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Mathematical analysis and numerical study of the true and spurious eigenequations for free vibration of annular plate using the BEM

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Abstract

In this paper, the complex-valued BEM for solving the eigenfrequencies of the annular plates is proposed. By employing the complex-valued BEM, the spurious eigenevalues in conjunction with the true eigenvalues are obtained for free vibration of the multiply-connected plate. We analytically and numerically examine the occurrence of the spurious eigenvalues in the continuous and discrete systems of an annular plate. For the continuous system, the degenerate kernels for the fundamental solution and the Fourier series expansion for the boundary density are employed to derive the true and spurious eigenequations analytically. The circulant is adopted to analytically derive the true and spurious eigenequation in the discrete system. It is found that the spurious eigenvalues parasitizing in the multiply-connected plate depend on the associated true eigenvalues of the simply-connected plate with a radius b which is the inner circle of the annular domain. Three methods (SVD updating technique, the Burton & Miller method and the CHIEF method) are adopted to suppress the occurrence of the spurious eigenvalues, and a clamped-clamped annular plate is demonstrated analytically for the discrete system in this paper. Several examples were demonstrated to check the validity of the formulation.

邊界元素法對於同心圓板自由振動真假特徵方程數學分析及數值研究

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摘要

本文以邊界元素法求解同心圓板之特徵頻率問題。在多連通問題使用複數型邊界元素法在求解過程中所伴隨而來的假根問題為此文章之討論重點。為解析假根產生之機制，本文在連續系統中採用退化核及富利葉級數來進行數學推導，離散系統中使用循環矩陣來說明假根之產生，同時以固立端邊界條件之同心圓板為例來解析證明。經解析發現在多連通問題所產生之假根為對應內徑 b 之單連通圓板問題之真根。文中並提出三種方法(奇異值分法之補充式技巧，Burton & Miller 方法及 CHIEF 方法)來克服假根之產生，最後本文藉由不同的數值算例，來驗證上述的理論推導之正確性。

1. Introduction

For membrane or acoustic problems, either the real-part or imaginary-part BEM

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results in spurious eigensolutions. Tai and Shaw [13] first employed BEM to solve membrane vibration using the complex-valued kernel. De Mey [5], Hutchinson and Wong [8] employed only the real-part kernel to solve the membrane and plate vibrations to avoid the complex-valued computation in sacrifice of occurrence of spurious eigensolutions. Wong and Hutchinson [7,9] have presented a direct BEM involving displacement, slope, moment and shear force. They were able to obtain numerical results for simply-connected and clamped plates by employing only the real-part BEM with obvious computational gains. However, this saving leads to the spurious eigenvalues in addition to the true ones in free vibration analysis. Niwa *et al.* [12] also stated that ``One must take care to use the complete Green's function for outgoing waves, as attempts to use just the real (singular) or imaginary part (regular) separately will not provide the complete spectrum". This criticism is not correct since the real-part BEM does not lose any true eigenvalues. The reason is that the real and imaginary-part kernels satisfy the Hilbert transform. Complete eigenspectrum is imbedded in either one, real or imaginary-part kernel. The Hilbert transform is the constraint in the frequency domain corresponding to the casual effect in the time-domain fundamental solutions. The physical meaning of the real-part kernel is the standing wave [6]. Tai and Shaw [13] claimed that spurious eigenvalues are not present if the complex-valued kernel is employed for the eigenproblem. However, it is true only for the problem with a simply-connected domain. For multiply-connected problems, spurious eigenequation occur even though the complex-valued BEM is utilized [3, 4]. This is the reason why Chen and his coworkers have developed many systematic techniques [1, 2] for sorting out the true and the spurious eigensolutions.

In this paper, the spurious eigensolution for the multiply-connected plate eigenproblem will be studied in the complex-valued BEM. Since any two equations in the plate formulation (4 equations) can be chosen, 6 (C_2^4) options can be considered. The occurring mechanism for the spurious eigensolution in the multiply-connected plate problem will be studied analytically in the continuous and discrete systems. Three methods (SVD updating technique, the Burton & Miller method and the CHIEF method) are adopted to suppress the occurrence of the spurious eigenvalues, and a clamped-clamped annular plate is demonstrated analytically for the discrete system.

2. Boundary integral equations for plate eigenproblems

The governing equation for the free flexural vibration of a uniform thin plate is written as follows:

$$\nabla^4 u(x) = \lambda^4 u(x), \quad x \in \Omega \tag{1}$$

where u is the lateral displacement, $\lambda^4 = \omega^2 \rho_0 h / D$, λ is the frequency parameter, ω is the circular frequency, ρ_0 is the surface density, D is the flexural rigidity expressed as

$D = Eh^3/12(1-\nu^2)$ in terms of Young's modulus E, Poisson ratio ν, the plate thickness h, and Ω is the domain of the thin plate. The integral equations for the domain point can be derived from the Rayleigh-Green identity as follows [10]:

$$\alpha u(x) = \int_B \{-U(s,x)v(s) + \Theta(s,x)m(s) - M(s,x)\theta(s) + V(s,x)u(s)\}\, dB(s), \tag{2}$$

$$\alpha\theta(x) = \int_B \{-U_\theta(s,x)v(s) + \Theta_\theta(s,x)m(s) - M_\theta(s,x)\theta(s) + V_\theta(s,x)u(s)\}\, dB(s), \tag{3}$$

$$\alpha m(x) = \int_B \{-U_m(s,x)v(s) + \Theta_m(s,x)m(s) - M_m(s,x)\theta(s) + V_m(s,x)u(s)\}\, dB(s), \tag{4}$$

$$\alpha v(x) = \int_B \{-U_v(s,x)v(s) + \Theta_v(s,x)m(s) - M_v(s,x)\theta(s) + V_v(s,x)u(s)\}\, dB(s), \tag{5}$$

where $\alpha = 1$ for $x \in \Omega$, $\alpha = 1/2$ for $x \in B$, $\alpha = 0$ for $x \in \Omega^e$, Ω^e is the complementary domain of Ω, B is the boundary, u, θ, m and v mean the displacement, slope, normal moment, effective shear force, s and x are the source and field points, respectively, U, Θ, M and V kernel functions will be elaborated on later. The kernel function $U(s,x)$ is the fundamental solution $U_c(s,x)$ which is

$$U_c(s,x) = \frac{[Y_0(\lambda r) + iJ_0(\lambda r) - \frac{2}{\pi}(K_0(\lambda r) + iI_0(\lambda r))]}{8\lambda^2} \tag{6}$$

where J_0 and I_0 denote the *zero*-order Bessel and modified Bessel functions of the first kind, Y_0 and K_0 denote the *zero*-order Bessel and modified Bessel functions of the second kind, $r \equiv |s-x|$ and $i^2 = -1$, respectively. The three kernels, $\Theta(s,x)$, $M(s,x)$ and $V(s,x)$, are defined as follows :

$$\Theta(s,x) = K_\theta(U(s,x)) \tag{7}$$

$$M(s,x) = K_m(U(s,x)) \tag{8}$$

$$V(s,x) = K_v(U(s,x)) \tag{9}$$

where $K_\theta(\cdot)$, $K_m(\cdot)$ and $K_v(\cdot)$ mean the operators which are defined as follows:

$$K_\theta(\cdot) = \frac{\partial(\cdot)}{\partial n} \tag{10}$$

$$K_m(\cdot) = \nu\nabla^2(\cdot) + (1-\nu)\frac{\partial^2(\cdot)}{\partial n^2} \tag{11}$$

$$K_v(\cdot) = \frac{\partial\nabla^2(\cdot)}{\partial n} + (1-\nu)\frac{\partial}{\partial t}\left(\frac{\partial^2(\cdot)}{\partial n \partial t}\right) \tag{12}$$

where n and t are the normal vector and tangential vector, respectively. The operators $K_\theta(\cdot)$, $K_m(\cdot)$ and $K_v(\cdot)$ can be applied to U, Θ, M and V kernels. The kernel functions can be expressed as:

$$\Theta(s,x) = \frac{\partial U(s,x)}{\partial n_s} \tag{13}$$

$$M(s,x) = \nu\nabla_s^2 U(s,x) + (1-\nu)\frac{\partial^2 U(s,x)}{\partial n_s^2} \tag{14}$$

$$V(s,x) = \frac{\partial\nabla_s^2 U(s,x)}{\partial n_s} + (1-\nu)\frac{\partial}{\partial t_s}\left(\frac{\partial^2 U(s,x)}{\partial n_s \partial t_s}\right) \tag{15}$$

The displacement, slope, normal moment and effective shear force are derived by

$$\theta(x) = K_\theta(u(x)) \tag{16}$$

$$m(x) = K_m(u(x)) \tag{17}$$

$$v(x) = K_v(u(x)) \tag{18}$$

3. Mathematical analysis for the multiply-connected plate

3.1 Continuous system

We consider an annular plate clamped on the outer circle B_1 ($u_1 = 0$ and $\theta_1 = 0$) and the inner circle B_2 ($u_2 = 0$ and $\theta_2 = 0$), where u_1, θ_1, u_2 and θ_2 are the displacement and slope on the B_1 and B_2, respectively. The radii of the outer and inner circles are a and b, respectively. The moment and shear force, m_1, m_2, v_1 and v_2, can be expanded into Fourier series by

$$m_1(s) = \sum_{n=0}^{\infty} (p_{1,n}^{cc} \cos(n\bar{\phi}) + q_{1,n}^{cc} \cos(n\bar{\phi})) \tag{19}$$

$$m_2(s) = \sum_{n=0}^{\infty} (p_{2,n}^{cc} \cos(n\bar{\phi}) + q_{2,n}^{cc} \cos(n\bar{\phi})) \tag{20}$$

$$v_1(s) = \sum_{n=0}^{\infty} (a_{1,n}^{cc} \cos(n\bar{\phi}) + b_{1,n}^{cc} \cos(n\bar{\phi})) \tag{21}$$

$$v_2(s) = \sum_{n=0}^{\infty} (a_{2,n}^{cc} \cos(n\bar{\phi}) + b_{2,n}^{cc} \cos(n\bar{\phi})) \tag{22}$$

where the superscript `` cc '' denotes the clamped-clamped case, $\bar{\phi}$ is the angle on the circular boundary, $a_{i,n}^{cc}$, $b_{i,n}^{cc}$, $p_{i,n}^{cc}$ and $q_{i,n}^{cc}$ $(i=1,2)$ are the undetermined Fourier coefficients on B_i $(i=1,2)$. When the field point locates on B_1, substitution of the Eqs.(19)-(22) into the Eqs.(2) and (3) yields

$$\begin{aligned} 0 = &-\int_{B_1} U(s_{B!}, x_{B!}) v_1(s) dB(s) - \int_{B_2} U(s_{B2}, x_{B!}) v_2(s) dB(s) \\ &+ \int_{B_1} \Theta(s_{B!}, x_{B!}) m_1(s) dB(s) + \int_{B_2} \Theta(s_{B2}, x_{B!}) m_2(s) dB(s) \end{aligned} \tag{23}$$

$$\begin{aligned} 0 = &-\int_{B_1} U_\theta(s_{B!}, x_{B!}) v_1(s) dB(s) - \int_{B_2} U_\theta(s_{B2}, x_{B!}) v_2(s) dB(s) \\ &+ \int_{B_1} \Theta_\theta(s_{B!}, x_{B!}) m_1(s) dB(s) + \int_{B_2} \Theta_\theta(s_{B2}, x_{B!}) m_2(s) dB(s) \end{aligned} \tag{24}$$

When the field point locates on B_2, substitution of the Eqs.(19)-(22) into the Eqs.(2) and (3) yields

$$\begin{aligned} 0 = &-\int_{B_1} U(s_{B!}, x_{B2}) v_1(s) dB(s) - \int_{B_2} U(s_{B2}, x_{B2}) v_2(s) dB(s) \\ &+ \int_{B_1} \Theta(s_{B!}, x_{B2}) m_1(s) dB(s) + \int_{B_2} \Theta(s_{B2}, x_{B2}) m_2(s) dB(s) \end{aligned} \tag{25}$$

$$\begin{aligned} 0 = &-\int_{B_1} U_\theta(s_{B!}, x_{B2}) v_1(s) dB(s) - \int_{B_2} U_\theta(s_{B2}, x_{B2}) v_2(s) dB(s) \\ &+ \int_{B_1} \Theta_\theta(s_{B!}, x_{B2}) m_1(s) dB(s) + \int_{B_2} \Theta_\theta(s_{B2}, x_{B2}) m_2(s) dB(s) \end{aligned} \tag{26}$$

The kernel functions, $U(s,x)$, $\Theta(s,x)$, $U_\theta(s,x)$ and $\Theta_\theta(s,x)$, can be expanded by using the expansion formulae

$$Y_0(\lambda r) = \begin{cases} \sum_{m=-\infty}^{\infty} Y_m(\lambda\bar{\rho}) J_m(\lambda\rho) \cos(m(\bar{\phi}-\phi)), & \bar{\rho} > \rho \\ \sum_{m=-\infty}^{\infty} Y_m(\lambda\rho) J_m(\lambda\bar{\rho}) \cos(m(\bar{\phi}-\phi)), & \rho > \bar{\rho} \end{cases} \tag{27}$$

$$K_0(\lambda r) = \begin{cases} \sum_{m=-\infty}^{\infty} K_m(\lambda\bar{\rho}) I_m(\lambda\rho) \cos(m(\bar{\phi}-\phi)), & \bar{\rho} > \rho \\ \sum_{m=-\infty}^{\infty} K_m(\lambda\rho) I_m(\lambda\bar{\rho}) \cos(m(\bar{\phi}-\phi)), & \rho > \bar{\rho} \end{cases} \tag{28}$$

$$J_0(\lambda r) = \begin{cases} \sum_{m=-\infty}^{\infty} J_m(\lambda\bar{\rho}) J_m(\lambda\rho) \cos(m(\bar{\phi}-\phi)), & \bar{\rho} > \rho \\ \sum_{m=-\infty}^{\infty} J_m(\lambda\rho) J_m(\lambda\bar{\rho}) \cos(m(\bar{\phi}-\phi)), & \rho > \bar{\rho} \end{cases} \tag{29}$$

$$I_0(\lambda r) = \begin{cases} \sum_{m=-\infty}^{\infty} (-1)^m I_m(\lambda\bar{\rho}) I_m(\lambda\rho) \cos(m(\bar{\phi}-\phi)), & \bar{\rho} > \rho \\ \sum_{m=-\infty}^{\infty} (-1)^m I_m(\lambda\rho) I_m(\lambda\bar{\rho}) \cos(m(\bar{\phi}-\phi)), & \rho > \bar{\rho} \end{cases} \tag{30}$$

where J_m and I_m denote the *mth*-order Bessel and modified Bessel functions of the first kind, Y_m and K_m denote the *mth*-order Bessel and modified Bessel functions of the second kind. $s=(\bar{\rho},\bar{\phi})$ and $x=(\rho,\phi)$ are the polar coordinates of s and x, respectively. By using the degenerate kernels into Eqs.(23)-(26) and the orthogonality condition of the Fourier series, the Fourier coefficients a_n and p_n satisfy

$$\begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} = [TM] \begin{Bmatrix} a_{i,n}^{cc} \\ a_{i,n}^{cc} \\ p_{i,n}^{cc} \\ p_{i,n}^{cc} \end{Bmatrix} \tag{31}$$

where

$$[TM] = \begin{bmatrix} \int_{B_1} U(s_{B1}, x_{B1}) \cos(n\phi) dB(s) & \int_{B_2} U(s_{B2}, x_{B1}) \cos(n\phi) dB(s) & \int_{B_1} \Theta(s_{B1}, x_{B1}) \cos(n\phi) dB(s) & \int_{B_2} \Theta(s_{B2}, x_{B1}) \cos(n\phi) dB(s) \\ \int_{B_1} U(s_{B1}, x_{B2}) \cos(n\phi) dB(s) & \int_{B_2} U(s_{B2}, x_{B2}) \cos(n\phi) dB(s) & \int_{B_1} \Theta(s_{B1}, x_{B2}) \cos(n\phi) dB(s) & \int_{B_2} \Theta(s_{B2}, x_{B2}) \cos(n\phi) dB(s) \\ \int_{B_1} U_\theta(s_{B1}, x_{B1}) \cos(n\phi) dB(s) & \int_{B_2} U_\theta(s_{B2}, x_{B1}) \cos(n\phi) dB(s) & \int_{B_1} \Theta_\theta(s_{B1}, x_{B1}) \cos(n\phi) dB(s) & \int_{B_2} \Theta_\theta(s_{B2}, x_{B1}) \cos(n\phi) dB(s) \\ \int_{B_1} U_\theta(s_{B1}, x_{B2}) \cos(n\phi) dB(s) & \int_{B_2} U_\theta(s_{B2}, x_{B2}) \cos(n\phi) dB(s) & \int_{B_1} \Theta_\theta(s_{B1}, x_{B2}) \cos(n\phi) dB(s) & \int_{B_2} \Theta_\theta(s_{B2}, x_{B2}) \cos(n\phi) dB(s) \end{bmatrix} \tag{32}$$

Also, the coefficients of $b_{i,n}^{cc}$ and $q_{i,n}^{cc}$ have the same relationship in the matrix form. For the existence of nontrivial solution for $a_{i,n}^{cc}$, $b_{i,n}^{cc}$, $p_{i,n}^{cc}$ and $q_{i,n}^{cc}$, the determinant of the matrix

$[TM]$ versus the eigenvalue must be zero. By using the properties of the determinants, we can decompose the Eq.(32) to

$$\det[TM] = \det(\,[Sb_n^{u\theta}][T_n^{cc}]) \tag{33}$$

where

$$[T_n^{cc}] = \begin{bmatrix} J_n(\lambda a) & J_n(\lambda b) & J_n'(\lambda a) & J_n'(\lambda b) \\ Y_n(\lambda a) & Y_n(\lambda b) & Y_n'(\lambda a) & Y_n'(\lambda b) \\ I_n(\lambda a) & I_n(\lambda b) & I_n'(\lambda a) & I_n'(\lambda b) \\ K_n(\lambda a) & K_n(\lambda b) & K_n'(\lambda a) & K_n'(\lambda b) \end{bmatrix} \tag{34}$$

and

$$[S_n^{u\theta}] = \begin{bmatrix} Y_n(\lambda a)+iJ_n(\lambda a) & 0 & K_n(\lambda a)+i(-1)^n I_n(\lambda a) & 0 \\ iJ_n(\lambda b) & J_n(\lambda b) & i(-1)^n I_n(\lambda a) & I_n(\lambda a) \\ \lambda(Y_n'(\lambda a)+iJ_n'(\lambda a)) & 0 & \lambda(K_n'(\lambda a)+i(-1)^n I_n'(\lambda a)) & 0 \\ i\lambda J_n'(\lambda b) & \lambda J_n'(\lambda b) & i\lambda I_n'(\lambda b) & \lambda(I_n'(\lambda b)) \end{bmatrix} \tag{35}$$

It is noted that the matrix $[T_n^{cc}]$ denotes the matrix of true eigenequation for the C-C case and the matrix $[S_n^{u\theta}]$ denotes the matrix of spurious eigenequation in the u,θ formulation. Zero determinant in the Eq.(33) implies that the eigenequation is,

$$\det[TM] = \det(\,[Sb_n^{u\theta}][T_n^{cc}]) = 0 \tag{36}$$

After comparing with the analytical solution for the annular plate [11,14,15], the former matrix $[S_n^{u\theta}]$ in the Eq.(36) results in the spurious eigenequation while the latter matrix $[T_n^{cc}]$ results in the true eigenequation. All the true and spurious equations for the multiply-connected plate in the complex-valued BEM are shown in Tables 1 and 2.

3.2 Discrete system

For the discrete system, the Eqs.(23)-(26) can be rewritten as

$$0 = [U11]\{v_1\} + [U12]\{v_2\} + [\Theta 11]\{m_1\} + [\Theta 12]\{m_2\} \tag{37}$$

$$0 = [U21]\{v_1\} + [U22]\{v_2\} + [\Theta 21]\{m_1\} + [\Theta 22]\{m_2\} \tag{38}$$

$$0 = [U11_\theta]\{v_1\} + [U12_\theta]\{v_2\} + [\Theta 11_\theta]\{m_1\} + [\Theta 12_\theta]\{m_2\} \tag{39}$$

$$0 = [U21_\theta]\{v_1\} + [U22_\theta]\{v_2\} + [\Theta 21_\theta]\{m_1\} + [\Theta 22_\theta]\{m_2\} \tag{40}$$

By assembling Eqs.(37)-(40) together, we have

$$\begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} = \left[SM_1^{cc}\right] \begin{Bmatrix} v_1 \\ v_2 \\ m_1 \\ m_2 \end{Bmatrix} \tag{41}$$

where

$$[SM_1^{cc}] = \begin{bmatrix} U11 & U12 & \Theta 11 & \Theta 12 \\ U21 & U22 & \Theta 21 & \Theta 22 \\ U11_\theta & U12_\theta & \Theta 11_\theta & \Theta 12_\theta \\ U21_\theta & U22_\theta & \Theta 21_\theta & \Theta 22_\theta \end{bmatrix} \tag{42}$$

For the existence of nontrivial solution of v_1, v_2, m_1 and m_2, the determinant of the matrix $[SM_1^{cc}]$ versus eigenvalue must be zero. Since the rotation symmetry is preserved for a circular boundary, the influence matrices for the discrete system are found to be circulants such that the eigenvalue can be analytical derived

$$\begin{aligned} \mu_\ell^{[U11]} = -\frac{\pi a}{4\lambda^2}\{&[Y_\ell(\lambda a)J_\ell(\lambda a) - \frac{2}{\pi}K_\ell(\lambda a)I_\ell(\lambda a)] \\ &+ i[J_\ell(\lambda a)J_\ell(\lambda a) - \frac{2}{\pi}(-1)^\ell I_\ell(\lambda a)I_\ell(\lambda a)]\} \\ &\ell = 0,\pm 1,\pm 2,\ldots,\pm(N-1), N \end{aligned} \tag{43}$$

Since the matrix $[U11]$ is symmetric circulants, it can be expressed by

$$\begin{aligned} [U11] &= \Phi \Sigma_U \Phi^{-1} \\ &= \Phi \begin{bmatrix} \mu_0^{[U11]} & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \mu_1^{[U11]} & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \mu_{-1}^{[U11]} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \mu_{N-1}^{[U11]} & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & \mu_{-(N-1)}^{[U11]} & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & \mu_N^{[U11]} \end{bmatrix}_{2N\times 2N} \Phi^{-1} \end{aligned} \tag{44}$$

where

$$\Phi = \frac{1}{\sqrt{2N}} \begin{bmatrix} 1 & 1 & 0 & \cdots & 1 & 0 & 1 \\ 1 & \cos(\frac{2\pi}{2N}) & \sin(\frac{2\pi}{2N}) & \cdots & \cos(\frac{2\pi(2n-1)}{2N}) & \sin(\frac{2\pi(N-1)}{2N}) & \cos(\frac{2\pi N}{2N}) \\ 1 & \cos(\frac{4\pi}{2N}) & \sin(\frac{4\pi}{2N}) & \cdots & \cos(\frac{4\pi(2n-1)}{2N}) & \sin(\frac{4\pi(N-1)}{2N}) & \cos(\frac{4\pi N}{2N}) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & \cos(\frac{2\pi(2N-1)}{2N}) & \sin(\frac{2\pi(2n-1)}{2N}) & \cdots & \cos(\frac{2\pi(2N-1)(N-1)}{2N}) & \sin(\frac{2\pi(2N-1)(N-1)}{2N}) & \cos(\frac{2\pi(2N-1)N}{2N}) \end{bmatrix}_{2N\times 2N} \quad (45)$$

Similarly, we can obtain the other eigenvalues of the influence matrices by using the properties of the circulants. By decomposing the influences matrix in Eq.(42), we have

$$[SM_1^{cc}] = \begin{bmatrix} \Phi & 0 & 0 & 0 \\ 0 & \Phi & 0 & 0 \\ 0 & 0 & \Phi & 0 \\ 0 & 0 & 0 & \Phi \end{bmatrix} \begin{bmatrix} \Sigma_{U11} & \Sigma_{U12} & \Sigma_{\Theta 11} & \Sigma_{\Theta 12} \\ \Sigma_{U21} & \Sigma_{U22} & \Sigma_{\Theta 21} & \Sigma_{\Theta 22} \\ \Sigma_{U11_\theta} & \Sigma_{U12_\theta} & \Sigma_{\Theta 11_\theta} & \Sigma_{\Theta 12_\theta} \\ \Sigma_{U21_\theta} & \Sigma_{U22_\theta} & \Sigma_{\Theta 21_\theta} & \Sigma_{\Theta 22_\theta} \end{bmatrix} \begin{bmatrix} \Phi & 0 & 0 & 0 \\ 0 & \Phi & 0 & 0 \\ 0 & 0 & \Phi & 0 \\ 0 & 0 & 0 & \Phi \end{bmatrix}^t \quad (46)$$

Since Φ is orthogonal, the determinant of $[SM_1^{cc}]_{8N\times 8N}$ is

$$\det[SM_1^{cc}] = \det \begin{bmatrix} \Sigma_{U11} & \Sigma_{U12} & \Sigma_{\Theta 11} & \Sigma_{\Theta 12} \\ \Sigma_{U21} & \Sigma_{U22} & \Sigma_{\Theta 21} & \Sigma_{\Theta 22} \\ \Sigma_{U11_\theta} & \Sigma_{U12_\theta} & \Sigma_{\Theta 11_\theta} & \Sigma_{\Theta 12_\theta} \\ \Sigma_{U21_\theta} & \Sigma_{U22_\theta} & \Sigma_{\Theta 21_\theta} & \Sigma_{\Theta 22_\theta} \end{bmatrix} \quad (47)$$

By employing the eigenvalues of each influence matrices for Eq.(47), we have

$$\det[SM_1^{cc}] = \prod_{\ell=-(N+1)}^{N} \det([Sb_\ell^{u\theta}][T_\ell^{cc}]) = 0 \quad (48)$$

Zero determinant in Eq.(48) implies that the eigenequation is

$$\det([Sb_\ell^{u\theta}][T_\ell^{cc}]) = 0 \quad (49)$$

The true eigenequation for a continuous system can be obtained by approaching N in the discrete system to infinity. The former part in the middle bracket of Eq.(49) is the spurious eigenequation while the latter part in the big bracket is found to be the true eigenequation. In this case, it is interesting to find that the true and spurious eigenequation are the same with those derived in the continuous system.

3.3 Study of the spurious eigenequation

After comparing the Eq.(36) in the continuous system with the Eq.(49) in the discrete system for for the annular plate, the same spurious eigenequation $\det[S_n^{u\theta}] = 0$ is embedded in the same (u, θ) formulation no matter what the boundary condition is.

By using the cofactor of the matrix $[S_n^{u\theta}]$ to simplify the zero determinant of the Eq.(35) for the spurious eigenequation, we have

$$\det[S_n^{u\theta}] = \det[Sa_n^{u\theta}] \det[Sb_n^{u\theta}] \quad (50)$$

where

$$[Sa_n^{u\theta}] = \begin{bmatrix} Y_n(\lambda a) + iJ_n(\lambda a) & K_n(\lambda a) + i(-1)^n I_n(\lambda a) \\ \lambda(Y_n'(\lambda a) + iJ_n'(\lambda a)) & \lambda(K_n'(\lambda a) + i(-1)^n I_n'(\lambda a)) \end{bmatrix} \quad (51)$$

and

$$[Sb_n^{u\theta}] = \begin{bmatrix} J_n(\lambda b) & I_n(\lambda b) \\ \lambda J_n'(\lambda b) & \lambda I_n'(\lambda b) \end{bmatrix} \quad (52)$$

It is found that the determinant of the matrix $[Sa_n^{u\theta}]$ in the Eq.(51) is never zero. The spurious eigenequation is the zero determinant of the matrix $[Sb_n^{u\theta}]$ in the Eq.(52). It is interesting that the zero determinant of the $[Sb_n^{u\theta}]$ in the u, θ formulation results in the true eigenequation of simply-connected clamped plate with a radius b. The spurious eigenvalues parasitizing in the u, θ BEM depend on the radius b which is the inner circle of the annular domain. In fact, the multiply-connected problem can be superimposed by two problems, one is an interior problem with B_2 boundary and the other is an exterior problem with B_1 boundary. The source which causes the appearance of the spurious eigenvalues stems from the exterior problem with the inner boundary even though the complex-valued kernels are employed as well as the membrane and acoustics behaves [3,4].

4. Extraction of the true eigenvalues using SVD updating technique in the discrete system

A conventional approach to detect the nonunique solution is the criterion of satisfying all Eqs.(2)-(5) at the same time. For the clamped-clamped annular plate, the Eqs.(4)-(5) reduce to

$$0=[U11_m]\{v_1\}+[U12_m]\{v_2\}+[\Theta11_m]\{m_1\}+[\Theta12_m]\{m_2\} \tag{53}$$

$$0=[U21_m]\{v_1\}+[U22_m]\{v_2\}+[\Theta21_m]\{m_1\}+[\Theta22_m]\{m_2\} \tag{54}$$

$$0=[U11_v]\{v_1\}+[U12_v]\{v_2\}+[\Theta11_v]\{m_1\}+[\Theta12_v]\{m_2\} \tag{55}$$

$$0=[U21_v]\{v_1\}+[U22_v]\{v_2\}+[\Theta21_v]\{m_1\}+[\Theta22_v]\{m_2\} \tag{56}$$

After rearranging the terms, Eqs.(53)-(56) can be assembled to

$$\begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} = \left[SM_2^{cc}\right] \begin{Bmatrix} v_1 \\ v_2 \\ m_1 \\ m_2 \end{Bmatrix} \tag{57}$$

where

$$[SM_2^{cc}] = \begin{bmatrix} U11 & U12 & \Theta11 & \Theta12 \\ U21 & U22 & \Theta21 & \Theta22 \\ U11_\theta & U12_\theta & \Theta11_\theta & \Theta12_\theta \\ U21_\theta & U22_\theta & \Theta21_\theta & \Theta22_\theta \end{bmatrix} \tag{58}$$

To obtain an overdetermined system, we can combine Eqs.(42) and (58) by using the updating term,

$$[C]\begin{Bmatrix} v \\ m \end{Bmatrix} = 0 \tag{59}$$

where

$$[C] = \begin{bmatrix} SM_1^{cc} \\ SM_2^{cc} \end{bmatrix}_{8N\times 4N} \tag{60}$$

Since the eigenequation is nontrival, the rank of the matrix $[C]$ must be smaller then $4N$, the $4N$ singular values for the matrix $[C]$ must have at least one zero value. The explicit form for the matrix $[C]$ can be decomposed into

$$[C] = \begin{bmatrix} \Phi & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \Phi & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \Phi & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \Phi & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \Phi & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \Phi & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \Phi & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Phi \end{bmatrix} \begin{bmatrix} \Sigma_{U11} & \Sigma_{U12} & \Sigma_{\Theta11} & \Sigma_{\Theta12} \\ \Sigma_{U21} & \Sigma_{U22} & \Sigma_{\Theta21} & \Sigma_{\Theta22} \\ \Sigma_{U11_\theta} & \Sigma_{U12_\theta} & \Sigma_{\Theta11_\theta} & \Sigma_{\Theta12_\theta} \\ \Sigma_{U21_\theta} & \Sigma_{U22_\theta} & \Sigma_{\Theta21_\theta} & \Sigma_{\Theta22_\theta} \\ \Sigma_{U11_m} & \Sigma_{U12_m} & \Sigma_{\Theta11_m} & \Sigma_{\Theta12_m} \\ \Sigma_{U21_m} & \Sigma_{U22_m} & \Sigma_{\Theta21_m} & \Sigma_{\Theta22_m} \\ \Sigma_{U11_v} & \Sigma_{U12_v} & \Sigma_{\Theta11_v} & \Sigma_{\Theta12_v} \\ \Sigma_{U21_v} & \Sigma_{U22_v} & \Sigma_{\Theta21_v} & \Sigma_{\Theta22_v} \end{bmatrix} \begin{bmatrix} \Phi & 0 & 0 & 0 \\ 0 & \Phi & 0 & 0 \\ 0 & 0 & \Phi & 0 \\ 0 & 0 & 0 & \Phi \end{bmatrix}^T \tag{61}$$

Based on the equivalence between the SVD technique and the least-squares method in mathematical essence, the zero determinant of the matrix $[C]^T[C]$ is implies the nontrivial solution. After a length derivation, the only possibility for the zero determinant of the matrix $[C]^T[C]$ is only the true eigenequation to be zero, such that

$$\det[T_\ell^{cc}] = 0 \tag{62}$$

This indicates that only the true eigenequation of the clamped circular plate is sorted out in the SVD updating matrix since the true eigenequation is simultaneously embedded in the six formulations. The result matches well with Eqs.(36) and (49) in the continuous and discrete systems.

4.2 The Burton & Miller method

In the exterior acoustics of Helmholtz equation by using the dual BEM, Burton & Miller utilized the product of hypersingular equation with an imaginary constant and added to the singular equation in dealing with fictitious-frequency problem which results in a non-uniqueness solution. By extending this concept to solve the spurious eigenequation in the complex-valued BEM, we have

$$([SM_1^{cc}]+i[SM_2^{cc}])\begin{Bmatrix} v_1 \\ v_2 \\ m_1 \\ m_2 \end{Bmatrix}=\begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{63}$$

4.3 CHIEF method

By adding the CHIEF point $(\rho < b)$ to solve the multiply-connected plate eigenproblem in null-field integral equation, we have

$$[SM_3^{cc}]=\begin{bmatrix} UC1 & UC2 & \Theta C1 & \Theta C2 \\ UC1_\theta & UC2_\theta & \Theta C1_\theta & \Theta C2_\theta \end{bmatrix}_{2N_c \times 8N} \tag{64}$$

where the index C denotes the CHIEF point in the null-field integral equation and the subscript $N_C (\geq 1)$ indicates the number of additional CHIEF points. The symbols, $UC1, UC2, \Theta C1_\theta, \Theta C2_\theta, UC1_\theta, UC2_\theta, \Theta C1_\theta$ and $\Theta C2_\theta$ mean the influence row vectors resulted from of the U, Θ, U_θ and Θ_θ kernels which is collocating the CHIEF point. Combining the Eqs.(42) and (64) together to obtain the overdetermined system, we can sort out the true eigenvalues.

5. Numerical results and discussions

We consider an annular plate with the outer radius of one meter $(a = 1\,m)$ and the inner radius of 0.5 meter $b = 0.5\,m$ of B_1 and B_2, respectively, and the Poisson ratio $1/3$. The outer and inner boundaries are both discretized into ten constant boundary elements, respectively. Three cases (C-C, S-S and F-F annular plates) were considered.

Figures 1.(a) ~ 1.(c) show the determinant of $[SM]$ versus frequency parameter λ for the three cases of annular plate using the complex-valued formulations (u, θ). Both the true and spurious eigenvalues occur simultaneously. After comparing with (a), (b) and (c) results, the spurious eigenvalues (6.392, 9.222 and 11.810) are obtained no matter what the boundary condition is. It reconfirms that the spurious eigenvalues depends on the formulation instead of the specified boundary condition. All the spurious eigenvalues satisfy the spurious eigenequation ($[Sb_n^{u\theta}]=0$ in Eq.(52)) in the Table 2. The spurious eigenequation of multiply-connected eigenproblem by using the u, θ formulation is found to be the true eigenequation of the simply-connected clamped plate with a radius b which is the inner radius of the annular plate [3,4].

Three methods, the SVD technique of updating term ($(u,\theta)+(m,v)$ formulation), the Burton & Miller method ($(u,m)+i\,(\theta,v)$ formulation) and CHIEF method (two points), the true eigenvalues were obtained as shown in Figures 1.(d)-(f).

All the numerical data of the true eigenvalues are satisfied the true eigenequation in the Table 1, and the eigenvalues agree well with the data in Leissa and Laura *et al.* [11,14,15]. It is worth mentioning that we provide the unified form of the true eigenequations for the three cases of annular plates in Table 1 instead of the separate form (n=0,1,2) [11]. The true eigenvalues are well compared with the Leissa's numerical results. However, the obtained eigenvalues according to the Leissa's eigenequation are not consistent to those in his book. The possible explanation is that the eigenequations in the Leissa's book for some cases were wrongly typed.

6. Conclusions

A complex-valued BEM formulation has been derived for the free vibration of annular plate. The true and spurious eigenequations were derived analytically by using the Fourier series, degenerate kernels and circulants in both the continuous and discrete systems. Since either two equations in the plate formulation (4 equations) can be chosen, six options can be considered. The occurrence of spurious eigenequation only depends on the formulation instead of the specified boundary condition, while the true eigenequation is independent of the formulation and is relevant to the specified boundary condition. It is interesting that the spurious eigenequation of multiply-connected plate eigenproblem by using the u, θ formulation is found to be the true eigenequation of simply-connected clamped plate with a radius b which is the inner radius of the annular plate. All the results are shown in the Tables 1 and 2. Three methods (SVD updating technique, the Burton & Miller method and the CHIEF method) were adopted to suppress the occurrence of the spurious eigenvalues, only the true eigenvalues obtained.

A C-C annular plate was demonstrated analytically to see the validity of the present method. Several examples of plates were illustrated to check the validity of the present formulations. Although the annular case lacks generality, it leads significant insight into the occurring mechanism of true and spurious eigenequation. Although the proof is only limited to the annular case, it is a great help to the researchers who may require analytical explanation for the reason why the spurious eigenevalues appears. The same algorithm in the discrete system can be applied to solve arbitrary-shaped plate numerically without any difficulty. Nevertheless, mathematical derivation in the continuous and discrete systems can not be done anaytically.

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Table 1. True eigenequations for the annular plate.

Cases	true eigenequation $[T_n]$
C-C	$\begin{bmatrix} J_n(\lambda a) & J_n(\lambda b) & J'_n(\lambda a) & J'_n(\lambda b) \\ Y_n(\lambda a) & Y_n(\lambda b) & Y'_n(\lambda a) & Y'_n(\lambda b) \\ I_n(\lambda a) & I_n(\lambda b) & I'_n(\lambda a) & I'_n(\lambda b) \\ K_n(\lambda a) & K_n(\lambda b) & K'_n(\lambda a) & K'_n(\lambda b) \end{bmatrix}$
S-S	$\begin{bmatrix} J_n(\lambda a) & J_n(\lambda b) & \alpha_n^J(\lambda a) & \alpha_n^J(\lambda b) \\ Y_n(\lambda a) & Y_n(\lambda b) & \alpha_n^Y(\lambda a) & \alpha_n^Y(\lambda b) \\ I_n(\lambda a) & I_n(\lambda b) & \alpha_n^I(\lambda a) & \alpha_n^I(\lambda b) \\ K_n(\lambda a) & K_n(\lambda b) & \alpha_n^K(\lambda a) & \alpha_n^K(\lambda b) \end{bmatrix}$
F-F	$\begin{bmatrix} \alpha_n^J(\lambda a) & \alpha_n^J(\lambda b) & \beta_n^J(\lambda a)+\frac{1-\nu}{b}\gamma_n^J(\lambda a) & \beta_n^J(\lambda b)+\frac{1-\nu}{b}\gamma_n^J(\lambda b) \\ \alpha_n^Y(\lambda a) & \alpha_n^Y(\lambda b) & \beta_n^Y(\lambda a)+\frac{1-\nu}{b}\gamma_n^Y(\lambda a) & \beta_n^Y(\lambda b)+\frac{1-\nu}{b}\gamma_n^Y(\lambda b) \\ \alpha_n^I(\lambda a) & \alpha_n^I(\lambda b) & \beta_n^I(\lambda a)+\frac{1-\nu}{b}\gamma_n^I(\lambda a) & \beta_n^I(\lambda b)+\frac{1-\nu}{b}\gamma_n^I(\lambda b) \\ \alpha_n^K(\lambda a) & \alpha_n^K(\lambda b) & \beta_n^K(\lambda a)+\frac{1-\nu}{b}\gamma_n^K(\lambda a) & \beta_n^K(\lambda b)+\frac{1-\nu}{b}\gamma_n^K(\lambda b) \end{bmatrix}$

Table 2. Spurious eigenequations for the annular plate.

	$[Sb_n]$	B.C. of the simply-connected plate
u, θ Eqs.(2) and (3)	$\begin{bmatrix} J_n(\lambda b) & I_n(\lambda b) \\ \lambda(J'_n(\lambda b)) & \lambda(I'_n(\lambda b)) \end{bmatrix}$	$u=0, \quad \theta=0$
u,m Eqs.(2) and (4)	$\begin{bmatrix} J_n(\lambda b) & I_n(\lambda a) \\ \alpha_n^J(\lambda b) & \alpha_n^I(\lambda b) \end{bmatrix}$	$u=0, \quad m=0$
u,v Eqs.(2) and (5)	$\begin{bmatrix} J_n(\lambda b) & I_n(\lambda b) \\ [\beta_n^J(\lambda b)+\frac{(1-\nu)}{b}\gamma_n^J(\lambda b)] & [\beta_n^I(\lambda b)+\frac{(1-\nu)}{b}\gamma_n^I(\lambda b)] \end{bmatrix}$	$u=0, \quad v=0$
θ,m Eqs.(3) and (4)	$\begin{bmatrix} \lambda J'_n(\lambda b) & \lambda I'_n(\lambda a) \\ \alpha_n^J(\lambda b) & \alpha_n^I(\lambda b) \end{bmatrix}$	$\theta=0, \quad m=0$
θ,v Eqs.(3) and (5)	$\begin{bmatrix} \lambda J'_n(\lambda b) & \lambda I'_n(\lambda a) \\ [\beta_n^J(\lambda b)+\frac{(1-\nu)}{b}\gamma_n^J(\lambda b)] & [\beta_n^I(\lambda b)+\frac{(1-\nu)}{b}\gamma_n^I(\lambda b)] \end{bmatrix}$	$\theta=0, \quad v=0$
m,v Eqs.(4) and (5)	$\begin{bmatrix} \alpha_n^J(\lambda b) & \alpha_n^I(\lambda b) \\ [\beta_n^J(\lambda b)+\frac{(1-\nu)}{b}\gamma_n^J(\lambda b)] & [\beta_n^I(\lambda b)+\frac{(1-\nu)}{b}\gamma_n^I(\lambda b) \end{bmatrix}$	$m=0, \quad v=0$

where $\alpha_n^{\aleph}(\lambda a)=\lambda^2\aleph''_n(\lambda a)+\nu[\frac{1}{a}\lambda\aleph'_n(\lambda a)-(\frac{n}{a})^2\aleph_n(\lambda a)]$,

$$\beta_n^{\aleph}(\lambda a)=\lambda^3\aleph'''_n(\lambda a)+\nu[\frac{1}{a}\lambda^2\aleph''_n(\lambda a)-(\frac{n}{a})^2\aleph'_n(\lambda a)-\frac{1}{a^2}\lambda\aleph'_n(\lambda a)+(\frac{2n^2}{a^3})\aleph_n(\lambda a)]$$

and $\gamma_n^{\aleph}=-n^2[\frac{1}{a^2}\aleph_n(\lambda a)+\frac{\lambda}{a}\aleph'(\lambda a)]$, $\aleph$ can be Y, J, K, I.

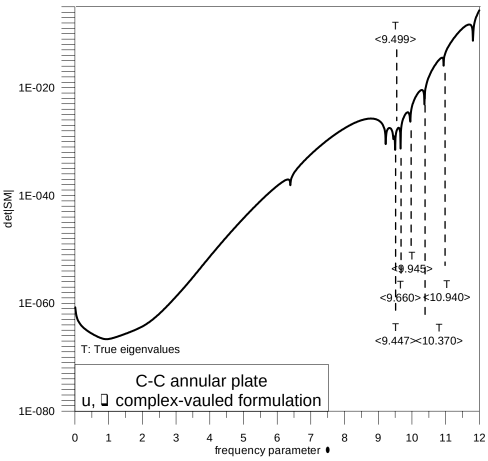

Figure 1.(a) $Det[SM^{cc}]$ v.s. λ (C-C annular plate)

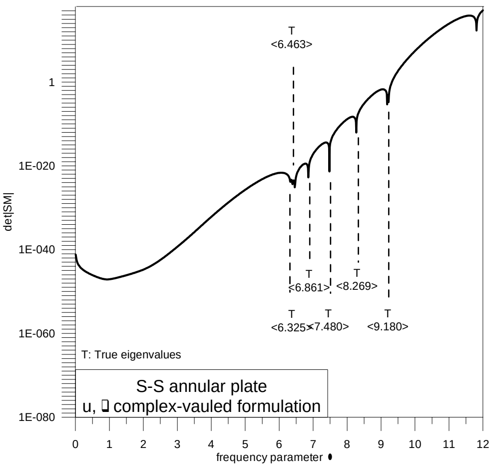

Figure 1.(b) $Det[SM^{SS}]$ v.s. λ (S-S annular plate)

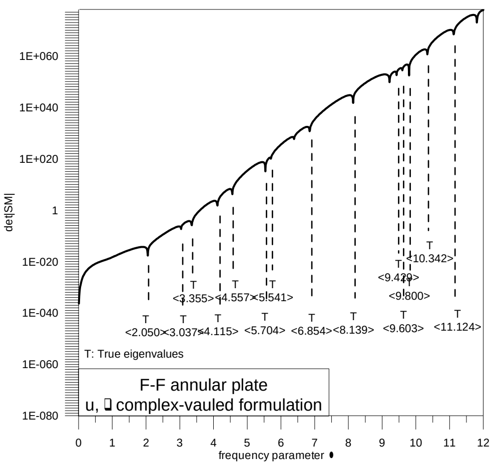

Figure 1.(c) $Det[SM^{FF}]$ v.s. λ (F-F annular plate)

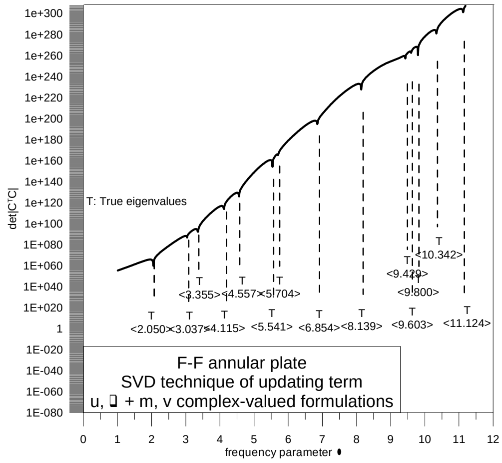

Figure 1.(d) SVD updating term (F-F annular plate)

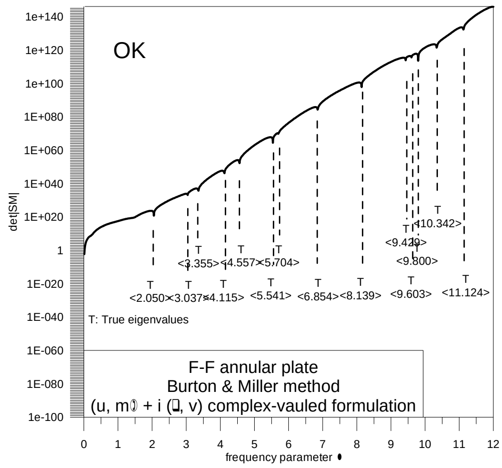

Figure 1.(e) The Burton & Miller method (F-F annular plate)

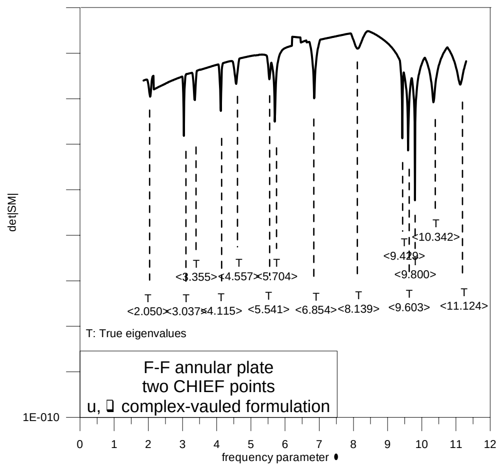

Figure 1.(f) The CHIEF method (F-F annular plate)