Conformal Mapping and Bipolar Coordinate for Eccentric Laplace Problems

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ABSTRACT: Boundary value problems on the eccentric annulus are quite complex and cannot directly be solved analytically using cartesian or polar coordinates. Many mathematical techniques have been used to solve such a problem by using conformal mapping and bipolar coordinate. In the literature, Carrier and Pearson [Partial differential equation-theory and technique. New York, Academic Press, 1976, pp 68—71], Muskhelishvili [Some basic problems of the mathematical theory of elasticity. Noordhoff, Groningen; 1953, pp 175—179], Ling [Torsion of an eccentric circular tube, Technical Report, No. 1, Chinese Bureau of Aeronautical Research, 1940], Timoshenko and Goordier [Theory of Elasticity. New York, McGraw-Hill; 1972, pp 196—202], Shen [Null-field approach for Laplace problems with circular boundaries using degenerate kernels, Master thesis, National Taiwan Ocean University, Keelung, Taiwan, 2005], Lebedev et al. [Worked Problems in Applied Mathematics. New York, Dover; 1965] have solved this kind of problems using similar techniques. By using transformation in a transformed plane in the complex variable theory, we can obtain the analytical solution easily. We focus on the connection between conformal mapping and curvilinear coordinates, and figure out the relation to take integration by way of mapping in the complex plane. All the transformations and curvilinear coordinates can be unified using the viewpoint of conformal mapping. Their relationship among available methods can be constructed by translation, stretching, rotation and inversion. Finally, an example of eccentric domain is solved by using various mappings and curvilinear coordinates and their relations are linked. Not only geometry transformation is addressed but also the solution of the Laplace equation is obtained. © 2009 Wiley Periodicals, Inc. Comput Appl Eng Educ; Published online in Wiley InterScience (www.interscience.wiley.com); DOI 10.1002/cae.20208

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INTRODUCTION

Based on the complex variable theory, a number of physical problems governed by Laplace equation in

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two independent variables, for example, steady-state heat condition, electrostatic potential and fluid flow, can be solved by using conformal mapping to obtain an analytical solution. Also we can formulate the same problems using different curvilinear coordinates to obtain a solution, for example, bipolar coordinate and elliptic coordinate. Carrier and Pearson [1] used a technique for solving certain kinds of potential problems, based on the bilinear transformation of conformal mapping. The eccentric case was mapped to an annular domain. For a polygonal shape, it can also be mapped to regular region by using the Schwarz-Christoffel transformation [2,3]. For the regular domain, it is easy to solve the Laplace equation subject to Dirichlet boundary condition in the polar or rectangular coordinate. Muskhelishvili [4] gave us a detailed description how an eccentric annulus can be mapped into concentric annulus using a simple form of linear fractional transformation. Shen [5], and Chen and Weng [7] also used the same method to solve eccentric annulus problems. Although a bilinear transformation was used, the mapping functions were not exactly the same between the one of Carrier and Pearson [1] and that of Muskhelishvili [4]. Some differences such as translation and stretching can be found [6,10]. Problems involving two nonconcentric boundaries usually require the use of the bipolar coordinate. Ling [8], Timoshenko and Goordier [9], and Lebedev et al. [11] all presented an analytic solution using the bipolar coordinate for the torsion of an eccentric bar. However, the mapping functions were also not exactly the same. One [8] is a cotangent function, another [11] is a hyperbolic tangent function and the other [9] is a hyperbolic cotangent function. After the bipolar coordinate is introduced, the special domain problem can be solved by using separation of variable in a regular domain. A typical example would be the electric field surrounding two parallel cylindrical conductors. Stephens and Casemore [12], proposed an alternative method to determine the negative stiffness coefficients for a large class of magnetic actuators. They solved the Dirichlet Laplace problems for the magnetomotive force in the actuator air gap, subject to periodic boundary conditions that can be represented by Fourier series. We can find that the bipolar coordinate is actually a kind of conformal mapping. Although Carrier and Pearson [1], Muskhelishvili [4], Ling [8], Timoshenko and Goordier [9], Shen [5], Stephens and Casemore [12] have solved the eccentric Laplace problems, their approaches are very similar but they are not fully equivalent.

This paper will link the relationship of various approaches from the viewpoint of conformal mapping.

The other parts of the present paper are arranged as follows. In Conformal Mapping Using Bilinear Function Section, the method of conformal mapping is reviewed and the analytic solution of the Laplace equation with an annular domain is reviewed. In Geometric Characterization of the Bipolar Coordinate Section, we not only describe the geometric characterization of bipolar coordinate but also derive the analytic solution of Laplace's equation. In An Illustrative Example Section, an example is demonstrated to link the relationship among many previous approaches based on the viewpoint of conformal transformation through translation, stretching, rotation and inversion. Finally, a conclusion is drawn in the fifth section.

CONFORMAL MAPPING USING BILINEAR FUNCTION

Definition of the bilinear transformation is given below:

$$z = \frac{aw + b}{cw + d} \tag{1}$$

where a,b,c,d are constants (in general, complex). Equation (1) is also called a linear fractional transformation or Möbius transformation [2]. Solving (1) for w in terms of z, we have, likewise bilinear, the inverse transformation:

$$w = \frac{-dz + b}{cz - a} \tag{2}$$

The bilinear transformation has a remarkable property that circles map to circles. The equation of any circle in the z plane is known to be of the following form:

$$A(x^2 + y^2) + Bx + Cy + D = 0 \tag{3}$$

where $A,B,C,$ and D are real constants. Since $x = ((z + \bar{z}) / 2)$, $y = ((z - \bar{z}) / 2i)$, $x^2 + y^2 = z\bar{z}$, Equation (3) can be reformulated to:

$$Az\bar{z} + Mz + \bar{M}\bar{z} + D = 0 \tag{4}$$

where M and $\bar{M}$ are conjugate complex constants. By representing z in terms of w using Equation (1) and substituting into Equation (4), we have

$$A_0 w\bar{w} + M_0 w + \bar{M}_0 \bar{w} + D_0 = 0 \tag{5}$$

where A_0 and D_0 are real, and M_0 and $\bar{M}_0$ are conjugate complex constants. This confirms that a circle in the z plane (Eq. 3) implies another circle in the w plane (Eq. 5). This provides one way to map an

eccentric case to an annulus case. For an eccentric annulus, a simple bilinear transformation will be studied here, as shown below:

$$z = \frac{w}{1 - aw} \tag{6}$$

where $z = x + iy$, $w = \rho e^{i\phi}$ and a is a real positive constant. We consider now two circles L_1 and L_2 in the z plane, which can be mapped to the two circles L_1' and L_2' with radii ρ_1 and ρ_2 in the w plane (Fig. 1). From Figure 1, we can find the circle L' in the w plane with the radius ρ. For the mapped annular case, we have ρ_1 and ρ_2 for inner and outer radii, respectively. Based on Equations (3) and (5), we can determine the origins and radii of L_1 and L_2 in the z plane as shown below:

$$c_i = \frac{a\rho_i^2}{1 - a^2\rho_i^2}, \quad i = 1, 2 \tag{7}$$

$$r_i = \frac{\rho_i}{1 - a^2\rho_i^2}, \quad i = 1, 2 \tag{8}$$

The complex function maps eccentric circles with radii of r_1 and r_2 in the z plane onto the annular region bounded by the inner radius of ρ_1 and outer radius of ρ_2 in the w plane as shown in Figure 1 where l is the distance between their centers. Therefore it is easy to determine the quantity a, appearing in (6), and the radii ρ_1, ρ_2 of the circles L_1' and L_2'. In real practice, three quantities r_1, r_2, and l are given in the z plane. For the w plane, ρ_1, ρ_2, and a satisfy the following three equations:
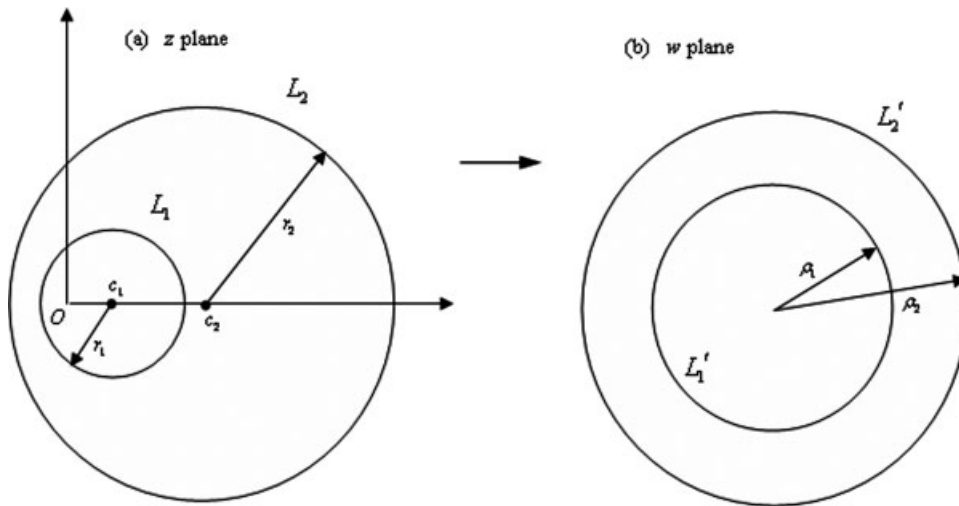
$$r_1 = \frac{\rho_1}{1 - a^2\rho_1^2} \tag{9}$$

$$r_2 = \frac{\rho_2}{1 - a^2\rho_2^2} \tag{10}$$

$$l = c_2 - c_1 = \frac{a\rho_2^2}{1 - a^2\rho_2^2} - \frac{a\rho_1^2}{1 - a^2\rho_1^2} \tag{11}$$

According to three equations (9–11), three unknown constants ρ_1, ρ_2, and a can be determined by

$$a = \frac{l}{\sqrt{(r_1^2 - r_2^2)^2 - 2l^2(r_1^2 + r_2^2) + l^4}} \tag{12}$$

$$\rho_1 = \frac{\sqrt{1 + 4r_1^2a^2} - 1}{2r_1a^2}, \quad \rho_2 = \frac{\sqrt{1 + 4r_2^2a^2} - 1}{2r_2a^2} \tag{13}$$

Then we can obtain an annulus with the radii of ρ_1 and ρ_2 from any eccentric annulus using mapping of Equation (6) with an appropriate value of a.

Derivation of an Analytical Solution for the Laplace Equation With an Annular Domain

By using the bilinear transformation, an eccentric domain can be mapped to an annulus domain. The Dirichlet problem for an annular domain can be described as shown below:

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta} = 0, \ r_1 < r < r_2 \tag{14}$$
$$u(r_1, \theta) = f_1(\theta), \ u(r_2, \theta) = f_2(\theta)$$

In addition, $u(r,\theta)$ must satisfy the periodicity condition. Accordingly, $f_1(\theta)$ and $f_2(\theta)$ must also be periodic with a period of 2π. Thus, the general solution is

Figure 1   Mapping of $w(z) = \frac{z + \frac{1}{4}}{z + 4}$ from the eccentric region to a concentric region.

$$u(r,\theta) = A_0 + B_0 \ln r + \sum_{n=1}^{\infty} [(A_n r^n + B_n r^{-n})$$

$$\cos n\theta + (C_n r^n + D_n r^{-n})\sin n\theta]$$

where the Fourier coefficients A_0, B_0, A_n, B_n, C_n, and D_n are found by solving the following six equations [13]:

$$A_0 + B_0 \ln r_1 = \frac{1}{2\pi}\int_0^{2\pi} f_1(\theta)d\theta \qquad (16)$$

$$A_n r_1^n + B_n r_1^{-n} = \frac{1}{\pi}\int_0^{2\pi} f_1(\theta)\cos n\theta d\theta \qquad (17)$$

$$C_n r_1^n + D_n r_1^{-n} = \frac{1}{\pi}\int_0^{2\pi} f_1(\theta)\sin n\theta d\theta \qquad (18)$$

$$A_0 + B_0 \ln r_2 = \frac{1}{2\pi}\int_0^{2\pi} f_2(\theta)d\theta \qquad (19)$$

$$A_n r_2^n + B_n r_2^{-n} = \frac{1}{\pi}\int_0^{2\pi} f_2(\theta)\cos n\theta d\theta \qquad (20)$$

$$C_n r_2^n + D_n r_2^{-n} = \frac{1}{\pi}\int_0^{2\pi} f_2(\theta)\sin n\theta d\theta \qquad (21)$$

Now we can obtain an analytical solution for the eccentric case by using the conformal mapping.

GEOMETRIC CHARACTERIZATION OF THE BIPOLAR COORDINATE

Bipolar coordinate (ξ,η) is defined by

$$x + iy = ic\cot\left(\frac{1}{2}\zeta\right), \quad \zeta = \xi + i\eta \qquad (22)$$

where $c > 0$. Equation (22) yields

$$x = c\frac{\sin h\eta}{\cosh\eta - \cos\xi}, \quad y = c\frac{\sin\xi}{\cosh\eta - \cos\xi} \qquad (23)$$

where $-\pi \leq \xi < \pi$, $-\infty < \eta < \infty$. By eliminating ξ in Equation (23), we obtain a circle with the center of $(c\cot h\eta, 0)$ and the radius of $c\csc h\eta$ as shown below:

$$(x - c\cot h\eta)^2 + y^2 = c^2 \csc h^2\eta \qquad (24)$$

Elimination of η from Equation (23) results in another circle with the center of $(0, c\cot\xi)$ and the radius of $c\csc\xi$ as given below:

$$x^2 + (y - c\cot\xi)^2 = c^2 \csc^2\xi \qquad (25)$$

From Equations (24) and (25), the bipolar coordinates are shown in Figure 2. Denoting by (Γ_1,φ_1) and (Γ_2,φ_2), the polar coordinates for the poles of $(\pm c,0)$, in Figure 3, respectively, we have

$$x + iy - c = \Gamma_1 e^{i\varphi_1}, x + iy + c = \Gamma_2 e^{i\varphi_2} \qquad (26)$$

$$\eta = \log\left(\frac{\Gamma_2}{\Gamma_1}\right), \xi = \varphi_1 - \varphi_2 \qquad (27)$$

It follows that a curve $\xi = $ constant is a family of circles passing through the poles $(\pm c,0)$. The curve of $\eta = $ constant shows a curve for which $\Gamma_2/\Gamma_1 = $ constant. The proof is shown in the Appendix 1. Family of circles are drawn in Figure 2. The outer radius R_1, inner radius R_2 and the displacement b are determined by Equation (24) as shown below:

$$R_1 = c\csch(\eta_1) \qquad (28)$$

$$R_2 = c\csch(\eta_2) \qquad (29)$$

$$b = c[\coth(\eta_1) - \coth(\eta_2)] \qquad (30)$$

To describe an eccentric circle in the bipolar coordinate, the three quantities, c, η_1, and η_2 should be determined as shown below:

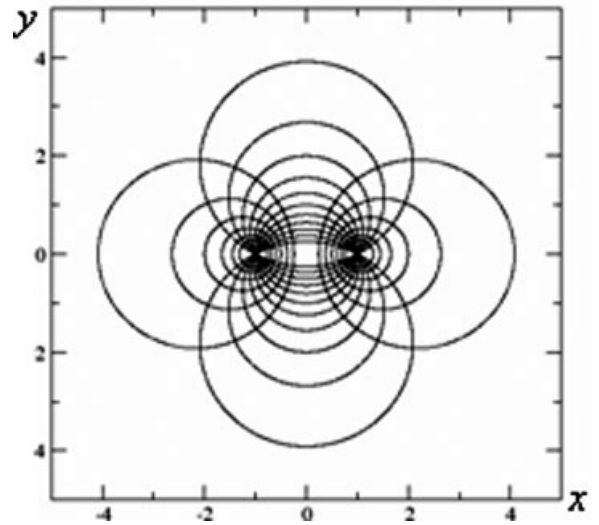$$c = \frac{\sqrt{R_1^4 + R_2^4 - 2R_1^2 R_2^2 - 2b^2(R_1^2 + R_2^2) + b^4}}{2b} \qquad (31)$$

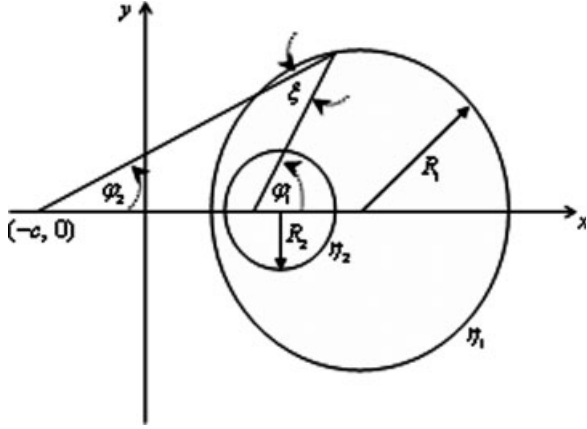**Figure 2** Bipolar coordinate system (poles located on $(\pm 1,0)$).

Figure 3   Geometry relation of bipolar coordinate.

$$\eta_1 = \sinh^{-1}\frac{c}{R_1} \tag{32}$$

$$\eta_2 = \sinh^{-1}\frac{c}{R_2} \tag{33}$$

Then we can describe an eccentric annulus by using the bipolar coordinate.

Viewpoint of Conformal Mapping for the Bipolar Coordinate

According to the inverse cotangent function [2] as shown below:

$$\cot^{-1}(z) = \frac{i}{2}\log\frac{z-i}{z+i} \tag{34}$$

Equation (22) can be rewritten as:

$$w(z) = \zeta = i\log\frac{z+c}{z-c} \tag{35}$$

Equation (35) indicates that the curvilinear bipolar coordinate stems from the conformal mapping of Equation (22). The coordinate ξ changes from $-\pi$ to π on crossing the segment of the x-axis joining the pole, its range for the whole plane being $-\pi$ to π. The transformation from Equation (35), where the principal branch of the logarithmic function is used, can be decomposed to two mappings,

$$Z = \frac{z+c}{z-c} \tag{36}$$

$$w = \text{Log}(Z) \tag{37}$$

More precisely, by expressing $Z = \mathbb{R}\exp(i\Theta)$ we obtain

$$\text{Log}(Z) = \ln\mathbb{R} + i\Theta, \quad (\mathbb{R} > 0, -\pi < \Theta < \pi) \tag{38}$$

We can understand that the point $Z = \mathcal{R}e^{i\Theta_0}$ $(0 < \Theta_0 < \pi)$ moves outward from the origin along the ray $\Theta = \Theta_0$, and its transformation is the point whose rectangular coordinate in the w plane is $(\ln\mathcal{R}, \Theta_0)$. That transformation evidently moves to the right along the entire length of the horizontal line $v = \Theta_0$. Based on this transformation, we can find that an eccentric annulus can be mapped onto a rectangular domain.

Derivation of an Analytical Solution for Eccentric Problems Using the Bipolar Coordinate

By using the bipolar coordinate, the eccentric domain can be mapped into the rectangular domain of (ξ,η). Note that there are discontinuities in this mapping at $\xi = 0$ and $\xi = \pi$. This is due to the symmetry in the geometry above and below the x-axis. Since the boundary condition is not necessarily symmetric to $(c,0)$ about this axis, the problem must be solved piecewise over the interval $0 < \xi < \pi$ and $-\pi < \xi < 0$. By using this transformation, the solution $f(x,y) = F(\xi,\eta)$ also satisfies the Laplace equation in the curvilinear coordinate as shown below:

$$F_{\xi\xi} + F_{\eta\eta} = 0 \tag{39}$$

The boundary conditions are given below in a general form as:

$$F(\eta_1, \xi) = F_1(\xi), \quad -\pi < \xi < \pi \tag{40}$$

$$F(\eta_2, \xi) = F_2(\xi), \quad -\pi < \xi < \pi \tag{41}$$

In addition to the specified boundary conditions, the following natural boundary conditions arise due to the periodic circular symmetry:

$$F_\xi(\eta, 0) = 0 \tag{42}$$

$$F_\xi(\eta, \pi) = 0 \tag{43}$$

The solution is obtained by using separation of variables which yields a series solution. This solution is valid on both piecewise intervals and is given by:

$$F(\eta, \xi) = a_0 + b_0\eta + \sum_{n=1}^{\infty}[(a_n e^{n\eta} + b_n e^{-n\eta})$$

$$\cos(n\xi) + (c_n e^{n\eta} + d_n e^{-n\eta})\sin(n\xi)]$$

where the Fourier coefficients a_0, b_0, a_n, b_n, c_n, and d_n are determined by solving the following six equations simultaneously,

$$a_0 + b_0 \eta_1 = \frac{1}{2\pi} \int_0^{2\pi} F_1(\xi) d\xi \qquad (45)$$

$$a_0 + b_0 \eta_2 = \frac{1}{2\pi} \int_0^{2\pi} F_2(\xi) d\xi \qquad (46)$$

$$a_n e^{n\eta_1} + b_n e^{-n\eta_1} = \frac{1}{\pi} \int_0^{2\pi} F_1(\xi) \cos(n\xi) d\xi \qquad (47)$$

$$a_n e^{n\eta_2} + b_n e^{-n\eta_2} = \frac{1}{\pi} \int_0^{2\pi} F_2(\xi) \cos(n\xi) d\xi \qquad (48)$$
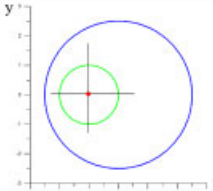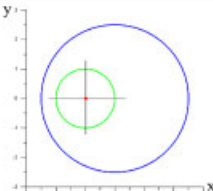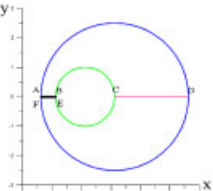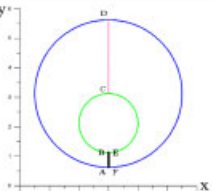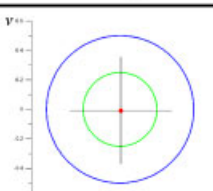
$$c_n e^{n\eta_1} + d_n e^{-n\eta_1} = \frac{1}{\pi} \int_0^{2\pi} F_1(\xi) \sin(n\xi) d\xi \qquad (49)$$
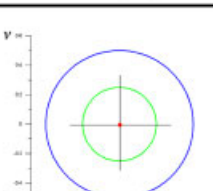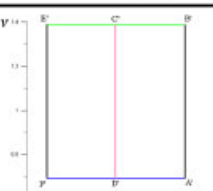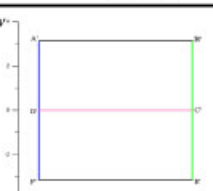
$$c_n e^{n\eta_2} + d_n e^{-n\eta_2} = \frac{1}{\pi} \int_0^{2\pi} F_2(\xi) \sin(n\xi) d\xi \qquad (50)$$

where $F_1(\xi)$ and $F_2(\xi)$ are the associated boundary conditions given piecewise over the intervals $-\pi < \xi < 0$ and $0 < \xi < \pi$ as shown in Equations (40) and (41).
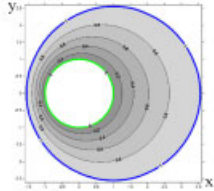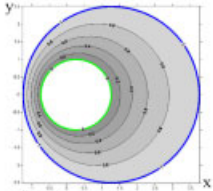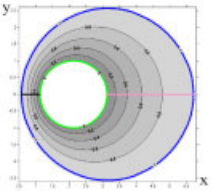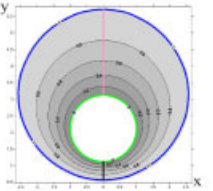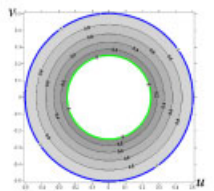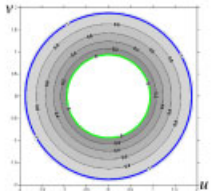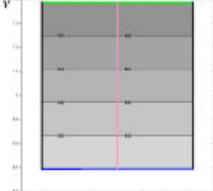
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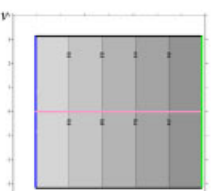
Suppose we want to determine a harmonic function ψ in an eccentric annular region lying between two nonconcentric circles, and satisfying $\psi = 0$ on the inner circle with a radius 1 and $\psi = 1$ on the outer circle with radius 5/2, where the center of outer circle located on the circumference of inner circle [1]. Several researchers, Carrier and Pearson [1], Muskhelishvili [4], Ling [8], Timoshenko and Goordier [9], Shen [5], Lebedev et al. [11] and Chen and Weng [7] have employed their approaches to solve this problem as shown in Tables 1 and 2. Although the result of Lebedev et al.'s [11] approach using the hyperbolic tangent function is not shown in Tables 1 and 2, it is different from the Timoshenko and Goordier formulation by inversion and rotation only. The geometry description of various formulations using the viewpoint of conformal mapping is shown in Table 1, and their relations are

Table 1 Geometry Transformation of Various Formulations Using the Viewpoint of Conformal Mapping

where $z = x + iy$ and $w = u + iv$.

Table 2 Analytic Solutions of Dirichlet Laplace Problems Using Different Formulations (Blue Curve $= 1$, Green Curve $= 0$)

Carrier & Pearson [1]		Muskhelishvili [2]		Chih-Bing Ling [3]		Timoshenko [4]
Bilinear mapping		Bilinear mapping		Bipolar coordinate		Bipolar coordinate
$z(w) = \dfrac{1-16w}{4(w-1)}$		$z(w) = \dfrac{15w}{15-4w}$		$z(w) = ic\cot\left(\dfrac{w}{2}\right)$		$z(w) = ic\coth\left(\dfrac{w}{2}\right)$

| Fig. 12 | Translation $\frac{1}{4}$ → | Fig. 14 | Translation $\frac{15}{8}$ → | Fig. 16 | Rotation i → | Fig. 18 |

| $w(z) = \dfrac{z+1/4}{z+4}$ | | $w(z) = \dfrac{z}{1+(4/15)z}$ | | $w(z) = i Log\left(\dfrac{z+15/8}{z-15/8}\right)$ | | $w(z) = Log\left(\dfrac{z+i(15/8)}{z-i(15/8)}\right)$ |

| Fig. 13 | Stretching $\frac{15}{4}$ → | Fig. 15 | Stretching $\frac{4}{15}$; rotation i; Inversion $\frac{1}{w}$; Take Log → | Fig. 17 | Rotation $-i$ → | Fig. 19 |

where $z = x + iy$ and $w = u + iv$.

connected through the operations of translation, stretching, rotation, and inversion. We present a case by using four methods in Table 1, and the mapping functions for each method are shown below:

Carrier and Pearson [1]:

$$w(z) = \frac{z+1/4}{z+4} \tag{51}$$

Muskhelishvili [4]:

$$w(z) = \frac{z}{1+(4/15)z} \tag{52}$$

Ling [8]:

$$w(z) = i Log\left(\frac{z+15/8}{z-15/8}\right) \tag{53}$$

Timoshenko and Goordier [9]:

$$w(z) = Log\left(\frac{z+i(15/8)}{z-i(15/8)}\right) \tag{54}$$

The eccentric annular problem needs three parameters to describe the geometry, eccentricity e, radii of large and small circles. We set the larger radius to be 5/2 and 1 for the smaller radius with the eccentricity 1. For different observer systems, we have four geometry descriptions for the same problem. After setting the observer system, each case can be mapped to a regular domain by using equations of (51)−(54), respectively. According to Table 1, Figure 4 can be obtained from Figure 6 by translating 1/4. By translating 15/8 again, we have Figure 8. By multiplying i

Table 3 Geometry Description

Carrier and Pearson [1]	Muskhelishvili [4]	Ling [8]	Timoshenko and Goordier [9]
$\lvert z_g\rvert = 1, \lvert z_b - 1\rvert = 5/2$	$\lvert z_g - 1/4\rvert = 1,$ $\lvert z_b - 5/4\rvert = 5/2$	$\eta_g = 0.693147,$ $\eta_b = 1.38629, c = 15/8,$ $\lvert z_g - 17/8\rvert = 1,$ $\lvert z_b - 25/8\rvert = 5/2$	$\xi_g = 0.693147, \xi_b = 1.38629,$ $c = 15/8, \lvert z_g - 17/8i\rvert = 1,$ $\lvert z_b - 25/8i\rvert = 5/2$
$\lvert w_g\rvert = 1/4, \lvert w_b\rvert = 1/2$	$\lvert w_g\rvert = 15/16, \lvert w_b\rvert = 15/8$	$A'(\pi, \ln(\eta_b)), B'(\pi, \ln(\eta_g)),$ $E'(\ln(\eta_g), \pi), F'(\ln(\eta_b), \pi),$	$A'(\ln(\xi_b), \pi), B'(\ln(\xi_g), \pi),$ $E'(\pi, \ln(\xi_g)), F'(\pi, \ln(\xi_b))$

where the subscripts g and b represent "green" curve and "blue" curve.

Table 4 Analytic Solutions

Carrier and Pearson [1]	Muskhelishvili [4]	Ling [8]	Timoshenko and Goordier [9]
$\psi(x,y) = \dfrac{1}{2\ln 2}\ln\left(\dfrac{16(x^2+y^2)+1+8x}{(x^2+y^2)+16+8x}\right)$	$\psi(x,y) = \dfrac{1}{\ln 2}\ln\left(\dfrac{256(x^2+y^2)}{225+16(x^2+y^2)+120x}\right)$	$\psi(x,y) = \dfrac{\cot h^{-1}\left(\dfrac{c^2+x^2+y^2}{2cx}\right)-1.38629}{0.693147-1.38629}$	$\psi(x,y) = \dfrac{\cot h^{-1}\left(\dfrac{c^2+x^2+y^2}{2cy}\right)-1.38629}{0.693147-1.38629}$
$\psi(u,v) = \dfrac{\ln(4(u^2+v^2))}{\ln 2}$	$\psi(u,v) = \dfrac{\ln\left(\dfrac{15}{16}\left(u^2+v^2\right)\right)}{\ln 2}$	$\psi(u,v) = \dfrac{v-1.38629}{0.693147-1.38629}$	$\psi(u,v) = \dfrac{u-1.38629}{0.693147-1.38629}$

(rotation of 90°) of Figure 8, we obtain Figure 10. By stretching 15/4 of Figure 5, we have Figure 7. By stretching 4/15, rotation i, inversion $1/w$ and taking $\mathrm{Log}(w)$ of Figure 7, we transform to a rectangular domain of Figure 9. By multiplying $-i$ (counter-clockwise rotation of 90°), Figure 11 can be obtained from Figure 9. Finally, it can be unified together through the operations of stretching, translation, rotation, inversion and taking Log. Analytic solutions are shown in Table 2 using various formulations and the contour plots are also shown to compare with each other. Table 3 shows the geometry description while Table 4 indicates the closed-form solution.

CONCLUSIONS

In this paper, various approaches including Carrier and Pearson [1], Muskhelishvili [4], Ling [8], Timoshenko and Goordier [9], and Lebedev et al. [11] for solving eccentric Laplace problems were reviewed. Based on the conformal mapping, all of them are unified together. The relations among them are constructed through the operations of translation, rotation, and inversion. One example was demonstrated to understand the relationship among various formulations.

APPENDIX

From Equation (22), we have

$$z = -c\coth(i\zeta) = -c\frac{\cos h(i\zeta)}{\sin h(i\zeta)} = -c\frac{e^{i\zeta}+e^{-i\zeta}}{e^{i\zeta}-e^{-i\zeta}}\tag{A1}$$

where $\zeta = ((\xi+i\eta)/2)$. Equation (A1) yields

$$(e^{i\zeta}-e^{-i\zeta})z = -c(e^{i\zeta}+e^{-i\zeta})$$

$$(z+c)e^{i\zeta} = (z-c)e^{-i\zeta}$$

$$e^{2i\zeta} = \frac{z-c}{z+c}$$

since $z-c = \Gamma_1 e^{i\varphi_1}$ and $z+c = \Gamma_2 e^{i\varphi_2}$

$$2i\zeta = \log\frac{\Gamma_1 e^{i\varphi_1}}{\Gamma_2 e^{i\varphi_2}}.\tag{A2}$$

According to the definition of $\zeta = ((\xi+i\eta)/2)$, we obtain

$$i\xi - \eta = \log\frac{\Gamma_1}{\Gamma_2} + i(\varphi_1 - \varphi_2)$$

Therefore, the equations of real and imaginary parts yield

$$\eta = \log \frac{\Gamma_2}{\Gamma_1}, \quad \xi = \varphi_1 - \varphi_2$$

This concludes the proof that (Γ_1/Γ_2) does not change for the constant η as shown in Figure 3.

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BIOGRAPHIES

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