On the equivalence of MFS and Trefftz method for Laplace problems

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Abstract

In this paper, it is proved that the two approaches for Laplace problems, known in the literature as the method of fundamental solutions (MFS) and the Trefftz method, are mathematically equivalent in spite of their essentially minor and apparent differences in the formulation. It is interesting to find that the T-complete set in the Trefftz method for the interior and exterior problems are imbedded in the degenerate kernels of MFS. By designing circular-domain and circular-hole problems, the unknown coefficients of each method correlate by a mapping matrix after considering the degenerate kernels for the fundamental solutions in the MFS and the T-complete function in the Trefftz method. The mapping matrix is composed of a rotation matrix and a geometric matrix depends on the source location. The degenerate scale for the Laplace equation appears using the MFS when the geometric matrix is singular. The occurring mechanism of the degenerate scale in the MFS is studied by using circulants. The ill-posed problem in the MFS also stems from the ill-conditioned geometric matrix when the source is distributed far away from the real boundary. Several examples, interior and exterior problems with either simply- or doubly-connected domain, were solved by using the Trefftz method and the MFS. The comparison of efficiency between the two methods was addressed.

Keywords: method of fundamental solutions (MFS), Trefftz method, T-complete set, degenerate kernels, mapping matrix, degenerate scale, ill-posed problem

1. Introduction

In 1926, Trefftz presented the Trefftz method for solving boundary value problems by superimposing the functions satisfying the governing equation, although various versions of Trefftz method, e.g., direct formulation and indirect formulations have been developed. The unknown coefficients are determined by matching the boundary condition. Many applications to the Helmholtz equation [3], the Navier equation [7,14] and the biharmonic equation [8] were done. Until recent years, the ill-posed nature in the method was noticed [1].

In the potential theory, it is well known that the method of fundamental solutions (MFS) can solve potential problems when a fundamental solution is known. This method was attributed to Kupradze (1964) [14] in Russia, extensive applications in solving a broad range of problems such as potential problems [3,9], acoustics [11], biharmonic problems [8] have been found. The MFS can be reviewed as an indirect boundary element method with concentrated source instead of distribution. The initial idea is to approximate the solution through a linear combination of fundamental solution with source located outside the domain of the problem. Moreover, it has certain advantages over BEM, e.g., no singularity and no boundary integrals are required. However, ill-posed behavior is inherent in the regular formulation. It can also be applied to acoustics [3], elasticity [7,14] and plate problems [8].

However, the link between the Trefftz method and the MFS was not discussed in detail to the authors' best knowledge. A similar case to link the DRBEM and the method of particular integral was done by Polyzos *et al.* [12]. In this paper, we solve the interior and exterior Laplace problems with a circular boundary and prove the mathematical equivalence between the Trefftz method and the MFS. Two mathematical tools are required. One is degenerate kernel for the expansion of the closed-form fundamental solution, the other is the Fourier series expansion for the boundary density. Circulants is employed to study the degenerate scale in the MFS. The ill-posed behavior of the MFS is addressed. Also, the efficiency between the Trefftz method and MFS is compared with each other under the same number of degrees of freedom.

2. Connection between the Trefftz method and the MFS for interior and exterior Laplace problems

Consider the two-dimensional Laplace problem with circular-domain (interior problem) or circular-hole (exterior problem) of radius a as shown in Fig.1 (a) and (b), respectively. The governing equation of the boundary value problem is the Laplace equation,

$$\nabla^2 u(x) = 0, \quad x \in D \tag{1}$$

where D is the domain, ∇^2 denotes the Laplacian operator and $u(x)$ is the potential function. The boundary condition is given by the Dirichlet type

$$u(x) = \bar{u}, \quad x \in B \tag{2}$$

By using the Fourier series expansion, the boundary condition $u(x)$ can be expressed as

$$u(a,f) = \bar{a}_0 + \sum_{n=1}^{N} \bar{a}_n \cos(nf) + \sum_{n=1}^{N} \bar{b}_n \sin(nf) \tag{3}$$

where $\bar{a}_0, \bar{a}_n, \bar{b}_n$ are the Fourier coefficients with respect to Fourier bases, $\cos(nf)$ and $\sin(nf)$, and f is the angle along the circular boundary.

2.1 Trefftz method

In the Trefftz method, the field solution $u(x)$ is superimposed by the T-complete functions, $u_j(x)$, as follows:

$$u(x) = \sum_{j=1}^{2N_T+1} w_j u_j(x) \tag{4}$$

where $2N_T+1$ is the number of T-complete functions, w_j is the unknown coefficient, $u_j(x)$ is the complementary set which satisfies the Laplace equation. For the interior problem, we choose 1, $r^n \sin(nf)$ and $r^n \cos(nf)$, $(n \in N \text{ and } 0 < r < a)$ to be the bases of the complementary set; and for the exterior problem we choose $\ln r$, $r^{-n}\sin(nf)$ and $r^{-n}\cos(nf)$ $(n \in N \text{ and } a < r < \infty)$ to be the T-complete functions in two-dimensional problem. Eq.(4) can be expressed by

$$u^{I}(r,f) = a_0 + \sum_{n=1}^{N_T} a_n r^n \cos(nf) + \sum_{n=1}^{N_T} b_n r^n \sin(nf),\ 0 < r < a \tag{5}$$

$$u^{E}(r,f) = a_0 \ln r + \sum_{n=1}^{N_T} a_n r^{-n} \cos(nf) + \sum_{n=1}^{N_T} b_n r^{-n} \sin(nf),\ a < r < \infty \tag{6}$$

where the superscripts '*I*' and '*E*' denote the interior and exterior problems, respectively. By matching the boundary condition at $r = a$, we have

$$u^{I}(a,f) = a_0 + \sum_{n=1}^{N_T} a_n a^n \cos(nf) + \sum_{n=1}^{N_T} b_n a^n \sin(nf). \tag{7}$$

$$u^{E}(a,f) = a_0 \ln a + \sum_{n=1}^{N_T} a_n a^{-n} \cos(nf) + \sum_{n=1}^{N_T} b_n a^{-n} \sin(nf). \tag{8}$$

After comparing the Eq.(3) with Eq.(7) for the interior problem, we obtain

$$a_0 = \bar{a}_0, \tag{9}$$

$$a_n = \frac{1}{a^n}\bar{a}_n, \qquad n = 1,2,...,N_T \tag{10}$$

$$b_n = \frac{1}{a^n}\bar{b}_n \qquad n = 1,2,...,N_T \tag{11}$$

For the exterior problem by using Eq.(8), we have

$$a_0 = \frac{1}{\ln a}\bar{a}_0, \tag{12}$$

$$a_n = a^n \bar{a}_n, \qquad n = 1,2,...,N_T \tag{13}$$

$$b_n = a^n \bar{b}_n, \qquad n = 1,2,...,N_T. \tag{14}$$

For the exterior problem, with the radius of $a = 1$, it is interesting to find the nonunique solution occurs since a_0 can not be determined in Eq.(12) by using the Trefftz method.

2.2 Method of fundamental solutions (MFS)

In the method of fundamental solutions, the field solution $u(x)$ can be superimposed by $U(x,s_j)$ as follows:

$$u(x) = \sum_{j=1}^{N_M} c_j U(x,s_j), \quad s_j \in D^e \tag{15}$$

where N_M is the number of source points in the MFS, c_j is the unknown coefficient, s and x are the source point and collocation point, respectively, D^e is the complementary domain and $U(x,s_j)$ is the fundamental solution with the symmetry property

$$U(x,s_j) = U(s_j,x) = \ln r. \tag{16}$$

where $r = \left|x - s_j\right|$.

In order to match Eq.(16), we have

$$u(x) = \sum_{j=1}^{N_M} c_j U(s_j, x), \quad s_j \in D^e. \tag{17}$$

The fundamental solution can be expressed by using the degenerate kernel

$$U(R,q;r,f) = \begin{cases} U^i(R,q;r,f) = \ln(R) - \sum_{n=1}^{\infty} \frac{1}{n}(\frac{r}{R})^n \cos(n(q-f)), & R > r \\ U^e(R,q;r,f) = \ln(r) - \sum_{n=1}^{\infty} \frac{1}{n}(\frac{R}{r})^n \cos(n(q-f)), & R < r \end{cases} \tag{18}$$

where the superscripts " i " and " e " denote the interior expression ($R > r$) and the exterior expression ($R < r$), $s = (R,q)$ and $x = (r,f)$ are the polar coordinates of s and x, respectively. By substituting the Eq.(18) into the kernel function

$$u^I(r,f) = \sum_{j=1}^{N_M} c_j [\ln(R) - \sum_{n=1}^{\infty} \frac{1}{n}(\frac{r}{R})^n \cos(n(q_j - f))], \quad 0 < r < a, \quad 0 < f < 2p \tag{19}$$

$$u^E(r,f) = \sum_{j=1}^{N_M} c_j [\ln(r) - \sum_{n=1}^{\infty} \frac{1}{n}(\frac{R}{r})^n \cos(n(q_j - f))] \quad a < r < \infty, \quad 0 < f < 2p \tag{20}$$

where $q_j = \frac{2p}{N_M} j$. Eqs.(19) and (20) in the MFS imply the T-complete set of Eqs.(7) and (8) in the Trefftz method for the interior and exterior problems, respectively. By employing the property of trigonometric function, Eq.(19) and Eq.(20) can be rewritten as

$$u^I(r,f) = \sum_{j=1}^{N_M} c_j \ln(R) - \sum_{n=1}^{\infty}[\sum_{j=1}^{N_M} c_j \frac{1}{n}(\frac{r}{R})^n \cos(nq_j)]\cos(nf) - \sum_{n=1}^{\infty}[\sum_{j=1}^{N_M} c_j \frac{1}{n}(\frac{r}{R})^n \sin(nq_j)]\sin(nf) \tag{21}$$

$$u^E(r,f) = \sum_{j=1}^{N_M} c_j \ln(r) - \sum_{n=1}^{\infty}[\sum_{j=1}^{N_M} c_j \frac{1}{n}(\frac{R}{r})^n \cos(nq_j)]\cos(nf) - \sum_{n=1}^{\infty}[\sum_{j=1}^{N_M} c_j \frac{1}{n}(\frac{R}{r})^n \sin(nq_j)]\sin(nf) \tag{22}$$

After comparing the Eq.(3) with Eq.(21) by truncating the higher order terms to N_M and matching the boundary condition, the interior problem yields:

$$\bar{a}_0 = \sum_{j=1}^{N_M} c_j \ln(R) \tag{23}$$

$$\frac{\bar{a}_n}{a^n} = -\sum_{j=1}^{N_M} c_j \frac{1}{n}(\frac{1}{R})^n \cos(nq_j), \quad n = 1,2,\ldots, N_M \tag{24}$$

$$\frac{\bar{b}_n}{a^n} = -\sum_{j=1}^{N_M} c_j \frac{1}{n}(\frac{1}{R})^n \sin(nq_j), \quad n = 1,2,\ldots, N_M. \tag{25}$$

For the exterior problem, we have

$$\frac{1}{\ln a}\bar{a}_0 = \sum_{j=1}^{N_M} c_j \tag{26}$$

$$a^n \bar{a}_n = -\sum_{j=1}^{N_M} c_j \frac{1}{n} (R)^n \cos(nq_j), \quad n = 1,2,..., N_M \tag{27}$$

$$a^n \bar{b}_n = -\sum_{j=1}^{N_M} c_j \frac{1}{n} (R)^n \sin(nq_j), \quad n = 1,2,..., N_M \tag{28}$$

2.3 Connection between the Trefftz method and MFS

We can compare the coefficients in the Trefftz method and in the MFS for interior and exterior problems. By setting $2N_T + 1 = N_M = 2N + 1$ under the request of the same number of degrees of freedom, the relationship between the unknown coefficients in the Trefftz method and the MFS can be written as

Interior problem:

$$a_0 = \sum_{j=1}^{2N+1} c_j \ln(R) \tag{29}$$

$$a_n = -\sum_{j=1}^{2N+1} c_j \frac{1}{n} (\frac{1}{R})^n \cos(nq_j), \quad n = 1,2,...2N+1 \tag{30}$$

$$b_n = -\sum_{j=1}^{2N+1} c_j \frac{1}{n} (\frac{1}{R})^n \sin(nq_j), \quad n = 1,2,...2N+1 \tag{31}$$

Exterior problem:

$$a_0 = \sum_{j=1}^{2N+1} c_j \tag{32}$$

$$a_n = -\sum_{j=1}^{2N+1} c_j \frac{1}{n} (R)^n \cos(nq_j), \quad n = 1,2,...2N+1 \tag{33}$$

$$b_n = -\sum_{j=1}^{2N+1} c_j \frac{1}{n} (R)^n \sin(nq_j), \quad n = 1,2,...2N+1 \tag{34}$$

After comparing the two solutions for the Trefftz method and the MFS, we can correlate as

$$\left\{ \underset{\sim}{u} \right\} = [K] \left\{ \underset{\sim}{v} \right\} \tag{35}$$

where

$$\begin{aligned} \underset{\sim}{u} &= \{a_0 \quad a_1 \quad b_1 \quad a_2 \quad b_2 \quad \cdots \quad a_N \quad b_N\}^T \\ \underset{\sim}{v} &= \{c_1 \quad c_2 \quad c_3 \quad c_4 \quad c_5 \quad \cdots \quad c_{2N} \quad c_{2N+1}\}^T \end{aligned} \tag{36}$$

and the matrix $[K]$ for the interior case is

$$
\left[K^{I}\right]=\begin{bmatrix}
\ln R & \ln R & \ln R & \cdots & \ln R \\
\frac{-1}{R}\cos(q_1) & \frac{-1}{R}\cos(q_2) & \frac{-1}{R}\cos(q_3) & \cdots & \frac{-1}{R}\cos(q_{2N+1}) \\
\frac{-1}{R}\sin(q_1) & \frac{-1}{R}\sin(q_2) & \frac{-1}{R}\sin(q_3) & \cdots & \frac{-1}{R}\sin(q_{2N+1}) \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\frac{-1}{N}(\frac{1}{R})^{N}\cos(Nq_1) & \frac{-1}{N}(\frac{1}{R})^{N}\cos(Nq_2) & \frac{-1}{N}(\frac{1}{R})^{N}\cos(Nq_3) & \cdots & \frac{-1}{N}(\frac{1}{R})^{N}\cos(Nq_{2N+1}) \\
\frac{-1}{N}(\frac{1}{R})^{N}\sin(Nq_1) & \frac{-1}{N}(\frac{1}{R})^{N}\sin(Nq_2) & \frac{-1}{N}(\frac{1}{R})^{N}\sin(Nq_3) & \cdots & \frac{-1}{N}(\frac{1}{R})^{N}\sin(Nq_{2N+1})
\end{bmatrix} \tag{37}
$$

and

$$
\left[K^{E}\right]=\begin{bmatrix}
1 & 1 & 1 & \cdots & 1 \\
-R\cos(q_1) & -R\cos(q_2) & \frac{-1}{R}\cos(q_3) & \cdots & -R\cos(q_{2N+1}) \\
-R\sin(q_1) & -R\sin(q_2) & \frac{-1}{R}\sin(q_3) & \cdots & -R\sin(q_{2N+1}) \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\frac{-1}{N}(R)^{N}\cos(Nq_1) & \frac{-1}{N}(R)^{N}\cos(Nq_2) & \frac{-1}{N}(R)^{N}\cos(Nq_3) & \cdots & \frac{-1}{N}(R)^{N}\cos(Nq_{2N+1}) \\
\frac{-1}{N}(R)^{N}\sin(Nq_1) & \frac{-1}{N}(R)^{N}\sin(Nq_2) & \frac{-1}{N}(R)^{N}\sin(Nq_3) & \cdots & \frac{-1}{N}(R)^{N}\sin(Nq_{2N+1})
\end{bmatrix} \tag{38}
$$

is for the exterior case. The relation of Eq.(35) was obtained to connect the Trefftz method and the MFS. We can decompose the matrix [K] into two parts, one is the matrix, $[T_R]$, which depends on the radius of the source distribution; the other is the matrix, $[T_q]$, which depends on the angle of the source point (Fig.1), as follows:

$$
[K]=[T_R][T_q] \tag{39}
$$

$$
[T_R]^{I}=\begin{bmatrix}
\ln(R) & 0 & 0 & 0 & \cdots & \cdots & 0 & 0 \\
0 & \frac{-1}{R} & 0 & 0 & \cdots & \cdots & 0 & 0 \\
0 & 0 & \frac{-1}{R} & 0 & \cdots & \cdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \frac{-1}{2}(\frac{1}{R})^{2} & \cdots & \cdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \frac{-1}{2}(\frac{1}{R})^{2} & \cdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \cdots & \ddots & 0 & 0 \\
0 & 0 & 0 & \cdots & \cdots & \cdots & \frac{-1}{N}(\frac{1}{R})^{N} & 0 \\
0 & 0 & 0 & \cdots & \cdots & \cdots & 0 & \frac{-1}{N}(\frac{1}{R})^{N}
\end{bmatrix}_{(2N+1)\times(2N+1)} \tag{40}
$$

is for the interior problem and

$$
[T_R]^{E}=\begin{bmatrix}
1 & 0 & 0 & 0 & \cdots & \cdots & 0 & 0 \\
0 & -R & 0 & 0 & \cdots & \cdots & 0 & 0 \\
0 & 0 & -R & 0 & \cdots & \cdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \frac{-1}{2}(R)^{2} & \cdots & \cdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \frac{-1}{2}(R)^{2} & \cdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \cdots & \ddots & 0 & 0 \\
0 & 0 & 0 & \cdots & \cdots & \cdots & \frac{-1}{N}(R)^{N} & 0 \\
0 & 0 & 0 & \cdots & \cdots & \cdots & 0 & \frac{-1}{N}(R)^{N}
\end{bmatrix}_{(2N+1)\times(2N+1)} \tag{41}
$$

is for the exterior problem

$$[T_q] = \begin{bmatrix} 1 & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & 1 \\ \cos(q_1) & \cos(q_2) & \cdots & \cdots & \cdots & \cdots & \cdots & \cos(q_{2N+1}) \\ \sin(q_1) & \sin(q_2) & \cdots & \cdots & \cdots & \cdots & \cdots & \sin(q_{2N+1}) \\ \cos(2q_1) & \cos(2q_2) & \cdots & \cdots & \cdots & \cdots & \cdots & \cos(2q_{2N+1}) \\ \sin(2q_1) & \sin(2q_2) & \cdots & \cdots & \cdots & \cdots & \cdots & \sin(2q_{2N+1}) \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \cos(Nq_1) & \cos(Nq_2) & \cdots & \cdots & \cdots & \cdots & \cdots & \cos(Nq_{2N+1}) \\ \sin(Nq_1) & \sin(Nq_2) & \cdots & \cdots & \cdots & \cdots & \cdots & \sin(Nq_{2N+1}) \end{bmatrix}_{(2N+1)\times(2N+1)} \tag{42}$$

In Eq.(42), it is interesting to find

$$\det[T_q] = \frac{(2N+1)^{N+\frac{1}{2}}}{2^N} \tag{43}$$

due to the orthogonal property as follows:

$$[T_q][T_q]^T = \begin{bmatrix} 2N+1 & 0 & \cdots & \cdots & 0 \\ 0 & \frac{2N+1}{2} & \cdots & \cdots & 0 \\ 0 & 0 & \frac{2N+1}{2} & \cdots & \vdots \\ \vdots & \vdots & \cdots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & \frac{2N+1}{2} \end{bmatrix}_{(2N+1)\times(2N+1)} \tag{44}$$

When the radius of the source location *R* moves far away from the real boundary and the number of the source points N_M become large, the condition number of [*K*] matrix deteriorates. This is the reason why the ill-posed behavior is inherent in the MFS. In the $[T_R]^I$ matrix, it becomes singular at radius of one ($\ln R = 0$ for $R = 1$) which results in a degenerate scale in the MFS. For the exterior Dirichlet problem of radius one ($\ln a = 0$ for $a = 1$), the nonunique solution exists in Eq.(8) where a_0 can not be determined. Even though the Trefftz method can not obtain the unique solution. This may explain why the transformation matrix between the MFS and the Trefftz method is nonsingular. A detailed study for the degenerate scale in the MFS due to numerical nonuniqueness is elaborated on in the next section.

2.4 Discussion of the degenerate scale in the MFS for Dirichlet problem

For the circular case with radius a , we can express $x = (r, f)$ and $s = (R, q)$ in terms of polar coordinate. Eqs.(19) and (20) can be expressed in terms of the degenerate kernel as below:

$$u^I(a,f) = \sum_{j=1}^{2N+1} c_j \left[\ln R - \sum_{n=1}^{\infty} \frac{1}{n} (\frac{a}{R})^n \cos(n(q_j - f)) \right] \tag{45}$$

$$u^E(a,f) = \sum_{j=1}^{2N+1} c_j \left[\ln a - \sum_{n=1}^{\infty} \frac{1}{n} (\frac{R}{a})^n \cos(n(q_j - f)) \right] \tag{46}$$

By matching the boundary condition, MFS yields the following algebraic equation

$$\underset{\sim}{U} \underset{\sim}{c_j} = \underset{\sim}{T_q} \begin{Bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{2N-1} \\ a_{2N} \end{Bmatrix} \tag{47}$$

Based on the circulants for the finite number degrees of freedom system by locating *2N* source points on a circular boundary, we have the influence matrix

$$[U] = \begin{bmatrix} z_0 & z_1 & z_2 & \cdots & z_{2N-1} \\ z_{2N-1} & z_0 & z_1 & \cdots & z_{2N-2} \\ z_{2N-2} & z_{2N-1} & z_0 & \cdots & z_{2N-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ z_1 & z_2 & z_3 & \cdots & z_0 \end{bmatrix}_{2N \times 2N} \tag{48}$$

where

$$z_m = U(R, q_m; r, 0), \qquad m = 0, 1, 2, \cdots, 2N-1 \tag{49}$$

in which $q_m = \frac{2pm}{2N}$, $f = 0$ without loss of generality.

The matrix U in Eq.(48) is found to be circulants since the rotation symmetry for the influence coefficients is considered. By introducing the following bases for the circulants $I, C_{2N}^1, C_{2N}^2, \cdots, C_{2N}^{2N-1}$, we can expand matrix A into

$$[U] = a_0 I + a_1 C_{2N}^1 + a_2 C_{2N}^2 + \cdots + a_{2N-1} C_{2N}^{2N-1} \tag{50}$$

where I is a unit matrix and

$$C_{2N}^1 = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \tag{51}$$

Based on the circulant theory, the spectral properties for the influence matrix, A, can be easily found as follows:

$$l_k = a_0 + a_1 a_k + a_2 a_k^2 + \cdots + a_{2N-1} a_k^{2N-1}, \quad k = 0, \pm 1, \pm 2, ... \pm N-1, N \tag{52}$$

where l_k and a_k are the eigenvalues for matrix A and C_{2N}^1, respectively. It is easily found that the eigenvalues for the circulants C_{2N}^1 is the root for $a^{2N} = 1$ as shown below:

$$a_k = e^{i\frac{2pk}{2N}}, \quad k = 0, \pm 1, \pm 2, ... \pm N-1, N \tag{53}$$

Substituting Eq.(53) into Eq.(52), we have

$$l_k = \sum_{m=0}^{2N-1} z_m a_k^m = \sum_{m=0}^{2N-1} z_m e^{i\frac{2pmk}{2N}} \tag{54}$$

According to the definition for z_m in Eq.(49), we have

$$z_m = z_{2N-m}, \quad m = 0, 1, 2, ..., 2N-1 \tag{55}$$

Substitution of Eq.(53) and (55) yields

$$l_k = \sum_{m=0}^{2N-1} z_m \cos(mk\Delta q) \tag{56}$$

Substituting z_m in the Eq.(49) into Eq.(56) and using the degenerate kernel of *U* in Eq.(18), the Reimann sum of infinite terms reduces to the following integral

$$l_k = \lim_{N\to\infty} \sum_{m=0}^{2N-1} U(m\Delta q, 0)\cos(mk\Delta q) \approx \frac{1}{r\Delta q}\int_0^{2p} U(q, 0)\cos(kq)\, r dq \tag{57}$$

By using the degenerate kernel for *U(s,x)* in Eq.(18) and the orthogonal conditions, Eq.(57) can be derived as

$$l_k = \begin{cases} 2N\ln R, & k=0 \\ \dfrac{-2N}{|k|}(\dfrac{a}{R})^{|k|}, & k=\pm 1, \pm 2, \cdots \pm (N-1), N \end{cases} \tag{58}$$

for the interior problem, and

$$l_k = \begin{cases} 2N\ln a, & k=0 \\ \dfrac{-2N}{|k|}(\dfrac{R}{a})^{|k|}, & k=\pm 1, \pm 2, \cdots \pm (N-1), N \end{cases} \tag{59}$$

for the exterior problem. Therefore, the determinant of matrix *U* can be represented to

$$\det[U^I] = (-2N)^{2N-1}\frac{2a\ln R}{R}\prod_{b=1}^{N-1}(\frac{a}{R})^{2b}\frac{1}{b^2} \tag{60}$$

$$\det[U^E] = (-2N)^{2N-1}\frac{2R\ln a}{a}\prod_{b=1}^{N-1}(\frac{R}{a})^{2b}\frac{1}{b^2} \tag{61}$$

According to Eqs.(60) and (61), we find that $\ln R$ and $\ln a$ are embedded in the determinant of influence matrices and the phenomena of degenerate scale still occur for the interior and exterior problems. Finally, it is obvious to examine the occurring mechanism of the degenerate scale through Eq.(60) and (61).

4. Numerical examples

Two examples, interior and exterior problems, are considered. Also, the doubly-connected problems with annular domain and eccentric case are solved by using the Trefftz method and the MFS.

4.1 Simply-connected problems

4.1.1 Interior problems

Given a Laplace Dirichlet problem with a circular domain of radius one, both the Trefftz method and the MFS are utilized to solve the problem as shown in Fig.2. The exact solution is shown in Fig.2(c). The sensitivity of N_T and N_M is also addressed. Good agreement can be made for the case of $N_T = 4$ (Fig.2(b)) and $N_M = 20$ (Fig.2(f)).

4.1.2 Exterior problems

An infinite domain with a circular hole is designed for the exterior Dirichlet problem. Since the radius one results in the mathematical nonuniqueness, the radius is chosen as two. The numerical results are shown in Fig.3 by using the Trefftz method (Fig.3(a), (b)) and the MFS (Fig.3(d), (e), (f)). Also, the exact solution is shown in Fig.3(c). The different numbers of N_T and N_M are tested for the convergence. Good agreement can be made for the case of $N_T = 4$ (Fig.3(b)) and $N_M = 20$ (Fig.3(f)). If the radius is one, both the Trefftz method and the MFS fail in real computation. No results are shown for the problem of radius one.

4.2 Doubly-connected problems

Consider the Laplace annular problem with radius a_1 and a_2 ($a_1 = 1$, $a_2 = 2.5$) as shown in Fig.4. By selecting the source location of $R_1 = 0.9$ and $R_2 = 2.6$ in the MFS, the results are shown in Fig.5. Good agreement is made after comparing with the exact solution in Fig.5 (b). And the Trefftz method has the same results shown in Fig.6.

For the eccentric case in Fig.7, the same techniques are used. By choosing the different source locations in the MFS, the results are shown in Fig.8. Good agreement is made after comparing with the exact solution in Fig.8 (d). In addition, the Trefftz method can obtain the good results in Fig.9.

5. Conclusions

In this paper, the proof of the mathematical equivalence for Laplace problem between the Treffz method and the MFS were derived successfully. It is interesting to find that the T-complete set in the Trefftz method for the interior and exterior problems are imbedded in the degenerate kernels of MFS. The degenerate scale appears when using the MFS based on the problem construction and the Trefftz method related to the T-complete function. The ill-posed problem in MFS also stems from the geometric matrix when the source is distributed far away from the real boundary. Both the Trefftz method and the MFS were employed to solve the interior and exterior problems with simply-connected and doubly-connected domain.

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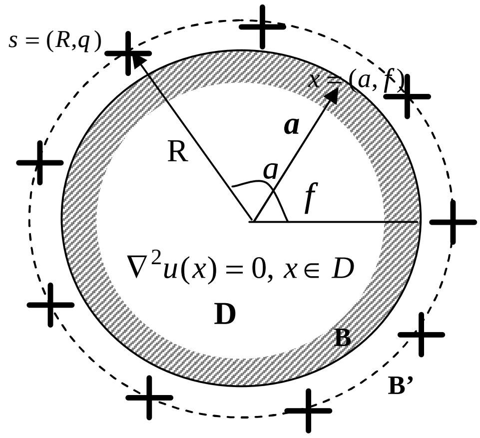


Fig.1(a) Definition sketch of polar coordinate for the interior Laplace equation with the circular domain (? is the source location of the MFS)

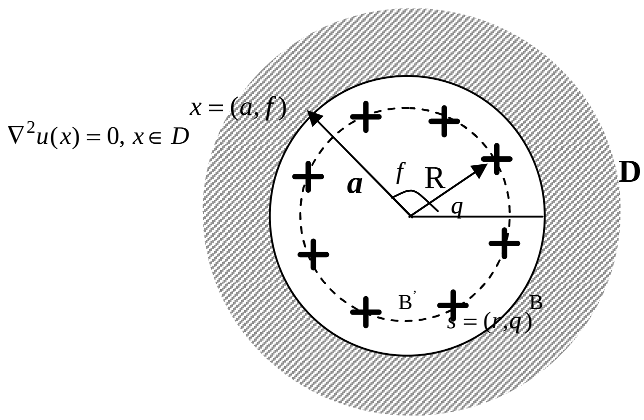
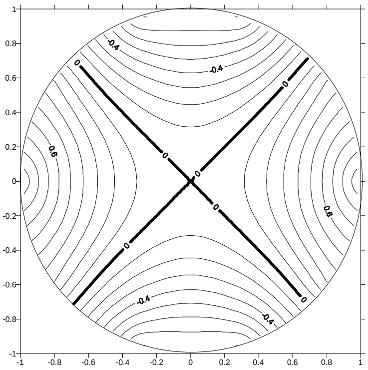
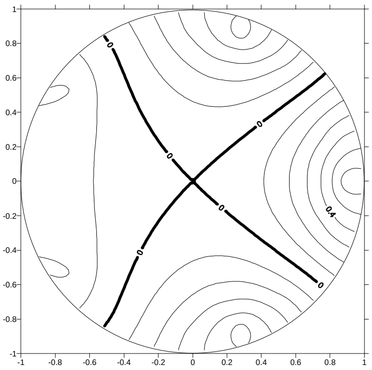


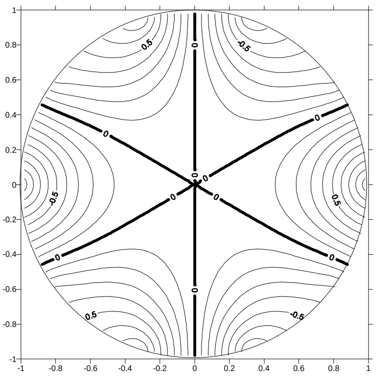
Fig.1(b) Definition sketch of polar coordinate for the exterior Laplace problem with the circular hole. (? is the source location of the MFS)



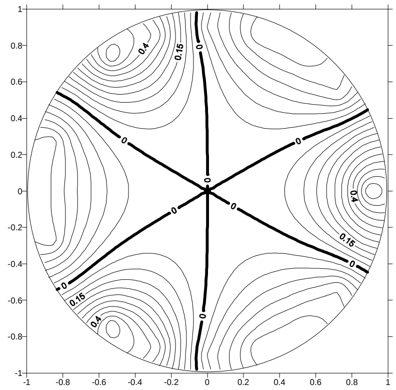
(a) $N_T = 2$



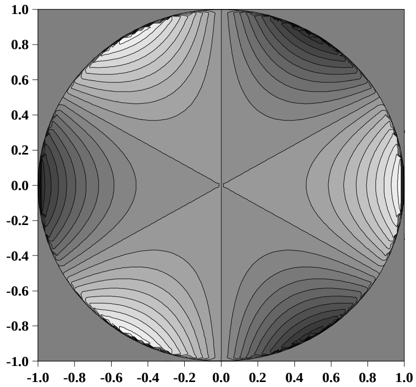
(d) $N_M = 5$



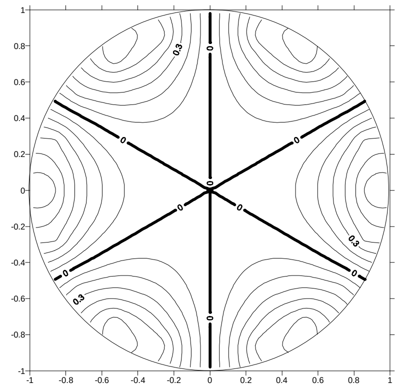
(b) $N_T = 4$



(e) $N_M = 9$



(c) Exact solution: $u(r, f) = r^3 \cos(3f)$

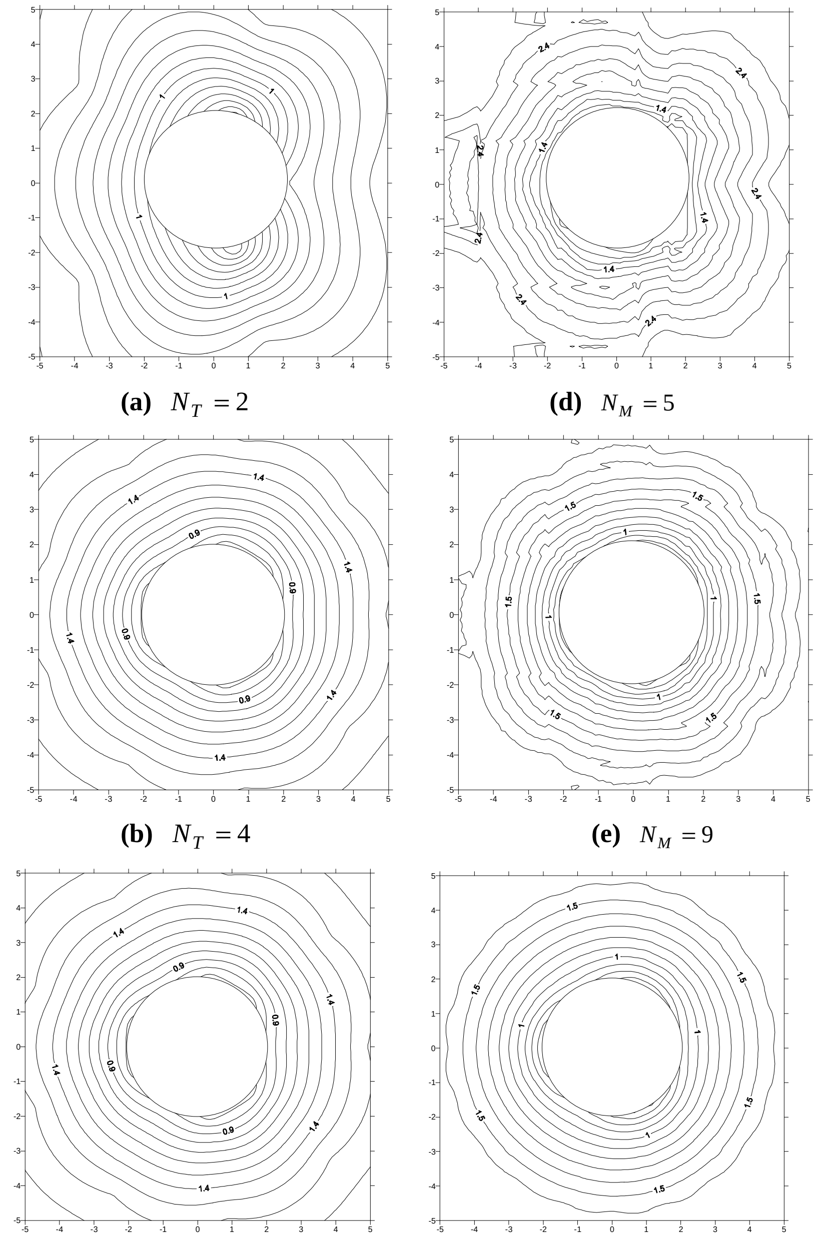


(f) $N_M = 20$

Fig. 2 The contour for the interior and exterior Laplace problems using the Trefftz method and the MFS (source location: R=1.1).

Trefftz method: (a) $N_T = 2$**, (b)** $N_T = 4$**, (c) Exact solution**

MFS : (d) $N_M = 5$**, (e)** $N_M = 9$**, (f)** $N_M = 20$



(a) $N_T = 2$ **(d)** $N_M = 5$

(b) $N_T = 4$ **(e)** $N_M = 9$

(c) Exact solution $u(r, f) = \ln r + \dfrac{1}{r^3}\cos(3f)$ **(f)** $N_M = 20$

Fig. 3 The contour for the exterior problem with radius $a = 2$ **using the Trefftz method and the MFS (source point: R=1.9).**

Trefftz method: (a) $N_T = 2$ **, (b)** $N_T = 4$ **, (c) Exact solution**

MFS : (d) $N_M = 5$ **, (e)** $N_M = 9$ **, (f)** $N_M = 20$

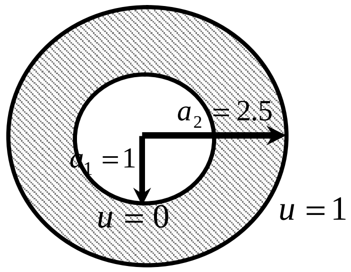
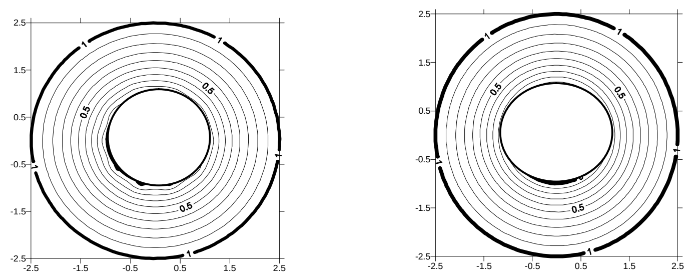


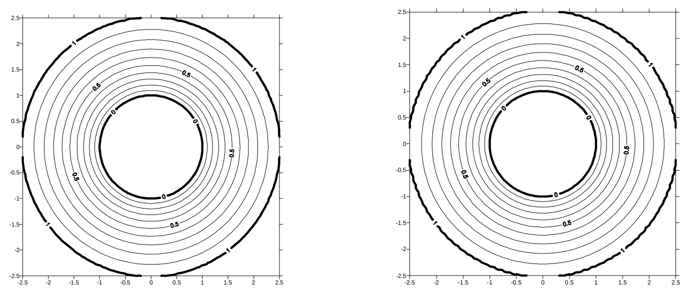
Fig. 4 Laplace problem with an annular domain



(a) Numerical solution **(b) Exact solution** $u(r,f)=\frac{\ln r}{\ln 2.5}$

Fig. 5 The contour for annular circle using the MFS

(inner: R_1=0.9, 20 points ; outer: R_2=2.6, 60 points)



(a) Numerical solution **(b) Exact solution** $u(r,f)=\frac{\ln r}{\ln 2.5}$

Fig. 6 The contour for annular circle using the Trefftz method (26 points)

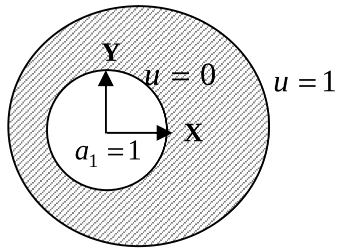
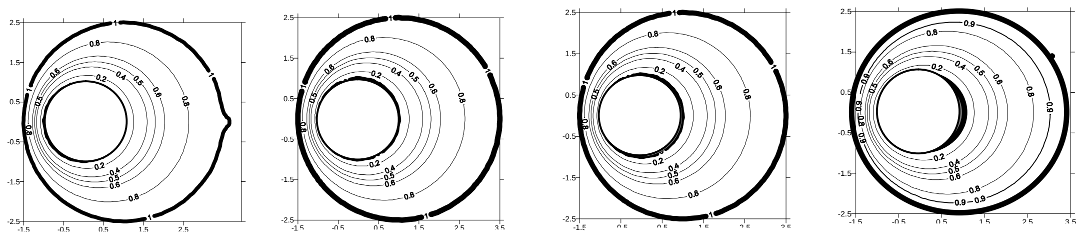


Fig.7 Laplace problem for eccentric circle

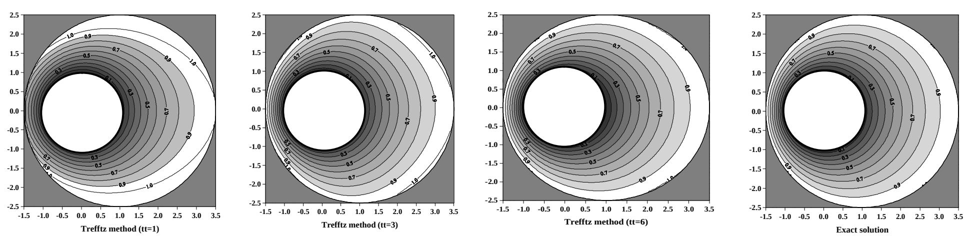


(a) R_1=0.9, R_2=2.6 **(b) R_1=0.9, R_2=3.0** **(c) R_1=0.9, R_2=4.0** **(d) Exact solution**

$$u(r,f) = \frac{1}{2\ln 2}\ln\{\frac{16r^2+1+8r\cos f}{r^2+16+8r\cos f}\}$$

Fig. 8 The contour for the eccentric case using the MFS

(inner: 20 points; outer: 60 points)



(a) 12 points **(b) 28 points** **(c) 52 points** **(d) Exact solution**

$$u(r,f) = \frac{1}{2\ln 2}\ln\{\frac{16r^2+1+8r\cos f}{r^2+16+8r\cos f}\}$$

Fig. 9 The contour for eccentric case using the Trefftz method