hat{u}(s,\tau) dB(s) d\tau$$
$$+ \int_0^t \int_V L(s,x;\tau,t)\ f(s,\tau) dV(s) d\tau$$

- **Series :**

$$U(s,x;\tau,t) = \lim_{N\to\infty} C(N,1) \{ \sum_{m=1}^{N} e^{-k\omega_m (t-\tau)} u_m(x)\ u_m(s) / N_m \}$$

$$T(s,x;\tau,t) = \lim_{N\to\infty} C(N,1) \{ \sum_{m=1}^{N} e^{-k\omega_m (t-\tau)} u_m(x)\ t_m(s) / N_m \}$$

$$L(s,x;\tau,t) = \lim_{N\to\infty} C(N,2) \{ \sum_{m=1}^{N} e^{-k\omega_m (t-\tau)} t_m(x)\ u_m(s) / N_m \}$$

$$M(s,x;\tau,t) = \lim_{N\to\infty} C(N,2) \{ \sum_{m=1}^{N} e^{-k\omega_m (t-\tau)} t_m(x)\ t_m(s) / N_m \}$$

$C(N,r)$: Cesaro operator with order r

Methods of Solution for Heat Conduction Problems with Time-Dependent Boundary Conditions

- **Series Solution**
- **Large Conductance Technique**

  Including high frequency modes
- **Quasi-static Decomposition**

  Mindlin and Goodman, C. L. Dym and H. Reismamn
- **Cesaro sum**
- **Stokes' Transformation**

Relations of Series Representation, Large Conductance Technique, Cesaro Sum, Quasi-static Decomposition and Stokes' Transformation

Motivations of Quasi-static Decomposition and Stokes' Transformation

Quasi-static decomposition

$$u(x,t) = \underline{\underline{U(x,t)}} + \sum_{n=0}^{N} q_n(t)\, u_n(x)$$

(Physical meaning)

Differentiation

$$u'(x,t) = U'(x,t) + \sum_{n=0}^{N} q_n(t)\, u_n'(x)$$

Stokes' transformation

$$u(x,t) \cong \sum_{n=0}^{N} \bar{q}_n(t)\, u_n(x)$$

Asymptotic analysis

$$= U(x,t) + \sum_{n=0}^{N} c_n(t)\, u_n(x) + \sum_{n=0}^{N} \bar{q}_n(t)\, u_n(x)$$

Integration

$$u'(x,t) = \sum_{n=0}^{N} \underline{\underline{b_n(t)\, u_n'(x)}} + \sum_{n=0}^{N} \bar{q}_n(t)\, u_n'(x)$$

$-\infty$ $\infty + F.P.$

(Mathematical way)

$$= U'(x,t) + \sum_{n=0}^{N} c_n(t)\, u_n'(x) + \sum_{n=0}^{N} \bar{q}_n(t)\, u_n'(x)$$

$-\infty$ $\infty + F.P.$

$$q_n(t) = c_n(t) + \bar{q}_n(t)$$

$$U(x,t) \cong -\sum_{n=0}^{N} c_n(t)\, u_n(x)$$

$$\sum_{n=0}^{N} b_n(t)\, u_n'(x) =$$ **Series representation for distribution on boundary**

Three Analytical Ways and One Simulation Technique to Introduce the Quasi-static Part

- **By Solving Boundary Value Problem Directly**

  Quasi-static decomposition method (Mindlin and Goodman)

- **By Integrating the Secondary Field Derived from Stokes' Transformation**

  Boundary terms are available

- **By Adding-and-Subtracting Technique Using Asymptotic Analysis**

  Series representation (Eringen and Suhubi)

- **Large Conductance Technique (MSC/NASTRAN)**

  Including high frequency modes

Cesaro Regularization Technique

- **Series Solution(Partial Sum)**

$$s_0 = a_0$$

$$s_1 = a_0 + a_1$$

$$s_2 = a_0 + a_1 + a_2$$

$$\vdots \quad \vdots$$

$$s_{N-1} = a_0 + a_1 + a_2 + \cdots + a_{N-1}$$

$$(\textit{partial} \;\; \textit{sum}) \quad s_N = a_0 + a_1 + a_2 + \cdots + a_{N-1} + a_N \quad (\textit{divergent}, \;\; N \to \infty)$$

$$\frac{s_0 + s_1 + \cdots + s_{N-1} + s_N}{N+1} = a_0 + \frac{N}{N+1} a_1 + \frac{N-1}{N+1} a_2 + \cdots + \frac{2}{N+1} a_{N-1} + \frac{1}{N+1} a_N \quad (\textit{convergent}, \;\; N \to \infty)$$

$$(\textit{Cesaro} \;\; \textit{sum}) \quad S_N = \frac{1}{N+1} \sum_{k=0}^{N} (N - k + 1) \;\; a_k \quad (\textit{moving} \quad \textit{average})$$

Stokes' Transformation --- Summation by Parts

- **Term by Term Differentiation Is Not Always Legal**
- **Boundary Term Is Present for Some Cases**

$$f'(x) = \frac{d}{dx}\left\{ f(x) \right\} = \frac{d}{dx}\left\{ \sum_{k=0}^{N} c_k\, u_k(x) \right\} = \sum_{k=0}^{N} c_k\, u_k'(x) + \underbrace{\sum_{k=0}^{N} b_k\, u_k'(x)}_{\text{Boundary term}}$$

if $\sum_{k=0}^{N} b_k\, u_k'(x) \neq 0$

- **Term by Term Differentiation Is Legal**

if $\sum_{k=0}^{N} b_k\, u_k'(x) = 0$

Why Cesaro sum can Extract the Finite Part of Divergent Series

$$\nabla u(x,t) = \sum_{l=0}^{N} \bar{q}_l^{\nabla}(t)\ \nabla u_l(x) = \sum_{l=0}^{N} \frac{1}{N_l\ \omega_l}\{\int_{B_u} u(s,t)\frac{\partial u_l(s)}{\partial n_s} dB(s)\}\nabla u_l(x) + \sum_{l=0}^{N}\bar{q}_l(\mathrm{t})\ \nabla u_l(x)$$

(convergent) **(divergent)** **(divergent)**

$C(N,2)$ operator $C(N,2)$ operator $C(N,2)$ operator

finite part = **zero** + **finite part**

(Stokes' transformation) ← **(Cesaro sum)**

Regularization Techniques --- Different Points of View

- **Divergent Integral (Hypersingular kernel) :**

$$H.P.V.\int M(s,x)\ u(s)dB(s)$$

- **Divergent Series (Dual series representation) :**

$$C(N,2)\ \{\sum_{m=0}^{N} \int_B t_n(s)\ u(s,t)dB(s)\ t_n(x)\}$$

- **Cesaro Sum (Arithmetic mean) :**

$$S_N(x,t) = C(N,1)\{\sum_{m=0}^{N} a_m(x,t)\} = \frac{s_0(x,t)+s_1(x,t)+\cdots+s_{N-1}(x,t)+s_N(x,t)}{N+1}$$

- **Reproducing Kernel (Fejer kernel) :**

$$F_{N+1}(x) = \frac{1}{2\pi(N+1)}\frac{\sin^2((N+1)x/2)}{\sin^2(x/2)}$$

- **Moving Average (MA model) :**

$$S_N(x,t) = \frac{1}{N+1}\sum_{m=0}^{N}(N-m+1)\ a_m(x,t)$$

- **Stokes' Transformation (Summation by parts) :**

$$f'(x) = \frac{d}{dx}\left[f(x)\right] = \frac{d}{dx}\left\{\sum_{k=0}^{N} c_k\ u_k(x)\right\} = \sum_{k=0}^{N} c_k\ u_k'(x) + \sum_{k=0}^{N} b_k\ u_k'(x)$$

Literature Review of Stokes' Transformation

- **Single Fourier Series :**

  Oscillating waves (Stokes', 1880)

  Stability of viscous fluid (Goldstein, 1936)

  Free vibration

  - **twisted beam (Budianskey and Diprima, 1960)**
  - **shell (Chung, 1981)**
  - **beam on viscoelastic foundation (Chuang and Wang, 1991)**

  Time Dependent Boundary Condition

  - **support excitation (Chen, Hong and Yeh, 1993)**
  - **heat conduction (present)**

- **Double Fourier Series**

  Static analysis of doubly curved shells (Chaudhuri and Kabir, 1993)

Relations of Hypersingular Kernel and Divergent Series

The Series Representation Solutions of the Three Analytical Formulations

Transient Responses at t=1 second

Temperature profile

Temperature gradient profile

Conclusions

- **Hypersingularity and Divergent Series Are Linked Based on Dual Integral Formulation**
- **Regularization Technique of Hypersingularity Is Transformed to**
  - **Cesaro sum**
  - **Stokes' transformation**
- **Stokes' Transformation**
  - **Free from calculating quasi-static solution**
  - **Accelerate convergence rate**
- **Why Cesaro Sum Can Extract Finite Part Is Proved by Stokes' Transformation and Distribution Theory**
- **The Transient Responses of Heat Conduction Problems with Time Dependent Boundary Conditions Have Been Solved.**