Mathematical analysis of true and spurious eigenvalues for annular plates using the method of fundamental solutions

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Abstract

In this paper, the method of fundamental solutions (MFS) for solving the eigenfrequencies of an annular plate is proposed. By employing the fundamental solutions, the coefficients of influence matrices are easily determined. Spurious eigensolutions in conjunction with the eigensolutions appear. It is found that spurious eigensolutions using the MFS depend on the location of the inner boundary where fictitious sources are distributed. To verify this finding, true and spurious eigenvalues for an annular plate are analytically studied using degenerate kernels and circulants. In order to obtain the true eigensolution, the singular value decomposition (SVD) updating technique and the Burton & Miller method are utilized to filter out the spurious eigensolutions. An annular plate is demonstrated analytically to see the validity of the present method.

Keywords: Annular plate; Method of fundamental solutions; Circulant; Degenerate kernel; SVD updating technique; Burton & Miller method

1. Introduction

The method of fundamental solutions (MFS) is a numerical technique as well as finite difference method (FDM), finite element method (FEM) and boundary element method (BEM). It is well known that the method of fundamental solutions can deal with many engineering problems when a fundamental solution is known. This method was attributed to Kupradze in 1964 [1]. The method of fundamental solutions can be applied to potential [2], Helmholtz [3], diffusion [4], biharmonic [5] and elasticity problems [1]. The method of fundamental solutions can be regarded as one kind of meshless method. The basic idea is to approximate the solution by a linear superposition of fundamental solution with sources located outside the domain of the problem. It has some advantages over boundary element method, *e.g.*, no boundary integrals, no singularity and mesh-free model.

In boundary element method, Tai and Shaw [6] first employed the complex-valued BEM to solve membrane vibration. De Mey [7], Yas'ko [8], Hutchinson and Wong [9] employed only the real-part kernel to solve the membrane and plate vibrations, respectively. Although the complex-valued computation is avoided, they faced the occurrence of spurious eigenequations. One has to investigate the mode shapes in order to identify and reject the spurious ones. If we usually need to look for the eigenmode as well as eigenvalue, the sorting for the spurious eigenvalues pay a small price by identifying the mode shapes. Chen *et al.* [10] commented that the detection of spurious modes may mislead the judgment of the true and spurious ones, since the true and spurious modes can have the same nodal line by observation in case of different eigenvalues. This is the reason why Chen and his coworkers have developed many systematic techniques, *e.g.*, dual formulation [10], domain partition [11], SVD updating technique [12], CHEEF method [13], for sorting out the true and the spurious eigenvalues. Spurious eigenvalues occur when real-part BEM, imaginary-part BEM and MRM are employed to solve the eigenproblem of simply connected domain. For multiply-connected problems, spurious eigenvalues still occur even though the complex-valued BEM is utilized. The occurrence of spurious eigenvalues and its treatment have been studied in the membrane and acoustic problems [14, 15].

In meshless method, Kang *et al*. proposed so-called nondimensional dynamic influence function (NDIF) to solve membrane [16] and plate vibration [17]. They also faced the problem of spurious eigensolutions. Therefore, they addressed the net approach to filter out the spurious eigenvalues. Later, Chen *et al*. commented that the NDIF is the special case of imaginary-part MFS for membrane [18] and plate [19]. Although MFS has been applied to solve many engineering problems, most of them are for cases with simply-connected domains. Chen *et al*. have tried to solve the eigenproblem of annular membrane and found that spurious eigenvalues also appear. We may wonder what happen for the plate case instead of membrane.

In this paper, the method of fundamental solutions for solving the eigenfrequencies of annular plate is proposed. The occurring mechanism of the spurious eigensolution of an annular plate is studied analytically. The degenerate kernels and circulants are employed to determine the spurious eigensolution. In order to filter out the spurious eigenvalues, singular value decomposition updating technique and Burton & Miller method are utilized. An annular case is demonstrated analytically to see the validity of the present method.

2. Formulation of annular problem using the method of fundamental solutions

The governing equation for an annular plate vibration in Fig.1 is the biharmonic equation as follows:

$$\nabla^4 u(x) = \lambda^4 u(x), \quad x \in \Omega, \tag{1}$$

where ∇^4 is the biharmonic operator, u is the lateral displacement, $\lambda^4 = \frac{\omega^2 \rho_0 h}{D}$, λ is the frequency parameter, ω is the circular frequency, ρ_0 is the surface density, D is the flexural rigidity expressed as $D = \frac{Eh^3}{12(1-\nu^2)}$ in terms of Young's modulus E, the Possion ratio ν and the plate thickness h, and Ω is the domain of the thin plate.

The kernel function $U_c(s,x)$ is the fundamental solution which satisfy

$$\nabla^4 U_c(s,x) - \lambda^4 U_c(s,x) = -\delta(x-s), \tag{2}$$

where $\delta(x-s)$ is the Dirac-Delta function, and s and x are the source and field points, respectively. We have

$$\begin{aligned} U_c(s,x) &= -\frac{i}{8\lambda^2}[H_0^{(1)}(\lambda r) - H_0^{(1)}(i\lambda r)] \\ &= \frac{1}{8\lambda^2}[Y_0(\lambda r) - iJ_0(\lambda r) + \frac{2}{\pi}K_0(\lambda r)], \end{aligned} \tag{3}$$

where $r \equiv |s-x|$, $i^2 = -1$, $H_0^{(1)}(\lambda r)$ is the first kind zeroth-order Hankel function, $J_0(\lambda r)$ and $Y_0(\lambda r)$ are the first kind and second kind zeroth-order Bessel functions, respectively, and $K_0(\lambda r)$ is the second kind zeroth-order modified Bessel function. Because the first kind modified Bessel function $I_0(\lambda r)$ is the homogeneous solution of the biharmonic operator, we can add it to the fundamental solution for satisfying the Hilbert transform of causal constraint. Then, the complete kernel function $U(s,x)$ is shown below:

$$U(s,x)=\frac{1}{8\lambda^2}[Y_0(\lambda r)-iJ_0(\lambda r)+\frac{2}{\pi}(K_0(\lambda r)-iI_0(\lambda r))]. \quad (4)$$

Based on the definition of MFS, we can represent the displacement field of plate vibration by

$$u(x_i)=\sum_{j=1}^{2N}P(s_j,x_i)\phi_j+\sum_{j=1}^{2N}Q(s_j,x_i)\varphi_j\,, \quad (5)$$

where $2N$ is the number of fictitious source nodes. ϕ_j and φ_j are the known densities with respect to P and Q. The two kernels (P and Q) are obtained from either the two of the kernel $U(s,x)$ and the following three kernels,

$$\Theta(s,x)=\aleph_\theta(U(s,x))\,, \quad (6)$$

$$M(s,x)=\aleph_m(U(s,x))\,, \quad (7)$$

$$V(s,x)=\aleph_v(U(s,x))\,, \quad (8)$$

where $\aleph_\theta(\cdot)$, $\aleph_m(\cdot)$ and $\aleph_v(\cdot)$ mean the operators which are defined as follows:

$$\aleph_\theta(\cdot)=\frac{\partial(\cdot)}{\partial n}\,, \quad (9)$$

$$\aleph_m(\cdot)=\nu\nabla^2(\cdot)+(1-\nu)\frac{\partial^2(\cdot)}{\partial n^2}\,, \quad (10)$$

$$\aleph_v(\cdot)=\frac{\partial\nabla^2(\cdot)}{\partial n}+(1-\nu)\frac{\partial}{\partial t}(\frac{\partial^2(\cdot)}{\partial n\partial t})\,, \quad (11)$$

where n and t are the normal vector and tangential vector, respectively. The operators in Eqs.(9), (10) and (11) can be applied to U, Θ, M and V kernel to generate sixteen kernels as shown in Fig. 2. Three operators can be also applied to Eq.(5), and we have

$$\theta(x)=\aleph_\theta(u(x)) \quad (12)$$

$$m(x)=\aleph_m(u(x)) \quad (13)$$

$$v(x)=\aleph_v(u(x)) \quad (14)$$

where θ, m and v denote the slope, normal moment and effective shear force, respectively. For the purpose of deriving the exact eigensolution, we consider the annular plate. The radii of inner and outer circles are a and b for the real boundary, respectively. The source strengths are distributed on the inner and outer fictitious boundaries of radii a' and b' in Fig.3, respectively. For demonstrating the validity of this approach, we consider the clamped case (u=0 and $\theta=0$) by using U and Θ kernels. We distributed $2N$ field points at each real boundary, and the same $2N$ sources are distributed on the fictitious boundary. By matching the boundary condition, Eq.(5) can be obtained and can be written in a matrix form as follows:

$$\begin{Bmatrix}0\\0\end{Bmatrix}=\begin{bmatrix}U11 & U12\\U21 & U22\end{bmatrix}\begin{Bmatrix}\phi1\\\phi2\end{Bmatrix}+\begin{bmatrix}\Theta11 & \Theta12\\\Theta21 & \Theta22\end{bmatrix}\begin{Bmatrix}\varphi1\\\varphi2\end{Bmatrix}, \quad (15)$$

where $\{\phi1\}$, $\{\phi2\}$, $\{\varphi1\}$ and $\{\varphi2\}$ are the generalized coefficients for B_1 and B_2 with dimension $2N\times1$, the matrices $[Uij]$ and $[\Theta ij]$ mean the influence matrices of U and Θ kernels which are obtained by collocating the field and source points on B_i and B'_j with a dimension $2N\times2N$, respectively. Similarly, the Eq.(12) can be rewritten as

$$\begin{Bmatrix}0\\0\end{Bmatrix}=\begin{bmatrix}U11_\theta & U12_\theta\\U21_\theta & U22_\theta\end{bmatrix}\begin{Bmatrix}\phi1\\\phi2\end{Bmatrix}+\begin{bmatrix}\Theta11_\theta & \Theta12_\theta\\\Theta21_\theta & \Theta22_\theta\end{bmatrix}\begin{Bmatrix}\varphi1\\\varphi2\end{Bmatrix}, \quad (16)$$

where the matrices $[Uij_\theta]$ and $[\Theta ij_\theta]$ mean the influence matrices of U_θ and Θ_θ kernels which are obtained by selecting the field and source points on B_i and B'_j with a dimension $2N\times2N$, respectively. By assembling the Eqs.(15) and (16) together, we have

$$[SM^{cc}]\begin{Bmatrix}\phi1\\\phi2\\\varphi1\\\varphi2\end{Bmatrix}=\{0\} \quad (17)$$

where the superscript "*cc*" denotes the clamped-clamped case and

$$[SM^{cc}]=\begin{bmatrix}U11 & U12 & \Theta11 & \Theta12\\U21 & U22 & \Theta21 & \Theta22\\U11_\theta & U12_\theta & \Theta11_\theta & \Theta12_\theta\\U21_\theta & U22_\theta & \Theta21_\theta & \Theta22_\theta\end{bmatrix}_{8N\times8N} \quad (18)$$

For the existence of nontrivial solution, the determinant of the matrix versus the eigenvlaue must be zero, *i.e.*,

$$\det[SM^{cc}]=0 \quad (19)$$

By plotting the determinant versus the frequency parameter, the curve drops at the positions of eigenvalues.

3. Mathematical analysis of true and spurious eigenvalues

For the kernel function, we can express $x=(\rho,\phi)$ and $s=(R,\theta)$ in terms of polar coordinate. The U kernel can be expressed by using degenerate kernels as shown below:

$$U^{I}(R,\theta;\rho,\phi)=\sum_{m=-\infty}^{\infty}\frac{1}{8\lambda^{2}}\{J_{m}(\lambda\rho)[Y_{m}(\lambda R)-iJ_{m}(\lambda R)] \\ +\frac{2}{\pi}(-1)^{m}I_{m}(\lambda\rho)[(-1)^{m}K_{m}(\lambda R)-iI_{m}(\lambda R)]\}\cos(m(\theta-\phi)),\quad R>\rho, \tag{20}$$

$$U^{E}(R,\theta;\rho,\phi)=\sum_{m=-\infty}^{\infty}\frac{1}{8\lambda^{2}}\{J_{m}(\lambda R)[Y_{m}(\lambda\rho)-iJ_{m}(\lambda\rho)] \\ +\frac{2}{\pi}(-1)^{m}I_{m}(\lambda R)[(-1)^{m}K_{m}(\lambda\rho)-iI_{m}(\lambda\rho)]\}\cos(m(\theta-\phi)),\quad R<\rho, \tag{21}$$

where the subscripts "I " and "E " denote the interior ($R>\rho$) and exterior domains ($R<\rho$), respectively.
Similarly, other kernels, Θ, U_{θ} and Θ_{θ}, are obtained as follows:

$$\Theta^{I}(R,\theta;\rho,\phi)=\sum_{m=-\infty}^{\infty}\frac{1}{8\lambda}\{J_{m}(\lambda\rho)[Y'_{m}(\lambda R)-iJ'_{m}(\lambda R)] \\ +\frac{2}{\pi}(-1)^{m}I_{m}(\lambda\rho)[(-1)^{m}K'_{m}(\lambda R)-iI'_{m}(\lambda R)]\}\cos(m(\theta-\phi)),\quad R>\rho, \tag{22}$$

$$\Theta^{E}(R,\theta;\rho,\phi)=\sum_{m=-\infty}^{\infty}\frac{1}{8\lambda}\{J'_{m}(\lambda R)[Y_{m}(\lambda\rho)-iJ_{m}(\lambda\rho)] \\ +\frac{2}{\pi}(-1)^{m}I'_{m}(\lambda R)[(-1)^{m}K_{m}(\lambda\rho)-iI_{m}(\lambda\rho)]\}\cos(m(\theta-\phi)),\quad R<\rho, \tag{23}$$

$$U_{\theta}^{I}(R,\theta;\rho,\phi)=\sum_{m=-\infty}^{\infty}\frac{1}{8\lambda}\{J'_{m}(\lambda\rho)[Y_{m}(\lambda R)-iJ_{m}(\lambda R)] \\ +\frac{2}{\pi}(-1)^{m}I'_{m}(\lambda\rho)[(-1)^{m}K_{m}(\lambda R)-iI_{m}(\lambda R)]\}\cos(m(\theta-\phi)),\quad R>\rho, \tag{24}$$

$$U_{E}^{I}(R,\theta;\rho,\phi)=\sum_{m=-\infty}^{\infty}\frac{1}{8\lambda^{2}}\{J_{m}(\lambda R)[Y'_{m}(\lambda\rho)-iJ'_{m}(\lambda\rho)] \\ +\frac{2}{\pi}(-1)^{m}I_{m}(\lambda R)[(-1)^{m}K'_{m}(\lambda\rho)-iI'_{m}(\lambda\rho)]\}\cos(m(\theta-\phi)),\quad R<\rho, \tag{25}$$

$$U^{I}(R,\theta;\rho,\phi)=\sum_{m=-\infty}^{\infty}\frac{1}{8}\{J'_{m}(\lambda\rho)[Y'_{m}(\lambda R)-iJ'_{m}(\lambda R)] \\ +\frac{2}{\pi}(-1)^{m}I'_{m}(\lambda\rho)[(-1)^{m}K'_{m}(\lambda R)-iI'_{m}(\lambda R)]\}\cos(m(\theta-\phi)),\quad R>\rho, \tag{26}$$

$$U^{E}(R,\theta;\rho,\phi)=\sum_{m=-\infty}^{\infty}\frac{1}{8}\{J'_{m}(\lambda R)[Y'_{m}(\lambda\rho)-iJ'_{m}(\lambda\rho)] \\ +\frac{2}{\pi}(-1)^{m}I'_{m}(\lambda R)[(-1)^{m}K'_{m}(\lambda\rho)-iI'_{m}(\lambda\rho)]\}\cos(m(\theta-\phi)),\quad R<\rho. \tag{27}$$

Since the rotation symmetry is preserved for a circular boundary, the sixteen influence matrices in Eqs.(15) and (16) are all symmetric circulants. We have the influence matrices [*U11*],

$$[U11]=\begin{bmatrix} a_0 & a_1 & a_2 & \cdots & a_{2N-2} & a_{2N-1} \\ a_{2N-1} & a_0 & a_1 & \cdots & a_{2N-3} & a_{2N-2} \\ a_{2N-2} & a_{2N-1} & a_0 & \cdots & a_{2N-4} & a_{2N-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_1 & a_2 & a_3 & \cdots & a_{2N-1} & a_0 \end{bmatrix} \tag{28}$$

where the elements of the first row can be obtained by

$$a_{j-i}=U11(s_j,x_i). \tag{29}$$

The matrix [*U11*] in Eq.(28) is found to be a circulant since the rotational symmetry for the influence coefficients is considered. By using the degenerate kernel and the orthogonal property, the eigenvalue of the matrices [*U11*], [*U12*], [*U21*] and [*U22*] can be obtained as follows:

$$\lambda_m^{[U11]}=\frac{N}{4\lambda^2}\{J_m(\lambda a')[Y_m(\lambda a)-iJ_m(\lambda a)]+ \\ \frac{2}{\pi}I_m(\lambda a')[K_m(\lambda a)-(-1)^m iI_m(\lambda a)]\}, \tag{30}$$

$$\lambda_m^{[U12]}=\frac{N}{4\lambda^2}\{J_m(\lambda a)[Y_m(\lambda b')-iJ_m(\lambda b')]+ \\ \frac{2}{\pi}I_m(\lambda a)[K_m(\lambda b')-(-1)^m iI_m(\lambda b')]\}, \tag{31}$$

$$\lambda_m^{[U21]}=\frac{N}{4\lambda^2}\{J_m(\lambda a')[Y_m(\lambda b)-iJ_m(\lambda b)]+ \\ \frac{2}{\pi}I_m(\lambda a')[K_m(\lambda b)-(-1)^m iI_m(\lambda b)]\}, \tag{32}$$

$$\lambda_m^{[U22]}=\frac{N}{4\lambda^2}\{J_m(\lambda b)[Y_m(\lambda b')-iJ_m(\lambda b')]+ \\ \frac{2}{\pi}I_m(\lambda b)[K_m(\lambda b')-(-1)^m iI_m(\lambda b')]\}. \tag{33}$$

where $m=0,\pm1,\pm2,\cdots,\pm(N-1),N$. Similarly, the eigenvalue of the other twelve matrices can be obtained. By using the similar transformation, we can decompose the [*U11*] matrix into

$$[U11]=\Phi\Sigma_{[U11]}\Phi^{H}, \tag{34}$$

where "H" is the transpose conjugate,

$$\Sigma_{[U11]}=diag(\lambda_0^{[U11]}\lambda_1^{[U11]}\lambda_{-1}^{[U11]}\cdots\lambda_{N-1}^{[U11]}\lambda_N^{[U11]}), \tag{35}$$

and

$$\Phi=\frac{1}{\sqrt{2N}}\begin{bmatrix} 1 & (e^{2\pi i/2N})^0 & (e^{-2\pi i/2N})^0 & \cdots & (e^{-2(N-1)\pi i/2N})^0 & (e^{2N\pi i/2N})^0 \\ 1 & (e^{2\pi i/2N})^1 & (e^{-2\pi i/2N})^1 & \cdots & (e^{-2(N-1)\pi i/2N})^1 & (e^{2N\pi i/2N})^1 \\ 1 & (e^{2\pi i/2N})^2 & (e^{-2\pi i/2N})^2 & \cdots & (e^{-2(N-1)\pi i/2N})^2 & (e^{2N\pi i/2N})^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & (e^{2\pi i/2N})^{2N-2} & (e^{-2\pi i/2N})^{2N-2} & \cdots & (e^{-2(N-1)\pi i/2N})^{2N-2} & (e^{2N\pi i/2N})^{2N-2} \\ 1 & (e^{2\pi i/2N})^{2N-1} & (e^{-2\pi i/2N})^{2N-1} & \cdots & (e^{-2(N-1)\pi i/2N})^{2N-1} & (e^{2N\pi i/2N})^{2N-1} \end{bmatrix}. \tag{36}$$

Similarly, the other fifteen matrices can be decomposed. Equation (18) can be decomposed and rearranged into

$$[SM^{cc}]=\begin{bmatrix} \Phi & 0 & 0 & 0 \\ 0 & \Phi & 0 & 0 \\ 0 & 0 & \Phi & 0 \\ 0 & 0 & 0 & \Phi \end{bmatrix}\begin{bmatrix} \Sigma_{[U11]} & \Sigma_{[U12]} & \Sigma_{[\Theta11]} & \Sigma_{[\Theta12]} \\ \Sigma_{[U21]} & \Sigma_{[U22]} & \Sigma_{[\Theta21]} & \Sigma_{[\Theta22]} \\ \Sigma_{[U11_\theta]} & \Sigma_{[U12_\theta]} & \Sigma_{[\Theta11_\theta]} & \Sigma_{[\Theta12_\theta]} \\ \Sigma_{[U21_\theta]} & \Sigma_{[U22_\theta]} & \Sigma_{[\Theta21_\theta]} & \Sigma_{[\Theta22_\theta]} \end{bmatrix}\begin{bmatrix} \Phi & 0 & 0 & 0 \\ 0 & \Phi & 0 & 0 \\ 0 & 0 & \Phi & 0 \\ 0 & 0 & 0 & \Phi \end{bmatrix}^{H}. \tag{37}$$

Since Φ is unitary, the determinant of [SM^{cc}] is

$$\det[SM^{cc}]=\det\begin{bmatrix} \Sigma_{[U11]} & \Sigma_{[U12]} & \Sigma_{[\Theta11]} & \Sigma_{[\Theta12]} \\ \Sigma_{[U21]} & \Sigma_{[U22]} & \Sigma_{[\Theta21]} & \Sigma_{[\Theta22]} \\ \Sigma_{[U11_\theta]} & \Sigma_{[U12_\theta]} & \Sigma_{[\Theta11_\theta]} & \Sigma_{[\Theta12_\theta]} \\ \Sigma_{[U21_\theta]} & \Sigma_{[U22_\theta]} & \Sigma_{[\Theta21_\theta]} & \Sigma_{[\Theta22_\theta]} \end{bmatrix}_{8N\times 8N}$$

$$=\prod_{m=-(N-1)}^{N}\det([T_m^{cc}][S_m^{U\Theta}]), \tag{38}$$

where

$$[T_m^{cc}] = \begin{bmatrix} J_m(\lambda a) & Y_m(\lambda a) & I_m(\lambda a) & K_m(\lambda a) \\ J_m(\lambda b) & Y_m(\lambda b) & I_m(\lambda b) & K_m(\lambda b) \\ J'_m(\lambda a) & Y'_m(\lambda a) & I'_m(\lambda a) & K'_m(\lambda a) \\ J'_m(\lambda b) & Y'_m(\lambda b) & I'_m(\lambda b) & K'_m(\lambda b) \end{bmatrix} \quad (39)$$

and

$$[S_m^{U\Theta}] = \begin{bmatrix} -iJ_m(\lambda a') & Y_m(\lambda b') - iJ_m(\lambda b') & -iJ'_m(\lambda a') & Y'_m(\lambda b') - iJ'_m(\lambda b') \\ J_m(\lambda a') & 0 & J'_m(\lambda a') & 0 \\ -(-1)^m i\frac{2}{\pi} I_m(\lambda a') & \frac{2}{\pi}[k_m(\lambda b') - (-1)^m iI_m(\lambda b')] & -(-1)^m i\frac{2}{\pi} I'_m(\lambda a') & \frac{2}{\pi}[k'_m(\lambda b') - (-1)^m iI'_m(\lambda b')] \\ \frac{2}{\pi} I_m(\lambda a') & 0 & \frac{2}{\pi} I'_m(\lambda a') & 0 \end{bmatrix} \quad (40)$$

It is noted that the matrix $[T_m^{cc}]$ denotes the matrix of true eigenequation for the *C-C* case and the matrix $[S_m^{U\Theta}]$ denotes the matrix of spurious eignequation in the U and Θ formulations. Zero determinant in the Eq.(38) implies that the eigenequation is

$$\det([T_m^{cc}][S_m^{U\Theta}]) = 0 \quad (41)$$

After comparing with the analytical solution for the annular plate [20], the former matrix $[T_m^{cc}]$ in the Eq.(41) results in the true eigenequation while the latter matrix $[S_m^{U\Theta}]$ results in the spurious eigenequation. The spurious eigenvalues occur when $\det[S_m^{U\Theta}] = 0$. The second matrix in Eq.(41) can be further decomposed into

$$\det[S_m^{U\Theta}] = \det[S_{a'}(a')]\det[S_{b'}(b')] = 0, \quad (42)$$

where

$$\det[S_{a'}(a')] = \frac{2}{\pi}\begin{vmatrix} J_m(\lambda a') & J'_m(\lambda a') \\ I_m(\lambda a') & I'_m(\lambda a') \end{vmatrix} \quad (43)$$

and

$$\det[S_{b'}(b')] = \frac{2}{\pi}\begin{vmatrix} Y_m(\lambda b') - iJ_m(\lambda b') & Y'_m(\lambda b') - iJ'_m(\lambda b') \\ K_m(\lambda b') - i(-1)^m I_m(\lambda b') & K'_m(\lambda b') - i(-1)^m I'_m(\lambda b') \end{vmatrix}. \quad (44)$$

Since $\det[S_{b'}(b')]$ is never zero, the spurious eigenequation depends on a'. It is noted that the spurious eigensolution happens to be true eigensolution of the clamped circular plate with a radius a'. Therefore, the positions of spurious eigenvalues for the annular problem depend on the location of inner fictitious boundary a' where the sources are distributed. Problems subject to different boundary conditions on the outer and inner boundaries (*C-S*, *C-F*, *S-C*, *S-S*, *S-F*, *F-C*, *F-S* and *F-F* in which *S* and *F* denote simply-supported and free boundary conditions, respectively) are also solved. All the results for different boundary conditions of the annular plate are shown in Table 1.

4. Treatment of spurious eigenvalues

4.1 SVD updating technique

In order to extract out the true eigenvalues, the SVD updating technique is utilized. In spite of the U and Θ formulations to obtain Eq.(16), we can also select the M and V formulations and obtain

$$[SM_1^{cc}]\begin{Bmatrix} \phi'1 \\ \phi'2 \\ \varphi'1 \\ \varphi'2 \end{Bmatrix} = \begin{bmatrix} M11 & M12 & V11 & V12 \\ M21 & M22 & V21 & V22 \\ M11_\theta & M12_\theta & V11_\theta & V12_\theta \\ M21_\theta & M22_\theta & V21_\theta & V22_\theta \end{bmatrix}\begin{Bmatrix} \phi'1 \\ \phi'2 \\ \varphi'1 \\ \varphi'2 \end{Bmatrix}. \quad (45)$$

where $\{\phi'1\}$, $\{\phi'2\}$, $\{\varphi'1\}$ and $\{\varphi'2\}$ are the generalized coefficients for B_1 and B_2 with dimension $2N \times 1$ using M and V formulations. By employing the relation in the degenerate kernels between direct and indirect methods, the SVD updating document (Indirect method) to extract out the true eigenequation is equivalent to the SVD updating term (Direct method). We have

$$[C] = \begin{bmatrix} (SM^{cc})^H \\ (SM_1^{cc})^H \end{bmatrix}, \quad (46)$$

For the existence of nontrivial solutions, the rank of the matrix $[C]$ must be smaller than *8N*. By using the property of Eq.(37), the matrix can be written as

$$[C] = \begin{bmatrix} \Phi & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \Phi & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \Phi & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \Phi & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \Phi & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \Phi & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \Phi & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Phi \end{bmatrix}\begin{bmatrix} \Sigma_{[U11]} & \Sigma_{[U21]} & \Sigma_{[U11_\theta]} & \Sigma_{[U21_\theta]} \\ \Sigma_{[U12]} & \Sigma_{[U22]} & \Sigma_{[U12_\theta]} & \Sigma_{[U22_\theta]} \\ \Sigma_{[\Theta11]} & \Sigma_{[\Theta21]} & \Sigma_{[\Theta11_\theta]} & \Sigma_{[\Theta21_\theta]} \\ \Sigma_{[\Theta12]} & \Sigma_{[\Theta22]} & \Sigma_{[\Theta12_\theta]} & \Sigma_{[\Theta22_\theta]} \\ \Sigma_{[M11]} & \Sigma_{[M21]} & \Sigma_{[M11_\theta]} & \Sigma_{[M21_\theta]} \\ \Sigma_{[M12]} & \Sigma_{[M22]} & \Sigma_{[M12_\theta]} & \Sigma_{[M22_\theta]} \\ \Sigma_{[V11]} & \Sigma_{[V21]} & \Sigma_{[V11_\theta]} & \Sigma_{[V21_\theta]} \\ \Sigma_{[V12]} & \Sigma_{[V22]} & \Sigma_{[V12_\theta]} & \Sigma_{[V22_\theta]} \end{bmatrix}\begin{bmatrix} \Phi^{-1} & 0 & 0 & 0 \\ 0 & \Phi^{-1} & 0 & 0 \\ 0 & 0 & \Phi^{-1} & 0 \\ 0 & 0 & 0 & \Phi^{-1} \end{bmatrix} \quad (47)$$

Based on the equivalence between the SVD technique and the least-squares method, we can obtain the true eigenequation

$$\begin{vmatrix} J_m(\lambda a) & Y_m(\lambda a) & I_m(\lambda a) & K_m(\lambda a) \\ J_m(\lambda b) & Y_m(\lambda b) & I_m(\lambda b) & K_m(\lambda b) \\ J'_m(\lambda a) & Y'_m(\lambda a) & I'_m(\lambda a) & K'_m(\lambda a) \\ J'_m(\lambda b) & Y'_m(\lambda b) & I'_m(\lambda b) & K'_m(\lambda b) \end{vmatrix} = 0 \cdot \quad (48)$$

This indicates that only the true eigenvalues for annular plate are imbedded in the SVD updating matrix.

4.2 Burton & Miller method

By employing the Burton & Miller method for dealing with fictitious frequency, we extend this concept to suppress the appearance of the spurious eigenvalue of the annular plate in the method of fundamental solutions.

By assembling the Eqs.(17) and (45) with an imaginary number, we have

$$\left[[SM^{cc}] + i[SM_1^{cc}]\right]\begin{Bmatrix}\psi 1 \\ \psi 2\end{Bmatrix} = \{0\}, \quad (49)$$

where the $\psi 1$ and $\psi 2$ are the mixed densities. Thus, the true eigenequation

$$\begin{vmatrix} J_m(\lambda a) & Y_m(\lambda a) & I_m(\lambda a) & K_m(\lambda a) \\ J_m(\lambda b) & Y_m(\lambda b) & I_m(\lambda b) & K_m(\lambda b) \\ J'_m(\lambda a) & Y'_m(\lambda a) & I'_m(\lambda a) & K'_m(\lambda a) \\ J'_m(\lambda b) & Y'_m(\lambda b) & I'_m(\lambda b) & K'_m(\lambda b) \end{vmatrix} = 0, \quad (50)$$

is obtained. After comparing Eq.(48) with Eq.(50), we can find that true eigenequations are the same either by using the SVD updating technique or by using the Burton & Miller method.

6. Conclusions

The mathematical analysis has shown that spurious eigenvalues occur by using degenerate kernels and circulants when the method of fundamental solutions is used to solve the eigenvalue of annular plates. The positions of spurious eigenvalues for the annular problem depend on the location of inner fictitious boundary where the sources are distributed. The spurious eigenvalues in the annular problem are found to be the true eigenvalues of the associated simply-connected problem bounded by the inner sources. We have employed the SVD updating technique and Burton & Miller method to filter out the spurious eigenvalues successfully.

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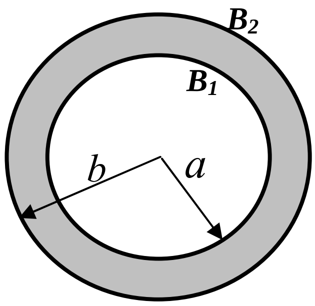

Fig. 1 An annular problem

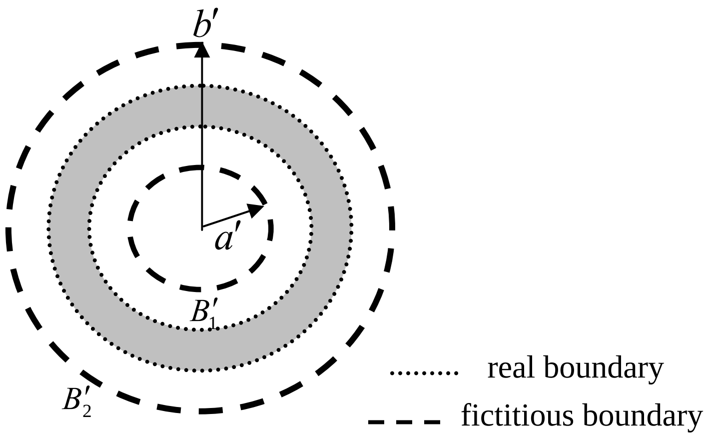

Fig. 3 Figure sketch for source distribution

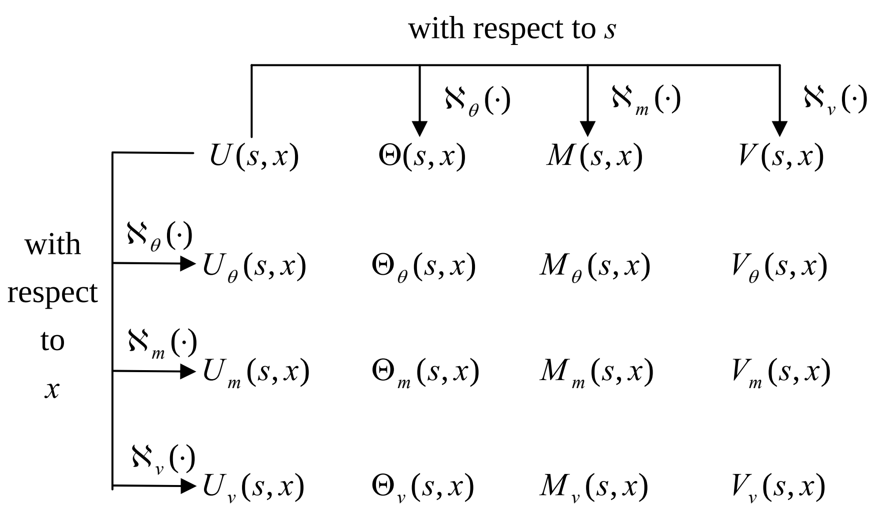

Fig. 2 The relation of sixteen kernels

Table 1 True eigenequations for the annular plate

Cases	**True eigenequation**
C-C	$\begin{vmatrix} J_m(\lambda a) & Y_m(\lambda a) & I_m(\lambda a) & K_m(\lambda a) \\ J_m(\lambda b) & Y_m(\lambda b) & I_m(\lambda b) & K_m(\lambda b) \\ J'_m(\lambda a) & Y'_m(\lambda a) & I'_m(\lambda a) & K'_m(\lambda a) \\ J'_m(\lambda b) & Y'_m(\lambda b) & I'_m(\lambda b) & K'_m(\lambda b) \end{vmatrix} = 0$
C-S	$\begin{vmatrix} J_m(\lambda a) & Y_m(\lambda a) & I_m(\lambda a) & K_m(\lambda a) \\ J_m(\lambda b) & Y_m(\lambda b) & I_m(\lambda b) & K_m(\lambda b) \\ \alpha_m^J(\lambda a) & \alpha_m^Y(\lambda a) & \alpha_m^I(\lambda a) & \alpha_m^K(\lambda a) \\ J'_m(\lambda b) & Y'_m(\lambda b) & I'_m(\lambda b) & K'_m(\lambda b) \end{vmatrix} = 0$
C-F	$\begin{vmatrix} \alpha_m^J(\lambda a) & \alpha_m^Y(\lambda a) & \alpha_m^I(\lambda a) & \alpha_m^K(\lambda a) \\ J_m(\lambda b) & Y_m(\lambda b) & I_m(\lambda b) & K_m(\lambda b) \\ \beta_m^J(\lambda a) & \beta_m^Y(\lambda a) & \beta_m^I(\lambda a) & \beta_m^K(\lambda a) \\ J'_m(\lambda b) & Y'_m(\lambda b) & I'_m(\lambda b) & K'_m(\lambda b) \end{vmatrix} = 0$
S-C	$\begin{vmatrix} J_m(\lambda a) & Y_m(\lambda a) & I_m(\lambda a) & K_m(\lambda a) \\ J_m(\lambda b) & Y_m(\lambda b) & I_m(\lambda b) & K_m(\lambda b) \\ J'_m(\lambda a) & Y'_m(\lambda a) & I'_m(\lambda a) & K'_m(\lambda a) \\ \alpha_m^J(\lambda b) & \alpha_m^Y(\lambda b) & \alpha_m^I(\lambda b) & \alpha_m^K(\lambda b) \end{vmatrix} = 0$
S-S	$\begin{vmatrix} J_m(\lambda a) & Y_m(\lambda a) & I_m(\lambda a) & K_m(\lambda a) \\ J_m(\lambda b) & Y_m(\lambda b) & I_m(\lambda b) & K_m(\lambda b) \\ \alpha_m^J(\lambda a) & \alpha_m^Y(\lambda a) & \alpha_m^I(\lambda a) & \alpha_m^K(\lambda a) \\ \alpha_m^J(\lambda b) & \alpha_m^Y(\lambda b) & \alpha_m^I(\lambda b) & \alpha_m^K(\lambda b) \end{vmatrix} = 0$
S-F	$\begin{vmatrix} \alpha_m^J(\lambda a) & \alpha_m^Y(\lambda a) & \alpha_m^I(\lambda a) & \alpha_m^K(\lambda a) \\ J_m(\lambda b) & Y_m(\lambda b) & I_m(\lambda b) & K_m(\lambda b) \\ \beta_m^J(\lambda a) & \beta_m^Y(\lambda a) & \beta_m^I(\lambda a) & \beta_m^K(\lambda a) \\ \alpha_m^J(\lambda b) & \alpha_m^Y(\lambda b) & \alpha_m^I(\lambda b) & \alpha_m^K(\lambda b) \end{vmatrix} = 0$
F-C	$\begin{vmatrix} J_m(\lambda a) & Y_m(\lambda a) & I_m(\lambda a) & K_m(\lambda a) \\ \alpha_m^J(\lambda b) & \alpha_m^Y(\lambda b) & \alpha_m^I(\lambda b) & \alpha_m^K(\lambda b) \\ J'_m(\lambda a) & Y'_m(\lambda a) & I'_m(\lambda a) & K'_m(\lambda a) \\ \beta_m^J(\lambda b) & \beta_m^Y(\lambda b) & \beta_m^I(\lambda b) & \beta_m^K(\lambda b) \end{vmatrix} = 0$

F-S	$\begin{vmatrix} J_m(\lambda a) & Y_m(\lambda a) & I_m(\lambda a) & K_m(\lambda a) \\ \alpha_m^J(\lambda b) & \alpha_m^Y(\lambda b) & \alpha_m^I(\lambda b) & \alpha_m^K(\lambda b) \\ \alpha_m^J(\lambda a) & \alpha_m^Y(\lambda a) & \alpha_m^I(\lambda a) & \alpha_m^K(\lambda a) \\ \beta_m^J(\lambda b) & \beta_m^Y(\lambda b) & \beta_m^I(\lambda b) & \beta_m^K(\lambda b) \end{vmatrix} = 0$
F-F	$\begin{vmatrix} \alpha_m^J(\lambda a) & \alpha_m^Y(\lambda a) & \alpha_m^I(\lambda a) & \alpha_m^K(\lambda a) \\ \alpha_m^J(\lambda b) & \alpha_m^Y(\lambda b) & \alpha_m^I(\lambda b) & \alpha_m^K(\lambda b) \\ \beta_m^J(\lambda a) & \beta_m^Y(\lambda a) & \beta_m^I(\lambda a) & \beta_m^K(\lambda a) \\ \beta_m^J(\lambda b) & \beta_m^Y(\lambda b) & \beta_m^I(\lambda b) & \beta_m^K(\lambda b) \end{vmatrix} = 0$

where

$$\alpha_m^J(\lambda a) = J''_m(\lambda a) + \frac{\nu}{\lambda a} J'_m(\lambda a) - \frac{m^2\nu}{(\lambda a)^2} J_m(\lambda a),$$

$$\alpha_m^Y(\lambda a) = Y''_m(\lambda a) + \frac{\nu}{\lambda a} Y'_m(\lambda a) - \frac{m^2\nu}{(\lambda a)^2} Y_m(\lambda a),$$

$$\alpha_m^I(\lambda a) = I''_m(\lambda a) + \frac{\nu}{\lambda a} I'_m(\lambda a) - \frac{m^2\nu}{(\lambda a)^2} I_m(\lambda a),$$

$$\alpha_m^K(\lambda a) = K''_m(\lambda a) + \frac{\nu}{\lambda a} K'_m(\lambda a) - \frac{m^2\nu}{(\lambda a)^2} K_m(\lambda a),$$

$$\beta_m^J(\lambda a) = J'''_m(\lambda a) + \frac{1}{\lambda a} J''_m(\lambda a) - \frac{1}{(\lambda a)^2} J'_m(\lambda a) + \frac{m^2(3-\nu)}{(\lambda a)^3} J_m(\lambda a) - \frac{m^2(2-\nu)}{(\lambda a)^2} J'_m(\lambda a),$$

$$\beta_m^Y(\lambda a) = Y'''_m(\lambda a) + \frac{1}{\lambda a} Y''_m(\lambda a) - \frac{1}{(\lambda a)^2} Y'_m(\lambda a) + \frac{m^2(3-\nu)}{(\lambda a)^3} Y_m(\lambda a) - \frac{m^2(2-\nu)}{(\lambda a)^2} Y'_m(\lambda a),$$

$$\beta_m^I(\lambda a) = I'''_m(\lambda a) + \frac{1}{\lambda a} I'''_m(\lambda a) - \frac{1}{(\lambda a)^2} I'_m(\lambda a) + \frac{m^2(3-\nu)}{(\lambda a)^3} I_m(\lambda a) - \frac{m^2(2-\nu)}{(\lambda a)^2} I'_m(\lambda a),$$

$$\beta_m^K(\lambda a) = K'''_m(\lambda a) + \frac{1}{\lambda a} K''_m(\lambda a) - \frac{1}{(\lambda a)^2} K'_m(\lambda a) + \frac{m^2(3-\nu)}{(\lambda a)^3} K_m(\lambda a) - \frac{m^2(2-\nu)}{(\lambda a)^2} K'_m(\lambda a).$$