中華民國振動與噪音工程學會
九十二年年會暨第十一屆學術研討會

# Mathematical analysis and numerical study of the true and spurious eigenequations for free vibration of plate using an imaginary-part BEM

# 虛部邊界元素法之板自由振動真假特徵方程之數學分析及數值研究

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# Outlines

1. Introduction
2. Boundary integral equations for plate eigenproblems
3. Mathematical analysis (Discrete system)
4. Treatments of the spurious eigenvalues
5. Conclusions

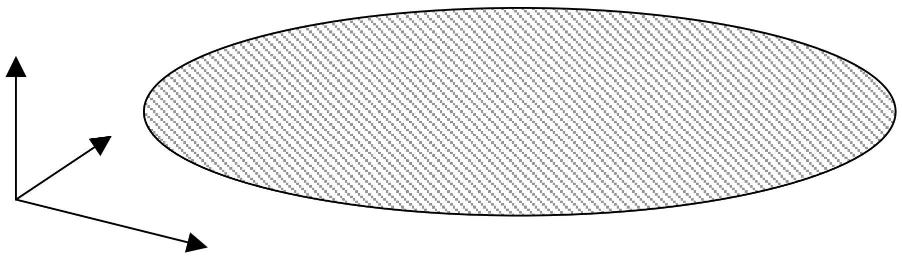
# Introduction

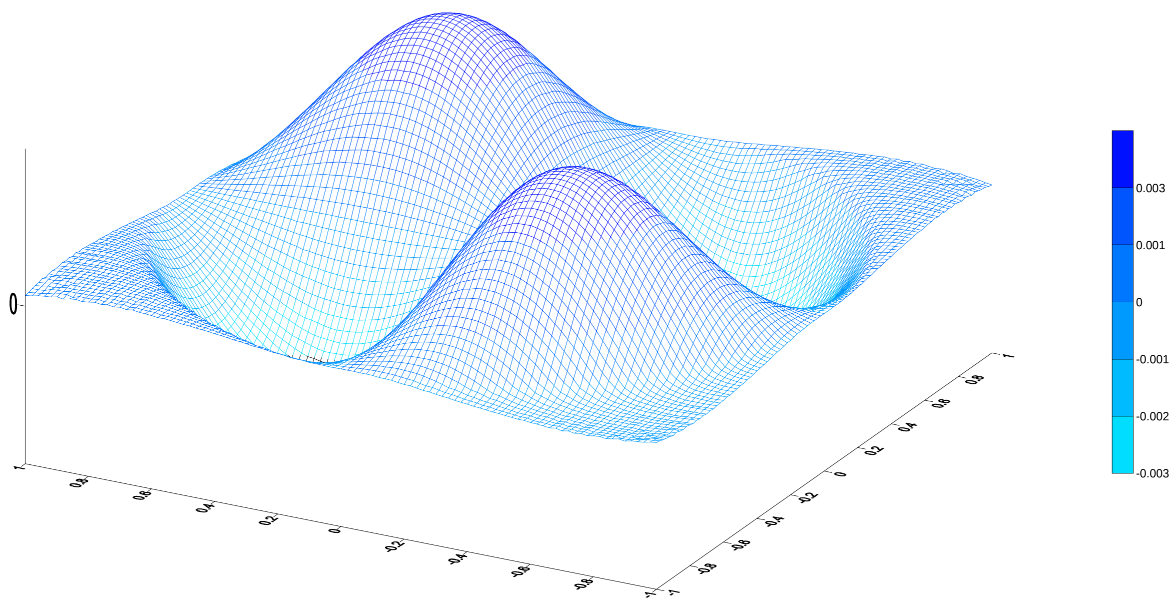
G.E. $$\nabla^4 u(x) = \lambda^4 u(x), x \in \Omega$$

$$\lambda^4 = \frac{\omega^2 \rho h}{D} \qquad D = \frac{Eh^3}{12(1-\nu^2)}$$

| | | | |
|---|---|---|---|
| $u$ | lateral displacement | $\lambda$ | frequency parameter |
| $\omega$ | circular frequency | $\rho$ | surface density |
| $h$ | plate thickness | $D$ | flexural rigidity |
| $E$ | Young's modulus | $\nu$ | Poisson ratio |
| $\Omega$ | domain | | |

# Free vibration of plate



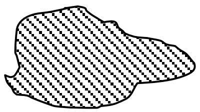


# Literature review

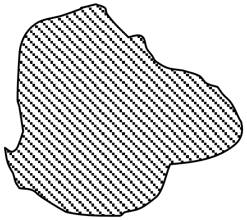
1. Tai and Shaw 1974 (complex-valued BEM)
2. De Mey 1976, Hutchinson and Wong 1979 (real-part kernel)
3. Wong and Hutchinson (real-part direct BEM program)
4. Shaw 1979, Hutchinson 1988, Niwa *et al*. 1982 (real-part kernel)
5. Tai and Shaw 1974, Chen *et al*. Proc. Roy. Soc. Lon. Ser. A, 2001, 2003 (multiply-connected problem)
6. Chen *et al*. (dual formulation, domain partition, SVD updating technique, CHEEF method)

# Motivation

Simply-connected problem



| | Real | Imaginary | Complex |
|---|---|---|---|
| Saving CPU time | Yes | Yes | No |
| Avoid singular integral | No | Yes | No |
| Spurious eigenvalues | Appear | Appear | No |



Real-part BEM (N.G.)
Imaginary-part BEM (N.G.)
Complex-valued (OK)

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# Boundary integral equations for plate eigenproblems

(1) Displacement

$$u(x) = \int_B \{-U(s,x)v(s) + \Theta(s,x)m(s) - M(s,x)\theta(s) + V(s,x)u(s)\}\, dB(s),$$

$K_\theta(\cdot)$

(2) Slope

$$\theta(x) = \int_B \{-U_\theta(s,x)v(s) + \Theta_\theta(s,x)m(s) - M_\theta(s,x)\theta(s) + V_\theta(s,x)u(s)\}\, dB(s),$$

$K_m(\cdot)$

(3) Normal moment

$$m(x) = \int_B \{-U_m(s,x)v(s) + \Theta_m(s,x)m(s) - M_m(s,x)\theta(s) + V_m(s,x)u(s)\}\, dB(s),$$

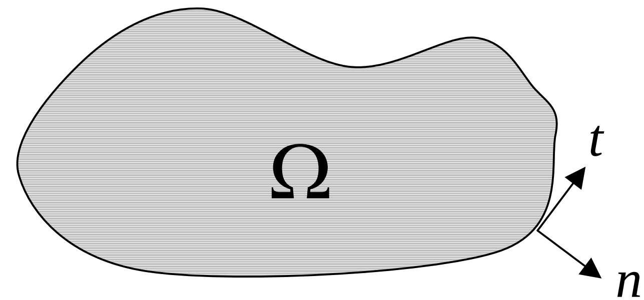
$K_v(\cdot)$

(4) Effective shear force

$$v(x) = \int_B \{-U_v(s,x)v(s) + \Theta_v(s,x)m(s) - M_v(s,x)\theta(s) + V_v(s,x)u(s)\}\, dB(s),$$

$x \in \Omega$

# Operators




Slope

$$K_\theta(\cdot) = \frac{\partial(\cdot)}{\partial n}$$

Normal moment

$$K_m(\cdot) = \nu\nabla^2(\cdot) + (1-\nu)\frac{\partial^2(\cdot)}{\partial n^2}$$

Effective shear force

$$K_v(\cdot) = \frac{\partial\nabla^2(\cdot)}{\partial n} + (1-\nu)\frac{\partial}{\partial t}\left(\frac{\partial^2(\cdot)}{\partial n\partial t}\right)$$

# Kernel functions

Fundamental solution

$$\nabla^4 U_c(s,x) - \lambda^4 U_c(s,x) = -\delta(x-s)$$

$$U_c(s,x) = \frac{1}{8\lambda^2}\left(Y_0(\lambda r) + iJ_0(\lambda r) + \frac{2}{\pi}(K_0(\lambda r) + iI_0(\lambda r))\right)$$

Kernel functions

$$U(s,x) = \mathrm{Im}[U_c]$$

$$\Theta(s,x) = K_\theta(U(s,x))$$

$$M(s,x) = K_m(U(s,x))$$

$$V(s,x) = K_v(U(s,x))$$

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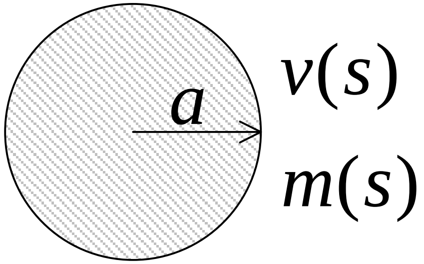
# Mathematical analysis (Discrete system)

For clamped circular plate (u=0 and q=0)

$$0=[U]\{v\}+[\Theta]\{m\}$$

$$0=[U_\theta]\{v\}+[\Theta_\theta]\{m\}$$




$$[SM]\begin{Bmatrix} v \\ m \end{Bmatrix}=0$$

$$[SM]=\begin{bmatrix} U & \Theta \\ U_\theta & \Theta_\theta \end{bmatrix}_{4N\times 4N}$$

# Expansion formulae

Degenerate kernels (separable kernels)

$$J_0(\lambda r) = \begin{cases} \sum_{m=-\infty}^{\infty} J_m(\lambda \bar{\rho}) J_m(\lambda \rho) \cos(m(\bar{\phi} - \phi)), & \bar{\rho} > \rho \\ \sum_{m=-\infty}^{\infty} J_m(\lambda \rho) J_m(\lambda \bar{\rho}) \cos(m(\bar{\phi} - \phi)), & \rho > \bar{\rho} \end{cases}$$

$$I_0(\lambda r) = \begin{cases} \sum_{m=-\infty}^{\infty} (-1)^m I_m(\lambda \bar{\rho}) I_m(\lambda \rho) \cos(m(\bar{\phi} - \phi)), & \bar{\rho} > \rho \\ \sum_{m=-\infty}^{\infty} (-1)^m I_m(\lambda \rho) I_m(\lambda \bar{\rho}) \cos(m(\bar{\phi} - \phi)), & \rho > \bar{\rho} \end{cases}$$

# Circulant

$$[U]=\begin{bmatrix} z_0 & z_1 & z_2 & \cdots & z_{2N-1} \\ z_{2N-1} & z_0 & z_1 & \cdots & z_{2N-2} \\ z_{2N-2} & z_{2N-1} & z_0 & \cdots & z_{2N-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ z_1 & z_2 & z_3 & z_{2N-1} & z_0 \end{bmatrix}_{2N\times 2N}$$

$$z_m=\int_{(m-\frac{1}{2})\Delta\bar{\phi}}^{(m+\frac{1}{2})\Delta\bar{\phi}}[-U(a,\bar{\phi},a,\phi)]a\,d\bar{\phi}\approx -U(a,\bar{\phi}_m,a,\phi)a\,\Delta\bar{\phi},$$

$$m=0,1,2,\ldots,2N-1$$

# The properties of the circulant

$$\mu_{\bullet}^{[U]} = \sum_{m=0}^{2N-1} z_m \alpha_{\bullet}^{m} = \sum_{m=0}^{2N-1} z_m e^{i\frac{2\pi m \bullet}{2N}},$$

$$\bullet = 0, \pm 1, \pm 2, \ldots, \pm (N-1), N$$

$$\alpha_{\bullet} = e^{i\frac{2\pi \bullet}{2N}}, \quad \bullet = 0, \pm 1, \pm 2, .., \pm N-1, N$$

$$or \quad \bullet = 0, 1, 2, .., 2N-1$$

$$\mu_{\bullet}^{[U]} = \lim_{N\to\infty} \sum_{m=0}^{2N-1} \cos(m \bullet \Delta\overline{\phi})[-U(a, \overline{\phi}_m, a, 0)] a \, \Delta\overline{\phi}$$

$$\approx \int_0^{2\pi} \cos(\bullet \overline{\phi})[-U(a, \overline{\phi}, a, 0)] a \, d\overline{\phi}$$

# Eigenvalues of the four matrices

$[U]$ $$\mu_{\bullet}^{[U]} = -\frac{\pi a}{4\lambda^2}[J_{\bullet}(\lambda a)J_{\bullet}(\lambda a) + \frac{2}{\pi}(-1)^{\bullet} I_{\bullet}(\lambda a)I_{\bullet}(\lambda a)]$$

$[\Theta]$ $$\mu_{\bullet}^{[\Theta]} = \frac{\pi a}{4\lambda}[J_{\bullet}(\lambda a)J'_{\bullet}(\lambda a) + \frac{2}{\pi}(-1)^{\bullet} I_{\bullet}(\lambda a)I'_{\bullet}(\lambda a)]$$

$[U_\theta]$ $$\kappa_{\bullet}^{[U]} = -\frac{\pi a}{4\lambda}[J'_{\bullet}(\lambda a)J_{\bullet}(\lambda a) + \frac{2}{\pi}(-1)^{\bullet} I'_{\bullet}(\lambda a)I_{\bullet}(\lambda a)]$$

$[\Theta_\theta]$ $$\kappa_{\bullet}^{[\Theta]} = \frac{\pi a}{4}[J'_{\bullet}(\lambda a)J'_{\bullet}(\lambda a) + \frac{2}{\pi}(-1)^{\bullet} I'_{\bullet}(\lambda a)I'_{\bullet}(\lambda a)]$$

$$\bullet = 0, \pm 1, \pm 2, \ldots, \pm(N-1), N$$

# Eigenvalue decomposition (similar matrices)

$$[U]=\Phi\Sigma_U\Phi^{-1}$$

$$=\Phi\begin{bmatrix} \mu_0^{[U]} & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \mu_1^{[U]} & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \mu_{-1}^{[U]} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \mu_{N-1}^{[U]} & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & \mu_{-(N-1)}^{[U]} & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & \mu_N^{[U]} \end{bmatrix}\Phi^{-1}$$

$$[\Theta]=\Phi\Sigma_\Theta\Phi^{-1}$$

$$=\Phi\begin{bmatrix} \mu_0^{[\Theta]} & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \mu_1^{[\Theta]} & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \mu_{-1}^{[\Theta]} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \mu_{N-1}^{[\Theta]} & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & \mu_{-(N-1)}^{[\Theta]} & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & \mu_N^{[\Theta]} \end{bmatrix}\Phi^{-1}$$

$$[U_\theta]=\Phi\Sigma_{U_\theta}\Phi^{-1}$$

$$=\Phi\begin{bmatrix} \kappa_0^{[U]} & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \kappa_1^{[U]} & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \kappa_{-1}^{[U]} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \kappa_{N-1}^{[U]} & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & \kappa_{-(N-1)}^{[U]} & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & \kappa_N^{[U]} \end{bmatrix}\Phi^{-1}$$

$$[\Theta_\theta]=\Phi\Sigma_{\Theta_\theta}\Phi^{-1}$$

$$=\Phi\begin{bmatrix} \kappa_0^{[\Theta]} & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \kappa_1^{[\Theta]} & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \kappa_{-1}^{[\Theta]} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \kappa_{N-1}^{[\Theta]} & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & \kappa_{-(N-1)}^{[\Theta]} & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & \kappa_N^{[\Theta]} \end{bmatrix}\Phi^{-1}$$

# Determinant

$$[SM]=\begin{bmatrix}\Phi\Sigma_U\Phi^{-1} & \Phi\Sigma_\Theta\Phi^{-1}\\ \Phi\Sigma_{U_\theta}\Phi^{-1} & \Phi\Sigma_{\Theta_\theta}\Phi^{-1}\end{bmatrix}$$

$$=\begin{bmatrix}\Phi & 0\\ 0 & \Phi\end{bmatrix}\begin{bmatrix}\Sigma_U & \Sigma_\Theta\\ \Sigma_{U_\theta} & \Sigma_{\Theta_\theta}\end{bmatrix}\begin{bmatrix}\Phi & 0\\ 0 & \Phi\end{bmatrix}^{-1}$$

$$\det[SM]=\det\begin{bmatrix}\Sigma_U & \Sigma_\Theta\\ \Sigma_{U_\theta} & \Sigma_{\Theta_\theta}\end{bmatrix}=\prod_{\bullet=-(N-1)}^{N}(\mu_\bullet^{[U]}\kappa_\bullet^{[\Theta]}-\mu_\bullet^{[\Theta]}\kappa_\bullet^{[U]})$$

# Eigenequations (imaginary-part BEM for clamped plate)

$$\det[SM]$$

$$= \prod_{\bullet=-(N-1)}^{N} \frac{-\pi a^2}{16\lambda^2}\{[J_\bullet(\lambda a)J_\bullet(\lambda a)+\frac{2}{\pi}(-1)^\bullet I_\bullet(\lambda a)I_\bullet(\lambda a)][J'_\bullet(\lambda a)J'_\bullet(\lambda a)+\frac{2}{\pi}(-1)^\bullet I'_\bullet(\lambda a)I'_\bullet(\lambda a)]$$

$$-[J_\bullet(\lambda a)J'_\bullet(\lambda a)+\frac{2}{\pi}(-1)^\bullet J_\bullet(\lambda a)I'_\bullet(\lambda a)][J'_\bullet(\lambda a)J_\bullet(\lambda a)+\frac{2}{\pi}(-1)^\bullet I'_\bullet(\lambda a)I_\bullet(\lambda a)]\}$$

$$= \prod_{\bullet=-(N-1)}^{N} (-1)^2 \frac{-\pi a^2}{16\lambda^2}[I_{\bullet+1}(\lambda a)J_\bullet(\lambda a)+I_\bullet(\lambda a)J_{\bullet+1}(\lambda a)]$$

$$\{I_{\bullet+1}(\lambda a)J_\bullet(\lambda a)+I_\bullet(\lambda a)J_{\bullet+1}(\lambda a)\}$$

$$[I_{\bullet+1}(\lambda a)J_\bullet(\lambda a)+I_\bullet(\lambda a)J_{\bullet+1}(\lambda a)]\ \{I_{\bullet+1}(\lambda a)J_\bullet(\lambda a)+I_\bullet(\lambda a)J_{\bullet+1}(\lambda a)\}=0$$

Spurious eigenequation

True eigenequation



MSVLAB
HRE, NTOU

# Comparisons of Leissa and present method

| | Leissa (Kitahara) | | Present method |
|---|---|---|---|
| Clamped | $I_{\bullet+1}J_{\bullet}+I_{\bullet}J_{\bullet+1}=0$ | $\equiv$ | $I_{\bullet+1}J_{\bullet}+I_{\bullet}J_{\bullet+1}=0$ |
| Simply-supported | $\frac{J_{\bullet+1}}{J_{\bullet}}+\frac{I_{\bullet+1}}{I_{\bullet}}=\frac{2\lambda}{(1-\nu)}$ | $\equiv$ | $(1-\nu)(I_{\bullet}J_{\bullet+1}+I_{\bullet+1}J_{\bullet})-2\lambda I_{\bullet}J_{\bullet}=0$ |
| Free | $\frac{\lambda^2 J_{\bullet}+(1-\nu)[\lambda J'_{\bullet}-\bullet^2 J_{\bullet}]}{\lambda^2 I_{\bullet}-(1-\nu)[\lambda I'_{\bullet}-\bullet^2 I_{\bullet}]}=\frac{\lambda^3 I'_{\bullet}+(1-\nu)\bullet^2[\lambda J'_{\bullet}-J_{\bullet}]}{\lambda^3 I'_{\bullet}-(1-\nu)\bullet^2[\lambda I'_{\bullet}-I_{\bullet}]}$ <br> $J'_{\bullet}$ | $\equiv$ | $\lambda(1-\nu)[-4\bullet^2(\bullet-1)I_{\bullet}J_{\bullet}-2\lambda^2 I_{\bullet+1}J_{\bullet+1}]$ $+2\bullet\lambda^2(1-\nu)(1-\bullet)(I_{\bullet+1}J_{\bullet}-I_{\bullet}J_{\bullet+1})$ $+[\bullet^2(1-\nu)^2(\bullet^2-1)+\lambda^4](I_{\bullet+1}J_{\bullet}+I_{\bullet}J_{\bullet+1})=0$ |

where $\bullet = 0,\pm1,\pm2,\pm3,\ldots\ldots$

# Spurious eigenequations using the imaginary-part BEM

| Eqs. number | Spurious eigenequation using the imaginary-part BEM |
|---|---|
| $u, \theta$ (1) and (2) | $I_{\bullet+1}J_{\bullet}+I_{\bullet}J_{\bullet+1}=0$ |
| $u, m$ (1) and (3) | $(1-\nu)(I_{\bullet}J_{\bullet+1}+I_{\bullet+1}J_{\bullet})-2\lambda\rho I_{\bullet}J_{\bullet}=0$ |
| $u, v$ (1) and (4) | $\bullet^{2}(1-\nu)(I_{\bullet}J_{\bullet+1}+I_{\bullet+1}J_{\bullet})-2\lambda\rho\bullet I_{\bullet}J_{\bullet}+\lambda^{2}\rho^{2}(I_{\bullet}J_{\bullet+1}-I_{\bullet+1}J_{\bullet})=0$ |
| $\theta, m$ (2) and (3) | $\bullet^{2}(1-\nu)(I_{\bullet}J_{\bullet+1}+I_{\bullet+1}J_{\bullet})-2\lambda\rho\bullet I_{\bullet}J_{\bullet}+\lambda^{2}\rho^{2}(I_{\bullet}J_{\bullet+1}-I_{\bullet+1}J_{\bullet})=0$ |
| $\theta, v$ (2) and (4) | $2\lambda\rho(\bullet^{2}I_{\bullet}J_{\bullet}-\lambda^{2}\rho^{2}I_{\bullet+1}J_{\bullet+1})+2\lambda^{2}\rho^{2}\bullet(I_{\bullet+1}J_{\bullet}-I_{\bullet}J_{\bullet+1})$ $-\bullet^{2}(1-\nu)](I_{\bullet+1}J_{\bullet}+I_{\bullet}J_{\bullet+1})=0$ |
| $m, v$ (3) and (4) | $\lambda\rho(1-\nu)[-4\bullet^{2}(\bullet-1)I_{\bullet}J_{\bullet}-2\lambda^{2}\rho^{2}I_{\bullet+1}J_{\bullet+1}]+2\bullet\lambda^{2}\rho^{2}(1-\nu)(1-\bullet)$ $(I_{\bullet+1}J_{\bullet}-I_{\bullet}J_{\bullet+1})+[\bullet^{2}(1-\nu)^{2}(\bullet^{2}-1)+\lambda^{4}\rho^{4}](I_{\bullet+1}J_{\bullet}+I_{\bullet}J_{\bullet+1})=0$ |

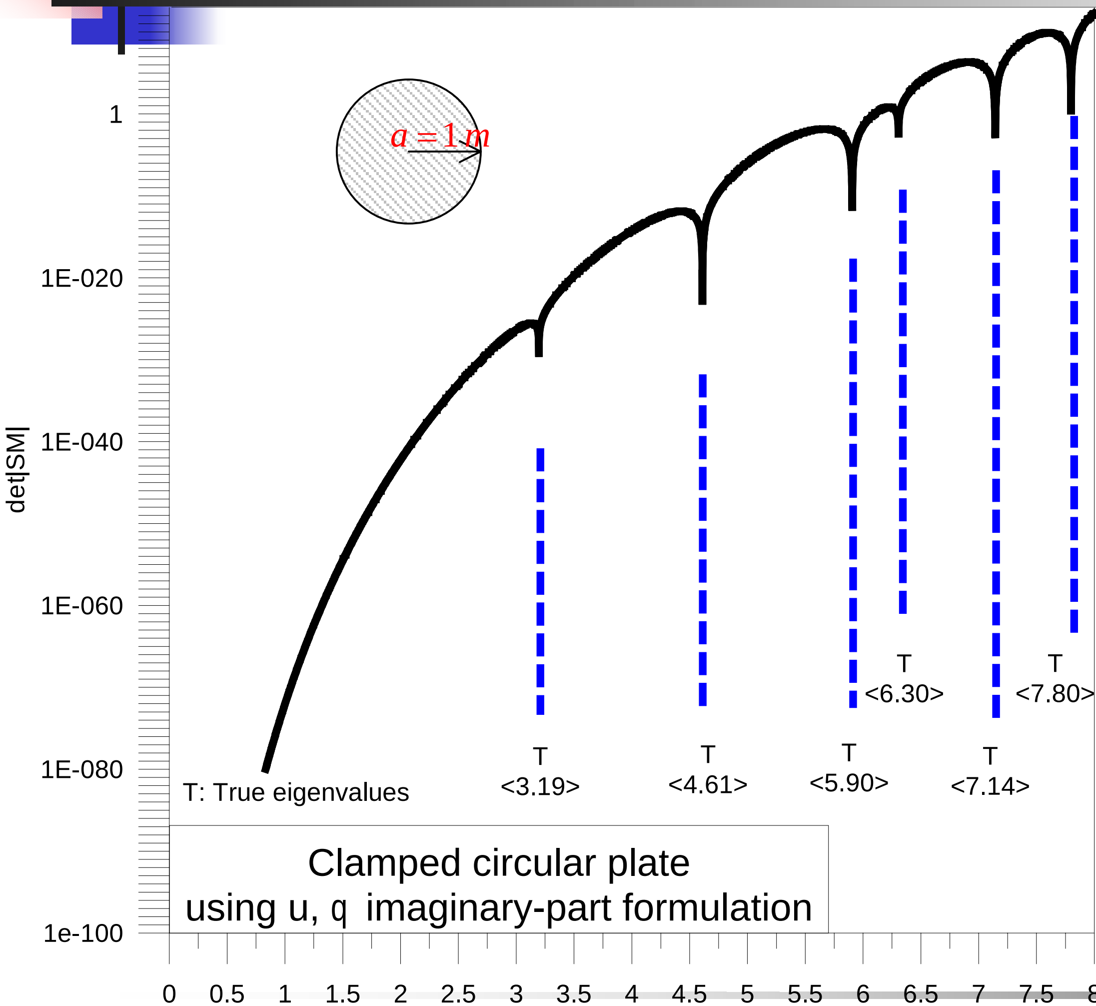
where $\bullet=0,\pm1,\pm2,\pm3,\ldots\ldots$

# True and spurious eigenequations

Imaginary-part BEM

| | True | Spurious |
|---|---|---|
| Plate (clamped) | $I_{\bullet+1}J_{\bullet}+I_{\bullet}J_{\bullet+1}=0$ | $I_{\bullet+1}J_{\bullet}+I_{\bullet}J_{\bullet+1}=0$ |
| Membrane (Dirichlet) | $J_{\bullet}=0$ | $J_{\bullet}=0$ |

# Determinant v.s frequency parameter (clamped)



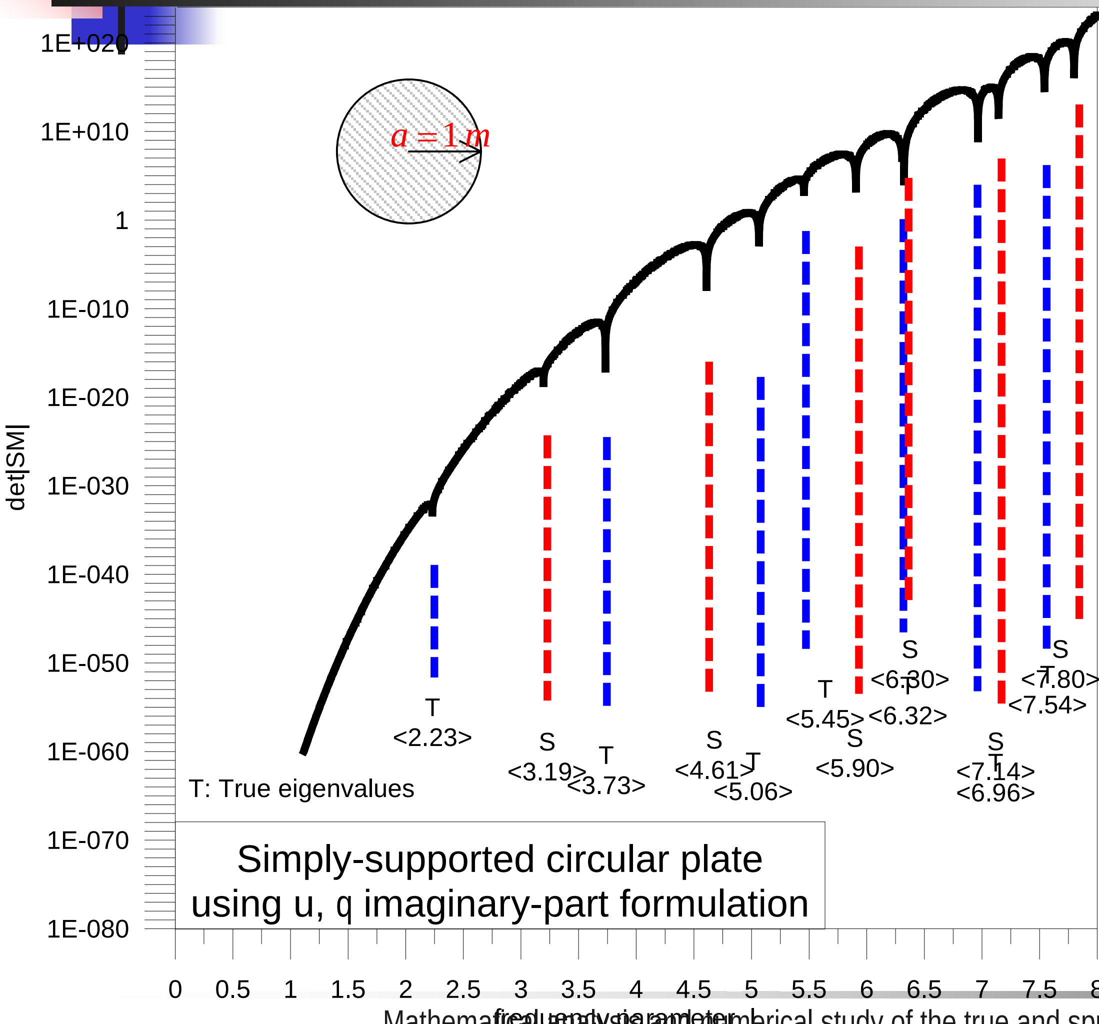

True eigenequation

$$\{I_{\bullet+1}(\lambda a)J_{\bullet}(\lambda a)+I_{\bullet}(\lambda a)J_{\bullet+1}(\lambda a)\}=0$$

Spurious eigenequation

$$\{I_{\bullet+1}(\lambda a)J_{\bullet}(\lambda a)+I_{\bullet}(\lambda a)J_{\bullet+1}(\lambda a)\}=0$$

# Determinant v.s frequency parameter simply-supported)



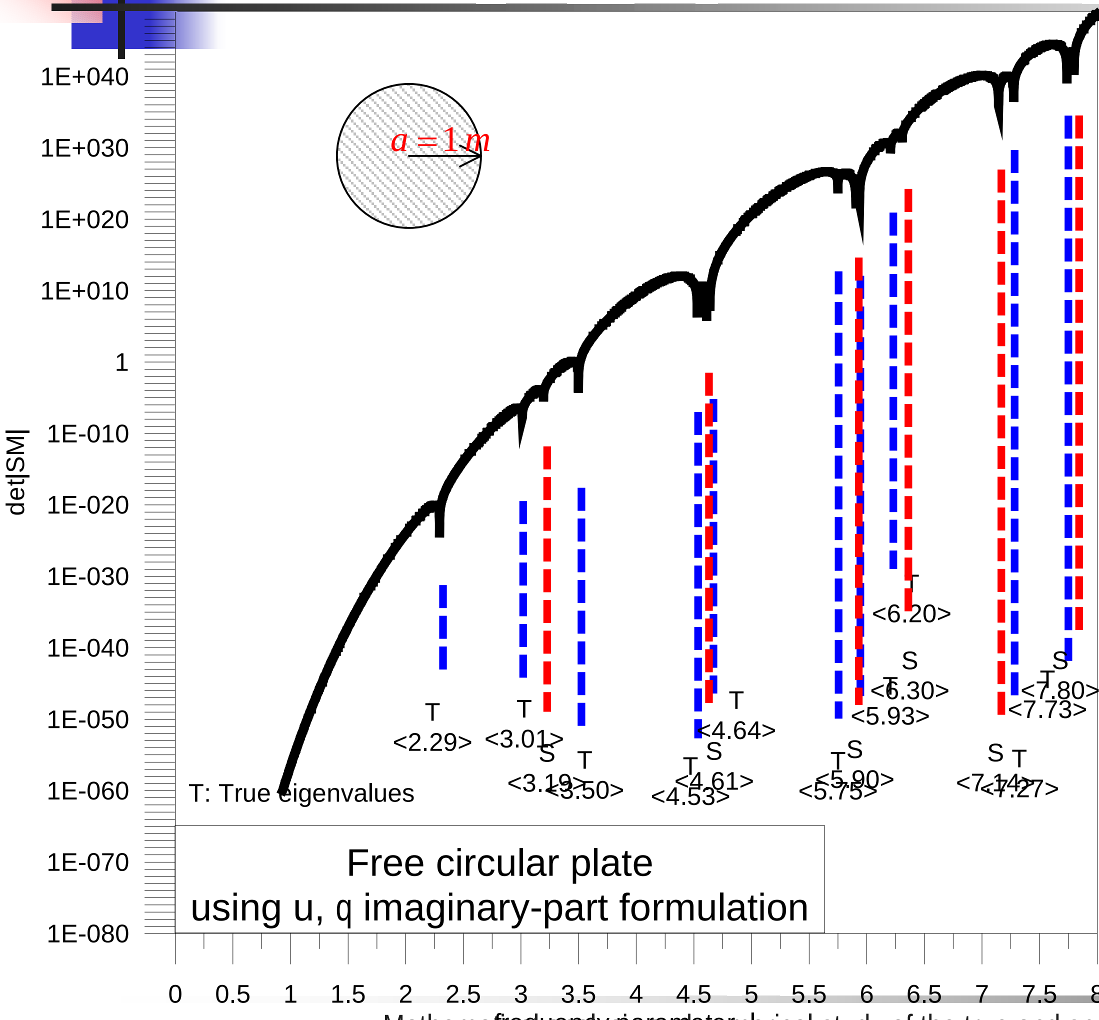

## True eigenequation

$$[(1-\nu)(I_{\bullet}(\lambda a)J_{\bullet+1}(\lambda a)+I_{\bullet+1}(\lambda a)J_{\bullet}(\lambda a)) - 2\lambda a I_{\bullet}(\lambda a)J_{\bullet}(\lambda a)] = 0$$

## Spurious eigenequation

$$\{I_{\bullet+1}(\lambda a)J_{\bullet}(\lambda a)+I_{\bullet}(\lambda a)J_{\bullet+1}(\lambda a)\} = 0$$

# Determinant v.s frequency parameter (free)




## True eigenequation

$$\lambda a(1-\nu)[-4\bullet^2(\bullet-1)I_\bullet(\lambda a)J_\bullet(\lambda a)-2\lambda^2a^2I_{\bullet+1}(\lambda a)J_{\bullet+1}(\lambda a)]$$
$$+2\bullet\lambda^2a^2(1-\nu)(1-\bullet)(I_{\bullet+1}(\lambda a)J_\bullet(\lambda a)-I_\bullet(\lambda a)J_{\bullet+1}(\lambda a))$$
$$+[\bullet^2(1-\nu)^2(\bullet^2-1)+\lambda^4a^4](I_{\bullet+1}(\lambda a)J_\bullet(\lambda a)+I_\bullet J_{\bullet+1}(\lambda a))=0$$

## Spurious eigenequation

$$\{I_{\bullet+1}(\lambda a)J_\bullet(\lambda a)+I_\bullet(\lambda a)J_{\bullet+1}(\lambda a)\}=0$$

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# SVD updating term (clamped plate)

u, q formulation

$$[SM_1^c]\begin{Bmatrix} v \\ m \end{Bmatrix} = 0 \quad [SM_1^c] = \begin{bmatrix} U & \Theta \\ U_\theta & \Theta_\theta \end{bmatrix}_{4N\times 4N}$$

m,v formulation

$$[SM_2^c]\begin{Bmatrix} v \\ m \end{Bmatrix} = 0 \quad [SM_2^c] = \begin{bmatrix} U_m & \Theta_m \\ U_v & \Theta_v \end{bmatrix}_{4N\times 4N}$$

SVD technique of updating term

$$[C]\begin{Bmatrix} v \\ m \end{Bmatrix} = 0 \qquad [C] = \begin{bmatrix} SM_1^c \\ SM_2^c \end{bmatrix}_{8N\times 4N}$$

# Determinant

$$[C]=\begin{bmatrix} SM_1^c \\ SM_2^c \end{bmatrix}=\begin{bmatrix} \Phi & 0 & 0 & 0 \\ 0 & \Phi & 0 & 0 \\ 0 & 0 & \Phi & 0 \\ 0 & 0 & 0 & \Phi \end{bmatrix}\begin{bmatrix} \Sigma_U & \Sigma_{U_\theta} \\ \Sigma_\Theta & \Sigma_{\Theta_\theta} \\ \Sigma_M & \Sigma_{M_\theta} \\ \Sigma_V & \Sigma_{V_\theta} \end{bmatrix}\begin{bmatrix} \Phi^{-1} & 0 \\ 0 & \Phi^{-1} \end{bmatrix}$$

$$\det\left[[C]^T[C]\right]=\prod_{\bullet=-(N-1)}^{N}[(\mu_\bullet^{[U]}\kappa_\bullet^{[\Theta]}-\kappa_\bullet^{[U]}\mu_\bullet^{[\Theta]})^2+(\mu_\bullet^{[U]}\varsigma_\bullet^{[\Theta]}-\varsigma_\bullet^{[U]}\mu_\bullet^{[\Theta]})^2+(\mu_\bullet^{[U]}\delta_\bullet^{[\Theta]}-\delta_\bullet^{[U]}\mu_\bullet^{[\Theta]})^2$$
$$+(\kappa_\bullet^{[U]}\varsigma_\bullet^{[\Theta]}-\varsigma_\bullet^{[U]}\kappa_\bullet^{[\Theta]})^2+(\kappa_\bullet^{[U]}\delta_\bullet^{[\Theta]}-\delta_\bullet^{[U]}\kappa_\bullet^{[\Theta]})^2+(\varsigma_\bullet^{[U]}\delta_\bullet^{[\Theta]}-\delta_\bullet^{[U]}\varsigma_\bullet^{[\Theta]})^2]$$

The only possibility for zero determinant

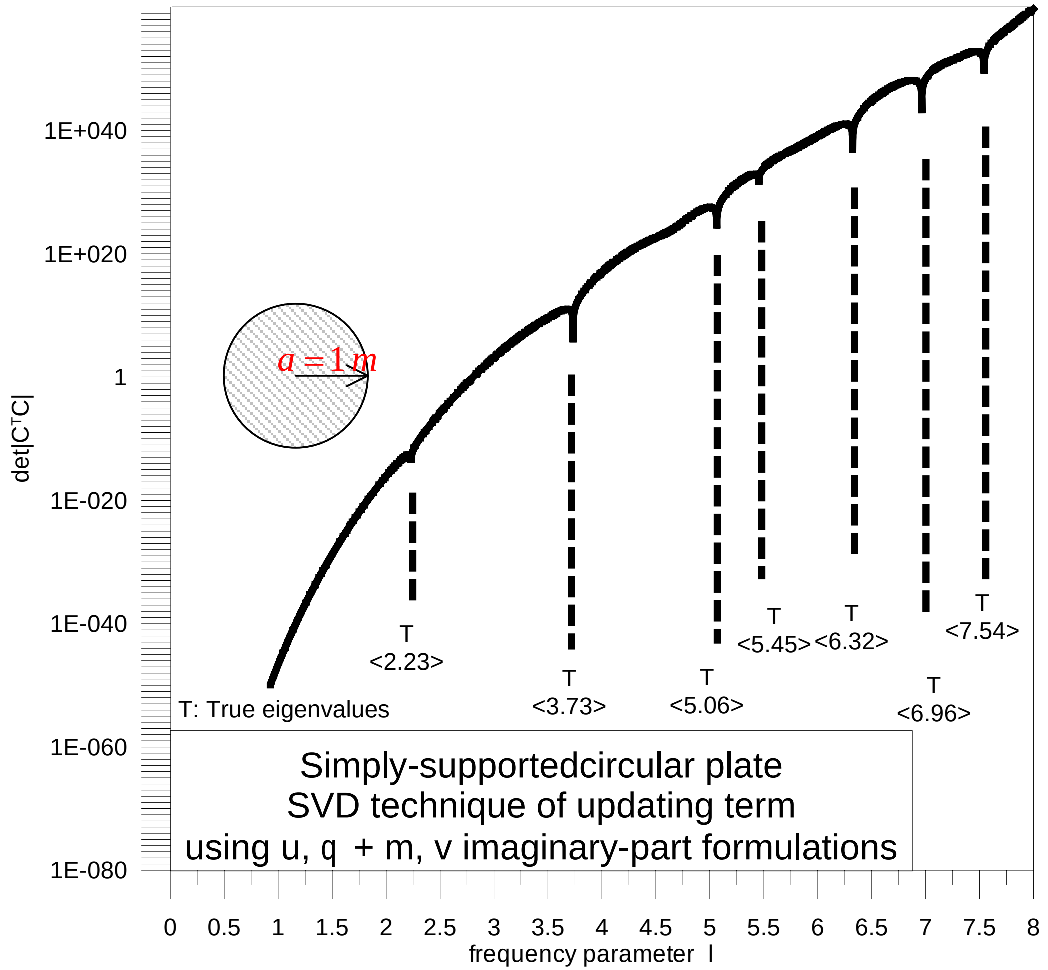
$(\mu_\bullet^{[U]}\kappa_\bullet^{[\Theta]}-\kappa_\bullet^{[U]}\mu_\bullet^{[\Theta]})^2=0,$ $(\mu_\bullet^{[U]}\delta_\bullet^{[\Theta]}-\delta_\bullet^{[U]}\mu_\bullet^{[\Theta]})^2=0,$

$(\mu_\bullet^{[U]}\varsigma_\bullet^{[\Theta]}-\varsigma_\bullet^{[U]}\mu_\bullet^{[\Theta]})^2=0,$ $(\kappa_\bullet^{[U]}\delta_\bullet^{[\Theta]}-\delta_\bullet^{[U]}\kappa_\bullet^{[\Theta]})^2=0,$

$(\kappa_\bullet^{[U]}\varsigma_\bullet^{[\Theta]}-\varsigma_\bullet^{[U]}\kappa_\bullet^{[\Theta]})^2=0,$ $(\varsigma_\bullet^{[U]}\delta_\bullet^{[\Theta]}-\delta_\bullet^{[U]}\varsigma_\bullet^{[\Theta]})^2=0.$

at the same time for the same $\bullet$.

# SVD updating term (imaginary-part BEM)



# Flowchart for clamped plate using the imag.-part BEM

G.E. $\nabla^4 u(x) = \lambda^4 u(x), \quad x \in \Omega$

↓

BIE

↓

BEM

↓ B.C. $u = 0 \quad \theta = 0$

$$0 = [U]\{v\} + [\Theta]\{m\}$$
$$0 = [U_\theta]\{v\} + [\Theta_\theta]\{m\}$$

→ Construction of [SM]

$$[SM] = \begin{bmatrix} U & \Theta \\ U_\theta & \Theta_\theta \end{bmatrix}_{4N \times 4N}$$

↓

$$\det[SM] = \prod_{\bullet=-(N-1)}^{N} (\mu_\bullet^{[U]} \kappa_\bullet^{[\Theta]} - \mu_\bullet^{[\Theta]} \kappa_\bullet^{[U]}) = 0$$

↓

True and spurious eigenvalues

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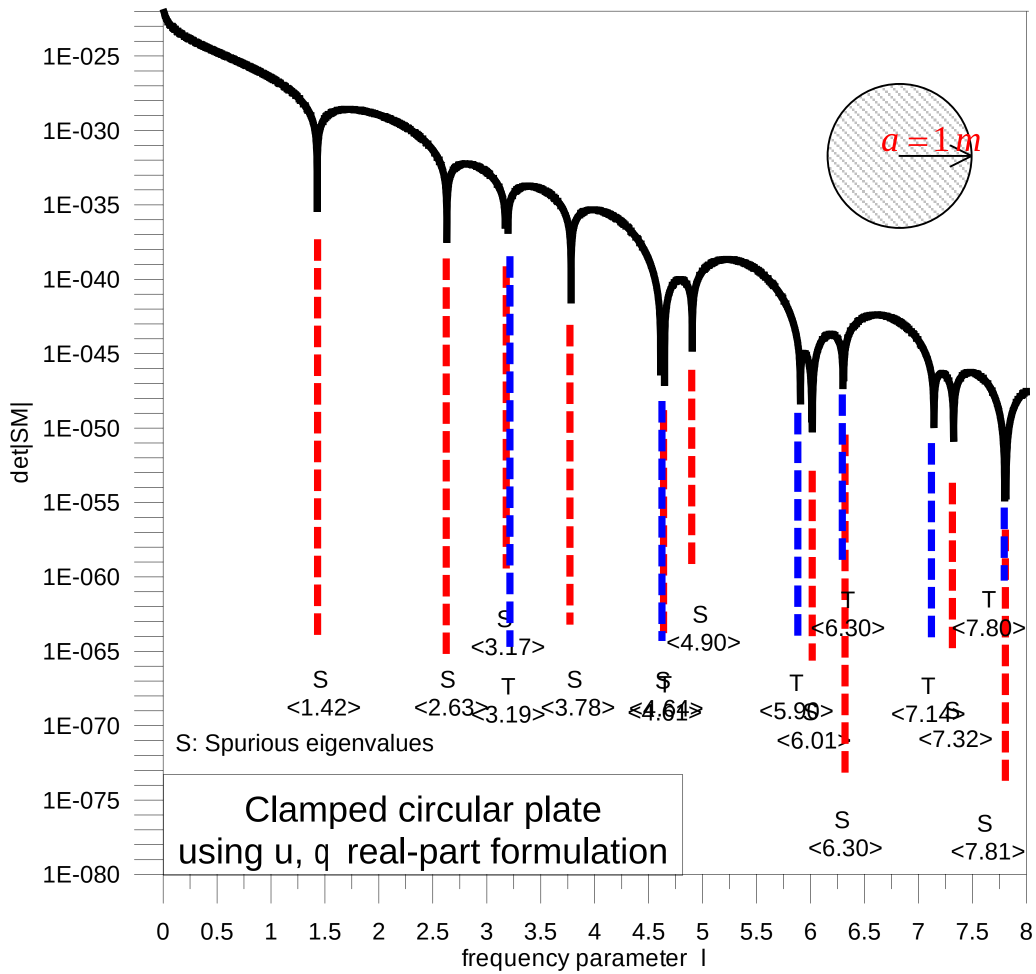
# Conclusions

1. The true eigenequations depend on the boundary condition.
2. The spurious eigenequations depend on the formulation.
3. The SVD updating term was successfully applied to suppress the spurious eigenvalues.

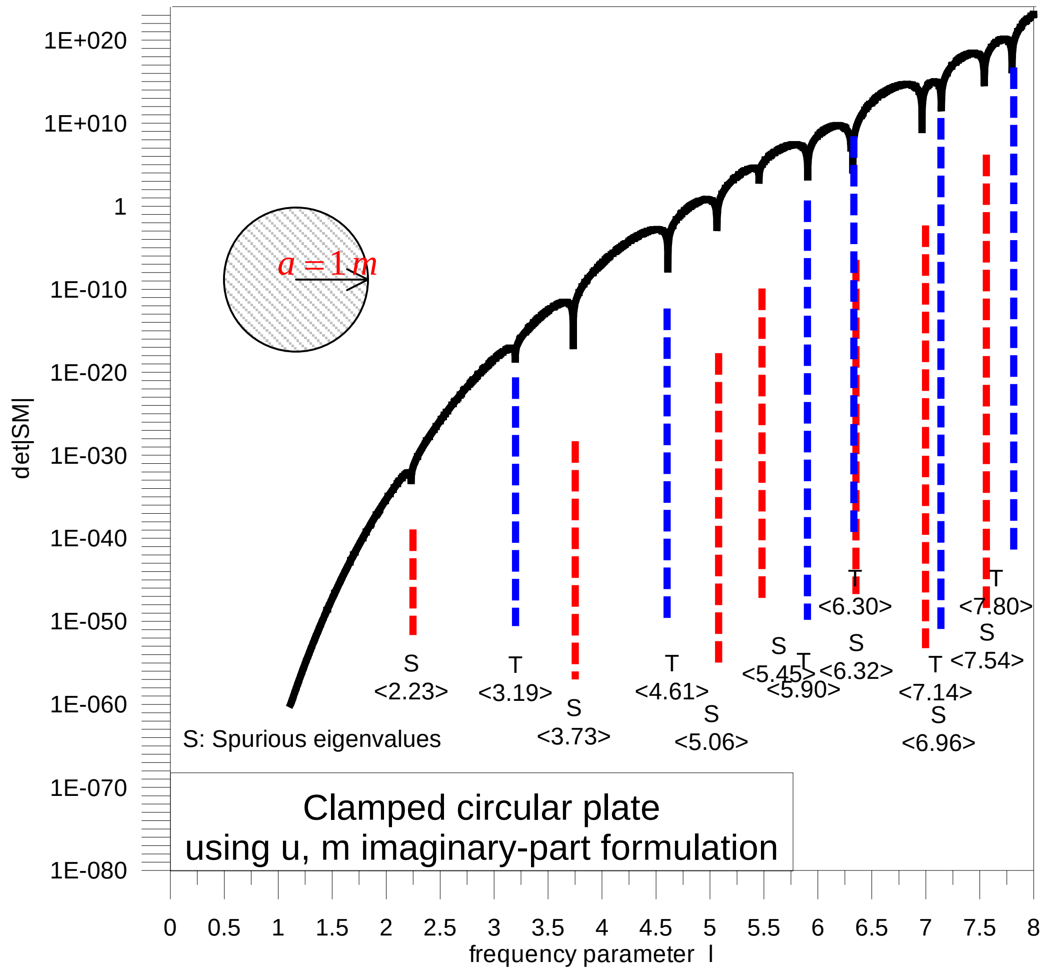
# The End

## Thanks for your kind attention

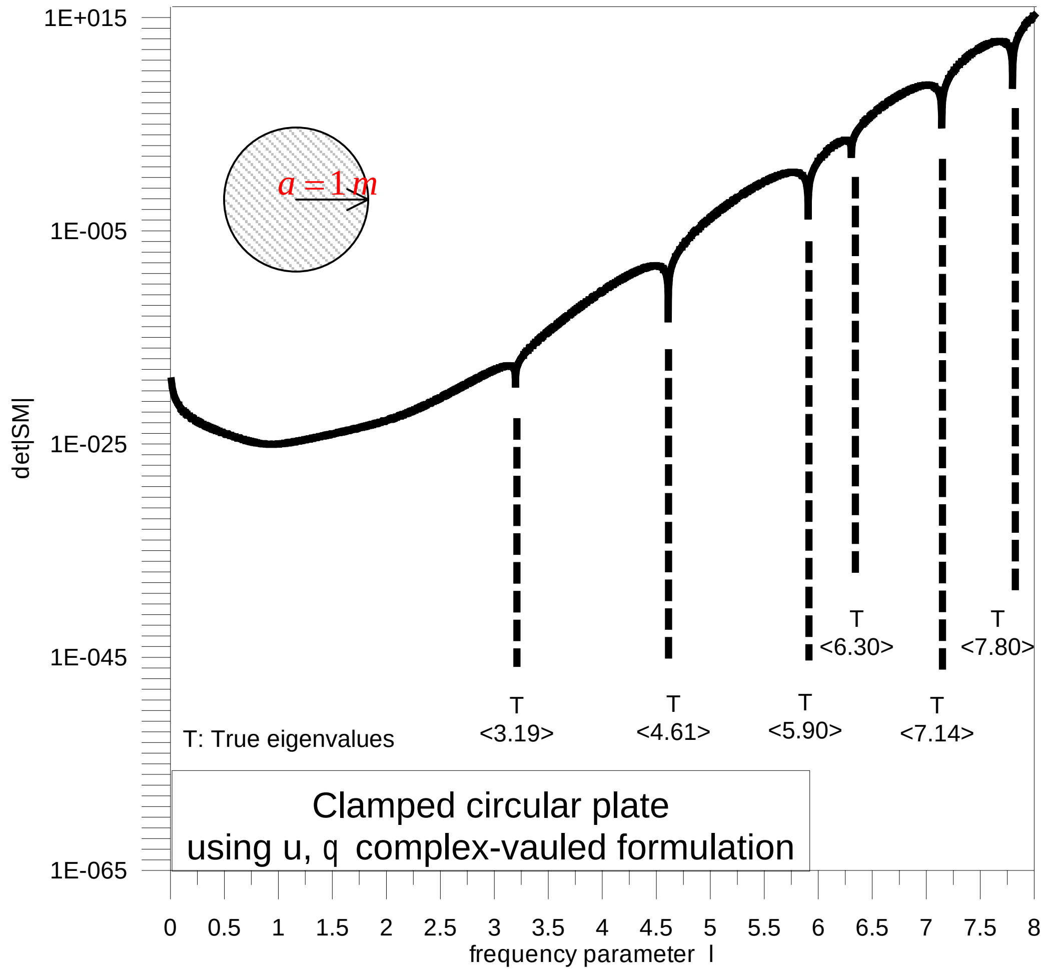
# Real-part BEM for simply-connected plate



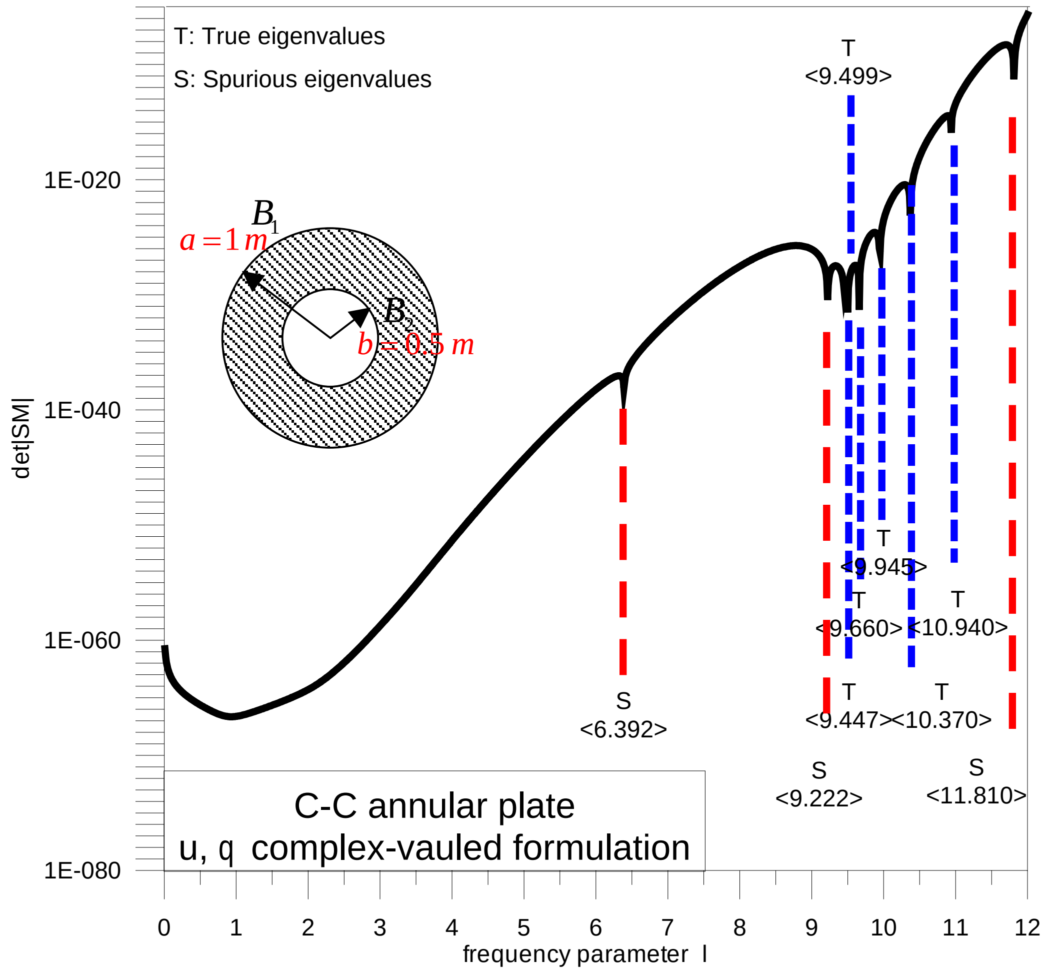
# Imaginary-part BEM for simply-connected plate



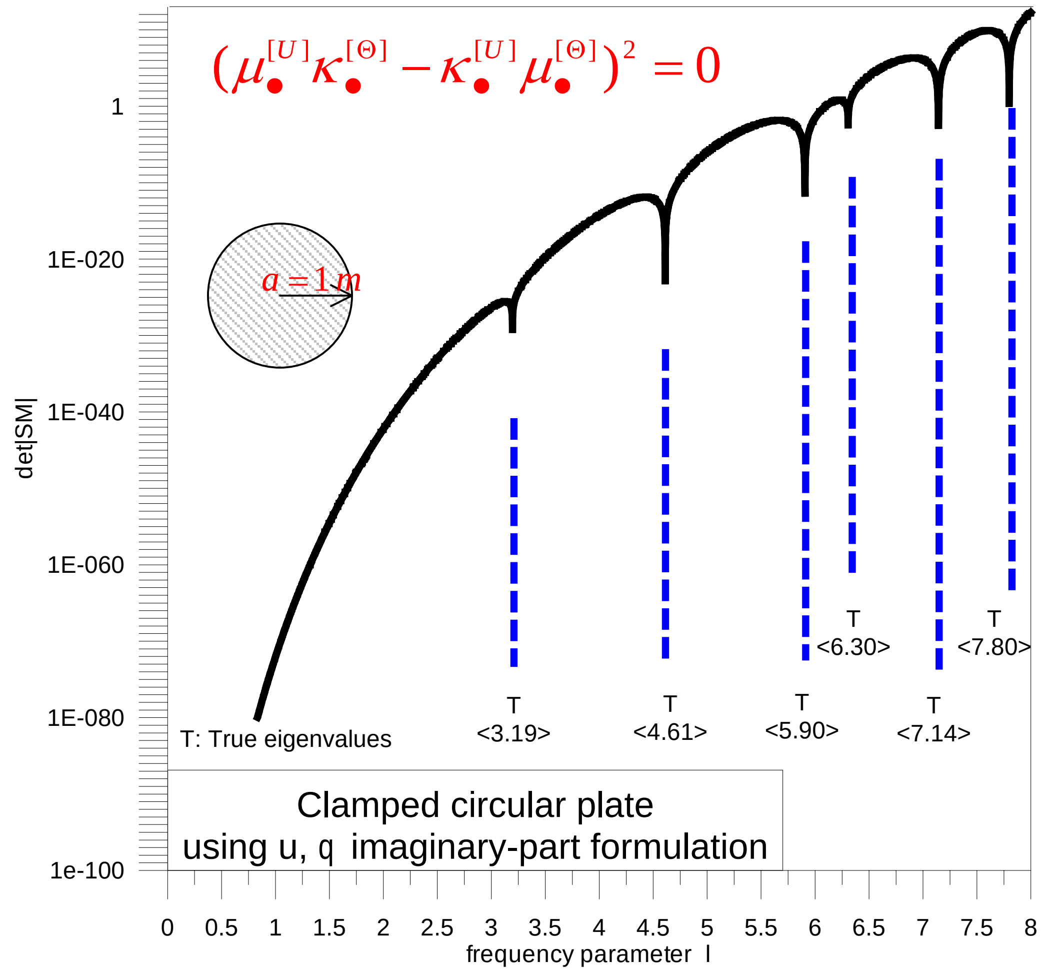
# Complex-valued BEM for simply-connected plate



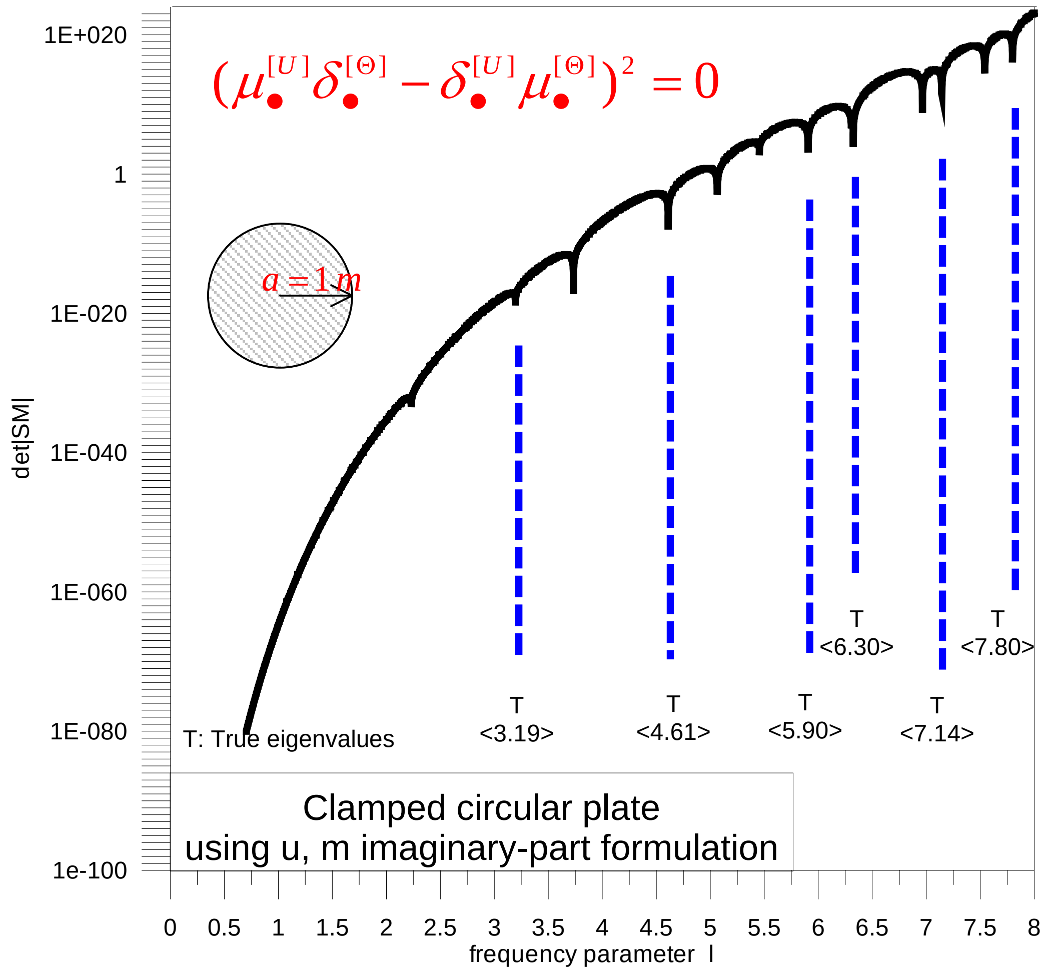
# Complex-valued BEM for multiply-connected plate



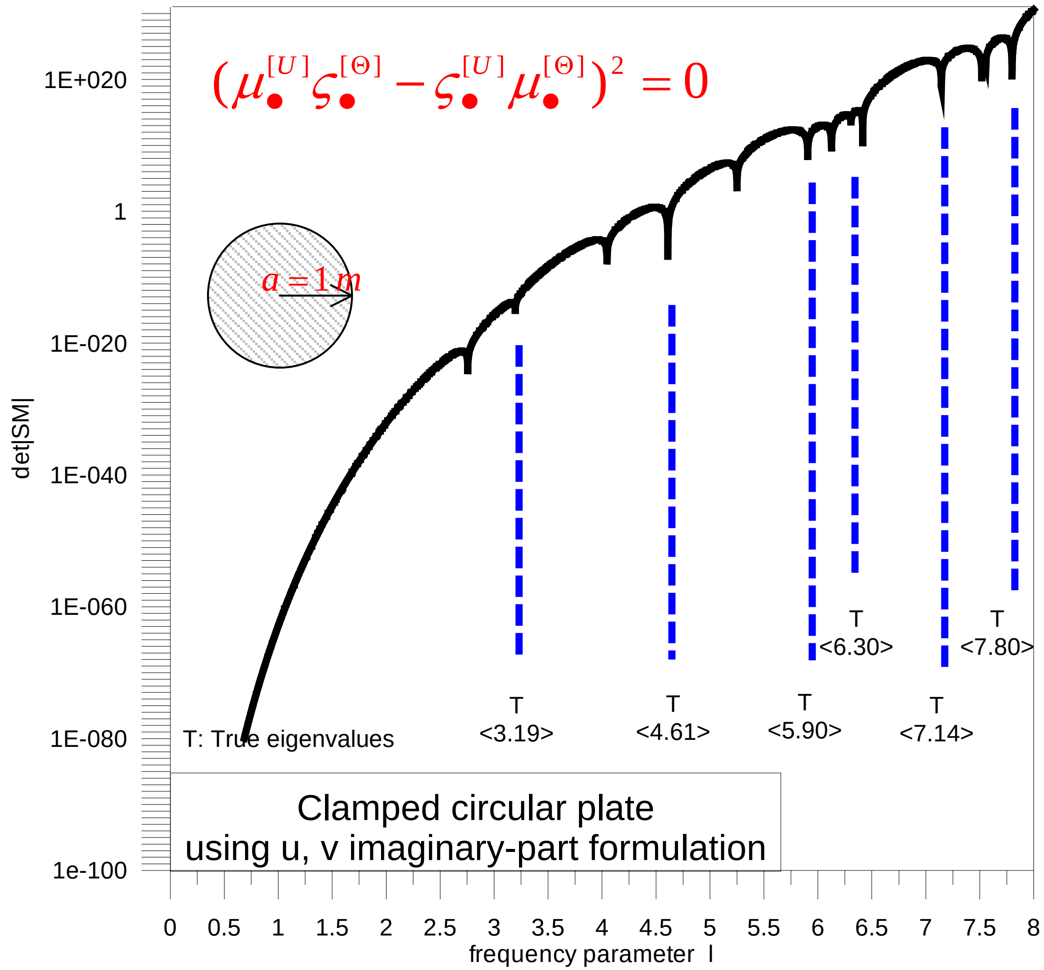
# Imaginary-part BEM (u, **q**) for clamped plate



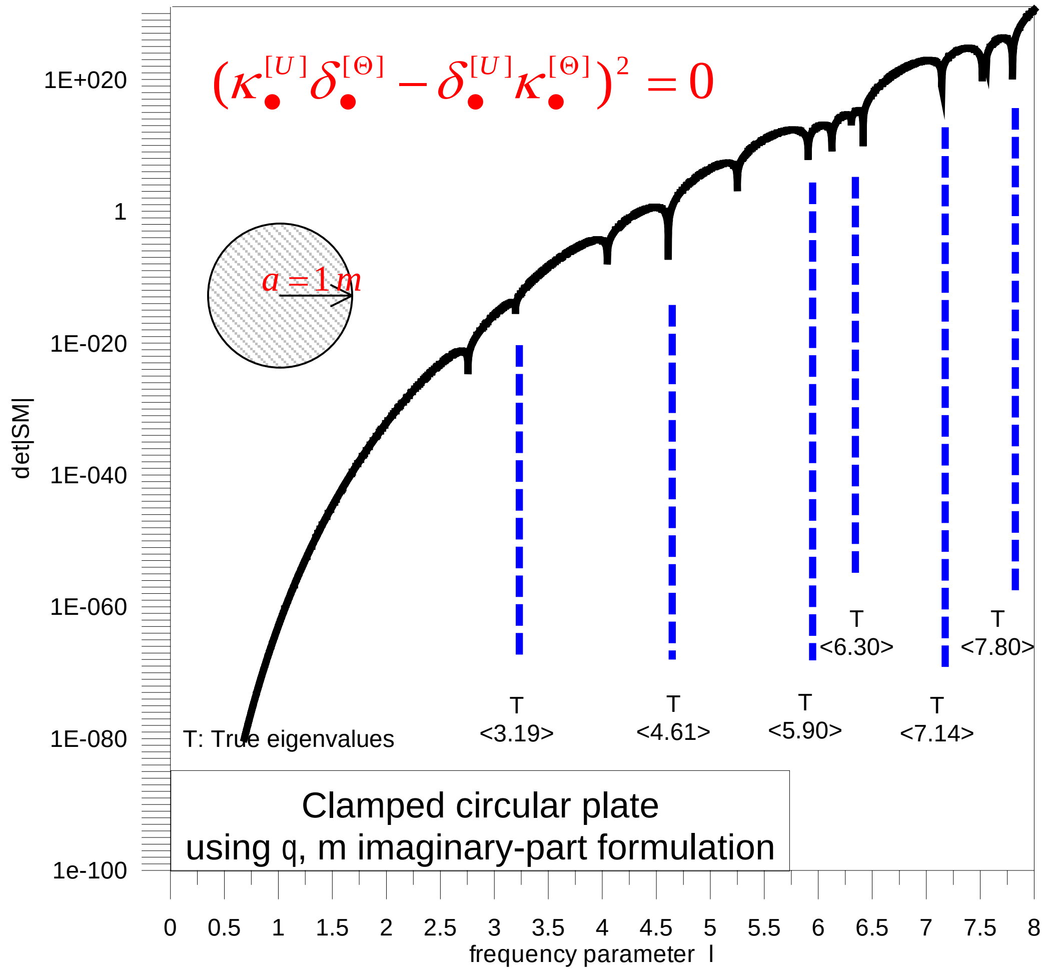
# Imaginary-part BEM (u, m) for clamped plate



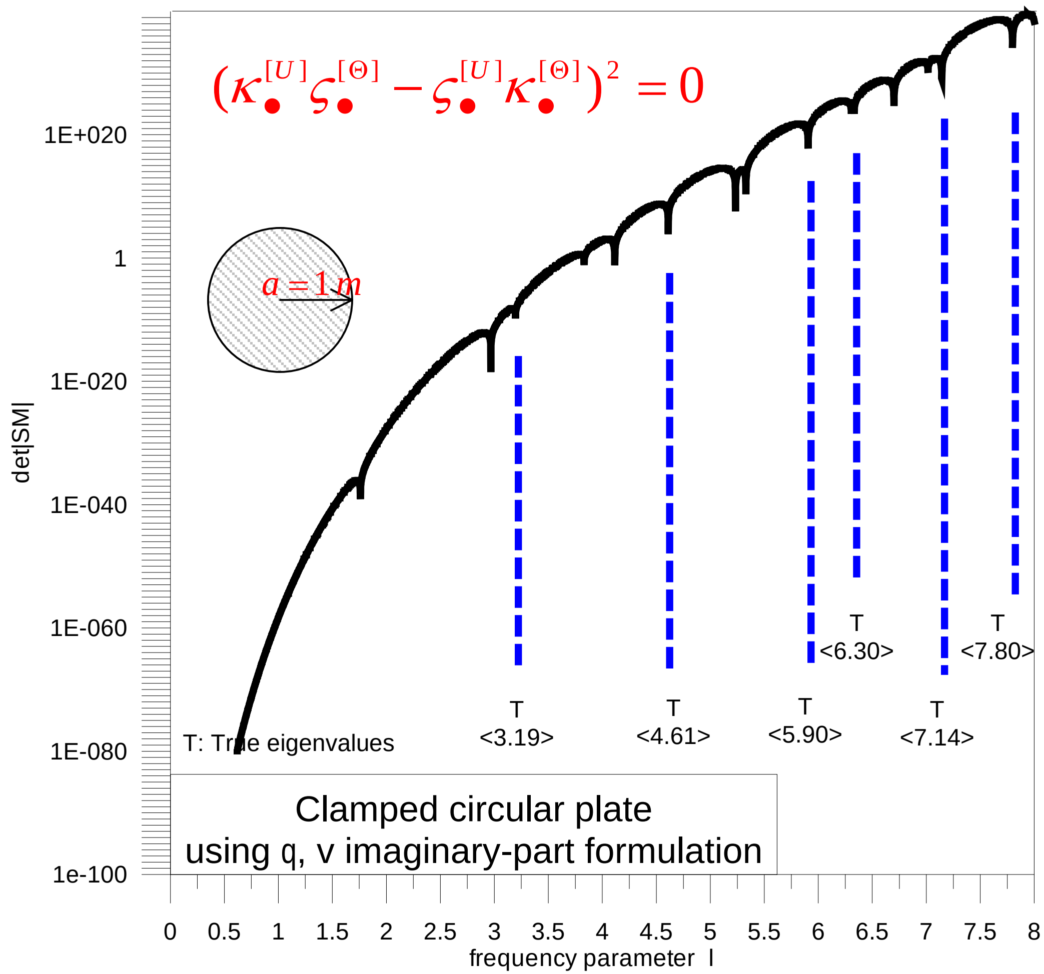
# Imaginary-part BEM (u,v) for clamped plate



# Imaginary-part BEM (q, m) for clamped plate



# Imaginary-part BEM (q, v) for clamped plate



# Imaginary-part BEM (m,v) for clamped plate