Mathematical analysis and numerical study of the true and spurious eigenequations for free vibration of plate using an imaginary-part BEM

S. Y. Lin, Y. T. Lee, W. Shen, J. T. Chen

Department of Harbor and River Engineering

National Taiwan Ocean University, Keelung, Taiwan

Abstract

In this paper, an imaginary-part BEM for solving the eigenfrequencies of plates is proposed for avoiding singularity and saving half effort in computation instead of using the complex-valued BEM. By employing the imaginary-part fundamental solution, the spurious eigenequations in conjunction with the true eigenequation are obtained for free vibration of plate. To verify this finding, the circulant is adopted to analytically derive the true and spurious eigenequations in the discrete system of a circular plate. In order to obtain the eigenvalues and boundary modes at the same time, the singular value decomposition (SVD) technique is utilized. The analytical solutions are derived in the discrete system. Three cases, clamped, simply-supported and free circular plates, are demonstrated analytically and numerically to see the validity of the present method. SVD updating technique is adopted to suppress the ocurrence of the spurious eigenvalues, and a clamped plate is demonstrated analytically for the discrete system in this paper.

Keywords: Imaginary-part BEM, Plate vibration, Spurious eigenvalue, Circulant, Degenerate kernel, SVD updating technique.

1. Introduction

For the simply-connected problems of interior acoustics and membrane, either the real-part or imaginary-part BEM results in spurious eigenequations. Tai and Shaw [23] first employed BEM to solve membrane vibration using a complex-valued kernel. De Mey [10, 11], Hutchinson and Wong [13] employed only the imaginary-part kernel to solve the membrane and plate vibrations free of the complex-valued computation in sacrifice of occurrence of spurious eigenvalues. Kamiya *et al.* [17, 18] and Yeih *et al.* [24] linked the relation of MRM and imaginary-part BEM independently. Wong and Hutchinson [14] have presented a direct BEM for plate vibration involving displacement, slope, moment and shear force. They were able to obtain numerical results for the clamped plates by employing only the imaginary-part BEM with obvious computational gains. However, this saving leads to the spurious eigenvalues in addition to the true ones in free vibration analysis. One has to investigate the mode shapes in order to identify and reject the spurious ones. Shaw [22] commented that only the imaginary-part approach was incorrect since the eigenequation must satisfy the real-part and imaginary-part equations at the same time. Hutchinson [14] replied that the claim of incorrectness was perhaps a little strong since the imaginary-part BEM does not miss any true eigenvalue although the solution is contaminated by spurious ones according to his numerical experience. If we need to look for the eigenmode as well as eigenvalue as usual, the sorting for the spurious eigenequations pays a small price by identifying the mode shapes. Chen *et al.* [3] commented that the spurious modes can be reasonable which may mislead the judgement of the true and spurious ones, since the true and spurious modes may have the same nodal line in case of different eigenvalues. This is the reason why Chen *et al.* have developed many systematic techniques [2, 3, 4, 5, 6,7], for sorting out the true and the spurious eigenvalues. Niwa *et al.* [21] also stated that ``One must take care to use the complete Green's function for outgoing waves, as attempts to use just the real (singular) or imaginary (regular) part separately will not provide the complete spectrum". As quoted from the reply of Hutchinson [13], this comment is not correct since the real-part BEM does not lose any true eigenvalue. The reason is that the real and imaginary-part kernels satisfy the Hilbert transform pair. Complete eigenspectrum is imbedded in either one, real or imaginary-part kernel. The Hilbert transform is the constraint in the frequency-domain fundamental solution corresponding to the casual effect in the time-domain fundamental solution. The physical meaning of the imaginary-part kernel is the standing wave [12]. Tai and Shaw [23] claimed that spurious eigenvalues are not present if the complex-valued kernel is employed for the eigenproblem. However, it is true only for the case of problem with a simply-connected domain [9, 10]. For multiply-connected problems, spurious eigenequation occur even though the complex-valued BEM is utilized.

In this paper, the spurious eigenequation for the plate eigenproblem will be studied in the imaginary-part BEM. First of all, the true and spurious eigenvalues will be examined for the simply-connected plate using the imaginary-part BEM. Since any two boundary integral equations in the plate formulation (4 equations) can be chosen, 6 (C_2^4) options can be considered. The occurring mechanism for the spurious eigenequation in the plate eigenproblem in each formulation will be studied analytically in the discrete system. For the discrete system, the degenerate kernels for the fundamental solution and circulants resulting from the circular boundary will be employed to determine the spurious eigenequation. Three types of plates subject to clamped, simply-supported and free boundary conditions will be illustrated to check the validity of the present formulations. Also, the SVD updating technique is adopted to suppress the ocurrence of the spurious eigenvalues for the free vibration of plate problem, and a clamped plate is demonstrated analytically for the discrete system in this paper.

2. Boundary integral equations for plate eigenproblems

The governing equation for free flexural vibration of a uniform thin plate is written as follows:

$$\nabla^4 u(x) = \lambda^4 u(x), \quad x \in \Omega \tag{1}$$

where u is the lateral displacement, $\lambda^4 = \omega^2 \rho_0 h / D$, λ is the frequency parameter, ω is the circular frequency, ρ_0 is the surface density, D is the flexural rigidity expressed as $D = Eh^3/12(1-\nu^2)$ in terms of Young's modulus E, Poisson ratio ν, and the plate thickness h, and Ω is the domain of the thin plate. The integral equations for the domain point can be derived from the Rayleigh-Green identity [24] as follows:

$$\begin{aligned} u(x) = \int_B \{ & -U(s,x)v(s) + \Theta(s,x)m(s) \\ & - M(s,x)\theta(s) + V(s,x)u(s)\}\, dB(s), \quad x \in \Omega \end{aligned} \tag{2}$$

$$\begin{aligned} \theta(x) = \int_B \{ & -U_\theta(s,x)v(s) + \Theta_\theta(s,x)m(s) \\ & - M_\theta(s,x)\theta(s) + V_\theta(s,x)u(s)\}\, dB(s), \quad x \in \Omega \end{aligned} \tag{3}$$

$$\begin{aligned} m(x) = \int_B \{ & -U_m(s,x)v(s) + \Theta_m(s,x)m(s) \\ & - M_m(s,x)\theta(s) + V_m(s,x)u(s)\}\, dB(s), \quad x \in \Omega \end{aligned} \tag{4}$$

$$\begin{aligned} v(x) = \int_B \{ & -U_v(s,x)v(s) + \Theta_v(s,x)m(s) \\ & - M_v(s,x)\theta(s) + V_v(s,x)u(s)\}\, dB(s), \quad x \in \Omega \end{aligned} \tag{5}$$

where B is the boundary, u, θ, m and v mean the displacement, slope, normal moment, effective shear force, s and x are the source and field points, respectively, U, Θ, M and V kernel functions will be elaborated on later. By moving the field point to the boundary, Eqs.(2)-(5) reduce to

$$\begin{aligned} \alpha\, u(x) = & -P.V.\int_B U(s,x)v(s)\, dB(s) + P.V.\int_B \Theta(s,x)m(s)\, dB(s) \\ & - P.V.\int_B M(s,x)\theta(s)\, dB(s) + P.V.\int_B V(s,x)u(s)\, dB(s), \quad x \in B \end{aligned} \tag{6}$$

$$\begin{aligned} \alpha\, \theta(x) = & -P.V.\int_B U_\theta(s,x)v(s)\, dB(s) + P.V.\int_B \Theta_\theta(s,x)m(s)\, dB(s) \\ & - P.V.\int_B M_\theta(s,x)\theta(s)\, dB(s) + P.V.\int_B V_\theta(s,x)u(s)\, dB(s), \quad x \in B \end{aligned} \tag{7}$$

$$\begin{aligned} \alpha\, m(x) = & -P.V.\int_B U_m(s,x)v(s)\, dB(s) + P.V.\int_B \Theta_m(s,x)m(s)\, dB(s) \\ & - P.V.\int_B M_m(s,x)\theta(s)\, dB(s) + P.V.\int_B V_m(s,x)u(s)\, dB(s), \quad x \in B \end{aligned} \tag{8}$$

$$\begin{aligned} \alpha\, v(x) = & -P.V.\int_B U_v(s,x)v(s)\, dB(s) + P.V.\int_B \Theta_v(s,x)m(s)\, dB(s) \\ & - P.V.\int_B M_v(s,x)\theta(s)\, dB(s) + P.V.\int_B V_v(s,x)u(s)\, dB(s), \quad x \in B \end{aligned} \tag{9}$$

where $P.V.$ denotes the principal value, and $\alpha = 1/2$ for a smooth boundary point. We consider only the imaginary-part kernel function $U(s,x)$ of the fundamental solution $U_c(s,x)$ which satisfies

$$\nabla^4 U_c(s,x) - \lambda^4 U_c(s,x) = \delta(x-s) \tag{10}$$

where $\delta(s-x)$ is the Dirac-Delta function. Considering the two singular solutions ($Y_0(\lambda r)$ and $K_0(\lambda r)$, which are the zeroth-order of second kind Bessel and modified Bessel functions, respectively) [14] and two regular solutions ($J_0(\lambda r)$ and $I_0(\lambda r)$, which are the zeroth-order of the first kind Bessel and modified Bessel functions, respectively) in the fundamental solution, we have

$$U_c(s,x) = \frac{1}{8\lambda^2}[(Y_0(\lambda r) + iJ_0(\lambda r)) - \frac{2}{\pi}(K_0(\lambda r) + iI_0(\lambda r))] \tag{11}$$

where $r \equiv |s-x|$ and $i^2 = -1$. The other three kernels, $\Theta(s,x)$, $M(s,x)$ and $V(s,x)$ are defined as follows :

$$\Theta(s,x) = K_\theta(U(s,x)) \tag{12}$$

$$M(s,x) = K_m(U(s,x)) \tag{13}$$

$$V(s,x) = K_v(U(s,x)) \tag{14}$$

where $K_\theta(\cdot)$, $K_m(\cdot)$ and $K_v(\cdot)$ mean the operators defined by

$$K_\theta(\cdot) = \frac{\partial(\cdot)}{\partial n} \tag{15}$$

$$K_m(\cdot) = \nu\nabla^2(\cdot) + (1-\nu)\frac{\partial^2(\cdot)}{\partial n^2} \tag{16}$$

$$K_v(\cdot) = \frac{\partial\nabla^2(\cdot)}{\partial n} + (1-\nu)\frac{\partial}{\partial t}\left(\frac{\partial^2(\cdot)}{\partial n \partial t}\right) \tag{17}$$

where n and t are the normal vector and tangential vector, respectively. The operators K_θ, K_m and K_v can be applied to U, Θ, M and V kernels. The kernel functions can be expressed as:

$$U(s,x) = \mathrm{Im}[U_c(s,x)] \tag{18}$$

$$\Theta(s,x) = K_\theta(U(s,x)) = \frac{\partial U(s,x)}{\partial n_s} \tag{19}$$

$$M(s,x) = K_m(U(s,x)) = \nu\nabla_s^2 U(s,x) + (1-\nu)\frac{\partial^2 U(s,x)}{\partial n_s^2} \tag{20}$$

$$V(s,x) = K_v(U(s,x)) = \frac{\partial\nabla_s^2 U(s,x)}{\partial n_s} + (1-\nu)\frac{\partial}{\partial t_s}\left(\frac{\partial^2 U(s,x)}{\partial n_s \partial t_s}\right) \tag{21}$$

The displacement, slope, normal moment and effective shear force are derived by

$$\theta(x) = K_\theta(u(x)) \tag{22}$$

$$m(x) = K_m(u(x)) \tag{23}$$

$$v(x) = K_v(u(x)) \tag{24}$$

Once the field point x locates outside the domain, the null-field BIEs of the direct method in Eqs.(6)-(9) yield

$$\begin{aligned} 0 = \int_B \{ & -U(s,x)v(s) + \Theta(s,x)m(s) \\ & - M(s,x)\theta(s) + V(s,x)u(s)\}\, dB(s), \quad x \in \Omega^e \end{aligned} \tag{25}$$

$$\begin{aligned} 0 = \int_B \{ & -U_\theta(s,x)v(s) + \Theta_\theta(s,x)m(s) \\ & - M_\theta(s,x)\theta(s) + V_\theta(s,x)u(s)\}\, dB(s), \quad x \in \Omega^e \end{aligned} \tag{26}$$

$$\begin{aligned} 0 = \int_B \{ & -U_m(s,x)v(s) + \Theta_m(s,x)m(s) \\ & - M_m(s,x)\theta(s) + V_m(s,x)u(s)\}\, dB(s), \quad x \in \Omega^e \end{aligned} \tag{27}$$

$$0=\int_B \{-U_v(s,x)v(s)+\Theta_v(s,x)m(s) - M_v(s,x)\theta(s)+V_v(s,x)u(s)\}\,dB(s), \quad x\in\Omega^e \tag{28}$$

where Ω^e is the complementary domain. Note that the null-field BIEs are not singular, since x and s never coincide.

When the boundary is discretized into $2N$ constant elements, the linear algebraic equations of Eqs.(6)-(9) can be obtained as follows:

$$0=[U]\{v\}+[\Theta]\{m\}+[M]\{\theta\}+[V]\{u\} \tag{29}$$

$$0=[U_\theta]\{v\}+[\Theta_\theta]\{m\}+[M_\theta]\{\theta\}+[V_\theta]\{u\} \tag{30}$$

$$0=[U_m]\{v\}+[\Theta_m]\{m\}+[M_m]\{\theta\}+[V_m]\{u\} \tag{31}$$

$$0=[U_v]\{v\}+[\Theta_v]\{m\}+[M_v]\{\theta\}+[V_v]\{u\} \tag{32}$$

where $[U]$, $[\Theta]$, $[M]$, $[V]$, $[U_\theta]$, $[\Theta_\theta]$, $[M_\theta]$, $[V_\theta]$, $[U_m]$, $[\Theta_m]$, $[M_m]$, $[V_m]$, $[U_v]$, $[\Theta_v]$, $[M_v]$ and $[V_v]$ are the sixteen influence matrices with a dimension $2N\times 2N$, $\{u\}$, $\{\theta\}$, $\{m\}$ and $\{v\}$ are the vectors of boundary data with a dimension $2N\times 1$.

3. Mathematical analysis for the true and spurious eigensolutions

Case 1. Clamped circular plate

For the clamped circular plate ($u=0$ and $\theta=0$) with a radius a, Eqs.(29) and (30) can be rewritten as

$$0=[U]\{v\}+[\Theta]\{m\} \tag{33}$$

$$0=[U_\theta]\{v\}+[\Theta_\theta]\{m\} \tag{34}$$

By assembling Eqs.(33) and (34) together, we have

$$[SM^c]\begin{Bmatrix} v \\ m \end{Bmatrix}=0 \tag{35}$$

where the superscript ``c" denotes the clamped case and

$$[SM^c]=\begin{bmatrix} U & \Theta \\ U_\theta & \Theta_\theta \end{bmatrix}_{4N\times 4N} \tag{36}$$

For the existence of nontrivial solution of $\begin{Bmatrix} v \\ m \end{Bmatrix}$, the determinant of the matrix versus eigenvalue must be zero.

Since the rotation symmetry is preserved for a circular boundary, the influence matrices for the discrete system are found to be circulants with the following forms into Eq.(33), we have

$$[U]=\begin{bmatrix} z_0 & z_1 & z_2 & \cdots & z_{2N-1} \\ z_{2N-1} & z_0 & z_1 & \cdots & z_{2N-2} \\ z_{2N-2} & z_{2N-1} & z_0 & \cdots & z_{2N-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ z_1 & z_2 & z_3 & z_{2N-1} & z_0 \end{bmatrix}_{2N\times 2N} \tag{37}$$

The coefficients of each element can be obtained by using degenerate kernel

$$z_m=\int_{(m-\frac{1}{2})\Delta\bar\phi}^{(m+\frac{1}{2})\Delta\bar\phi}[-U(a,\bar\phi,a,\phi)]a\,d\bar\phi \approx -U(a,\bar\phi_m,a,\phi)a\,\Delta\bar\phi, \quad m=0,1,2,\dots,2N-1 \tag{38}$$

where $\Delta\bar\phi=2\pi/2N$, $\bar\phi_m=m\Delta\bar\phi$. The kernel functions, $U(s,x)$, $\Theta(s,x)$, $U_\theta(s,x)$ and $\Theta_\theta(s,x)$, can be expanded by using the expansion formulae

$$J_0(\lambda r)=\begin{cases} J_0^i(\lambda r)=\sum_{m=-\infty}^{\infty} J_m(\lambda\bar\rho)J_m(\lambda\rho)\cos(m(\bar\phi-\phi)), & \bar\rho>\rho \\ J_0^e(\lambda r)=\sum_{m=-\infty}^{\infty} J_m(\lambda\rho)J_m(\lambda\bar\rho)\cos(m(\bar\phi-\phi)), & \rho>\bar\rho \end{cases} \tag{39}$$

$$I_0(\lambda r)=\begin{cases} I_0^i(\lambda r)=\sum_{m=-\infty}^{\infty} (-1)^m I_m(\lambda\bar\rho)I_m(\lambda\rho)\cos(m(\bar\phi-\phi)), & \bar\rho>\rho \\ I_0^e(\lambda r)=\sum_{m=-\infty}^{\infty} (-1)^m I_m(\lambda\rho)I_m(\lambda\bar\rho)\cos(m(\bar\phi-\phi)), & \rho>\bar\rho \end{cases} \tag{40}$$

where J_m and I_m denote the *mth*-order Bessel and modified Bessel functions of the first kind. The superscripts ``$i$" and ``e" denote the interior point ($\bar\rho>\rho$) and the exterior point ($\bar\rho<\rho$), $s=(\bar\rho,\bar\phi)$ and $x=(\rho,\phi)$ are the polar coordinates of s and x, respectively. In this case, $\bar\rho=\rho=a$ for the circular plate with a radius a. Similarly, the other kernels can also be expanded into degenerate forms. By introducing the following bases for circulants, I, $[C_{2N}]^1$, $[C_{2N}]^2$, $[C_{2N}]^3$,…, $[C_{2N}]^{2N-1}$, we can expand matrix $[U]$ into

$$[U]=z_0 I+z_1[C_{2N}]^1+z_2[C_{2N}]^2+\dots+z_{2N-1}[C_{2N}]^{2N-1} \tag{41}$$

where

$$[C_{2N}]=\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 0 \end{bmatrix}_{2N\times 2N} \tag{42}$$

Based on the similar properties for the matrices of $[U]$ and $[C_{2N}]$, we have

$$\mu_\ell^{[U]}=z_0+z_1\alpha_\ell+z_2\alpha_\ell^2+\dots+z_{2N-1}\alpha_\ell^{2N-1}, \quad \ell=0,1,2,\dots,2N-1 \tag{43}$$

where $\mu_\ell^{[U]}$ and α_ℓ are the eigenvalues for $[U]$ and $[C_{2N}]$, respectively. It is easily found that the eigenvalues for the circulants $[C_{2N}]$, are the roots for $\alpha^{2N}=1$ as shown below:

$$\alpha_\ell=e^{i\frac{2\pi\ell}{2N}}, \quad \ell=0,\pm1,\pm2,..,\pm N-1,N \quad or \quad \ell=0,1,2,..,2N-1 \tag{44}$$

The eigenvector for the circulant $[C_{2N}]$ is

$$\{\phi_\ell\}=\begin{Bmatrix} 1 \\ \alpha_\ell \\ \alpha_\ell^2 \\ \vdots \\ \alpha_\ell^{2N-1} \end{Bmatrix}_{2N\times 1} \tag{45}$$

Substituting Eq.(44) into Eq.(43), we have

$$\mu_\ell^{[U]}=\sum_{m=0}^{2N-1} z_m\alpha_\ell^m=\sum_{m=0}^{2N-1} z_m e^{i\frac{2\pi m\ell}{2N}}, \quad \ell=0,\pm1,\pm2,\dots,\pm(N-1),N \tag{46}$$

According to the definition for z_m in Eq.(38), we have

$$z_m=z_{2N-m}, \quad m=0,1,2,\dots,2N-1 \tag{47}$$

Substitution of Eq.(47) into Eq.(46) yields

$$\mu_\ell^{[U]}=z_0+(-1)^\ell z_N+\sum_{m=1}^{N-1}(\alpha_\ell^m+\alpha_\ell^{2N-m})z_m=\sum_{m=0}^{2N-1}\cos(m\ell\Delta\bar\phi)z_m \tag{48}$$

Substituting Eq.(38) into Eq.(48) for $\phi=0$ without loss of generality, the Riemann sum of infinite terms reduces to the following integral

$$
\begin{aligned}
\mu_\ell^{[U]} &= \lim_{N\to\infty}\sum_{m=0}^{2N-1}\cos(m\ell\Delta\bar{\phi})[-U(a,\bar{\phi}_m;a,0)] \\
&\approx \int_0^{2\pi}\cos(\ell\bar{\phi})[-U(a,\bar{\phi};a,0)]a\,d\bar{\phi}
\end{aligned}
\tag{49}
$$

By using the degenerate kernel for $U(s,x)$ and the orthogonal conditions of Fourier series, Eq.(49) reduces to

$$
\begin{aligned}
\mu_\ell^{[U]} &= -\frac{\pi a}{4\lambda^2}[J_\ell(\lambda a)J_\ell(\lambda a)-\frac{2}{\pi}I_\ell(\lambda a)I_\ell(\lambda a)] \\
&\ell = 0,\pm1,\pm2,\ldots,\pm(N-1),N
\end{aligned}
\tag{50}
$$

Similarly, we have

$$
\begin{aligned}
\mu_\ell^{[\Theta]} &= \frac{\pi a}{4\lambda}[J_\ell(\lambda a)J'_\ell(\lambda a)-\frac{2}{\pi}I_\ell(\lambda a)I'_\ell(\lambda a)] \\
&\ell = 0,\pm1,\pm2,\ldots,\pm(N-1),N
\end{aligned}
\tag{51}
$$

$$
\begin{aligned}
\kappa_\ell^{[U]} &= -\frac{\pi a}{4\lambda}[J'_\ell(\lambda a)J_\ell(\lambda a)-\frac{2}{\pi}I'_\ell(\lambda a)I_\ell(\lambda a)] \\
&\ell = 0,\pm1,\pm2,\ldots,\pm(N-1),N
\end{aligned}
\tag{52}
$$

$$
\begin{aligned}
\kappa_\ell^{[\Theta]} &= \frac{\pi a}{4}[J'_\ell(\lambda a)J'_\ell(\lambda a)-\frac{2}{\pi}I'_\ell(\lambda a)I'_\ell(\lambda a)] \\
&\ell = 0,\pm1,\pm2,\ldots,\pm(N-1),N
\end{aligned}
\tag{53}
$$

where $\mu_\ell^{[\Theta]}$, $\kappa_\ell^{[U]}$ and $\kappa_\ell^{[\Theta]}$ are the eigenvalues of $[\Theta]$, $[U_\theta]$ and $[\Theta_\theta]$ matrices, respectively. Since the four matrices $[U]$, $[\Theta]$, $[U_\theta]$ and $[\Theta_\theta]$ are all symmetric circulants, they can be expressed by

$$
[U] = \Phi\Sigma_U\Phi^{-1}
= \Phi\begin{bmatrix}
\mu_0^{[U]} & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & \mu_1^{[U]} & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & \mu_{-1}^{[U]} & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & \mu_{N-1}^{[U]} & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & \mu_{-(N-1)}^{[U]} & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & \mu_N^{[U]}
\end{bmatrix}_{2N\times 2N}\Phi^{-1}
\tag{54}
$$

$$
[\Theta] = \Phi\Sigma_\Theta\Phi^{-1}
= \Phi\begin{bmatrix}
\mu_0^{[\Theta]} & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & \mu_1^{[\Theta]} & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & \mu_{-1}^{[\Theta]} & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & \mu_{N-1}^{[\Theta]} & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & \mu_{-(N-1)}^{[\Theta]} & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & \mu_N^{[\Theta]}
\end{bmatrix}_{2N\times 2N}\Phi^{-1}
\tag{55}
$$

$$
[U_\theta] = \Phi\Sigma_{U_\theta}\Phi^{-1}
= \Phi\begin{bmatrix}
\kappa_0^{[U]} & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & \kappa_1^{[U]} & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & \kappa_{-1}^{[U]} & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & \kappa_{N-1}^{[U]} & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & \kappa_{-(N-1)}^{[U]} & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & \kappa_N^{[U]}
\end{bmatrix}_{2N\times 2N}\Phi^{-1}
\tag{56}
$$

$$
[\Theta_\theta] = \Phi\Sigma_{\Theta_\theta}\Phi^{-1}
= \Phi\begin{bmatrix}
\kappa_0^{[\Theta]} & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & \kappa_1^{[\Theta]} & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & \kappa_{-1}^{[\Theta]} & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & \kappa_{N-1}^{[\Theta]} & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & \kappa_{-(N-1)}^{[\Theta]} & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & \kappa_N^{[\Theta]}
\end{bmatrix}_{2N\times 2N}\Phi^{-1}
\tag{57}
$$

where

$$
\Phi = \frac{1}{\sqrt{2N}}
\begin{bmatrix}
1 & 1 & 0 & \cdots & 1 & 0 & 1 \\
1 & \cos(\frac{2\pi}{2N}) & \sin(\frac{2\pi}{2N}) & \cdots & \cos(\frac{2\pi(2n-1)}{2N} & \sin(\frac{2\pi(N-1)}{2N} & \cos(\frac{2\pi N}{2N}) \\
1 & \cos(\frac{4\pi}{2N}) & \sin(\frac{4\pi}{2N}) & \cdots & \cos(\frac{4\pi(2n-1)}{2N} & \sin(\frac{4\pi(N-1)}{2N} & \cos(\frac{4\pi N}{2N}) \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
1 & \cos(\frac{2\pi(2N-1)}{2N}) & \sin(\frac{2\pi(2n-1)}{2N}) & \cdots & \cos(\frac{2\pi(2N-1)(N-1)}{2N}) & \sin(\frac{2\pi(2N-1)(N-1)}{2N}) & \cos(\frac{2\pi(2N-1)N}{2N})
\end{bmatrix}_{2N\times 2N}
\tag{58}
$$

By employing Eqs.(54)-(57) for Eq.(36), we have

$$
[SM^c] = \begin{bmatrix}
\Phi\Sigma_U\Phi^{-1} & \Phi\Sigma_\Theta\Phi^{-1} \\
\Phi\Sigma_{U_\theta}\Phi^{-1} & \Phi\Sigma_{\Theta_\theta}\Phi^{-1}
\end{bmatrix}_{4N\times 4N}
\tag{59}
$$

Eq.(59) can be reformulated into

$$
[SM^c] = \begin{bmatrix} \Phi & 0 \\ 0 & \Phi \end{bmatrix}
\begin{bmatrix} \Sigma_U & \Sigma_\Theta \\ \Sigma_{U_\theta} & \Sigma_{\Theta_\theta} \end{bmatrix}
\begin{bmatrix} \Phi & 0 \\ 0 & \Phi \end{bmatrix}^{-1}
\tag{60}
$$

Since Φ is orthogonal ($\det|\Phi| = \det|\Phi^{-1}| = 1$), the determinant of $[SM^c]_{4N\times 4N}$ is

$$
\det[SM^c] = \det\begin{bmatrix} \Sigma_U & \Sigma_\Theta \\ \Sigma_{U_\theta} & \Sigma_{\Theta_\theta} \end{bmatrix}
= \prod_{\ell=-(N-1)}^{N}(\mu_\ell^{[U]}\kappa_\ell^{[\Theta]} - \mu_\ell^{[\Theta]}\kappa_\ell^{[U]})
\tag{61}
$$

By employing Eqs.(50)-(53) for Eq.(61), we have

$$
\begin{aligned}
\det[SM^c] = &\prod_{\ell=-(N-1)}^{N}\frac{\pi^2a^2}{16\lambda^2} \\
&\{[J_\ell(\lambda a)J_\ell(\lambda a)-\frac{2}{\pi}I_\ell(\lambda a)I_\ell(\lambda a)][J'_\ell(\lambda a)J'_\ell(\lambda a)-\frac{2}{\pi}I'_\ell(\lambda a)I'_\ell(\lambda a)] \\
&-[J_\ell(\lambda a)J'_\ell(\lambda a)-\frac{2}{\pi}I_\ell(\lambda a)I'_\ell(\lambda a)][J'_\ell(\lambda a)J_\ell(\lambda a)-\frac{2}{\pi}I'_\ell(\lambda a)I_\ell(\lambda a)]\}
\end{aligned}
\tag{62}
$$

Eq.(62) can be simplified into

$$
\begin{aligned}
\det[SM^c] = \prod_{\ell=-(N-1)}^{N}&\frac{\pi^2a^2}{8\lambda^2}[I_{\ell+1}(\lambda a)J_\ell(\lambda a)-I_\ell(\lambda a)J_{\ell+1}(\lambda a)] \\
&\{I_{\ell+1}(\lambda a)J_\ell(\lambda a)+I_\ell(\lambda a)J_{\ell+1}(\lambda a)\}
\end{aligned}
\tag{63}
$$

Zero determinant in Eq.(63) implies that the eigenequation is

$$
\begin{aligned}
&[I_{\ell+1}(\lambda a)J_\ell(\lambda a)+I_\ell(\lambda a)J_{\ell+1}(\lambda a)] \\
&\{I_{\ell+1}(\lambda a)J_\ell(\lambda a)+I_\ell(\lambda a)J_{\ell+1}(\lambda a)\} = 0, \quad \ell = 0,\pm1,\pm2,\cdots,\pm(N-1),N
\end{aligned}
\tag{64}
$$

After comparing with the analytical solution for the clamped circular plate [19], the true eigenequation for a continuous system can be obtained by approaching N in the the discrete system to infinity. The former part in the middle bracket of Eq.(64) is the spurious eigenequation while the latter part in the big bracket is found to be the true eigenequation. In this case, it is interesting to find that the true and spurious eigenequation are the same. We can also comment that no spurious eigenvalue occurs although the spurious multiplicity appears.

Case 2. Simply-supported circular plate

For the simply-supported circular plate ($u = 0$ and $m = 0$) with a radius a , we have

$$
[SM^s] = \begin{bmatrix} U & M \\ U_\theta & M_\theta \end{bmatrix}_{4N\times 4N}
\tag{65}
$$

where the superscript `` s '' denotes the simply-supported case. Since the rotation symmetry is preserved for a circular boundary, the eigenvalues of the influence matrices for the discrete system can be found by using circulants, we have

$$\mu_\ell^{[M]} = -\frac{\pi a}{4\lambda^2}[J_\ell(\lambda a)\alpha_\ell^J(\lambda a) - \frac{2}{\pi}I_\ell(\lambda a)\alpha_\ell^I(\lambda a)] \quad \ell = 0,\pm1,\pm2,\ldots,\pm(N-1),N \tag{66}$$

$$\kappa_\ell^{[M]} = -\frac{\pi a}{4\lambda}[J'_\ell(\lambda a)\alpha_\ell^J(\lambda a) - \frac{2}{\pi}I'_\ell(\lambda a)\alpha_\ell^I(\lambda a)] \quad \ell = 0,\pm1,\pm2,\ldots,\pm(N-1),N \tag{67}$$

where $\mu_\ell^{[M]}$ and $\kappa_\ell^{[M]}$ are the eigenvalues of $[M]$ and $[M_\theta]$ matrices, respectively, and

$$\alpha_n^J = \lambda^2 J''_n(\lambda a) + \nu[\frac{1}{a}\lambda J'_n(\lambda a) - (\frac{n}{a})^2 J_n(\lambda a)] \tag{68}$$

$$\alpha_n^I = \lambda^2 I''_n(\lambda a) + \nu[\frac{1}{a}\lambda I'_n(\lambda a) - (\frac{n}{a})^2 I_n(\lambda a)] \tag{69}$$

Since the two matrices $[M]$ and $[M_\theta]$ are all symmetric circulants, they can be expressed by

$$[M] = \Phi \Sigma_M \Phi^T \tag{70}$$

$$[M_\theta] = \Phi \Sigma_{M_\theta} \Phi^T \tag{71}$$

By employing Eqs.(54), (56), (70) and (71) for Eq.(65), we have

$$[SM^s] = \begin{bmatrix} \Phi & 0 \\ 0 & \Phi \end{bmatrix}\begin{bmatrix} \Sigma_U & \Sigma_M \\ \Sigma_{U_\theta} & \Sigma_{M_\theta} \end{bmatrix}\begin{bmatrix} \Phi & 0 \\ 0 & \Phi \end{bmatrix}^T \tag{72}$$

Since Φ is orthogonal, the determinant of $[SM^s]_{4N\times4N}$ is

$$\det[SM^s] = \det\begin{bmatrix} \Sigma_U & \Sigma_M \\ \Sigma_{U_\theta} & \Sigma_{M_\theta} \end{bmatrix} = \prod_{\ell=-(N-1)}^{N}(\mu_\ell^{[U]}\kappa_\ell^{[M]} - \mu_\ell^{[M]}\kappa_\ell^{[U]}) \tag{73}$$

By employing Eqs.(50), (52), (66) and (67) for Eq.(73), we have

$$\begin{aligned}\det[SM^s] = \prod_{\ell=-(N-1)}^{N} \frac{\pi^2 a^2}{16\lambda^3} & \{[J_\ell(\lambda a)J_\ell(\lambda a) - \frac{2}{\pi}I_\ell(\lambda a)I_\ell(\lambda a)][J'_\ell(\lambda a)\alpha_\ell^J(\lambda a) - \frac{2}{\pi}I'_\ell(\lambda a)\alpha_\ell^I(\lambda a)] \\ & -[J_\ell(\lambda a)\alpha_\ell^J(\lambda a) - \frac{2}{\pi}I_\ell(\lambda a)\alpha_\ell^I(\lambda a)][J'_\ell(\lambda a)J_\ell(\lambda a) - \frac{2}{\pi}I'_\ell(\lambda a)I_\ell(\lambda a)]\}\end{aligned} \tag{74}$$

Eq.(74) can be simplified into

$$\begin{aligned}\det[SM^s] = \prod_{\ell=-(N-1)}^{N} \frac{\pi a}{8\lambda^2} & [I_{\ell+1}(\lambda a)J_\ell(\lambda a) - I_\ell(\lambda a)J_{\ell+1}(\lambda a)] \\ & \{(1-\nu)I_\ell(\lambda a)J_{\ell+1}(\lambda a) + I_{\ell+1}(\lambda a)J_\ell(\lambda a) - 2\lambda a I_\ell(\lambda a)J_\ell(\lambda a)\}\end{aligned} \tag{75}$$

Zero determinant in Eq.(75) implies that the eigenequation is

$$\begin{aligned}& [I_{\ell+1}(\lambda a)J_\ell(\lambda a) - I_\ell(\lambda a)J_{\ell+1}(\lambda a)] \\ & \{(1-\nu)I_\ell(\lambda a)J_{\ell+1}(\lambda a) + I_{\ell+1}(\lambda a)J_\ell(\lambda a) - 2\lambda a I_\ell(\lambda a)J_\ell(\lambda a)\} = 0, \\ & \ell = 0,\pm1,\pm2,\cdots,\pm(N-1),N\end{aligned} \tag{76}$$

After comparing with the analytical solution for the simply-supported circular plate [19], the true eigenequation for a continuous system can be obtained by approaching N in the discrete system to infinity. The former part in Eq.(76) inside the middle bracket is the spurious eigenequation while the latter part inside the big bracket is found to be the true eigenequation.

Case 3. Free circular plate

For the free circular plate ($m = 0$ and $v = 0$) with a radius a , we have

$$[SM^f] = \begin{bmatrix} M & V \\ M_\theta & V_\theta \end{bmatrix}_{4N\times4N} \tag{77}$$

where the superscript `` f " denotes the free case. Since the rotation symmetry is preserved for a circular boundary, the eigenvalues of the influence matrices for the discrete system can be found by using circulants, we have

$$\begin{aligned}\mu_\ell^{[V]} = -\frac{\pi a}{4\lambda^2}&[J_\ell(\lambda a)\beta_\ell^J(\lambda a) - \frac{2}{\pi}I_\ell(\lambda a)\beta_\ell^I(\lambda a) \\ &+\frac{1-\nu}{a}[J_\ell(\lambda a)\gamma_\ell^J(\lambda a) - \frac{2}{\pi}I_\ell(\lambda a)\gamma_\ell^I(\lambda a)] \\ &\ell = 0,\pm1,\pm2,\ldots,\pm(N-1),N\end{aligned} \tag{78}$$

$$\begin{aligned}\kappa_\ell^{[V]} = -\frac{\pi a}{4\lambda}&[J'_\ell(\lambda a)\beta_\ell^J(\lambda a) - \frac{2}{\pi}I'_\ell(\lambda a)\beta_\ell^I(\lambda a) \\ &+\frac{1-\nu}{a}[J'_\ell(\lambda a)\gamma_\ell^J(\lambda a) - \frac{2}{\pi}I'_\ell(\lambda a)\gamma_\ell^I(\lambda a)] \\ &\ell = 0,\pm1,\pm2,\ldots,\pm(N-1),N\end{aligned} \tag{79}$$

where $\mu_\ell^{[V]}$ and $\kappa_\ell^{[V]}$ are the eigenvalues of $[V]$ and $[V_\theta]$ matrices, respectively, and

$$\beta_n^J = \lambda^3 J'''_n(\lambda a) + \nu[\frac{1}{a}\lambda^2 J''_n(\lambda a) - (\frac{n}{a})^2 J'_n(\lambda a) - \frac{1}{a^2}\lambda J'_n(\lambda a) + (\frac{2n^2}{a^3})J_n(\lambda a)] \tag{80}$$

$$\beta_n^I = \lambda^3 I'''_n(\lambda a) + \nu[\frac{1}{a}\lambda^2 I''_n(\lambda a) - (\frac{n}{a})^2 I'_n(\lambda a) - \frac{1}{a^2}\lambda I'_n(\lambda a) + (\frac{2n^2}{a^3})I_n(\lambda a)] \tag{81}$$

$$\gamma_n^J = -n^2[\frac{1}{a^2}J_n(\lambda a) + \frac{\lambda}{a}J'(\lambda a)] \tag{82}$$

$$\gamma_n^I = -n^2[\frac{1}{a^2}I_n(\lambda a) + \frac{\lambda}{a}I'(\lambda a)] \tag{83}$$

Since the two matrices $[V]$ and $[V_\theta]$ are all symmetric circulants, they can be expressed by

$$[V] = \Phi \Sigma_V \Phi^T \tag{84}$$

$$[V_\theta] = \Phi \Sigma_{V_\theta} \Phi^T \tag{85}$$

By employing Eqs.(70), (71), (84) and (85) for Eq.(77), we have

$$[SM^f] = \begin{bmatrix} \Phi & 0 \\ 0 & \Phi \end{bmatrix}\begin{bmatrix} \Sigma_M & \Sigma_V \\ \Sigma_{M_\theta} & \Sigma_{V_\theta} \end{bmatrix}\begin{bmatrix} \Phi & 0 \\ 0 & \Phi \end{bmatrix}^T \tag{86}$$

Since Φ is orthogonal, the determinant of $[SM^f]_{4N\times4N}$ is

$$\det[SM^f] = \det\begin{bmatrix} \Sigma_M & \Sigma_V \\ \Sigma_{M_\theta} & \Sigma_{V_\theta} \end{bmatrix} = \prod_{\ell=-(N-1)}^{N}(\mu_\ell^{[M]}\kappa_\ell^{[V]} - \mu_\ell^{[V]}\kappa_\ell^{[M]}) \tag{87}$$

By employing Eqs.(66), (67), (78) and (79) for Eq.(87), we have

$$\begin{aligned}\det[SM^f] = \prod_{\ell=-(N-1)}^{N} \frac{\pi^2 a^2}{16\lambda^3} & \{[J_\ell(\lambda a)J_\ell(\lambda a) - \frac{2}{\pi}I_\ell(\lambda a)I_\ell(\lambda a)][J'_\ell(\lambda a)\alpha_\ell^J(\lambda a) - \frac{2}{\pi}I'_\ell(\lambda a)\alpha_\ell^I(\lambda a)] \\ & -[J_\ell(\lambda a)\alpha_\ell^J(\lambda a) - \frac{2}{\pi}I_\ell(\lambda a)\alpha_\ell^I(\lambda a)][J'_\ell(\lambda a)J_\ell(\lambda a) - \frac{2}{\pi}I'_\ell(\lambda a)I_\ell(\lambda a)]\}\end{aligned} \tag{88}$$

Eq.(88) can be simplified into

$$\det[SM^f] = \prod_{\ell=-(N-1)}^{N} \frac{\pi a}{8\lambda^2}[I_{\ell+1}(\lambda a)J_\ell(\lambda a) - I_\ell(\lambda a)J_{\ell+1}(\lambda a)] \{\lambda a(1-\nu)[-4\ell^2(\ell-1)]I_\ell(\lambda a)J_\ell(\lambda a) - 2\lambda^2 a^2 I_{\ell+1}(\lambda a)J_{\ell+1}(\lambda a) + 2\ell\lambda^2 a^2(1-\nu)(1-\ell)(I_{\ell+1}(\lambda a)J_\ell(\lambda a) - I_\ell(\lambda a)J_{\ell+1}(\lambda a)) + [\ell^2(1-\nu)^2(\ell^2-1) + \lambda^4 a^4](I_{\ell+1}(\lambda a)J_\ell(\lambda a) + I_\ell(\lambda a)J_{\ell+1}(\lambda a))\} = 0 \tag{89}$$

Zero determinant in Eq.(89) implies that the eigenequation is

$$[I_{\ell+1}(\lambda a)J_\ell(\lambda a) - I_\ell(\lambda a)J_{\ell+1}(\lambda a)] \{\lambda a(1-\nu)[-4\ell^2(\ell-1)I_\ell(\lambda a)J_\ell(\lambda a) - 2\lambda^2 a^2 I_{\ell+1}(\lambda a)J_{\ell+1}(\lambda a)] + 2\ell\lambda^2 a^2(1-\nu)(1-\ell)(I_{\ell+1}(\lambda a)J_\ell(\lambda a) - I_\ell(\lambda a)J_{\ell+1}(\lambda a)) + [\ell^2(1-\nu)^2(\ell^2-1) + \lambda^4 a^4](I_{\ell+1}(\lambda a)J_\ell(\lambda a) + I_\ell(\lambda a)J_{\ell+1}(\lambda a))]\} = 0, \quad \ell = 0, \pm1, \pm2, \cdots, \pm(N-1), N \tag{90}$$

After comparing with the analytical solution for the free circular plate [19], the true eigenequation for a continuous system can be obtained by approaching N in the discrete system to infinity. The former part in Eq.(90) inside the middle bracket is the spurious eigenequation while the latter part inside the big bracket is the true eigenequation. After comparing Eq.(64) with Eqs.(76) and (90), the same spurious eigenequation ($[I_{\ell+1}(\lambda a)J_\ell(\lambda a) + I_\ell(\lambda a)J_{\ell+1}(\lambda a)] = 0$) is embedded in the u, θ formulation no matter what the boundary condition is. This reconfirms that spurious eigenequation depends on the formulation instead of the specified boundary condition. It is noted that the true eigenequation of free plate does not agree with that of the Leissa result [19]. However, the same true eigenvalues are obtained numerically between the present and Leissa's results. After finding the eigenvalues according to the Leissa's eigenequation, the eigenvalues are not consistent in his book. The possible explanation is that the eigenequation in the Leissa's book for the free case was wrongly typed. After careful check, the eigenequation in the Leissa's book was a misprint where the I index in the numerator of the right hand side of the equation should be J.

Since any two equations in the plate formulation (Eqs.(29)-(32)) can be chosen, 6 (C_2^4) options of the formulation can be considerd. If we choose different formulae for either one of the the clamped, simply-supported or free circular plate cases, we can obtain the same true eigenequation but different spurious eigensolutions. At the same time, either clamped or simply-supported circular plate results in the same spurious eigenequation, once we use the same formulation. The occurrence of spurious eigensolution only depends on the formulation instead of the boundary condition. True eigenequation depends on the specified boundary condition instead of the formulation. All the spurious eigenequations are summarized in Table 1 for the six formulations.

4. Extraction of the true eigenvalues using SVD updating technique in the discrete system

A conventional approach to detect the nonunique solution is the criterion of satisfying all Eqs.(29)-(32) at the same time. For the clamped plate ($u = 0$ and $\theta = 0$), the Eqs.(29)-(32) reduce to

$$0 = [U]\{v\} + [\Theta]\{m\}, \tag{91}$$

$$0 = [U_\theta]\{v\} + [\Theta_\theta]\{m\}, \tag{92}$$

$$0 = [U_m]\{v\} + [\Theta_m]\{m\}, \tag{93}$$

$$0 = [U_v]\{v\} + [\Theta_v]\{m\}, \tag{94}$$

After rearranging the terms, Eqs.(91) and (92) can be assembled to

$$[SM_1]\begin{Bmatrix} v \\ m \end{Bmatrix} = 0, \tag{95}$$

where

$$[SM_1] = \begin{bmatrix} U & \Theta \\ U_\theta & \Theta_\theta \end{bmatrix}_{4N\times4N}. \tag{96}$$

Similarly, Eqs.(93) and (94) yield

$$[SM_2]\begin{Bmatrix} v \\ m \end{Bmatrix} = 0, \tag{97}$$

where

$$[SM_2] = \begin{bmatrix} U_m & \Theta_m \\ U_v & \Theta_v \end{bmatrix}_{4N\times4N}. \tag{98}$$

Since the imaginary-part BEM misses the real-part information, we can reconstruct the independment equation by differentiation. To obtain an overdetermined system, we can combine Eqs.(95) and (97) by using the updating term,

$$[C]\begin{Bmatrix} v \\ m \end{Bmatrix} = 0, \tag{99}$$

where

$$[C] = \begin{bmatrix} SM_1 \\ SM_2 \end{bmatrix}_{8N\times4N}. \tag{100}$$

Since the eigenequation is nontrival, the rank of the matrix $[C]$ must be smaller then $4N$, the $4N$ singular values for the matrix $[C]$ must have at least one zero value. The explicit form for the matrix $[C]$ can be decomposed into

$$[C] = \begin{bmatrix} \Phi & 0 & 0 & 0 \\ 0 & \Phi & 0 & 0 \\ 0 & 0 & \Phi & 0 \\ 0 & 0 & 0 & \Phi \end{bmatrix} \begin{bmatrix} \Sigma_U & \Sigma_\Theta \\ \Sigma_{U\theta} & \Sigma_{\Theta_\theta} \\ \Sigma_{Um} & \Sigma_{\Theta m} \\ \Sigma_{Uv} & \Sigma_{\Theta_m} \end{bmatrix} \begin{bmatrix} \Phi^T & 0 \\ 0 & \Phi^T \end{bmatrix}. \tag{101}$$

Based on the equivalence between the SVD technique and the least-squares method in mathematical essence, the least square form leads to

$$[C]^T[C] = \begin{bmatrix} \Phi & 0 \\ 0 & \Phi \end{bmatrix} [D]_{4N\times4N} \begin{bmatrix} \Phi & 0 \\ 0 & \Phi \end{bmatrix}^T \tag{102}$$

where

$$[D] = \begin{bmatrix} \Sigma_U & \Sigma_{U\theta} & \Sigma_{Um} & \Sigma_{Uv} \\ \Sigma_\Theta & \Sigma_{\Theta_\theta} & \Sigma_{\Theta m} & \Sigma_{\Theta v} \end{bmatrix} \begin{bmatrix} \Sigma_U & \Sigma_\Theta \\ \Sigma_{U\theta} & \Sigma_{\Theta_\theta} \\ \Sigma_{Um} & \Sigma_{\Theta m} \\ \Sigma_{Uv} & \Sigma_{\Theta_v} \end{bmatrix}. \tag{103}$$

If the determinant of the matrix $[C]^T[C]$ is zero, we can obtain the nontrival solution. Since Φ is orthogonal, the determinant of the matrix $[C]^T[C]$ is equal to the determinant of the matrix $[D]$. By calculating the determinant of the matrix $[D]$, we have

$$\det[D] = \prod_{\ell=-(N-1)}^{N} [(\mu_\ell^{[U]}\kappa_\ell^{[\Theta]} - \mu_\ell^{[\Theta]}\kappa_\ell^{[U]})^2 + (\mu_\ell^{[U]}\zeta_\ell^{[\Theta]} - \mu_\ell^{[\Theta]}\zeta_\ell^{[U]})^2 + (\mu_\ell^{[U]}\delta_\ell^{[\Theta]} - \mu_\ell^{[\Theta]}\delta_\ell^{[U]})^2 + (\kappa_\ell^{[U]}\zeta_\ell^{[\Theta]} - \kappa_\ell^{[\Theta]}\zeta_\ell^{[U]})^2 + (\kappa_\ell^{[U]}\delta_\ell^{[\Theta]} - \kappa_\ell^{[\Theta]}\delta_\ell^{[U]})^2 + (\zeta_\ell^{[U]}\delta_\ell^{[\Theta]} - \zeta_\ell^{[\Theta]}\delta_\ell^{[U]})^2] \quad (104)$$

where $\zeta_\ell^{[U]}$, $\zeta_\ell^{[\Theta]}$, $\delta_\ell^{[\Theta]}$ and $\delta_\ell^{[U]}$ are the eigenvalues of the matrices $[U_m]$, $[\Theta_m]$, $[U_v]$ and $[\Theta_v]$, respectively. The only possibility for the zero determinant of the matrix $[D]$ occurs when the six terms $(\mu_\ell^{[U]}\kappa_\ell^{[\Theta]} - \mu_\ell^{[\Theta]}\kappa_\ell^{[U]})$, $(\mu_\ell^{[U]}\zeta_\ell^{[\Theta]} - \mu_\ell^{[\Theta]}\zeta_\ell^{[U]})$, $(\mu_\ell^{[U]}\delta_\ell^{[\Theta]} - \mu_\ell^{[\Theta]}\delta_\ell^{[U]})$, $(\kappa_\ell^{[U]}\zeta_\ell^{[\Theta]} - \kappa_\ell^{[\Theta]}\zeta_\ell^{[U]})$, $(\kappa_\ell^{[U]}\delta_\ell^{[\Theta]} - \kappa_\ell^{[\Theta]}\delta_\ell^{[U]})$ and $(\zeta_\ell^{[U]}\delta_\ell^{[\Theta]} - \zeta_\ell^{[\Theta]}\delta_\ell^{[U]})$ are all zeros at the same time for the same ℓ. Here we can find that the six terms result in the six different spurious eigenequations as shown in Table 1, and the same true eigenequation is commonly imbedded in the six formulations. The only possibility for the zero determinant of the matrix $[D]$ is only the common term (true eigenequation) to be zero, such that

$$\{I_{\ell+1}(\lambda a)J_\ell(\lambda a) + J_{\ell+1}(\lambda a)I_\ell(\lambda a)\} = 0, \quad \ell = 0, \pm1, \pm2, \cdots, \pm(N-1), N \quad (105)$$

This indicates that only the true eigenequation of the clamped circular plate is sorted out in the SVD updating matrix since the true eigenequation is simultaneously embedded in the six formulations. The result matches well with Eq.(64) in the discrete system.

5. Numerical results and discussions

Circular plate (clamped, simply-supported and free boundary conditions)

A circular plate with a radius ($a = 1\,m$) is considered. The boundary is discretized into ten constant elements. Since any two equations in the plate formulation (Eqs.(29)-(32)) can be chosen, 6 (C_2^4) options of the formulation can be considered. Figures 1.(a)-(f) show the determinant of $[SM]$ versus frequency parameter λ for the clamped circular plate using the six formulations. We find that the true eigenvalues depends on the specified boundary condition instead of the formulation. Figures 2.(a)-(c) show the determinant of $[SM]$ versus λ using the formualtion (*e.g.* u, θ formulation) to solve plates subject to different boundary conditions.

Figures 2.(d)-(f) show the determinant of the $[C]^T[C]$ versus λ for the clamped, simply-supported and free circular plates using the imaginary-part formulation in conjunction with the SVD technique of updating term. It is found that the spurious eigenvalues are filtered out and only the true eigenvalues appear as predicted in Eq.(104) for the clamped case. The occurrence of spurious eigenvalues only depends on the formulation instead of the specified boundary condition. All the resluts are summarized in Table 1, and the eigenvalues agree well with the data in Leissa [19].

6. Conclusions

An imaginary-part BEM formulation has been derived for the free vibration of plate problems. For a circular plate, the true and spurious eigenvalues and eigenequations were derived analytically by using the degenerate kernel and circulants in the discrete systems. Since either two equations in the plate formulation (4 equations) can be chosen, $C_2^4(6)$ options can be considered. The occurrence of spurious eigenequation only depends on the formulation instead of the specified boundary condition, while the true eigenequation is independent of the formulation and is relevant to the specified boundary condition. All the spurious eigenequations are shown in Table 1. Three cases were demonstrated analytically and numerically to see the validity of the present method. Also, the SVD updating technique is adopted to suppress the ocurrence of the spurious eigenvalues for the clamped plate. Although the circular case lacks generality, it leads significant insight into the occurring mechanism of true and spurious eigenequation. Although the proof is only limited to the circular plate, it is a great help to the researchers who may require analytical explanation about the appearance of the spurious eigenequation. The same algorithm in the discrete system can be applied to solve the arbitrary-shaped plate numerically without any difficulty. Nevertheless, mathematical derivation in continuous and discrete systems can not be done anaytically.

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虛部邊界元素法之板自由振動真假特徵方程之數學分析及數值研究

林盛益, 李應德, 沈文成, 陳正宗

國立台灣海洋大學河海工程學系

摘要

本文以虛部邊界元素法求解一固定圓板之特徵頻率問題以節省數值運算時間與避免奇異性。使用虛部邊界元素法在求解板自由振動過程中所伴隨而來的真假特徵方程爲此文章之討論重點。爲證明假根產生之機制，本文在於離散系統中利用退化核及循環矩陣來探討解析一圓形板的真假特徵方程。文中以一圓板（固定端，簡支撐及自由邊界條件）爲例在離散系統中來說明，並以驗證此方法之正確性。最後則提出採用「奇異值分解法-補充式」之技巧來克服假根之產生，並以一固定端圓板爲例在離散系統下說明並解析。

關鍵字: 虛部邊界元素法，板振動，假根，循環矩陣，退化核，奇異值分解法-補充式技巧

Table 1. **Spurious eigenequations in the six formulations by using the imaginary-part BEM**

	Spurious eigenequations for the imaginary-part BEM
u, θ formulation Eq. (2) and Eq. (3)	$I_{\ell+1}J_{\ell}+I_{\ell}J_{\ell+1}=0$
u,m formulation Eq. (2) and Eq. (4)	$(1-\nu)(I_{\ell}J_{\ell+1}+I_{\ell+1}J_{\ell})-2\lambda\rho I_{\ell}J_{\ell}=0$
u,v formulation Eq. (2) and Eq. (5)	$\ell^2(1-\nu)(I_{\ell}J_{\ell+1}+I_{\ell+1}J_{\ell})-2\lambda\rho\,\ell\,I_{\ell}J_{\ell}+\lambda^2\rho^2(I_{\ell}J_{\ell+1}-I_{\ell+1}J_{\ell})=0$
θ,m formulation Eq. (3) and Eq. (4)	$\ell^2(1-\nu)(I_{\ell}J_{\ell+1}+I_{\ell+1}J_{\ell})-2\lambda\rho\,\ell\,I_{\ell}J_{\ell}+\lambda^2\rho^2(I_{\ell}J_{\ell+1}-I_{\ell+1}J_{\ell})=0$
θ,v formulation Eq. (3) and Eq. (5)	$2\lambda\rho(\ell^2 I_{\ell}J_{\ell}-\lambda^2\rho^2 I_{\ell+1}J_{\ell+1})+2\lambda^2\rho^2\,\ell\,(I_{\ell+1}J_{\ell}-I_{\ell}J_{\ell+1})-\ell^2(1-\nu)](I_{\ell+1}J_{\ell}+I_{\ell}J_{\ell+1})=0$
m,v formulation Eq. (4) and Eq. (5)	$\lambda\rho(1-\nu)[-4\ell^2(\ell-1)I_{\ell}J_{\ell}-2\lambda^2\rho^2 I_{\ell+1}J_{\ell+1}]+2\ell\lambda^2\rho^2(1-\nu)(1-\ell)(I_{\ell+1}J_{\ell}-I_{\ell}J_{\ell+1})$ $+[\ell^2(1-\nu)^2(\ell^2-1)+\lambda^4\rho^4](I_{\ell+1}J_{\ell}+I_{\ell}J_{\ell+1})=0$

where $\ell=0,\pm1,\pm2,\pm3,\ldots$

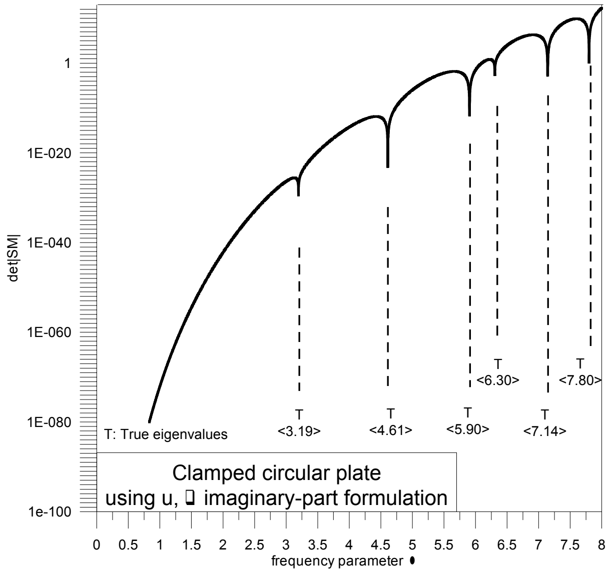

Figure 1.(a) $Det[SM^c]$ v.s. λ (u, θ formulation)

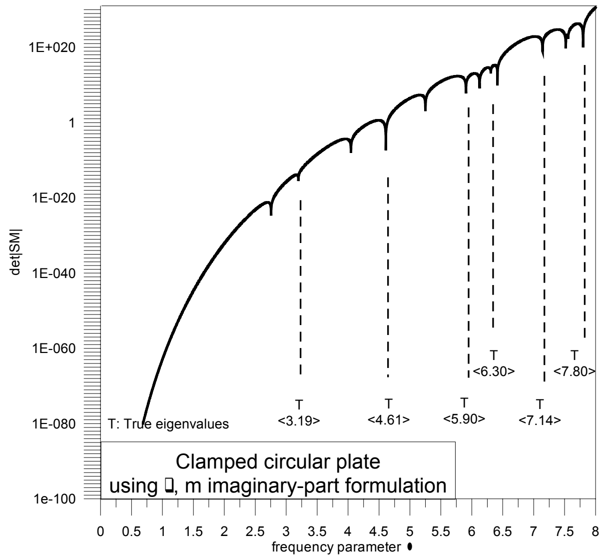

Figure 1.(d) $Det[SM^c]$ v.s. λ (θ, m formulation)

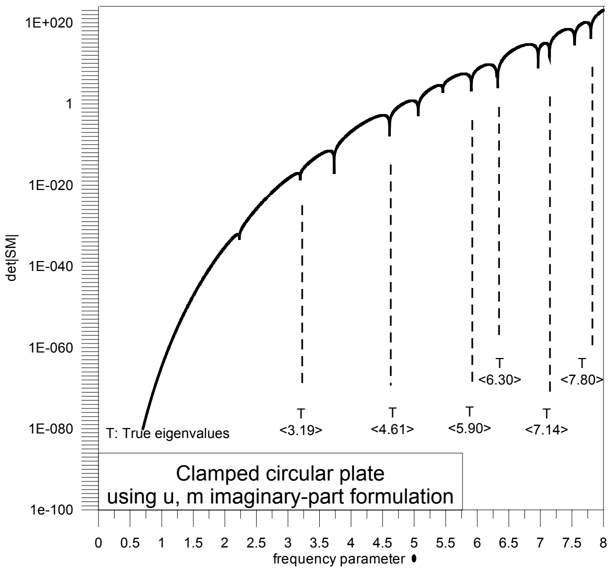

Figure 1.(b) $Det[SM^c]$ v.s. λ (u, m formulation)

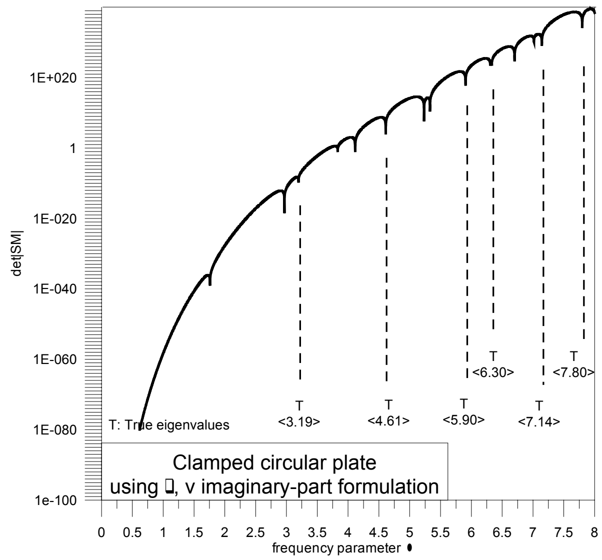

Figure 1.(e) $Det[SM^c]$ v.s. λ (θ, v formulation)

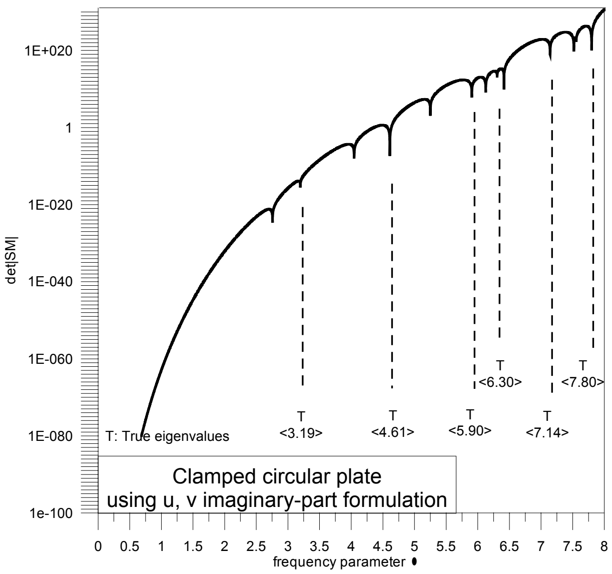

Figure 1.(c) $Det[SM^c]$ v.s. λ (u, v formulation)

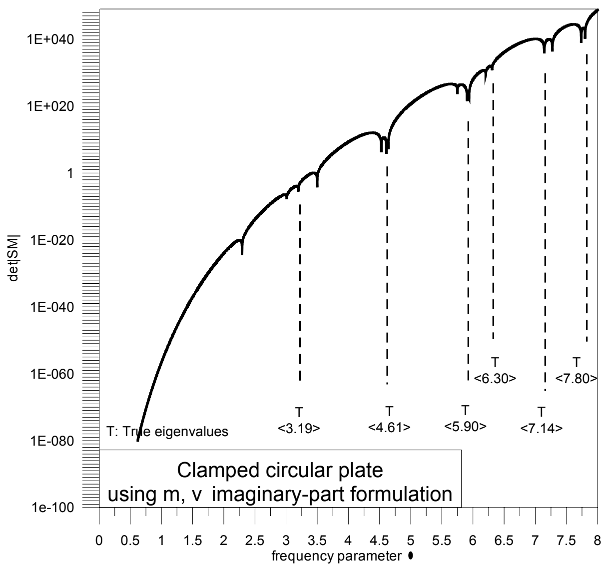

Figure 1.(d) $Det[SM^c]$ v.s. λ (m, v formulation)

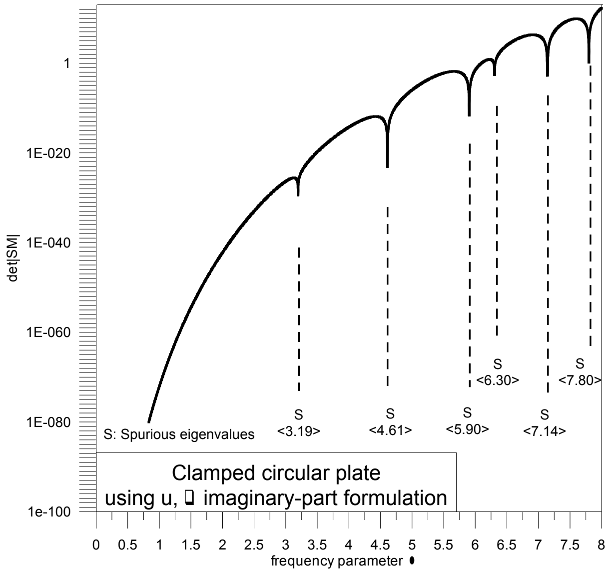

Figure 2.(a) $Det[SM^c]$ v.s. λ (u, θ formulation)

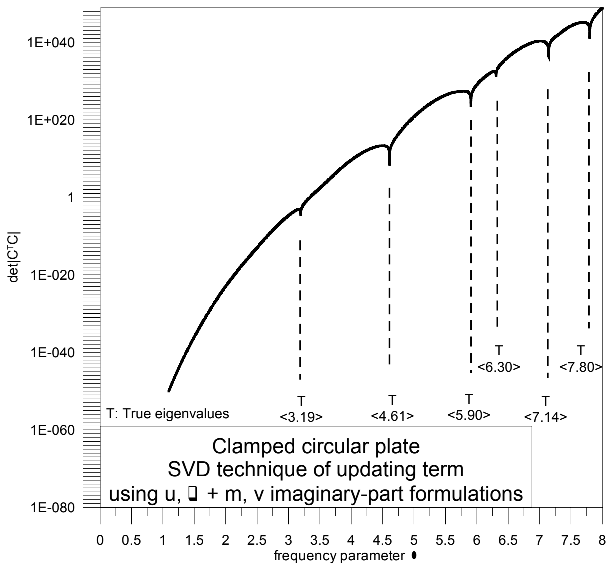

Figure 2.(d) SVD updating term (clamped)

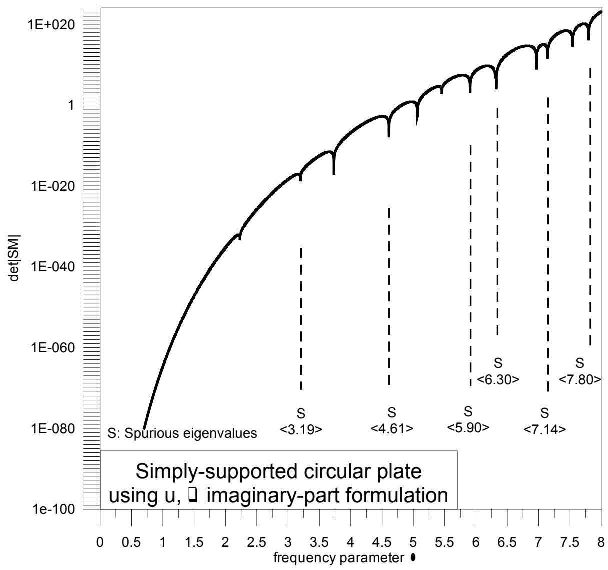

Figure 2.(b) $Det[SM^s]$ v.s. λ (u, θ formulation)

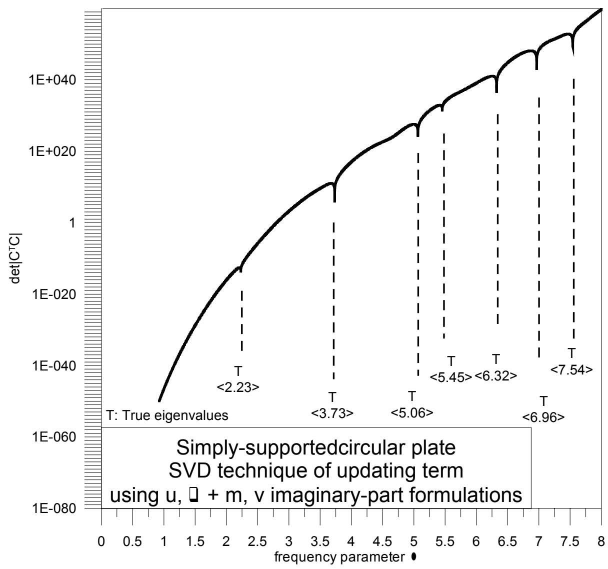

Figure 2.(e) SVD updating term for (simply-supported)

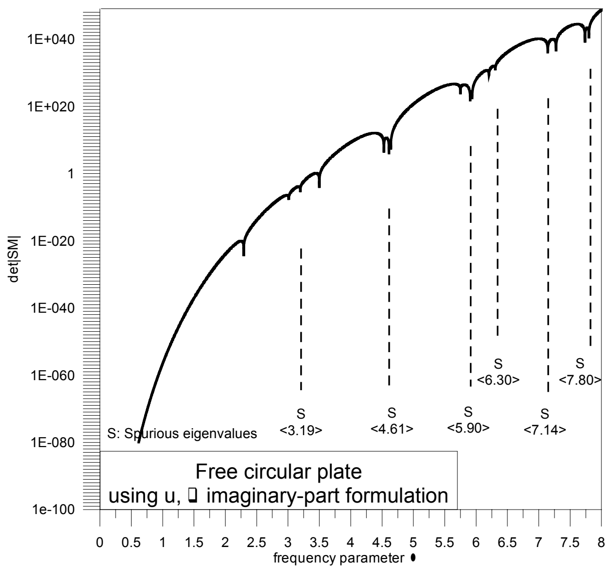

Figure 2.(c) $Det[SM^f]$ v.s. λ (u, θ formulation)

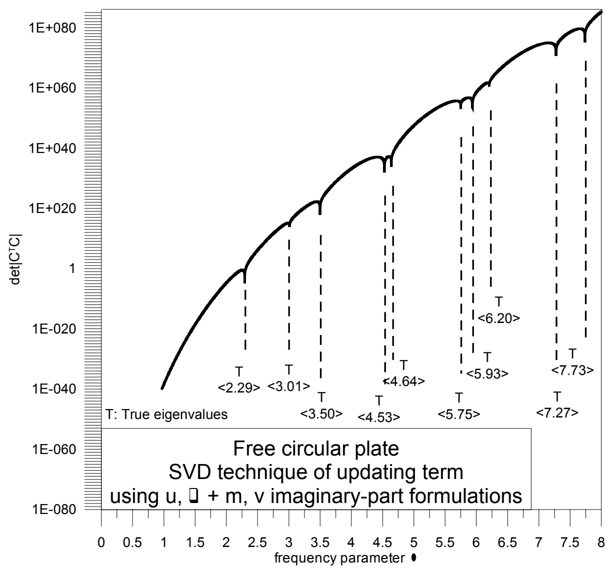

Figure 2.(f) SVD updating term (free)