中國土木水利工程學會
九十二年「電子計算機於土木水利工程應用」研討會

Mathematical analysis and numerical study of the true and spurious eigenequations for free vibration of annular plate using the BEM

邊界元素法對於同心圓板自由振動真假特徵方程數學分析及數值研究

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Outlines

1. Introduction
2. BEM for the free vibration of multiply-connected plate
3. Treatments of the spurious eigenvalues for multiply-connected plate
4. Conclusions

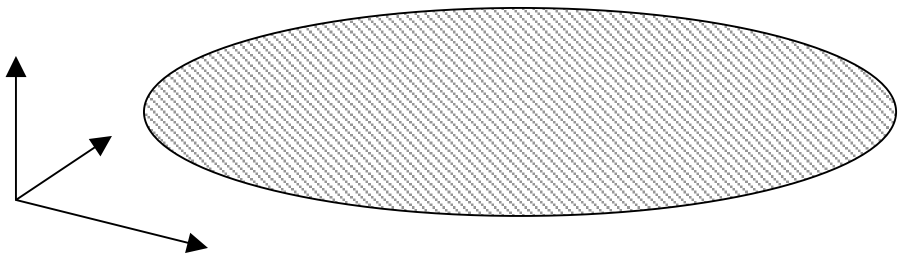
Introduction

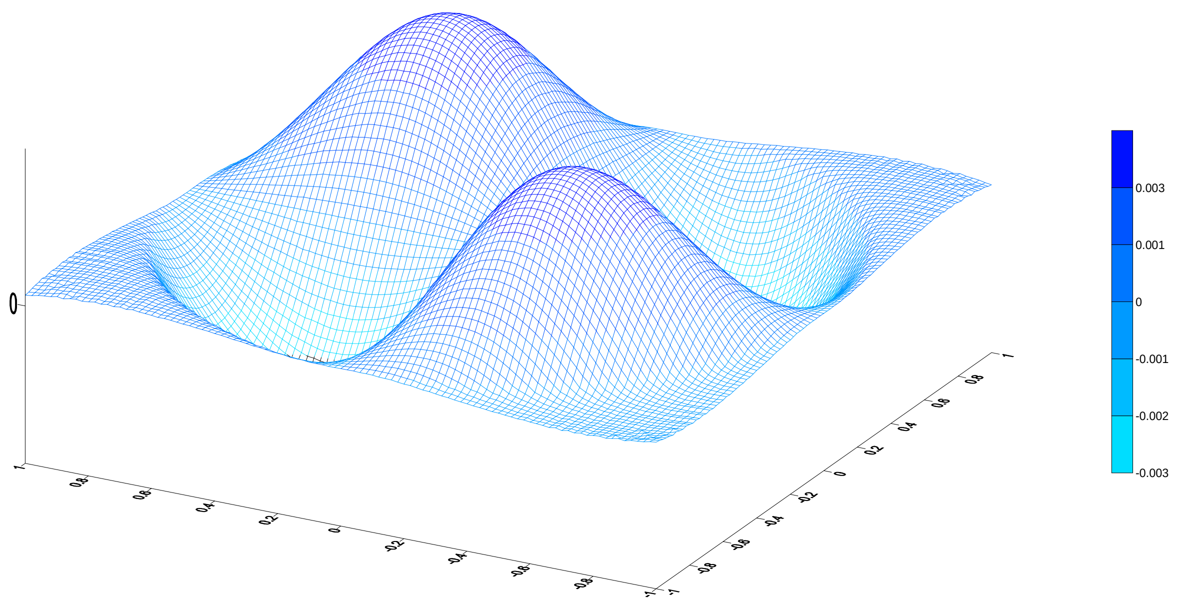
G.E. $$\nabla^4 u(x) = \lambda^4 u(x), x \in \Omega$$

$$\lambda^4 = \frac{\omega^2 \rho h}{D} \qquad D = \frac{Eh^3}{12(1-\nu^2)}$$

u	lateral displacement	λ	frequency parameter
ω	circular frequency	ρ	surface density
h	plate thickness	D	flexural rigidity
E	Young's modulus	ν	Poisson ratio
Ω	domain		

Free vibration of plate



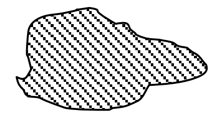


Literature review

1. Tai and Shaw 1974 (complex-valued BEM)
2. De Mey 1976, Hutchinson and Wong 1979 (real-part kernel)
3. Wong and Hutchinson (real-part direct BEM program)
4. Shaw 1979, Hutchinson 1988, Niwa *et al*. 1982 (real-part kernel)
5. Tai and Shaw 1974, Chen *et al*. Proc. Roy. Soc. Lon. Ser. A, 2001, 2003 (multiply-connected problem)
6. Chen *et al*. (dual formulation, domain partition, SVD updating technique, CHEEF method)

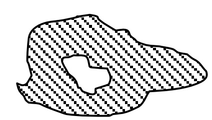
Motivation

Simply-connected problem



	Real	Imaginary	Complex
Saving CPU time	Yes	Yes	No
Avoid singular integral	No	Yes	No
Spurious eigenvalues	Appear	Appear	No

Multiply-connected problem

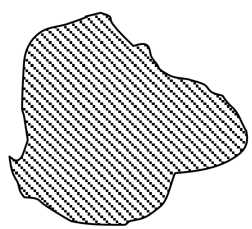


	Complex
Spurious eigenvalues	Appear

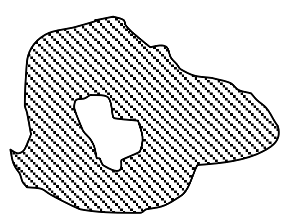
Spurious eigenvalues

Spurious eigenvalues occur in two aspects:

1. Simply-connected eigenproblem by using the real-part or imaginary-part BEM.
2. Multiply-connected eigenproblem by using the complex-valued BEM.



	Real-part BEM	(N.G.)
	Imaginary-part BEM	(N.G.)
	Complex-valued	(OK)



	Complex-valued	(N.G.)

1. Introduction
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Boundary integral equations for plate eigenproblems

(1) Displacement

$$u(x)=\int_B\{-U(s,x)v(s)+\Theta(s,x)m(s)-M(s,x)\theta(s)+V(s,x)u(s)\}\,dB(s),$$

$K_\theta(\cdot)$

(2) Slope

$$\theta(x)=\int_B\{-U_\theta(s,x)v(s)+\Theta_\theta(s,x)m(s)-M_\theta(s,x)\theta(s)+V_\theta(s,x)u(s)\}\,dB(s),$$

$K_m(\cdot)$

(3) Normal moment

$$m(x)=\int_B\{-U_m(s,x)v(s)+\Theta_m(s,x)m(s)-M_m(s,x)\theta(s)+V_m(s,x)u(s)\}\,dB(s),$$

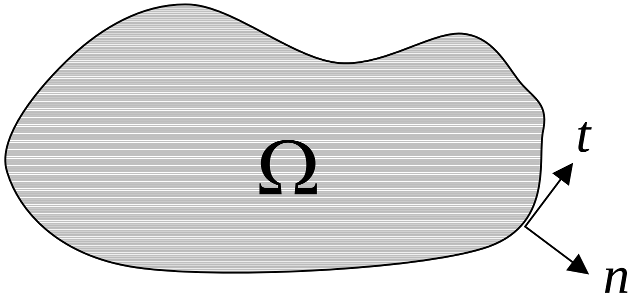
$K_v(\cdot)$

(4) Effective shear force

$$v(x)=\int_B\{-U_v(s,x)v(s)+\Theta_v(s,x)m(s)-M_v(s,x)\theta(s)+V_v(s,x)u(s)\}\,dB(s),$$

$x\in\Omega$

Operators



Slope

$$K_\theta(\cdot) = \frac{\partial(\cdot)}{\partial n}$$

Normal moment

$$K_m(\cdot) = \nu\nabla^2(\cdot) + (1-\nu)\frac{\partial^2(\cdot)}{\partial n^2}$$

Effective shear force

$$K_v(\cdot) = \frac{\partial\nabla^2(\cdot)}{\partial n} + (1-\nu)\frac{\partial}{\partial t}\left(\frac{\partial^2(\cdot)}{\partial n\partial t}\right)$$

Kernel functions

Fundamental solution

$$\nabla^4 U_c(s,x) - \lambda^4 U_c(s,x) = -\delta(x-s)$$

$$U_c(s,x) = \frac{1}{8\lambda^2}(Y_0(\lambda r) + iJ_0(\lambda r) + \frac{2}{\pi}(K_0(\lambda r) + iI_0(\lambda r)))$$

Kernel functions

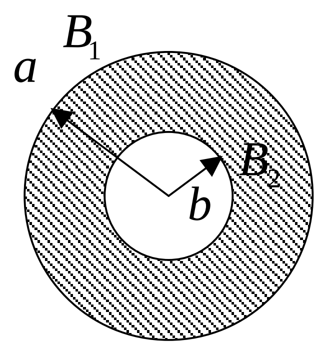
$$\Theta(s,x) = K_\theta(U(s,x))$$

$$M(s,x) = K_m(U(s,x))$$

$$V(s,x) = K_v(U(s,x))$$

Mathematical analysis (Continuous system)

For clamped-clamped annular plate



$$u_1 = 0, \quad \theta_1 = 0, \quad u_2 = 0, \quad \theta_2 = 0$$

$$m_1(s) = \sum_{n=0}^{\infty} (p_{1,n}^{cc} \cos(n\bar{\phi}) + q_{1,n}^{cc} \cos(n\bar{\phi}))$$

$$m_2(s) = \sum_{n=0}^{\infty} (p_{2,n}^{cc} \cos(n\bar{\phi}) + q_{2,n}^{cc} \cos(n\bar{\phi}))$$

$$v_1(s) = \sum_{n=0}^{\infty} (a_{1,n}^{cc} \cos(n\bar{\phi}) + b_{1,n}^{cc} \cos(n\bar{\phi}))$$

$$v_2(s) = \sum_{n=0}^{\infty} (a_{2,n}^{cc} \cos(n\bar{\phi}) + b_{2,n}^{cc} \cos(n\bar{\phi}))$$

Expansion formulae

Degenerate kernels (separable kernels)

$$K_0(\lambda r)=\begin{cases}\sum_{m=-\infty}^{\infty}K_m(\lambda\bar{\rho})I_m(\lambda\rho)\cos(m(\bar{\phi}-\phi)), & \bar{\rho}>\rho\\ \sum_{m=-\infty}^{\infty}K_m(\lambda\rho)I_m(\lambda\bar{\rho})\cos(m(\bar{\phi}-\phi)), & \rho>\bar{\rho}\end{cases}$$

$$J_0(\lambda r)=\begin{cases}\sum_{m=-\infty}^{\infty}J_m(\lambda\bar{\rho})J_m(\lambda\rho)\cos(m(\bar{\phi}-\phi)), & \bar{\rho}>\rho\\ \sum_{m=-\infty}^{\infty}J_m(\lambda\rho)J_m(\lambda\bar{\rho})\cos(m(\bar{\phi}-\phi)), & \rho>\bar{\rho}\end{cases}$$

$$Y_0(\lambda r)=\begin{cases}\sum_{m=-\infty}^{\infty}Y_m(\lambda\bar{\rho})J_m(\lambda\rho)\cos(m(\bar{\phi}-\phi)), & \bar{\rho}>\rho\\ \sum_{m=-\infty}^{\infty}Y_m(\lambda\rho)J_m(\lambda\bar{\rho})\cos(m(\bar{\phi}-\phi)), & \rho>\bar{\rho}\end{cases}$$

$$I_0(\lambda r)=\begin{cases}\sum_{m=-\infty}^{\infty}(-1)^m I_m(\lambda\bar{\rho})I_m(\lambda\rho)\cos(m(\bar{\phi}-\phi)), & \bar{\rho}>\rho\\ \sum_{m=-\infty}^{\infty}(-1)^m I_m(\lambda\rho)I_m(\lambda\bar{\rho})\cos(m(\bar{\phi}-\phi)), & \rho>\bar{\rho}\end{cases}$$

Relationship

$$[TM]\begin{Bmatrix} a_{1,n}^{cc} \\ a_{2,n}^{cc} \\ p_{1,n}^{cc} \\ p_{2,n}^{cc} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$[TM] = \begin{bmatrix} \int_{B_1} U(s_{B1},x_{B1})\cos(n\phi)dB(s) & \int_{B_2} U(s_{B2},x_{B1})\cos(n\phi)dB(s) & \int_{B_1} \Theta(s_{B1},x_{B1})\cos(n\phi)dB(s) & \int_{B_2} \Theta(s_{B2},x_{B1})\cos(n\phi)dB(s) \\ \int_{B_1} U(s_{B1},x_{B2})\cos(n\phi)dB(s) & \int_{B_2} U(s_{B2},x_{B2})\cos(n\phi)dB(s) & \int_{B_1} \Theta(s_{B1},x_{B2})\cos(n\phi)dB(s) & \int_{B_2} \Theta(s_{B2},x_{B2})\cos(n\phi)dB(s) \\ \int_{B_1} U_\theta(s_{B1},x_{B1})\cos(n\phi)dB(s) & \int_{B_2} U_\theta(s_{B2},x_{B1})\cos(n\phi)dB(s) & \int_{B_1} \Theta_\theta(s_{B1},x_{B1})\cos(n\phi)dB(s) & \int_{B_2} \Theta_\theta(s_{B2},x_{B1})\cos(n\phi)dB(s) \\ \int_{B_1} U_\theta(s_{B1},x_{B2})\cos(n\phi)dB(s) & \int_{B_2} U_\theta(s_{B2},x_{B2})\cos(n\phi)dB(s) & \int_{B_1} \Theta_\theta(s_{B1},x_{B2})\cos(n\phi)dB(s) & \int_{B_2} \Theta_\theta(s_{B2},x_{B2})\cos(n\phi)dB(s) \end{bmatrix}$$

Eigenequations (C-C annular plate)

$$\det[TM] = \det[S_n^{u\theta}]\det[T_n] = 0$$

True eigenequation

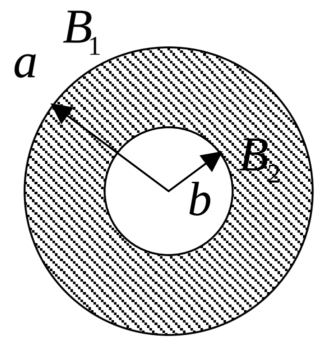
$$\det[T_n^{cc}] = \det\begin{bmatrix} J_n(\lambda a) & J_n(\lambda b) & J_n'(\lambda a) & J_n'(\lambda b) \\ Y_n(\lambda a) & Y_n(\lambda b) & Y_n'(\lambda a) & Y_n'(\lambda b) \\ I_n(\lambda a) & I_n(\lambda b) & I_n'(\lambda a) & I_n'(\lambda b) \\ K_n(\lambda a) & K_n(\lambda b) & K_n'(\lambda a) & K_n'(\lambda b) \end{bmatrix} = 0$$

Spurious eigenequation

$$\det[S_n^{u\theta}] = \det\begin{bmatrix} Y_n(\lambda a)+iJ_n(\lambda a) & 0 & K_n(\lambda a)+i(-1)^n I_n(\lambda a) & 0 \\ iJ_n(\lambda b) & J_n(\lambda b) & i(-1)^n I_n(\lambda b) & I_n(\lambda b) \\ \lambda(Y_n'(\lambda a)+iJ_n'(\lambda a)) & 0 & \lambda(K_n'(\lambda a)+i(-1)^n I_n'(\lambda a)) & 0 \\ i\lambda J_n'(\lambda b) & \lambda J_n'(\lambda b) & i\lambda I_n'(\lambda b) & \lambda I_n'(\lambda b) \end{bmatrix} = 0$$

Mathematical analysis (Discrete system)

For clamped-clamped annular plate



$$0=[U11]\{v_1\}+[U12]\{v_2\}+[\Theta 11]\{m_1\}+[\Theta 12]\{m_2\}$$

$$0=[U21]\{v_1\}+[U22]\{v_2\}+[\Theta 21]\{m_1\}+[\Theta 22]\{m_2\}$$

$$0=[U11_\theta]\{v_1\}+[U12_\theta]\{v_2\}+[\Theta 11_\theta]\{m_1\}+[\Theta 12_\theta]\{m_2\}$$

$$0=[U21_\theta]\{v_1\}+[U22_\theta]\{v_2\}+[\Theta 21_\theta]\{m_1\}+[\Theta 22_\theta]\{m_2\}$$

$$\begin{Bmatrix}0\\0\\0\\0\end{Bmatrix}=\left[SM^{cc}\right]\begin{Bmatrix}v_1\\v_2\\m_1\\m_2\end{Bmatrix} \qquad \left[SM^{cc}\right]=\begin{bmatrix}U11 & U12 & \Theta 11 & \Theta 12\\ U21 & U22 & \Theta 21 & \Theta 22\\ U11_\theta & U12_\theta & \Theta 11_\theta & \Theta 12_\theta\\ U21_\theta & U22_\theta & \Theta 21_\theta & \Theta 22_\theta\end{bmatrix}$$

Eigenequations (C-C annular plate)

$$\det\left[SM^{cc}\right]=\prod_{\bullet=-(N-1)}^{N}\left(\det[S_{\bullet}^{u\theta}]\ \det[T_{\bullet}^{cc}]\right)=0$$

True eigenequation

$$\det[T_{\bullet}^{cc}]=\det\begin{bmatrix} J_{\bullet}(\lambda a) & J_{\bullet}(\lambda b) & J'_{\bullet}(\lambda a) & J'_{\bullet}(\lambda b) \\ Y_{\bullet}(\lambda a) & Y_{\bullet}(\lambda b) & Y'_{\bullet}(\lambda a) & Y'_{\bullet}(\lambda b) \\ I_{\bullet}(\lambda a) & I_{\bullet}(\lambda b) & I'_{\bullet}(\lambda a) & I'_{\bullet}(\lambda b) \\ K_{\bullet}(\lambda a) & K_{\bullet}(\lambda b) & K'_{\bullet}(\lambda a) & K'_{\bullet}(\lambda b) \end{bmatrix}=0$$

Spurious eigenequation

$$\det[S_{\bullet}^{u\theta}]=\det\begin{bmatrix} Y_{\bullet}(\lambda a)+iJ_{\bullet}(\lambda a) & 0 & K_{\bullet}(\lambda a)+i(-1)^{\bullet}I_{\bullet}(\lambda a) & 0 \\ iJ_{\bullet}(\lambda b) & J_{\bullet}(\lambda b) & i(-1)^{\bullet}I_{\bullet}(\lambda b) & I_{\bullet}(\lambda b) \\ \lambda(Y'_{\bullet}(\lambda a)+iJ'_{\bullet}(\lambda a)) & 0 & \lambda(K'_{n}(\lambda a)+i(-1)^{n}I'_{\bullet}(\lambda a)) & 0 \\ i\lambda J'_{\bullet}(\lambda b) & \lambda J'_{\bullet}(\lambda b) & i\lambda I'_{\bullet}(\lambda b) & \lambda I'_{\bullet}(\lambda b) \end{bmatrix}=0$$

Physical meaning of the spurious eigenequation

Spurious eigenequation of the u, **q** formulation

(multiply-connected: The radii of the outer and inner circles are a and b)

$$\det[S_n^{u\theta}] = \det[Sa_n^{u\theta}]\det[Sb_n^{u\theta}]$$

$$[Sa_n^{u\theta}] = \begin{bmatrix} Y_n(\lambda a) + iJ_n(\lambda a) & K_n(\lambda a) + i(-1)^n I_n(\lambda a) \\ \lambda(Y_n'(\lambda a) + iJ_n'(\lambda a)) & \lambda(K_n'(\lambda a) + i(-1)^n I_n'(\lambda a)) \end{bmatrix} \qquad \det[Sa_n^{u\theta}] \neq 0$$

$$[Sb_n^{u\theta}] = \begin{bmatrix} J_n(\lambda b) & I_n(\lambda b) \\ \lambda J_n'(\lambda b) & \lambda I_n'(\lambda b) \end{bmatrix} \qquad \det[Sb_n^{u\theta}] = 0$$

$$\det[Sb_n^{u\theta}] = \{I_{n+1}(\lambda b)J_n(\lambda b) + I_n(\lambda b)J_{n+1}(\lambda b)\} = 0$$

True eigenequation of the clamped case

(simply-connected plate: The radius is b)

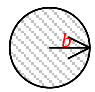
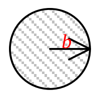
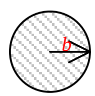
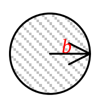
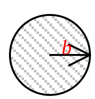
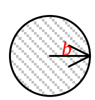
$$\{I_{n+1}(\lambda b)J_n(\lambda b) + I_n(\lambda b)J_{n+1}(\lambda b)\} = 0$$

True eigenequations for the annular plate

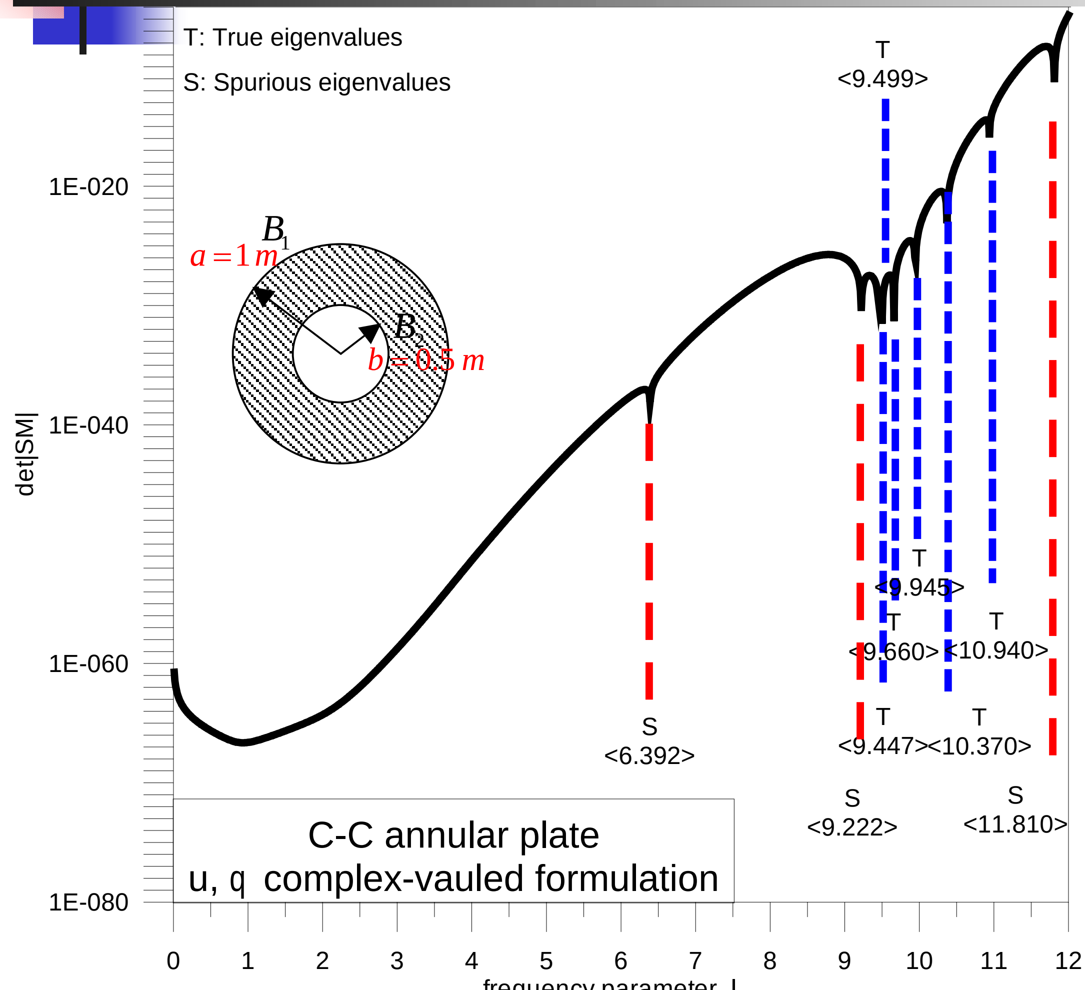
B.C.	$[T_n]$
C-C	$\begin{bmatrix} J_n(\lambda a) & J_n(\lambda b) & J'_n(\lambda a) & J'_n(\lambda b) \\ Y_n(\lambda a) & Y_n(\lambda b) & Y'_n(\lambda a) & Y'_n(\lambda b) \\ I_n(\lambda a) & I_n(\lambda b) & I'_n(\lambda a) & I'_n(\lambda b) \\ K_n(\lambda a) & K_n(\lambda b) & K'_n(\lambda a) & K'_n(\lambda b) \end{bmatrix}$
S-S	$\begin{bmatrix} J_n(\lambda a) & J_n(\lambda b) & \alpha_n^J(\lambda a) & \alpha_n^J(\lambda b) \\ Y_n(\lambda a) & Y_n(\lambda b) & \alpha_n^Y(\lambda a) & \alpha_n^Y(\lambda b) \\ I_n(\lambda a) & I_n(\lambda b) & \alpha_n^I(\lambda a) & \alpha_n^I(\lambda b) \\ K_n(\lambda a) & K_n(\lambda b) & \alpha_n^K(\lambda a) & \alpha_n^K(\lambda b) \end{bmatrix}$
F-F	$\begin{bmatrix} \alpha_n^J(\lambda a) & \alpha_n^J(\lambda b) & \beta_n^J(\lambda a)+\frac{1-\nu}{b}\gamma_n^J(\lambda a) & \beta_n^J(\lambda b)+\frac{1-\nu}{b}\gamma_n^J(\lambda b) \\ \alpha_n^Y(\lambda a) & \alpha_n^Y(\lambda b) & \beta_n^Y(\lambda a)+\frac{1-\nu}{b}\gamma_n^Y(\lambda a) & \beta_n^Y(\lambda b)+\frac{1-\nu}{b}\gamma_n^Y(\lambda b) \\ \alpha_n^I(\lambda a) & \alpha_n^I(\lambda b) & \beta_n^I(\lambda a)+\frac{1-\nu}{b}\gamma_n^I(\lambda a) & \beta_n^I(\lambda b)+\frac{1-\nu}{b}\gamma_n^I(\lambda b) \\ \alpha_n^K(\lambda a) & \alpha_n^K(\lambda b) & \beta_n^K(\lambda a)+\frac{1-\nu}{b}\gamma_n^K(\lambda a) & \beta_n^K(\lambda b)+\frac{1-\nu}{b}\gamma_n^K(\lambda b) \end{bmatrix}$

$n = 0,1,2,\ldots$

Spurious eigenequations for the annular plate

Eqs. number	Spurious eigenequations for the annular plate		
u, θ (1) and (2)		$\begin{bmatrix} J_n(\lambda b) & I_n(\lambda b) \\ \lambda(J_n'(\lambda b)) & \lambda(I_n'(\lambda b)) \end{bmatrix}$	$u=0, \quad \theta=0$
u, m (1) and (3)		$\begin{bmatrix} J_n(\lambda b) & I_n(\lambda b) \\ \alpha_n^J(\lambda b) & \alpha_n^I(\lambda b) \end{bmatrix}$	$u=0, \quad m=0$
u, v (1) and (4)		$\begin{bmatrix} J_n(\lambda b) & I_n(\lambda b) \\ [\beta_n^J(\lambda b)+\frac{(1-\nu)}{b}\gamma_n^J(\lambda b)] & [\beta_n^I(\lambda b)+\frac{(1-\nu)}{b}\gamma_n^I(\lambda b)] \end{bmatrix}$	$u=0, \quad v=0$
θ, m (2) and (3)		$\begin{bmatrix} \lambda J_n'(\lambda b) & \lambda I_n'(\lambda b) \\ \alpha_n^J(\lambda b) & \alpha_n^I(\lambda b) \end{bmatrix}$	$\theta=0, \quad m=0$
θ, v (2) and (4)		$\begin{bmatrix} \lambda J_n'(\lambda b) & \lambda I_n'(\lambda b) \\ [\beta_n^J(\lambda b)+\frac{(1-\nu)}{b}\gamma_n^J(\lambda b)] & [\beta_n^I(\lambda b)+\frac{(1-\nu)}{b}\gamma_n^I(\lambda b)] \end{bmatrix}$	$\theta=0, \quad v=0$
m, v (3) and (4)		$\begin{bmatrix} \alpha_n^J(\lambda b) & \alpha_n^I(\lambda b) \\ [\beta_n^J(\lambda b)+\frac{(1-\nu)}{b}\gamma_n^J(\lambda b)] & [\beta_n^I(\lambda b)+\frac{(1-\nu)}{b}\gamma_n^I(\lambda b) \end{bmatrix}$	$m=0, \quad v=0$

Determinant v.s frequency parameter (C-C)



True eigenequation

$$\det[T_n^{cc}] = \det\begin{bmatrix} J_n(\lambda a) & J_n(\lambda b) & J'_n(\lambda a) & J'_n(\lambda b) \\ Y_n(\lambda a) & Y_n(\lambda b) & Y'_n(\lambda a) & Y'_n(\lambda b) \\ I_n(\lambda a) & I_n(\lambda b) & I'_n(\lambda a) & I'_n(\lambda b) \\ K_n(\lambda a) & K_n(\lambda b) & K'_n(\lambda a) & K'_n(\lambda b) \end{bmatrix} = 0$$

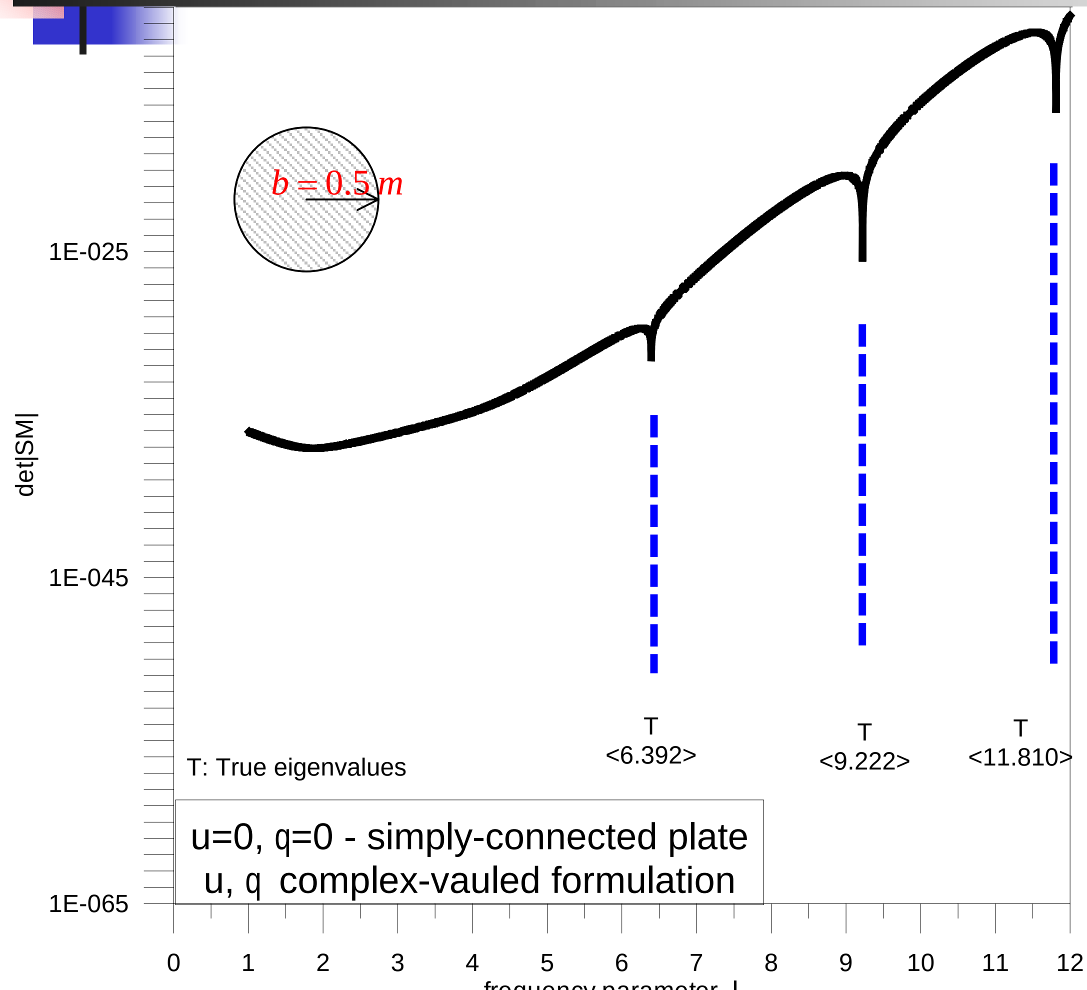
Spurious eigenequation

$$\det[S_n^{u\theta}] = \det\begin{bmatrix} Y_n(\lambda a)+iJ_n(\lambda a) & 0 & K_n(\lambda a)+i(-1)^n I_n(\lambda a) & 0 \\ iJ_n(\lambda b) & J_n(\lambda b) & i(-1)^n I_n(\lambda b) & I_n(\lambda b) \\ \lambda(Y'_n(\lambda a)+iJ'_n(\lambda a)) & 0 & \lambda(K'_n(\lambda a)+i(-1)^n I'_n(\lambda a)) & 0 \\ i\lambda J'_n(\lambda b) & \lambda J'_n(\lambda b) & i\lambda I'_n(\lambda b) & \lambda I'_n(\lambda b) \end{bmatrix} = 0$$

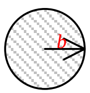
imply

$$\det[Sb_n^{u\theta}] = \det\begin{bmatrix} J_n(\lambda b) & I_n(\lambda b) \\ \lambda J'_n(\lambda b) & \lambda I'_n(\lambda b) \end{bmatrix} = 0$$

Physical meaning of the spurious eigenequation

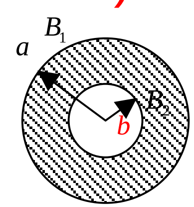


True eigenequation (clamped)
(simply-connected plate)



$$\{I_{n+1}(\lambda b)J_{\bullet}(\lambda b)+I_{\bullet}(\lambda b)J_{n+1}(\lambda b)\}=0$$

Spurious eigenequation (u, q)
(multiply-connected)



$$\det[Sb_n^{u\theta}] = \det\begin{bmatrix} J_n(\lambda b) & I_n(\lambda b) \\ \lambda J_n'(\lambda b) & \lambda I_n'(\lambda b) \end{bmatrix} = 0$$

Flowchart for C-C plate using the complex-valued BEM

G.E. $\nabla^4 u(x) = \lambda^4 u(x), \quad x \in \Omega$

↓

BIE

↓

BEM

↓ B.C. $u_1 = 0 \quad \theta_1 = 0 \quad u_2 = 0 \quad \theta_2 = 0$

$$0 = [U11]\{v_1\} + [U12]\{v_2\} + [\Theta 11]\{m_1\} + [\Theta 12]\{m_2\}$$
$$0 = [U21]\{v_1\} + [U22]\{v_2\} + [\Theta 21]\{m_1\} + [\Theta 22]\{m_2\}$$
$$0 = [U11_\theta]\{v_1\} + [U12_\theta]\{v_2\} + [\Theta 11_\theta]\{m_1\} + [\Theta 12_\theta]\{m_2\}$$
$$0 = [U21_\theta]\{v_1\} + [U22_\theta]\{v_2\} + [\Theta 21_\theta]\{m_1\} + [\Theta 22_\theta]\{m_2\}$$

Construction of [SM] →

$$[SM] = \begin{bmatrix} U11 & U12 & \Theta 11 & \Theta 12 \\ U21 & U22 & \Theta 21 & \Theta 22 \\ U11_\theta & U12_\theta & \Theta 11_\theta & \Theta 12_\theta \\ U21_\theta & U22_\theta & \Theta 21_\theta & \Theta 22_\theta \end{bmatrix}$$

↓

$$\det[SM] = \prod_{\bullet=-(N-1)}^{N} (\det[S_\bullet^{u\theta}] \det[T_\bullet]) = 0$$

↓

True and spurious eigenvalues

1. Introduction
2. BEM for the free vibration of multiply-connected plate
3. Treatments of the spurious eigenvalues for multiply-connected plate
4. Conclusions

Treatments

1. SVD updating term
2. The Burton & Miller concept
3. The CHIEF concept

SVD updating term (C-C annular plate)

u, q formulation

$$[SM_1^c]\begin{Bmatrix} v \\ m \end{Bmatrix} = 0$$

m,v formulation

$$[SM_2^c]\begin{Bmatrix} v \\ m \end{Bmatrix} = 0$$

SVD technique of updating term

$$[C]\begin{Bmatrix} v \\ m \end{Bmatrix} = 0 \qquad [C] = \begin{bmatrix} SM_1^c \\ SM_2^c \end{bmatrix}_{8N\times 4N}$$

The Burton & Miller concept (C-C annular plate)

u, q formulation

$$[SM_1^c]\begin{Bmatrix} v \\ m \end{Bmatrix} = 0$$

m,v formulation

$$[SM_2^c]\begin{Bmatrix} v \\ m \end{Bmatrix} = 0$$

The Burton & Miller concept

$$([SM_1^c] + i[SM_2^c])\begin{Bmatrix} v \\ m \end{Bmatrix} = 0$$

The CHEEF concept (clamped plate)

u, q formulation

$$\left[SM_1^c\right]\begin{Bmatrix} v \\ m \end{Bmatrix} = 0$$

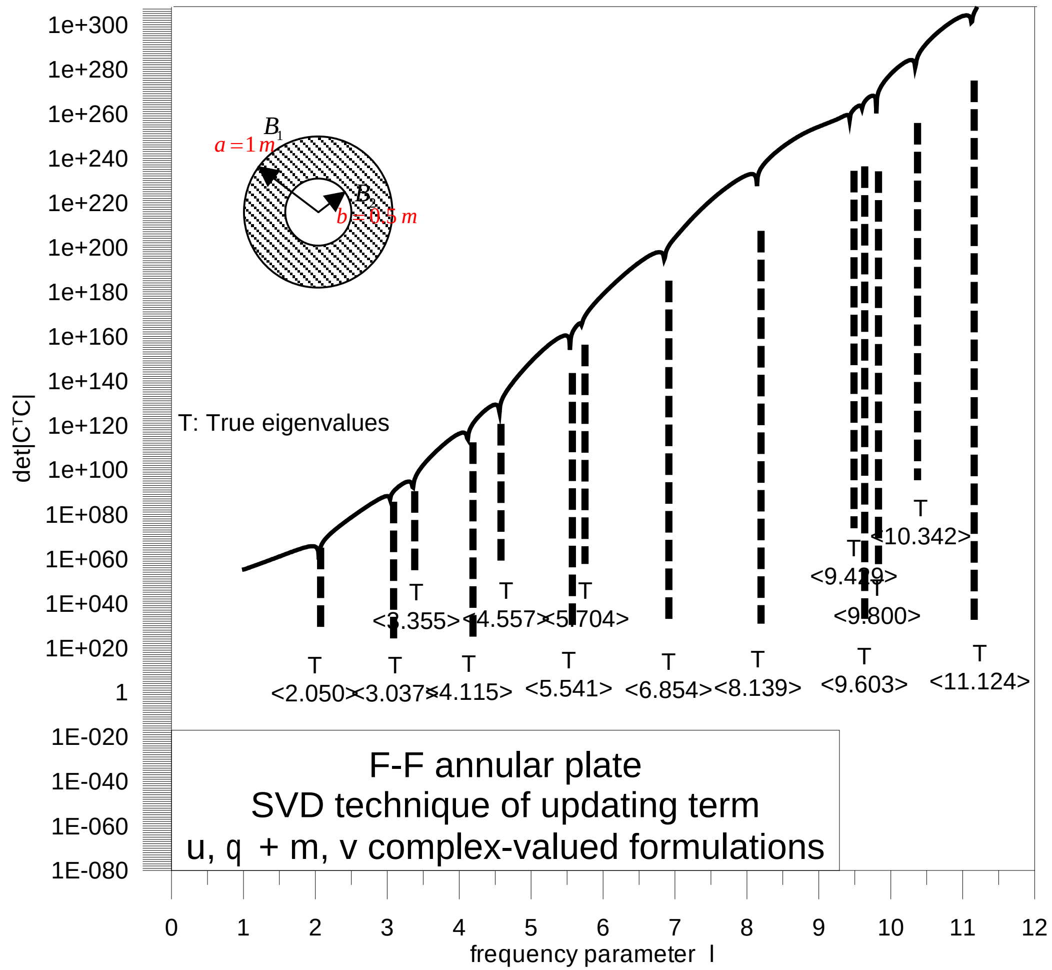
CHIEF points

$$[SM_3^{cc}] = \begin{bmatrix} UC1 & UC2 & \Theta C1 & \Theta C2 \\ UC1_\theta & UC2_\theta & \Theta C1_\theta & \Theta C2_\theta \end{bmatrix}_{2N_c \times 8N}$$

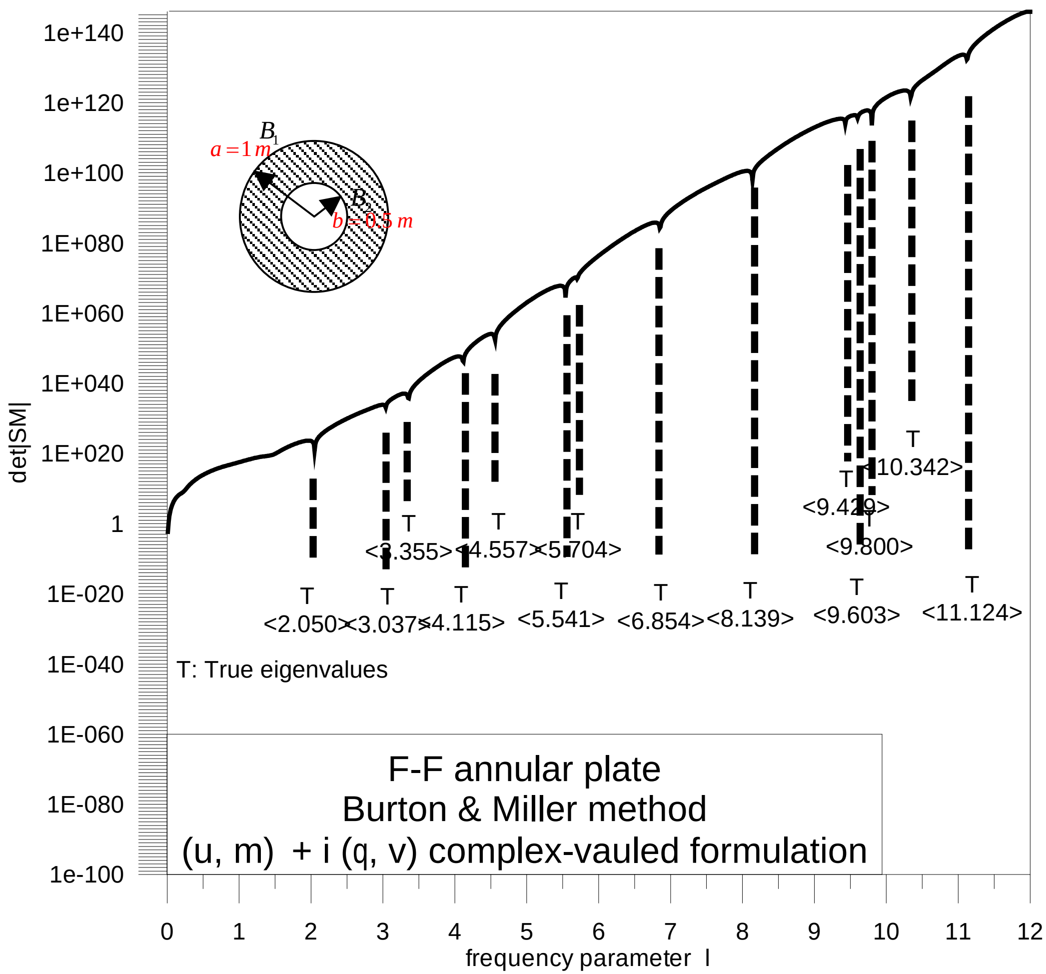
$$[C^*]\begin{Bmatrix} v \\ m \end{Bmatrix} = 0$$

$$[C^*] = \begin{bmatrix} U11 & U12 & \Theta 11 & \Theta 12 \\ U21 & U22 & \Theta 21 & \Theta 22 \\ U11_\theta & U12_\theta & \Theta 11_\theta & \Theta 12_\theta \\ U21_\theta & U22_\theta & \Theta 21_\theta & \Theta 22_\theta \\ UC1 & UC2 & \Theta C1 & \Theta C2 \\ UC1_\theta & UC2_\theta & \Theta C1_\theta & \Theta C2_\theta \end{bmatrix}_{2(4N+N_c)\times 8N}$$

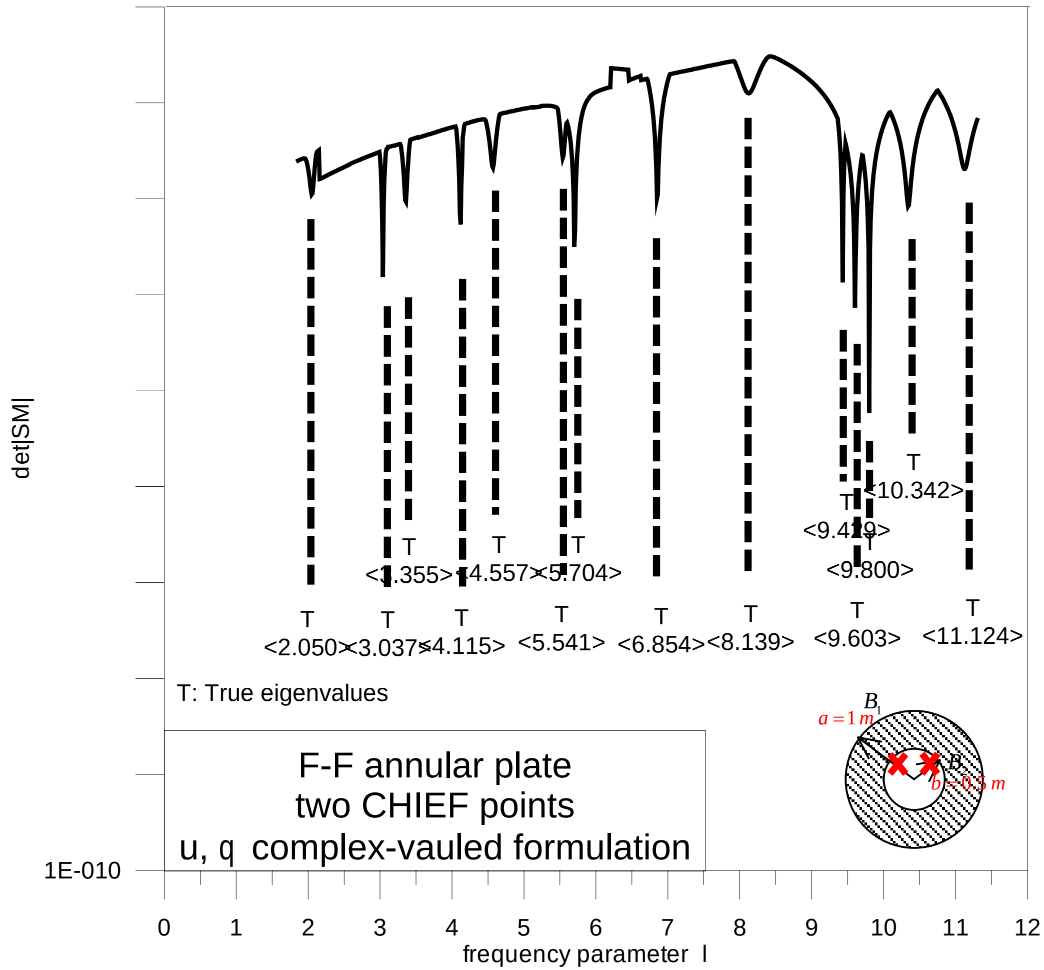
SVD updating term



The Burton & Miller concept



The CHIEF concept



1. Introduction
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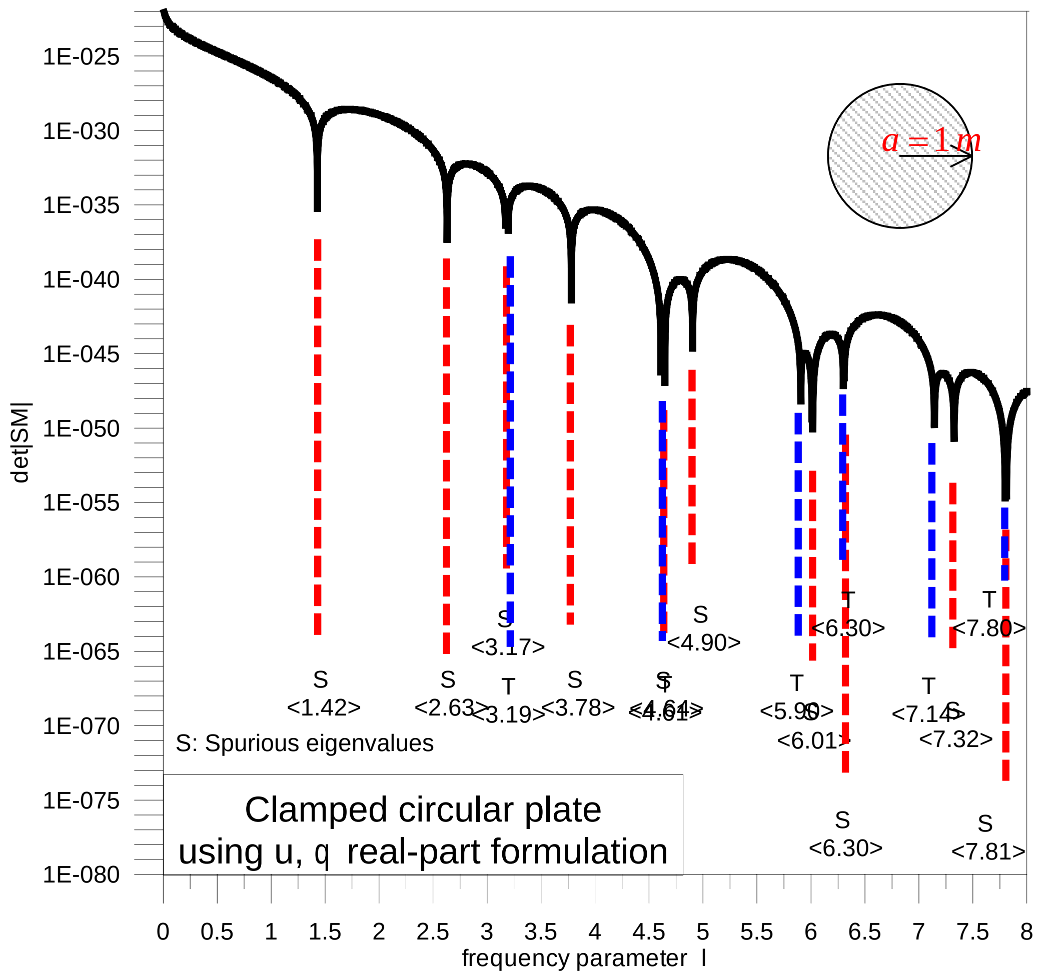
Conclusions

1. The true and spurious eigenequations depend on the B. C. and formulation, respectively.
2. The spurious eigenvalue in multiply-connected plate is the true eigenvalue of the associated simply-connected problem.
3. We provide the general form of the true eigenequation for the annular plates instead of the separate form.
4. The SVD updating term, Burton & Miller method and CHIEF method, were successfully applied to suppress the spurious eigenvalues.

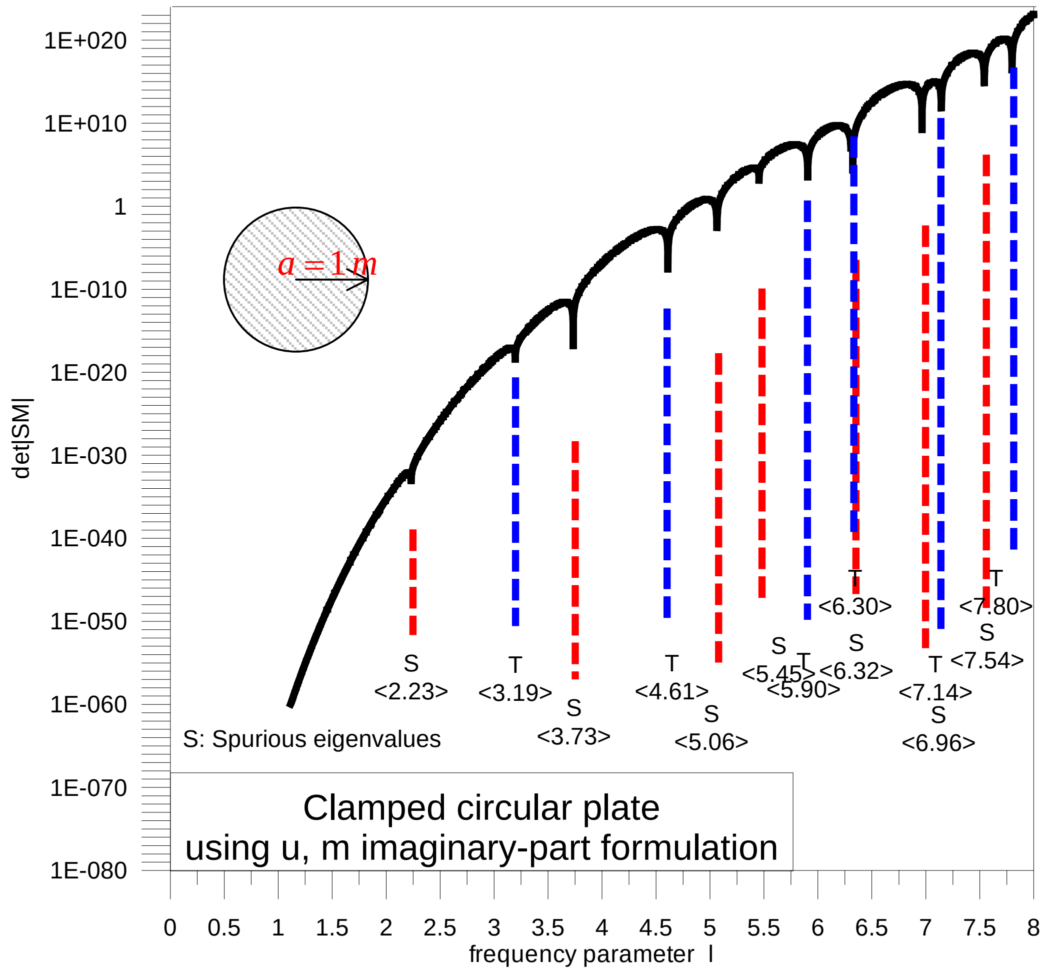
The End

Thanks for your kind attention

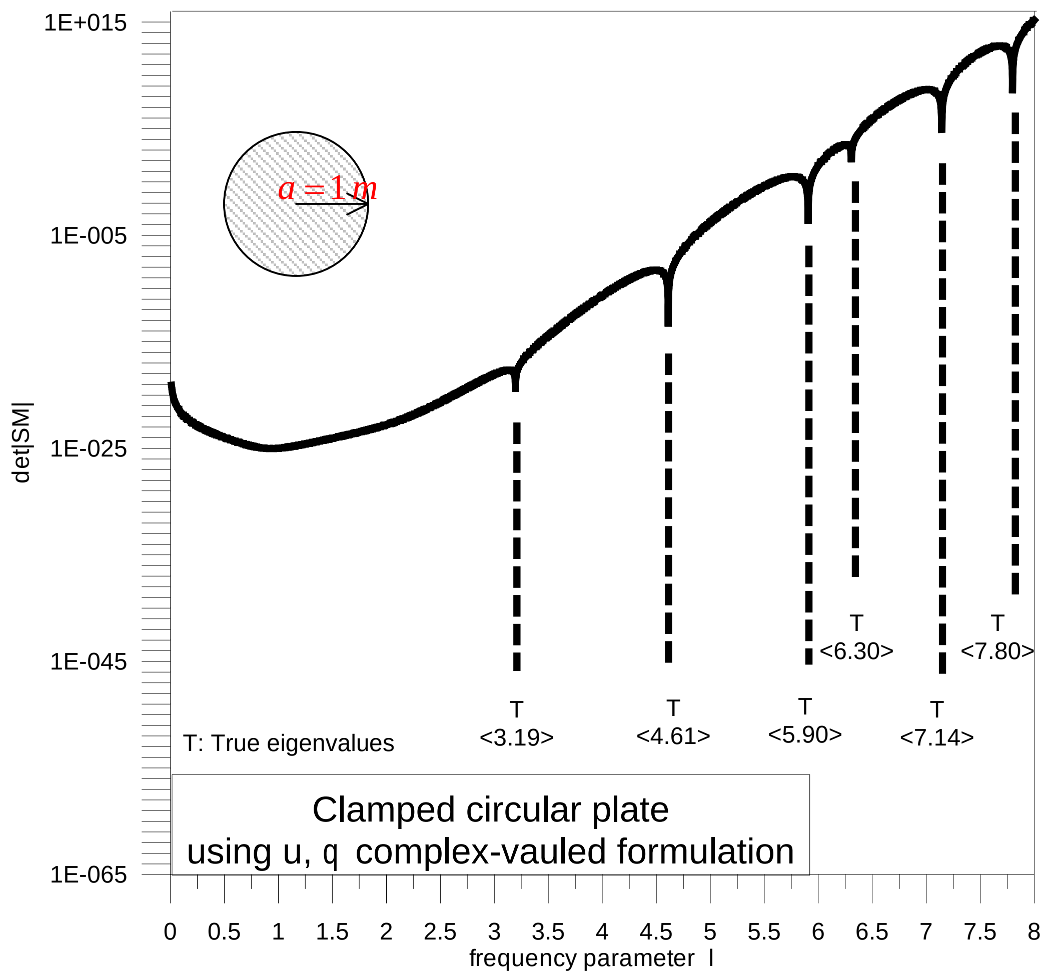
Real-part BEM for simply-connected plate



Imaginary-part BEM for simply-connected plate



Complex-valued BEM for simply-connected plate



Complex-valued BEM for multiply-connected plate