A new meshless method for eigenproblems using radial basis function

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Abstract

In this lecture, a new meshless method for solving eigenproblems using the radial basis function (RBF) is proposed. By employing the imaginary-part fundamental solution as the RBF, the diagonal and off-diagonal coefficients of the influence matrices are easily determined. True eigensolutions in conjunction with spurious eigensolutions occur at the same time. To verify this finding, the circulant is adopted to analytically derive the true and spurious eigenequations in the discrete system for a circular domain. In order to obtain the true and spurious eigenvalues, the singular value decomposition (SVD) technique of updating technique is utilized. Several examples, including 2-D and 3-D interior acoustics and plate eigenproblems, are demonstrated analytically and numerically to see the validity of the present method.

Keywords: Circulant, Helmholtz problem, Biharmonic problem, Acoustic cavity, Membrane, Plate vibration, Eigenproblem

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1. Introduction

Mesh generation of a complicated geometry is always time consuming in the stage of model creation for engineers in dealing with engineering problems by employing numerical methods, e.g., the finite difference method (FDM), finite element method (FEM) and boundary element method (BEM). In the last decade, researchers have paid attention to the meshless method without employing the concept of element. Several meshless methods have also been reported in the literature, for example, the domain-based methods including the element-free Galerkin method [1], the reproducing kernel method [2], and boundary-based methods including the boundary node method [3], the meshless local Petrov-Galerkin approach [4], the local boundary integral equation method [5] and the RBF approach [6].

Integral equations and BEM have been utilized to solve the boundary value problems for a long time. Several approaches, e.g., the complex-valued BEM, the method of fundamental solution, the dual reciprocity method (DRM), the method of particular integral [7], multiple reciprocity method (MRM), the real-part BEM and imaginary-part BEM [8], have been developed for eigenproblems. To solve the problem by using the complex-valued BEM, the influence coefficient matrix would be complex arithematics [9]. Therefore, Tai and Shaw [10] employed only the real-part kernel to solve the eigenvalue problems in sacrifice of appearance of spurious eigenvalues. To avoid the singular and hypersingular integrals, De Mey [11] used imaginary-part fundamental solution to solve the eigenproblems and also encountered the problem of eigensolution.

In the meshless method, Kang *et al.* proposed the NDIF (Non-dimensional Dynamic Influence Function) method to solve eigenproblems of membranes [12], acoustic cavities [13], and plates [14]. Later, Chen *et al.* commented that the NDIF method is a special case of imaginary-part BEM after lumping the distribution of density function for membrane vibrations [15], acoustics [16], and plate [17,18].

Nevertheless, spurious eigensolutions are inherent in the imaginary-part BEM, the real-part BEM, the MRM and the meshless method. Numerically speaking, the spurious eigensolutions result from the rank deficiency of the influence coefficient matrix. Rank deficiency in BEM formulation, e.g., spurious eigenvalue, fictitious frequency, degenerate boundary and degenerate scale had been discussed particularly in the plenary lecture of Chen *et al.* [19]. This implies the fewer number of constraint equations making the solution space larger. Mathematically speaking, the spurious eigensolutions for interior problems arise from the source of “improper approximation of the null space of operator”. Two sources of rank deficiency in the influence matrices can be classified, one is the spurious solution due to incompleteness for the representation of the solution and the other is true solution due to the nontrivial eigensolution. The spurious eigensolution stems from the numerical method and has no physical meaning. Chen and his coworkers have developed several techniques for overcoming the spurious eigenvalues, e.g., dual formulation [20], domain partition [21], SVD updating techniques [22] and CHEEF [23] method for sorting out the true and the spurious eigenvalues.

In this lecture, we will employ the imaginary-part fundamental solution as RBF to solve the 2-D acoustic [24], 3-D acoustic [25] and plate eigenproblems [26]. The main difference between the present formulation and the method of fundamental solution is that we adopt only the imaginary-part fundamental solution instead of employing the complex-valued singular kernel. Another point is that we can distribute the strength along the real boundary. In solving the problem numerically, elements are not required and only boundary nodes are necessary. Both the boundary and source points are distributed on the real boundary only. The occurrence of spurious eigenvalues and the remedies will be discussed in this lecture. For the case of circular plate, the eigensolutions will be analytically derived in the discrete system by using circulants. Several examples, 2-D circular cavity, 3-D circular cavity and circular plates, will be demonstrated to see the validity of the present formulation.

2. Meshless formulation using radial basis function of the imaginary-part fundamental solution

The governing equation for the interior eigenproblem is

$$\cdots u(x) = m\, u(x), \quad x \in \Omega, \tag{1}$$

where $\cdots$ is the differential operator, and

$$\cdots = \begin{cases} \nabla^2, & \textit{for the acoustic eigenproblem,} \\ \nabla^4, & \textit{for the plate eigenproblem,} \end{cases} \tag{2}$$

m is the eigenvalue,

$$m = \begin{cases} -k^2, & \textit{for the acoustic eigenproblem,} \\ l^4, & \textit{for the plate eigenproblem,} \end{cases} \tag{3}$$

u is the potential, ∇^2 is the Laplacian operator, ∇^4 is the biharmonic operator, k is the wave number which is the angular frequency over the speed of sound, ? is the frequency parameter, $l^4 = \frac{w^2 r_0 h}{D}$, w is the circular frequency, r_0 is the surface density, D is the flexural rigidity expressed as $D = \frac{Eh^3}{12(1-n^2)}$ in terms of Young's modulus E, the Poisson ratio ? , the plate thickness h, and O is the domain of the interest. The boundary conditions can be the Neumann or Dirichlet type for the acoustic eigenproblem, and the clamped, simply-supported or free boundary for the plate eigenproblem.

The radial basis function is defined by

$$G(x,s)=j\,(|s-x|) \tag{4}$$

where x and s are the collocation and source points, respectively. The Euclidean norm $|s-x|$ is referred to as the radial distance between the collocation and source points. The two-point function ($j\,(|s-x|)$) is called the RBF since it depends on the radial distance between x and s. By considering the imaginary-part fundamental solution as RBF, we have

$$U(s,x)=\begin{cases} \mathrm{Im}\{\dfrac{e^{ikr}}{2ki}\}, & \textit{for the } 1-D \textit{ Laplace operator},\\ \mathrm{Im}\{i\,H_0^{(1)}(kr)\}, & \textit{for the } 2-D \textit{ Laplace operator},\\ \mathrm{Im}\{\dfrac{e^{ikr}}{r}\}, & \textit{for the } 3-D \textit{ Laplace operator},\\ \mathrm{Im}\{\dfrac{i}{8l}(H_0^{(1)}(lr)+H_0^{(2)}(ilr))\}, & \textit{for the } 2-D \textit{ biharmonic operator } [27], \end{cases} \tag{5}$$

in which $r\equiv|s-x|$ is the distance between the source and collocation points, $H_0^{(1)}$ and $H_0^{(2)}$ are the first kind and second kind zeroth-order Hankel functions, respectively. We can choose the three kernels as follows:

$$\Theta(s,x)=K_q(U(s,x)), \tag{6}$$

$$M(s,x)=K_m(U(s,x)), \tag{7}$$

$$V(s,x)=K_v(U(s,x)), \tag{8}$$

where $K_q(\cdot)$, $K_m(\cdot)$ and $K_v(\cdot)$ mean the operators which are defined as follows:

$$K_q(\cdot)=\frac{\partial(\cdot)}{\partial n}, \tag{9}$$

$$K_m(\cdot)=n\,\nabla^2(\cdot)+(1-n)\frac{\partial^2(\cdot)}{\partial n^2}, \tag{10}$$

$$K_v(\cdot)=\frac{\partial\nabla^2(\cdot)}{\partial n}+(1-n)\frac{\partial}{\partial t}\left(\frac{\partial^2(\cdot)}{\partial n\partial t}\right), \tag{11}$$

where n and t are the normal vector and tangential vector, respectively. For the acoustic or membrane eigenproblem, the field solution can be represented by

Single-layer (essential) potential approach:

$$u(x_i)=\sum_j U(s_j,x_i)f_j, \tag{12}$$

$$t(x_i)=\sum_j \frac{\partial U(s_j,x_i)}{\partial n_x}f_j, \tag{13}$$

where f_j is the unknown density of the essential potential.

Double-layer (natural) potential approach:

$$u(x_i)=\sum_j \Theta(s_j,x_i)y_j, \tag{14}$$

$$t(x_i) = \sum_j \frac{\partial \Theta(s_j, x_i)}{\partial n_x} y_j \ , \tag{15}$$

where y_j is the unknown density of the natural potential. For the plate problem, the field solution is

$$u(x) = \sum_{j=1}^{2N} P(s_j, x) p_j + \sum_{j=1}^{2N} Q(s_j, x) q_j \ , \tag{16}$$

where p_j and q_j are the unknown coefficients. By applying the three operators in Eqs.(9)-(11) to Eq.(16), we have

$$q(x) = K_q(u(x)) = [P_q]\{p\} + [Q_q]\{q\}, \tag{17}$$

$$m(x) = K_m(u(x)) = [P_m]\{p\} + [Q_m]\{q\}, \tag{18}$$

$$v(x) = K_v(u(x)) = [P_v]\{p\} + [Q_v]\{q\}, \tag{19}$$

where q , m and v denote the slope, normal moment and effective shear force, respectively. Since the two kernels (*P* and *Q*) are obtained from any two combinations of the four kernels (*U*, T , *M* and *V*), six (C_2^4) formulations can be considered. By matching the boundary conditions and plotting the determinant (or minimum singular value) versus the frequency parameter l (or wave number k), we can determine the unknown coefficients (f and y ; *p* and *q*) and the eigenvalue (l or *k*) from the drop location.

3. Analytical study for the eigensolution of a circular membrane, spherical cavity and plate eigenproblems in the discrete system

The *U* kernel can be expressed in terms of degenerate kernels as shown below [30]:

$$U(s,x) = \begin{cases} U^I(s,x) = \sum_j^{\infty} X_j(x) S_j(s), & x \in \Omega^I, \\ U^E(s,x) = \sum_j^{\infty} X_j(s) S_j(x), & x \in \Omega^E \end{cases} \tag{20}$$

where the superscripts "*I*" and "*E*" denote the interior and exterior domains separated by the boundary, respectively.

3.1 Acoustic or membrane eigenproblem

For the 2-D circular acoustic cavity or membrane, the degenerate kernel of imaginary-part fundamental solution can be shown as below:

$$U(s,x) = \begin{cases} U^I(R,q;\, r,f) = \sum_{m=-\infty}^{\infty} J_m(kR) J_m(kr)\cos(m(f-q)), & R > r, \\ U^E(R,q;\, r,f) = \sum_{m=-\infty}^{\infty} J_m(kr) J_m(kR)\cos(m(f-q)), & R < r, \end{cases} \tag{21}$$

where $x = (r, f)$ and $s = (R, q)$ in terms of the polar coordinate and J_m denotes the *m*th order Bessel function. Since the rotation symmetry is preserved for a circular boundary, the influence matrix is denoted by [U] of the circulants with the elements,

$$U_{ij} = U(r, q_j; r, f_i), \tag{22}$$

where f_i and q_j are the angles of observation and boundary points, respectively. By superimposing 2N lumped strength along the boundary, we have the influence matrix,

$$[U] = \begin{bmatrix} a_0 & a_1 & a_2 & \cdots & a_{2N-2} & a_{2N-1} \\ a_{2N-1} & a_0 & a_1 & \cdots & a_{2N-3} & a_{2N-2} \\ a_{2N-2} & a_{2N-1} & a_0 & \cdots & a_{2N-4} & a_{2N-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_1 & a_2 & a_3 & \cdots & a_{2N-1} & a_0 \end{bmatrix} \tag{23}$$

where the elements of the first row can be obtained by

$$U_{ij} = U(s_j, x_i). \tag{24}$$

The matrix [U] in Eq.(23) is found to be a circulant [28] since the rotational symmetry for the influence coefficients is considered. Here, the field and source points are both collocated on the boundary with a radius a ($r = R = a$) by matching the boundary condition. By using the degenerate kernel and the orthogonal property, the eigenvalue of the matrix [U] is obtained as follows:

$$k_\ell^{[U]} = 2N\, J_\ell(ka) J_\ell(ka), \tag{25}$$

where $\ell = 0, \pm 1, \pm 2, \cdots, \pm(N-1), N$. Similarly, the eigenvalues of matrices, $[\Theta]$, $[U_q]$ and $[\Theta_q]$, are determined as follows:

$$k_\ell^{[\Theta]} = 2Nk\, J'_\ell(ka) J_\ell(ka), \tag{26}$$

$$k_\ell^{[U_q]} = 2Nk\, J_\ell(ka) J'_\ell(ka), \tag{27}$$

$$k_\ell^{[\Theta_q]} = 2Nk^2\, J'_\ell(ka) J'_\ell(ka). \tag{28}$$

3.1.1 Dirichlet problem

For the Dirichlet problem, a nontrivial solution of $\{f\}$ in Eq.(12) implies

$$\det[U] = 0. \tag{29}$$

By employing the SVD technique, we can decompose the [U] matrix into

$$[U] = \Phi\Sigma_U\Phi^T = \Phi\begin{bmatrix} l_0^{[U]} & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & l_1^{[U]} & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & l_{-1}^{[U]} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & l_{N-1}^{[U]} & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & l_{-(N-1)}^{[U]} & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & l_N^{[U]} \end{bmatrix}_{2N\times 2N} \Phi^T, \qquad (30)$$

where Φ is the orthogonal matrix as shown below:

$$\Phi = \frac{1}{\sqrt{2N}}\begin{bmatrix} 1 & 1 & 0 & \cdots & 1 & 0 & 1 \\ 1 & \cos(\frac{2p}{2N}) & \sin(\frac{2p}{2N}) & \cdots & \cos(\frac{2p(N-1)}{2N}) & \sin(\frac{2p(N-1)}{2N}) & \cos(\frac{2pN}{2N} \\ 1 & \cos(\frac{4p}{2N}) & \sin(\frac{4p}{2N}) & \cdots & \cos(\frac{4p(N-1)}{2N}) & \sin(\frac{4p(N-1)}{2N}) & \cos(\frac{4pN}{2N} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & \cos(\frac{2p(2N-2)}{2N}) & \sin(\frac{2p(2N-2)}{2N}) & \cdots & \cos(\frac{p(4N-4)(N-1)}{2N}) & \sin(\frac{p(4N-4)(N-1)}{2N}) & \cos(\frac{p(4N-4}{2N} \\ 1 & \cos(\frac{2p(2N-1)}{2N}) & \sin(\frac{2p(2N-1)}{2N}) & \cdots & \cos(\frac{p(4N-2)(N-1)}{2N}) & \sin(\frac{p(4N-2)(N-1)}{2N}) & \cos(\frac{p(4N-2}{2N} \end{bmatrix} \qquad (31)$$

Thus, we can obtain the eigenequation $J_\ell(ka)J_\ell(ka) = 0$ by using the single-layer potential approach. Similarly, we can obtain $J'_\ell(ka)J_\ell(ka) = 0$ by using the double-layer potential approach.

3.1.2 Neumann problem

By the same way, we can also obtain the eigenequation $J_\ell(ka)J'_\ell(ka) = 0$ by using the single-layer potential (essential) approach and $J'_\ell(ka)J'_\ell(ka) = 0$ by using the double-layer (natural) potential approach for the Neumann problem. These results are shown in Table 1. Similarly, the approach can be extended to deal with 3-D acoustic eigenproblem. The results are shown in Table 2.

3.2 Plate problem

For the circular plate, the *U* kernels can be expressed in terms of degenerate kernels as shown below:

$$U(s,x) = \begin{cases} U^I(q,f) = \dfrac{1}{8l^2}\displaystyle\sum_{m=-\infty}^{\infty}[J_m(lR)J_m(lr) + (-1)^m I_m(lR)I_m(lr)]\cos(m(q-f)), & R > \\ U^E(q,f) = \dfrac{1}{8l^2}\displaystyle\sum_{m=-\infty}^{\infty}[J_m(lr)J_m(lR) + (-1)^m I_m(lr)I_m(lR)]\cos(m(q-f)), & R < \end{cases} \qquad (32)$$

where the I_m denotes the *m*th order modified Bessel function. The field points and source points are distributed on the boundary with a radius *a* ($R = r = a$) by matching the boundary condition. By using the degenerate kernel and the circulants, the eigenvalue of the matrix [*U*] can be obtained as follows:

$$k_\ell^{[U]} = \frac{N}{4l^2}[J_\ell(la)J_\ell(la) + (-1)^\ell I_\ell(la)I_\ell(la)], \tag{33}$$

where $\ell = 0, \pm 1, \pm 2, \cdots, \pm(N-1), N$. Similarly, the eigenvalues of $[T]$, [U?] and [T?] are obtained as follows:

$$k_\ell^{[\Theta]} = \frac{N}{4l}[J'_\ell(la)J_\ell(la) + (-1)^\ell I'_\ell(la)I_\ell(la)], \tag{34}$$

$$k_\ell^{[U_q]} = \frac{N}{4l}[J_\ell(la)J'_\ell(la) + (-1)^\ell I_\ell(la)I'_\ell(la)], \tag{35}$$

$$k_\ell^{[\Theta_q]} = \frac{N}{4}[J'_\ell(la)J'_\ell(la) + (-1)^\ell I'_\ell(la)I'_\ell(la)]. \tag{36}$$

By employing the SVD technique, the matrices $[U]$, $[T]$, [U?] and [T?] can be decomposed. We consider the U and T kernels as P and Q for the clamped case (u=0 and ? =0). Combination of Eq.(16) with Eq.(17) yields

$$\begin{bmatrix} U & \Theta \\ U_q & \Theta_q \end{bmatrix}_{4N\times 4N} \begin{Bmatrix} p \\ q \end{Bmatrix} = \{0\}. \tag{37}$$

In order to obtain the nontrivial solution, Eq.(37) can be reduced to

$$\det[SM_e]_{4N\times 4N} = \det\begin{bmatrix} U & \Theta \\ U_q & \Theta_q \end{bmatrix}_{4N\times 4N} = 0, \tag{38}$$

where the subscript "*e*" denotes the essential potential approach. By employing the SVD technique, Eq.(38) is decomposed as shown below:

$$[SM_e]_{4N\times 4N} = \begin{bmatrix} \Phi\Sigma_U\Phi^T & \Phi\Sigma_\Theta\Phi^T \\ \Phi\Sigma_{U_q}\Phi^T & \Phi\Sigma_{\Theta_q}\Phi^T \end{bmatrix}_{4N\times 4N}. \tag{39}$$

Equation (39) can be reformulated into

$$[SM_e] = \begin{bmatrix} \Phi & 0 \\ 0 & \Phi \end{bmatrix} \begin{bmatrix} \Sigma_U & \Sigma_\Theta \\ \Sigma_{U_q} & \Sigma_{\Theta_q} \end{bmatrix} \begin{bmatrix} \Phi & 0 \\ 0 & \Phi \end{bmatrix}^T. \tag{40}$$

Since F is orthogonal, the determinant of $[SM]_{4N\times 4N}$ is

$$\begin{aligned} \det[SM] &= \det\begin{bmatrix} \Sigma_U & \Sigma_\Theta \\ \Sigma_{U_q} & \Sigma_{\Theta_q} \end{bmatrix} \\ &= \prod_{\ell=-(N=1)}^{N} \frac{(-1)^\ell N^2}{16 l^2}[J'_\ell(la)I_\ell(la) - J_\ell(la)I'_\ell(la)] \\ &\qquad \{J'_\ell(la)I_\ell(la) - J_\ell(la)I'_\ell(la)\} = 0, \end{aligned} \tag{41}$$

for the clamped case. After using the differential property of Bessel function, Eq.(41) can be reduced to

$$[J_{\ell+1}(la)I_\ell(la)+J_\ell(la)I_{\ell+1}(la)]\{J_{\ell+1}(la)I_\ell(la)+J_\ell(la)I_{\ell+1}(la)\}=0, \quad \ell=0,\pm1,\pm2,\cdots,\pm(N-1),N. \tag{42}$$

By comparing with the exact solution [29], we can obtain that the zero term in the big bracket is the true eigenequation and the zero term in the middle bracket is the spurious eigenequation. It is interesting that the true and spurious eigenequations are the same. Similarly, the eigenequation of simply-supported and free boundary can be obtained as follows:

$$\begin{aligned}&[J_\ell(la)I_{\ell+1}(la)+I_\ell(la)J_{\ell+1}(la)]\\&\{(-1+n)(J_\ell(la)I_{\ell+1}(la)+J_\ell(la)I_{\ell+1}(la))+2laJ_\ell(la)I_\ell(la)\}=0,\\&\ell=0,\pm1,\pm2,\cdots,\pm(N-1),N,\end{aligned} \tag{43}$$

and

$$\begin{aligned}&[J_\ell(la)I_{\ell+1}(la)+I_\ell(la)J_{\ell+1}(la)]\\&\{(\ell^2(\ell^2-1)(-1+n)^2+l^4a^4)(J_{\ell+1}(la)I_\ell(la)+J_\ell(la)I_{\ell+1}(la))\\&+2\ell l^2a^2(1-\ell)(-1+n)(J_{\ell+1}(la)I_\ell(la)-J_\ell(la)I_{\ell+1}(la))\\&+la(-1+n)(2l^2a^2J_{\ell+1}(la)I_{\ell+1}(la)+4\ell^2(-1+\ell)J_\ell(la)I_\ell(la)\}=0,\\&\ell=0,\pm1,\pm2,\cdots,\pm(N-1),N,\end{aligned} \tag{44}$$

, respectively. It is found that the eigensolution of the clamped case is embedded in the Eqs.(43) and (44). After comparing with the exact solution [29], the present approach results in the spurious eigensolution $[J_\ell(la)I_{\ell+1}(la)+I_\ell(la)J_{\ell+1}(la)]=0$ for all the boundary conditions. At the same time, we obtain the true eigenequation

$$(-1+n)(J_\ell(la)I_{\ell+1}(la)+J_\ell(la)I_{\ell+1}(la))+2laJ_\ell(la)I_\ell(la)=0, \tag{45}$$

for the simply-supported case and

$$\begin{aligned}&(\ell^2(\ell^2-1)(-1+n)^2+l^4a^4)(J_{\ell+1}(la)I_\ell(la)+J_\ell(la)I_{\ell+1}(la))\\&+2\ell l^2a^2(1-\ell)(-1+n)(J_{\ell+1}(la)I_\ell(la)-J_\ell(la)I_{\ell+1}(la))\\&+la(-1+n)(2l^2a^2J_{\ell+1}(la)I_{\ell+1}(la)+4\ell^2(-1+\ell)J_\ell(la)I_\ell(la)=0,\end{aligned} \tag{46}$$

for the free case. We can choose different kernels for P and Q and obtain different results for the spurious eigenequations as shown in Table 3 and Table 4. By comparing the Table 1 with Table 3, we find that the occurrence of spurious eigenequation only depends on the formulation instead of the specified boundary condition and the true eigenequation depends on the specified boundary condition instead of the formulation. It is also found that spurious and true eigenequations are the same not only when using the essential potential approach for the problem of essential boundary conditions ($JJ=0$, $jj=0$ and $AA=0$) but also the natural potential approach for the problem of natural boundary conditions ($J'J'=0$, $j'j'=0$ and $BB=0$) as shown in Tables 1, 2 and 3.

4. Treatment of the spurious eigenvalues using singular value decomposition updating techniques for circular cases

In order to extract out the true eigenvalues, the singular value decomposition (SVD) updating technique is utilized.

4.1 Acoustic or membrane eigenproblem

For the 2-D acoustic cavity or membrane subject to the Dirichlet boundary condition, spurious eigenvalues appear when the double-layer potential approach is employed. In order to extract out the true eigenvalues and filter out the spurious ones, the SVD updating technique is utilized. By employing the double-layer potential approach, the Dirichlet problem can be formulated as shown below:

$$[\Theta]\{y\} = 0. \tag{47}$$

Since the imaginary-part formulation is incomplete in the solution representation, additional constraints are required to filter out the spurious eigenvalue. To provide the additional constraint, the single-layer potential approach can be formulated as

$$[U]\{f\} = 0. \tag{48}$$

To obtain an overdetermined system, Eqs.(47) and (48) are both required. By using the relation in the degenerate kernels between the direct method and the indirect method [24], the SVD updating term to extract out the true eigenequation (for the direct method) is equivalent to the SVD updating document (for the indirect method). We have

$$[C] = \begin{bmatrix} [\Theta]^T \\ [U]^T \end{bmatrix}, \tag{49}$$

where the superscript "*T*" denote the transpose. By using the SVD technique and least square smethod for the matrix [*C*], we can easily extract out the true eigenequation ($J_\ell(ka) = 0$) for the Dirichlet problem. Similarly, we can also find the true eigenequation ($J'_\ell(ka) = 0$) for the Neumann problem. This approach can also be extended to deal with the 3-D acoustic eigenproblem. The true eigenequations, $j_\ell(ka) = 0$ for the Dirichlet case and $j'_\ell(ka) = 0$ for the Neumann case, are obtained.

4.2 Plate eigenproblem

In Table 4, we can find that the spurious eigenvalues occur when we use the *M* and *V* kernels as *P* and *Q* for the clamped boundary condition. The formulation is shown below:

$$[SM_n]\begin{Bmatrix} p' \\ q' \end{Bmatrix} = \begin{bmatrix} M & V \\ M_q & V_q \end{bmatrix}\begin{Bmatrix} p' \\ q' \end{Bmatrix} = \{0\}, \tag{50}$$

where the subscript "*n*" denotes the nature potential approach and p' and q' are the unknown coefficients corresponding to the *M* and *V* kernels, respectively. Similarly, the additional constraint by using the *U* and T kernels as *P* and *Q* is considered. The combined matrix is shown below:

$$[C]=\begin{bmatrix}(SM_n)^T\\(SM_e)^T\end{bmatrix}. \tag{51}$$

By using the SVD technique and least squares method, we can obtain the true eigenequation ($J_{\ell+1}(la)I_\ell(la)+J_\ell(la)I_{\ell+1}(la)=0$) for the clamped boundary condition. This indicates that only the true eigenvalues of the clamped circular plate is imbedded in the SVD updating matrix. Similarly, the true eigenequations for the simply-supported plate and free plate are also obtained respectively as follows:

$$(-1+n)(J_\ell(la)I_{\ell+1}(la)+J_\ell(la)I_{\ell+1}(la))+2laJ_\ell(la)I_\ell(la)=0 \tag{52}$$

for the simply-supported plate and

$$\begin{aligned}&(\ell^2(\ell^2-1)(-1+n)^2+l^4a^4)(J_{\ell+1}(la)I_\ell(la)+J_\ell(la)I_{\ell+1}(la))\\&+2\ell l^2a^2(1-\ell)(-1+n)(J_{\ell+1}(la)I_\ell(la)-J_\ell(la)I_{\ell+1}(la))\\&+la(-1+n)(2l^2a^2J_{\ell+1}(la)I_{\ell+1}(la)+4\ell^2(-1+\ell)J_\ell(la)I_\ell(la)=0\end{aligned} \tag{53}$$

for the free plate.

5. Numerical results

5.1 Acoustic or membrane eigenproblem

A 2-D circular cavity or membrane with a radius 1 *m* is considered. Fig.1(a) shows the minimum singular value s_1 versus the wave number *k* for the Dirichlet problem by using the single-layer potential (essential potential) approach. No spurious eigenvalue appears since the true and spurious eigenvalues ($J_n(k)=0$) are the same. Fig.1(b) shows the minimum singular value s_1 versus the wave number *k* for the Dirichlet problem by using the double-layer potential (natural potential) approach. As predicted analytically, the spurious eigenvalues ($J'_n(k)=0$) appear in Fig.1(b). The SVD updating technique is employed to extract out the true eigenvalues when using the double-layer potential approach as shown in Fig.1(c). After the treatment of using SVD updating technique, the drop of the spurious eigenvalues disappears. Good agreement is made, only the true eigenvalues ($J_n(k)=0$) are obtained.

For the 3-D acoustic cavity, the similar results are shown in Fig.2(a)-2(c). Fig.2(a) shows the minimum singular value s_1 versus the wave number *k* for the Dirichlet problem by using the single-layer potential approach. No spurious eigenvalue appears since the true and spurious eigenvalues ($j_n(k)=0$) are the same. Fig.2(b) shows the minimum singular value s_1 versus the wave number *k* for the Dirichlet approach by using the double-layer potential approach. As predicted analytically, the spurious eigenvalues ($j'_n(k)=0$) appear in Fig.2(b). When the SVD updating technique is used, we can also find that the spurious eigenvalues disappear in Fig.2(c). Good agreement is made, only the true eigenvalues ($j_n(k)=0$) are obtained.

5.2 Plate eigenproblem

A circular plate with a radius 1 *m* and the Poisson ratio $n = 1/3$ is considered. Fig.3(a) shows the determinant of the versus the frequency parameter *?* for the circular clamped plate by using the U and T kernels. No spurious eigenvalue appears since the true and spurious eigenvalues ($A = 0$) are the same. Fig.3(b) shows the determinant versus the frequency parameter *?* for the circular clamped plate by using M and V kernels. As predicted analytically, the spurious eigenvalues ($B = 0$) appear in Fig.3(b). Similarly, the SVD updating technique was used to extract out the true eigenvalues for the clamped plate eigenproblem in Fig.3(c). We can find that no spurious eigenvalue occurs. Good agreement is made, only the true eigenvalues ($A = 0$) are obtained.

6. Conclusions

We have developed a meshless method for the membrane, acoustic and plate eigenproblem by using the imaginary-part kernel, which was chosen as the RBF to represent the solution. Neither boundary elements nor singularities are required. It is interesting to find that the true spurious eigensolution is contaminated by spurious one. We also find that the spurious eigenequation only depends on the formulation instead of the specified boundary condition and the true eigenequation depends on the specified boundary condition instead of the formulation. In order to extract out the true eigenequation, the SVD technique was successfully utilized to overcome the problem of spurious eigenvalues. Although only circular, spherical cavities and plate was treated in the present approach, the same algorithm in the discrete system can be applied to solve arbitrary-shaped acoustic cavity and plate numerically without any difficulty.

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Table 1 The true and spurious eigenequations of 2-D acoustic problem by using the single-layer and double-layer potential approaches

	u = 0 (Dirichlet)	**t = 0 (Neumann)**
Essential potential (Single-layer potential)	$[J_\ell(la)]\{J_\ell(la)\}=0$	$[J_\ell(la)]\{J'_\ell(la)\}=0$
Natural potential (Double-layer potential)	$[J'_\ell(la)]\{J_\ell(la)\}=0$	$[J'_\ell(la)]\{J'_\ell(la)\}=0$

Table 2 The true and spurious eigenequations of 3-D acoustic problem by using the single-layer and double-layer potential approaches

	u = 0 (Dirichlet)	**t = 0 (Neumann)**
Essential potential (Single-layer potential)	$[j_\ell(la)]\{j_\ell(la)\}=0$	$[j_\ell(la)]\{j'_\ell(la)\}=0$
Natural potential (Double-layer potential)	$[j'_\ell(la)]\{j_\ell(la)\}=0$	$[j'_\ell(la)]\{j'_\ell(la)\}=0$

where j_m and j'_m are the *m*-th order spherical Bessel functions of the first kind and its derivative, respectively.

Table 3 True and spurious eigenequations for the circular plate by using the essential and natural potential approaches

	$u=0$ **and** $q=0$ **(Clamped boundary)**	$m=0$ **and** $v=0$ **(Free boundary)**
Essential formulation (*U* and Θ formulation)	$[A]\{A\}=0$	$[A]\{B\}=0$
Natural formulation (*M* and *V* formulation)	$[B]\{A\}=0$	$[B]\{B\}=0$

where $A=J_{\ell+1}(la)I_\ell(la)+J_\ell(la)I_{\ell+1}(la)$ and

$$
\begin{aligned}
B=&(\ell^2(\ell^2-1)(-1+n)^2+l^4a^4)(J_{\ell+1}(l\,a)I_\ell(l\,a)+J_\ell(l\,a)I_{\ell+1}(l\,a))\\
&+2\ell l^2a^2(1-\ell)(-1+n)(J_{\ell+1}(l\,a)I_\ell(l\,a)-J_\ell(l\,a)I_{\ell+1}(l\,a))\\
&+l\,a(-1+n)(2l^2a^2J_{\ell+1}(l\,a)I_{\ell+1}(l\,a)+4\ell^2(-1+\ell)J_\ell(l\,a)I_\ell(l\,a)=0,
\end{aligned}
$$

The term inside [ ] and { } mean the spurious and true eigenequation, respectively.

Table 4 True and spurious eigenequations for the clamped, simply-supported and free plates using the imaginary-part fundamental solution

		Clamped boundary	**Simply-supported boundary**	**Free boundary**
U T	**True**	$J_{\ell+1}(l\,a)I_\ell(l\,a)+J_\ell(l\,a)I_{\ell+1}(l\,a)=0$	$\frac{J_{\ell+1}(l\,a)}{J_\ell(l\,a)}+\frac{I_{\ell+1}(l\,a)}{I_\ell(l\,a)}=\frac{2l\,a}{1-n}$	*
	Spurious	$J_{\ell+1}(l\,a)I_\ell(l\,a)+J_\ell(l\,a)I_{\ell+1}(l\,a)=0$	$J_{\ell+1}(l\,a)I_\ell(l\,a)+J_\ell(l\,a)I_{\ell+1}(l\,a)=0$	$J_{\ell+1}(l\,a)I_\ell(l\,a)+J_\ell(l\,a)I_{\ell+1}(l\,a)=0$
M V	**True**	$J_{\ell+1}(l\,a)I_\ell(l\,a)+J_\ell(l\,a)I_{\ell+1}(l\,a)=0$	$\frac{J_{\ell+1}(l\,a)}{J_\ell(l\,a)}+\frac{I_{\ell+1}(l\,a)}{I_\ell(l\,a)}=\frac{2l\,a}{1-n}$	*
	Spurious	*	*	*
U M	**True**	$J_{\ell+1}(l\,a)I_\ell(l\,a)+J_\ell(l\,a)I_{\ell+1}(l\,a)=0$	$\frac{J_{\ell+1}(l\,a)}{J_\ell(l\,a)}+\frac{I_{\ell+1}(l\,a)}{I_\ell(l\,a)}=\frac{2l\,a}{1-n}$	*
	Spurious	$\frac{J_{\ell+1}(l\,a)}{J_\ell(l\,a)}+\frac{I_{\ell+1}(l\,a)}{I_\ell(l\,a)}=\frac{2l\,a}{1-n}$	$\frac{J_{\ell+1}(l\,a)}{J_\ell(l\,a)}+\frac{I_{\ell+1}(l\,a)}{I_\ell(l\,a)}=\frac{2l\,a}{1-n}$	$\frac{J_{\ell+1}(l\,a)}{J_\ell(l\,a)}+\frac{I_{\ell+1}(l\,a)}{I_\ell(l\,a)}=\frac{2l\,a}{1-n}$
U V	**True**	$J_{\ell+1}(l\,a)I_\ell(l\,a)+J_\ell(l\,a)I_{\ell+1}(l\,a)=0$	$\frac{J_{\ell+1}(l\,a)}{J_\ell(l\,a)}+\frac{I_{\ell+1}(l\,a)}{I_\ell(l\,a)}=\frac{2l\,a}{1-n}$	*
	Spurious	$\ell^2(-1+n)(J_{\ell+1}(l\,a)I_\ell(l\,a)+J_\ell(l\,a)I_{\ell+1}(l\,a))$ $-l^2r^2(J_{\ell+1}(l\,a)I_\ell(l\,a)-J_\ell(l\,a)I_{\ell+1}(l\,a)$ $=-2\ell\,l\,aJ_\ell(l\,a)I_\ell(l\,a)$	$\ell^2(-1+n)(J_{\ell+1}(l\,a)I_\ell(l\,a)+J_\ell(l\,a)I_{\ell+1}(l\,a))$ $-l^2r^2(J_{\ell+1}(l\,a)I_\ell(l\,a)-J_\ell(l\,a)I_{\ell+1}(l\,a)$ $=-2\ell\,l\,aJ_\ell(l\,a)I_\ell(l\,a)$	$\ell^2(-1+n)(J_{\ell+1}(l\,a)I_\ell(l\,a)+J_\ell(l\,a)I_{\ell+1}(l\,a))$ $-l^2r^2(J_{\ell+1}(l\,a)I_\ell(l\,a)-J_\ell(l\,a)I_{\ell+1}(l\,a)$ $=-2\ell\,l\,aJ_\ell(l\,a)I_\ell(l\,a)$
T M	**True**	$J_{\ell+1}(l\,a)I_\ell(l\,a)+J_\ell(l\,a)I_{\ell+1}(l\,a)=0$	$\frac{J_{\ell+1}(l\,a)}{J_\ell(l\,a)}+\frac{I_{\ell+1}(l\,a)}{I_\ell(l\,a)}=\frac{2l\,a}{1-n}$	*
	Spurious	$\ell^2(-1+n)(J_{\ell+1}(l\,a)I_\ell(l\,a)+J_\ell(l\,a)I_{\ell+1}(l\,a))$ $-l^2r^2(J_{\ell+1}(l\,a)I_\ell(l\,a)-J_\ell(l\,a)I_{\ell+1}(l\,a)$ $=-2\ell\,l\,aJ_\ell(l\,a)I_\ell(l\,a)$	$\ell^2(-1+n)(J_{\ell+1}(l\,a)I_\ell(l\,a)+J_\ell(l\,a)I_{\ell+1}(l\,a))$ $-l^2r^2(J_{\ell+1}(l\,a)I_\ell(l\,a)-J_\ell(l\,a)I_{\ell+1}(l\,a)$ $=-2\ell\,l\,aJ_\ell(l\,a)I_\ell(l\,a)$	$\ell^2(-1+n)(J_{\ell+1}(l\,a)I_\ell(l\,a)+J_\ell(l\,a)I_{\ell+1}(l\,a))$ $-l^2r^2(J_{\ell+1}(l\,a)I_\ell(l\,a)-J_\ell(l\,a)I_{\ell+1}(l\,a)$ $=-2\ell\,l\,aJ_\ell(l\,a)I_\ell(l\,a)$

TV	**True**	$J_{\ell+1}(l\,a)I_{\ell}(l\,a)+J_{\ell}(l\,a)I_{\ell+1}(l\,a)=0$	$\frac{J_{\ell+1}(l\,a)}{J_{\ell}(l\,a)}+\frac{I_{\ell+1}(l\,a)}{I_{\ell}(l\,a)}=\frac{2l\,a}{1-n}$	*
	Spurious	$\ell^2(-1+n)(J_{\ell+1}(l\,a)I_{\ell}(l\,a)+J_{\ell}(l\,a)I_{\ell+1}(l\,a)) -2\ell l^2a^2(J_{\ell+1}(l\,a)I_{\ell}(l\,a)-J_{\ell}(l\,a)I_{\ell+1}(l\,a)) +2l\,a(l^2a^2J_{\ell+1}(l\,a)I_{\ell+1}(l\,a)-\ell^2J_{\ell}(l\,a)I_{\ell}(l\,a))=0$	$\ell^2(-1+n)(J_{\ell+1}(l\,a)I_{\ell}(l\,a)+J_{\ell}(l\,a)I_{\ell+1}(l\,a)) -2\ell l^2a^2(J_{\ell+1}(l\,a)I_{\ell}(l\,a)-J_{\ell}(l\,a)I_{\ell+1}(l\,a)) +2l\,a(l^2a^2J_{\ell+1}(l\,a)I_{\ell+1}(l\,a)-\ell^2J_{\ell}(l\,a)I_{\ell}(l\,a))=0$	$\ell^2(-1+n)(J_{\ell+1}(l\,a)I_{\ell}(l\,a)+J_{\ell}(l\,a)I_{\ell+1}(l\,a)) -2\ell l^2a^2(J_{\ell+1}(l\,a)I_{\ell}(l\,a)-J_{\ell}(l\,a)I_{\ell+1}(l\,a)) +2l\,a(l^2a^2J_{\ell+1}(l\,a)I_{\ell+1}(l\,a)-\ell^2J_{\ell}(l\,a)I_{\ell}(l\,a))=0$

*
$$(\ell^2(\ell^2-1)(-1+n)^2+l^4a^4)(J_{\ell+1}(l\,a)I_{\ell}(l\,a)+J_{\ell}(l\,a)I_{\ell+1}(l\,a))+2\ell l^2a^2(1-\ell)(-1+n)(J_{\ell+1}(l\,a)I_{\ell}(l\,a)-J_{\ell}(l\,a)I_{\ell+1}(l\,a))$$
$$+\,l\,a(-1+n)(2l^2a^2J_{\ell+1}(l\,a)I_{\ell+1}(l\,a)+4\ell^2(-1+\ell)J_{\ell}(l\,a)I_{\ell}(l\,a))=0$$

2-D circular cavity | 3-D spherical cavity | Clamped circular plate

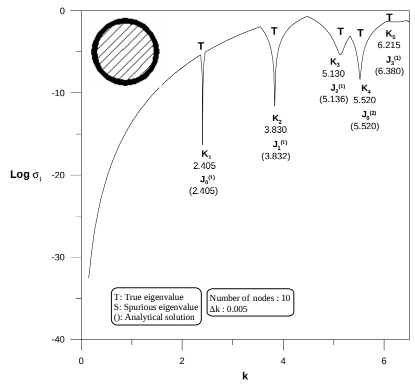

Fig.1(a)

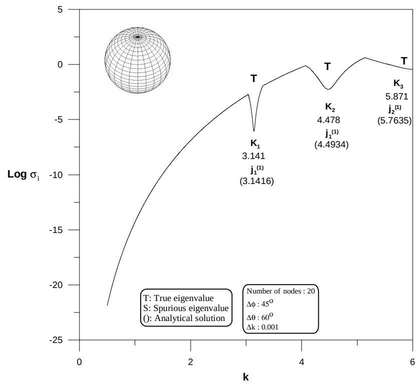

Fig.2(a)

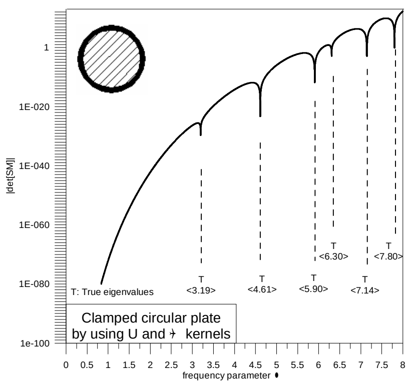

Fig.3(a)

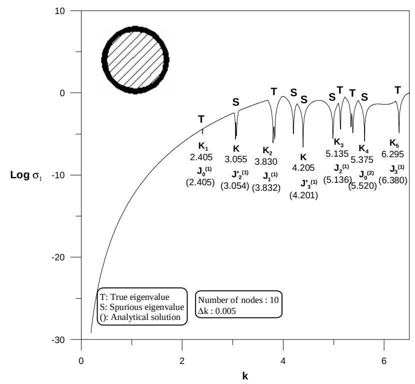

Fig.1(b)

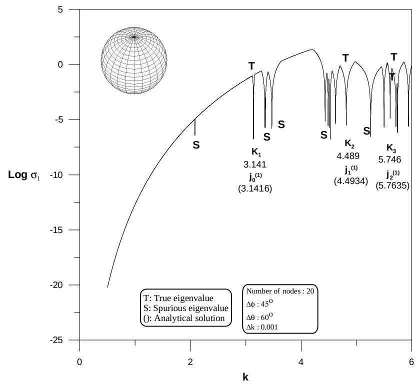

Fig.2(b)

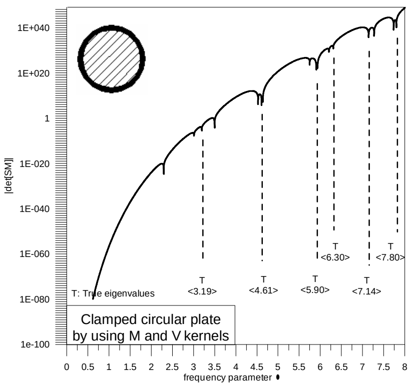

Fig.3(b)

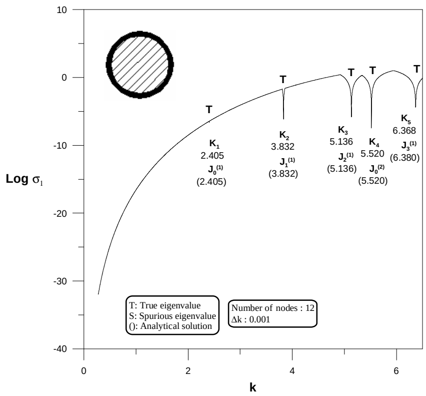

Fig.1(c)

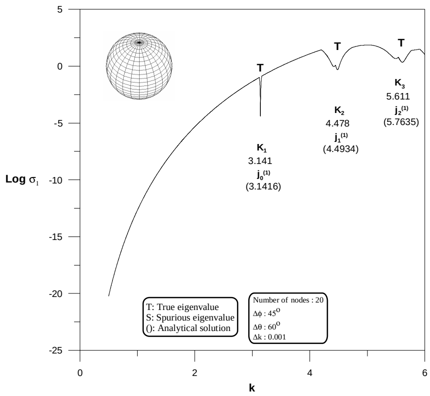

Fig.3(c)

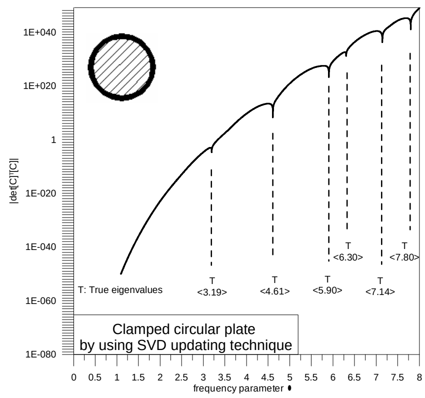

Fig.3(c)

Fig.1 The first minimum singular value versus the wave number for the 2-D circular cavity eigenproblem.

Fig.2 The first minimum singular value versus the wave number for the 3-D spherical cavity eigenproblem.

Fig.3 The determinant versus the frequency parameter for the clamped plate eigenproblem.

(a) Essential potential approach for the problem of the essential boundary condition.

(b) Natural potential approach for the problem of the essential boundary condition.

(c) Extraction of true eigenvalues of (b) by using the SVD updating technique.