Effective Condition Number and Condition Number for the First Kind Boundary Integral Equations by Mechanical Quadrature Methods

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Abstract

In the previous papers [14-17], we have constructed mechanical quadrature methods for solving the boundary integral equations of the first kind. The methods possess high accuracy $O(h_0^3)$ and low computing complexities, where $h_0 = \max_{1\leq m\leq d} h_m$ and h_m $(m = 1, ..., d)$ is the mesh witdth of the corresponding to curved edge Γ_m, because the generation of discrete matrix need not calculate any singular integrals. This paper aims at exploring the stability analysis based on the effective condition number (Cond_eff) and the condition number (Cond.). We first propose the new computational formulas for the effective condition number, and then estimate the upper and lower bounds of eigenvalues of the discrete matrix $\mathbf{K}_h$. Moreover, we derive that Cond. $= O(h^{-1})$ and Cond_eff $= O(h^{-1})$. Both Cond. and Cond_eff display an excellent stability of the numerical methods. Hence, we conclude that mechanical quadrature methods provide not only high accuracy algorithms $O(h^3)$, but also excellent stability.

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Numerical experiments are reported to support the stability analysis and the error estimates.

Keyword. Stability analysis, condition number, effective condition number, boundary integral equation, mechanical quadrature method.

1 Introduction

Consider the linear algebraic equations $\mathbf{K}_h\mathbf{x} = \mathbf{b}$ resulting from the first kind boundary integral equations (BIEs) of Laplace's equation by mechanical quadrature methods (MQM), where $\mathbf{K}_h$ is nonsingular and symmetric. The condition number is defined by Cond=$|\lambda_1|/|\lambda_n|$, where $|\lambda_1|$ and $|\lambda_n|$ are the maximal and the minimal values of $|\lambda_i|$ ($i = 1, ..., n$), respectively, and λ_i are the eigenvalues of matrix $\mathbf{K}_h \in R^{n\times n}$. The definition of condition number was given in Wilkinson [25], and then used in many books and papers, see Atkinson [1] and [3], Golub and van Loan [11] and Parlett [19]. The condition number is used to provide bounds of relative errors from the perturbation of both $\mathbf{K}_h$ and $\mathbf{b}$. However, in practical applications, the true relative errors may be smaller, or even much smaller than the worst Cond. Such a case was first studied in Chan and Foulser [4], Christiansen and Hansen [7], Christiansen and Saranen [8], and Huang and Li [13] recently, and is called the effective condition number. In this paper, we propose the computational formulas for the effective condition number (Cond_eff).

MQMs possess high accuracy $O(h_0^3)$ and low computing complexities[14−17], where $h_0 = \max_{1\leq m\leq d} h_m$ and h_m ($m = 1, ..., d$) is the mesh width of the corresponding to curved edge Γ_m. The generation of discrete matrix $\mathbf{K}_h$ need not calculate any singular integrals. Especially, for concave polygons Ω, the solution at a concave corner point of $\partial\Omega$ has singularities, which heavily dampen the accuracy of numerical solutions. The accuracy of Galerkin methods[23,24] is only $O(h^{1+\varepsilon})$ ($0 < \varepsilon < 1$) and the accuracy of collocation methods (CMs)[27] is even lower. In contrast, the accuracy of MQMs is as high as $O(h_0^3)$. In addition, the CMs[27] are greatly restricted in practice, since the interior angle θ of Ω can only be in $\theta \in (29.85^0,\ 330.15^0)$. Moreover, for MQMs, by extrapolations and splitting extrapolations (SEMs), the higher precision of numerical solutions and a posteriori error estimates can be achieved. In fact, the quadrature method was first proposed for an integral equation with a logarithmic kernel in Christiansen [6] in 1971, called the modified quadrature method, and its analysis was given in Saranen [20], to yield the $O(h^2)$ convergence rate. In [14-17], we propose the new quadrature methods called the MQMs, to yield the high $O(h_0^3)$ convergence rate.

The stability is a severe issue for numerical solutions of the first kind BIEs. Although the stability analysis for the modified quadrature method was given in [7,8], it is important to provide stability analysis for the new MQMs. In this paper, the effective condition number is applied to the first kind BIEs for Laplace's equation on arbitrary plane domains by MQMs, and the bounds of Cond_eff and Cond are derived in detail. We obtain that Cond.=$O(\bar{h}^{-1})$ and Cond_eff=$O(\bar{h}^{-1})$, which display an excellent stability of MQMs, where $\bar{h} = \min_{1\le m\le d} h_m$ $(m = 1, ..., d)$.

This paper is organized as follows. In next section, the effective condition number and its computational formulas are introduced. In Section 3, the bounds of Cond_eff and Cond. are derived for the typical BIEs of the first kind. In Sections 4 and 5, the stability analysis based on Cond_eff and Cond is made for closed smooth curves Γ, curved polygons Γ and open contours Γ. In Section 6, an analysis and comparisons are made for Cond_eff and Cond., and in Section 7, some numerical examples are reported to support the stability analysis, and numerical results show the significance of MQMs. In the last section, a few remarks are made.

2 Effective Condition Number

Consider the linear algebraic equations

$$\mathbf{K}_h\mathbf{x} = \mathbf{b}, \tag{2.1}$$

where the $\mathbf{x} \in R^n$ and $\mathbf{b} \in R^n$ are the unknown and known vectors, respectively. The condition number is defined by

$$\text{Cond.} = |\lambda_1|/|\lambda_n|, \tag{2.2}$$

where λ_i $(i = 1, ..., n)$ are the singular values of matrix $\mathbf{K}_h \in R^{n\times n}$ in the descending order in magnitude: $|\lambda_1| \ge ... \ge |\lambda_n| > 0$. When there occurs a perturbation of $\mathbf{b}$ or $\mathbf{K}_h$, the errors of $\mathbf{x}$ also exist as

$$\mathbf{K}_h(\mathbf{x} + \mathbf{\Delta x}) = \mathbf{b} + \mathbf{\Delta b}. \tag{2.3}$$

The values of Cond. are used to measure the relative errors of $\mathbf{x}$, given by

$$\frac{||\mathbf{\Delta x}||}{||\mathbf{x}||} \le \text{cond.}\ \frac{||\mathbf{\Delta b}||}{||\mathbf{b}||}, \tag{2.4}$$

where $||\mathbf{x}||$ is the Euclidean norm and the matrix norm $||\mathbf{K}_h|| = \sup_{\mathbf{x}\ne\mathbf{0}} \frac{||\mathbf{Kx}||}{||\mathbf{x}||}$. Note that the equality of (2.4) occurs only at very rare cases. In practical applications, the vector $\mathbf{b}$ varies within a certain region, and the true relative errors from the perturbation of $\mathbf{b}$ or $\mathbf{K}_h$ may be smaller, or even much smaller than Cond. given in (2.2).

Below we briefly provide the algorithms for the effective condition number. Details are given in [13]. Let the matrix $\mathbf{K}_h \in R^{n\times n}$ be real symmetric. The eigenvectors $\mathbf{u}_i$ satisfy $\mathbf{K}_h\mathbf{u}_i = \lambda_i\mathbf{u}_i$, where $\{\mathbf{u}_i\}$ are orthogonal, with

$$\mathbf{u}_i^T\mathbf{u}_j = \delta_{ij}, \tag{2.5}$$

where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$. In [7,11] the effective condition number is defined by

$$\text{Cond\_eff}=\frac{||\mathbf{b}||}{|\sigma_n|||\mathbf{x}||} = \frac{1}{|\sigma_n|}\frac{||\mathbf{b}||}{\sqrt{\sum_{i=1}^{n}\frac{\beta_i^2}{\sigma_i^2}}}. \tag{2.6}$$

where $\beta_i = \mathbf{u}_i^T \mathbf{b}$.

3 The typical BIE of the first kind

3.1 Description of Algorithms

We have constructed the MQMs for solving[14–17]:

$$\mathbf{A}v = f, \tag{3.1}$$

where the unknown $v(t)$ and known $f(t)$ are smooth periodic functions on $[0, 2\pi)$ with the period 2π, and the boundary integral operator $\mathbf{A}$ is defined by

$$(\mathbf{A}v)(t) = \int_0^{2\pi} a(t,\tau)v(\tau)d\tau,\ t \in [0, 2\pi] \tag{3.2}$$

with the integral kernel $a(t,\tau) = -\frac{1}{2\pi}\ln|2e^{-1/2}\sin\frac{t-\tau}{2}|$.

Let $\{\tau_j = jh,\ h = 2\pi/n,\ n \in N,\ j = 1, ..., n\}$ be the mesh set. Using the quadrature rules[21], we construct its Nyström approximate operator:

$$(\mathbf{A}_h v)(t) = h\sum_{j=1, t\neq\tau_j}^{n} a(t,\tau_j)v(\tau_j) - \frac{1}{2\pi}\ln(\frac{e^{-1/2}h}{2\pi})v(t)h, \tag{3.3}$$

and

$$E_n(\mathbf{A}) = 2\sum_{\mu=1}^{l-1}\frac{1}{(2\mu)!}\xi'(-2\mu)v^{(2\mu)}(t)h^{2\mu+1} + O(h^{2l}),\ \text{as } h \to 0, \tag{3.4}$$

where $\mathbf{E}_n(\mathbf{A}) = (\mathbf{A}_h v)(t) - (\mathbf{A}v)(t)$, and $v \in \tilde{C}^{2l}[0, 2\pi] = \{v(t)|v^{(\mu)}(t) \in C[0, 2\pi]$, and $v^{(\mu)}(t + 2\pi) = v^{(\mu)}(t),\ \mu = 0, 1\ , ..., 2l\}$. In (3.4) $\xi(t)$ is the Riemann zeta function. Using quadrature rules (3.3), we obtain the linear algebraic equations

$$(\mathbf{A}_h v_h)(t_i) = f(t_i),\ i = 1, ..., n. \tag{3.5}$$

In [17] we have proved that there exist the unique solutions for (3.5) such that $|v_h(t_i) - v(t_i)| = O(h^3)$ $(i = 1, ..., n)$ as $v(t) \in \tilde{C}^4[0, 2\pi]$. Moreover, we have derived in [17]

$$v(t) - v_h(t) = \sum_{\mu=1}^{2} w_\mu(t)h^{2\mu+1} + O(h^6),\ t \in \{t_i\},\ \text{as } v(t) \in \tilde{C}^6[0, 2\pi],$$

where $w_\mu(t) \in \tilde{C}[0, 2\pi]$ $(\mu = 1, 2)$ are independent of h, and $v_h(t)$ and $v(t)$ are the solutions of (3.5) and (3.1) at $t = t_j$ respectively. Hence, for MQMs, the superconvergence $O(h^6)$ can be achieved by Richardson's extrapolations[17].

3.2 Condition number

From (3.2), (3.3) and (3.5), we obtain that the discrete matrix $\mathbf{A}_h$ is a real symmetric matrix. In the subsection we will derive the upper and the lower bounds of the eigenvalues λ_i $(i = 0, 1, ..., n-1)$ of $\mathbf{A}_h$. First we give two Lemmas.

Lemma 3.1. Let n be an arbitrary positive integer number. Then

$$\Pi_{j=1}^{n-1} \sin^j(\frac{j\pi}{n}) = \frac{n^{n/2}}{2^{n(n-1)/2}}. \tag{3.6}$$

Especially, when $n = 2k$,

$$\Pi_{j=1}^{k-1} \sin\frac{j\pi}{2k} = \frac{\sqrt{k}}{2^{k-1}}, \tag{3.7}$$

and when $n = 2k-1$,

$$\Pi_{j=1}^{k-1} \sin\frac{j\pi}{2k-1} = \frac{\sqrt{2k-1}}{2^{k-1}}. \tag{3.8}$$

Proof. Let $w_k = e^{2k\pi i/n}$ and $\bar{w}_k = e^{-2k\pi i/n}$ $(k = 0, 1, ..., n-1)$ be a pair of conjugate complex numbers. Denote

$$V = \begin{vmatrix} 1 & 1 & ... & 1 & 1 \\ w_0 & w_1 & ... & w_{n-2} & w_{n-1} \\ w_0^2 & w_1^2 & ... & w_{n-2}^2 & w_{n-1}^2 \\ ... & ... & ... & ... & ... \\ w_0^{n-1} & w_1^{n-1} & ... & w_{n-2}^{n-1} & w_{n-1}^{n-1} \end{vmatrix}, \ \bar{V} = \begin{vmatrix} 1 & 1 & ... & 1 & 1 \\ \bar{w}_0 & \bar{w}_1 & ... & \bar{w}_{n-2} & \bar{w}_{n-1} \\ \bar{w}_0^2 & \bar{w}_1^2 & ... & \bar{w}_{n-2}^2 & \bar{w}_{n-1}^2 \\ ... & ... & ... & ... & ... \\ \bar{w}_0^{n-1} & \bar{w}_1^{n-1} & ... & \bar{w}_{n-2}^{n-1} & \bar{w}_{n-1}^{n-1} \end{vmatrix}.$$

Using the principal character of Vandermode determinants, we have

$$V\bar{V} = \Pi_{0\le j<k\le n-1}(w_k - w_j)\Pi_{0\le j<k\le n-1}(\bar{w}_k - \bar{w}_j)$$

$$= \Pi_{0\le j<k\le n-1}(2 - 2\cos(\frac{2(k-j)\pi}{n})) = \Pi_{0\le j<k\le n-1}(2\sin(\frac{(k-j)\pi}{n}))^2. \tag{3.9}$$

Based on the product rule of determinants, we obtain

$$(V\bar{V})_{(k+1,j+1)} = w_0^k \cdot 1 + w_1^k \cdot \bar{w}_j + ... + w_{n-1}^k \cdot \bar{w}_j^{n-1}$$

$$= 1 + e^{\frac{2(k-j)\pi i}{n}} + e^{\frac{4(k-j)\pi i}{n}} + ... + e^{\frac{2(n-)(k-j)\pi i}{n}}$$

$$= \begin{cases} n, \text{ as } k = j, \\ 0, \text{ as } k \neq j, \end{cases}$$

where $(V\bar{V})_{(k+1,j+1)}$ is the entry at the $(k+1)$th row and the $(j+1)$th column of the determinant $V\bar{V}$. Hence

$$V\bar{V} = n^n. \tag{3.10}$$

From (3.9) and (3.10), we have

$$\Pi_{0\leq j<k\leq n-1}(2\sin(\frac{(k-j)\pi}{n})) = n^{\frac{n}{2}}.$$

Since $\sin\frac{(n-k)\pi}{n} = \sin\frac{k\pi}{n}$ $(k = 1, ..., n-1)$, we conclude

$$\Pi_{j=1}^{n-1}\sin^j(\frac{j\pi}{n}) = \frac{n^{n/2}}{2^{n(n-1)/2}}.$$

This is the first result (3.6). Especially, when $n = 2k$, since

$$\Pi_{j=1}^{2k-1}\sin^j(\frac{j\pi}{2k}) = [\Pi_{j=1}^{k-1}\sin(\frac{j\pi}{2k})]^{2k} = \frac{(2k)^k}{2^{k(2k-1)}},$$

we have

$$\Pi_{j=1}^{k-1}\sin\frac{j\pi}{2k} = \frac{\sqrt{k}}{2^{k-1}}.$$

Also when $n = 2k-1$, since $\sin\frac{j\pi}{2k-1} = \sin(\frac{(2k-1-j)\pi}{2k-1})$ and

$$\Pi_{j=1}^{2k-2}\sin^j(\frac{j\pi}{2k-1}) = [\Pi_{j=1}^{k-1}\sin(\frac{j\pi}{2k-1})]^{2k-1} = \frac{(2k-1)^{\frac{2k-1}{2}}}{2^{(k-1)(2k-1)}},$$

we have

$$\Pi_{j=1}^{k-1}\sin\frac{j\pi}{2k-1} = \frac{\sqrt{2k-1}}{2^{k-1}},$$

and complete the proof of Lemma 3.1.

Lemma 3.2. Let λ_i be the eigenvalues of discrete real symmetric matrix $\mathbf{A}_h$. Then there exist two positive constant c_1 and c_2 independent of h such that

$$c_1 = \lambda_0 > \lambda_1 > ... > \lambda_{n-1} \geq c_2 h. \tag{3.11}$$

Proof. From (3.2) and (3.3), we conclude that A_h is a circular matrix and consists of the entries:

$$\begin{cases} a_0 = \frac{-h}{2\pi}\ln|\frac{e^{-1/2}h}{2\pi}|, \\ a_j = \frac{-h}{2\pi}\ln|2e^{-1/2}\sin\frac{jh}{2}|, j = 1, ..., n-1. \end{cases}$$

Moreover, based on the theory of circular matrix[9], eigenvalues λ_k of matrix A_h can be expressed by

$$\lambda_k = \sum_{j=0}^{n-1} a_j \varepsilon_k^j,\ k = 0, ..., n-1,\ \text{and}\ \varepsilon_k = \exp(2k\pi i/n), i = \sqrt{-1}.$$

Firstly, consider the maximal eigenvalue

$$\lambda_0 = \frac{-h}{2\pi}[\ln|\frac{e^{-1/2}h}{2\pi}| + \sum_{j=1}^{n-1} \ln|2e^{-1/2}\sin\frac{jh}{2}|].$$

Based on Lemma 3.1, there exist the following equalities

$$\sum_{j=1}^{2m-1} \ln|2\sin\frac{j\pi}{2m}| = \sum_{j=1}^{m-1} \ln|2\sin\frac{j\pi}{2m}| + \sum_{j=m+1}^{2m-1} \ln|2\sin\frac{j\pi}{2m}| + \ln 2$$

$$= 2\ln\sqrt{m} + \ln 2 = \ln n, \text{ as } n = 2m,$$

and

$$\sum_{j=1}^{2m} \ln|2\sin\frac{j\pi}{2m+1}| = \sum_{j=1}^{m} \ln|2\sin\frac{j\pi}{2m+1}| + \sum_{j=m+1}^{2m} \ln|2\sin\frac{j\pi}{2m+1}|$$

$$= 2\ln\sqrt{2m+1} = \ln n, \text{ as } n = 2m+1,$$

to give

$$\lambda_0 = -\frac{1}{n}[-\ln n - \frac{n}{2} + \ln n] = \frac{1}{2}.$$

This gives the upper bound in (3.11). Secondly, consider λ_k $(k = 1, ..., n-1)$, i.e.,

$$\lambda_k = \frac{-h}{2\pi}[\ln|\frac{e^{-1/2}h}{2\pi}| + \sum_{j=1}^{n-1} \cos\frac{2k\pi j}{2n} \ln|2e^{-1/2}\sin\frac{j\pi}{n}|]$$

$$= \frac{-1}{n}[\frac{-1}{2} - \ln n - \frac{1}{2}\sum_{j=1}^{n-1} \cos\frac{2k\pi j}{2n} + \sum_{j=1}^{n-1} \cos\frac{2k\pi j}{2n} \ln|2\sin\frac{j\pi}{n}|]$$

$$= \frac{-1}{n}[-\ln n + \sum_{j=1}^{n-1} \ln|2\sin\frac{\pi j}{n}| - \sum_{j=1}^{n-1} \sin^2\frac{k\pi j}{n} \ln|2\sin\frac{j\pi}{n}|]$$

$$= \frac{1}{n}\ln n - \frac{1}{2n}\ln n + \frac{1}{2n}\sum_{j=1}^{n-1} \sin^2\frac{k\pi j}{n} \ln|2\sin\frac{j\pi}{n}|.$$

Using the remainder expression of Euler-Maclaurin's formula[10] yields

$$\lambda_k = \frac{1}{n}\sum_{j=1}^{n-1} \sin^2\frac{k\pi j}{n} \ln|2\sin\frac{j\pi}{n}| = \int_0^1 \sin^2 k\pi x \ln|2\sin\pi x|dx + O(h^{2k+1})$$

$$= \frac{1}{2}\int_0^1 \ln|2\sin\pi x|dx - \frac{1}{2}\int_0^1 \cos(2k\pi x)\ln|2\sin\pi x|dx + O(h^{2k+1})$$

$$= \frac{1}{2}\ln 2 + \frac{1}{2}\int_0^1 \ln|\sin\pi x|dx - \frac{1}{2}\int_0^1 \cos(2k\pi x)[-\sum_{p=1}^{\infty} \frac{1}{p}\cos(2p\pi x)]dx + O(h^{2k+1})$$

$$= \frac{1}{4k} + O(h^{2k+1}) > \frac{1}{4k} + O(h^{2k+1}) > \frac{1}{4n} + O(h^{2k+1}) = c_2 h.$$

This gives the lower bound in (3.11), and completes the proof of Lemma 3.2.

Based on Lemma 3.2 and (2.2), we obtain the following theorem immediately.

Theorem 3.3. Let $\mathbf{A}_h$ be the discrete real symmetric matrix $\mathbf{A}_h$ in (3.5) defined by the quadrature rules (3.3). Then the condition number of $\mathbf{A}_h$ has the bound

$$\text{Cond.} = O(h^{-1}). \tag{3.12}$$

3.3 Effective condition number

Based on the definition of effective condition number (2.6), we need estimate the upper bound of known vector $\mathbf{b} = (f(t_1),\ ...,\ f(t_n))^T$ and the lower bound of unknown vector $\mathbf{x} = (v_h(t_1),\ ...,\ v_h(t_n))^T$ in 2-norm. From [17], when the solution $v(t) \in \tilde{C}^4[0, 2\pi]$ of (3.1), we have $|v_h(t_i) - v(t_i)| = O(h^3)$ $(i = 1, ..., n)$, which implies that there exist two positive constants c_1 and c_2 independent of h such that

$$c_1 \leq |v_h(t_i)| \leq c_2,\ i = 1, ..., n.$$

Hence, the solution vector $\mathbf{x} = (v_h(t_1),\ ...,\ v_h(t_n))^T$ satisfies

$$\tilde{c}_1 h^{-0.5} = c_1\sqrt{n} \leq ||\mathbf{x}|| = \{\sum_{i=1}^{n} |v_h(t_i)|^2\}^{1/2} \leq c_2\sqrt{n} = \tilde{c}_2 h^{-0.5}, \tag{3.13}$$

where $\tilde{c}_1 = \sqrt{2\pi}c_1$ and $\tilde{c}_2 = \sqrt{2\pi}c_2$. Next, we derive the upper bound of known vector $\mathbf{b} = (f(t_1),\ ...,\ f(t_n))^T$. From [2,23,26], if $v(t) \in \tilde{C}^k[0, 2\pi]$, then $f(t) \in \tilde{C}^{k+1}[0, 2\pi]$. Based on the theory of circular matrix[9], there exists an unitary matrix $\mathbf{U}$ such that

$$\mathbf{U}\mathbf{A}_h\mathbf{U}^T = \text{diag}(\lambda_0,\ ...,\ \lambda_{n-1}).$$

From the principal character of unitary matrix $\mathbf{U}$, we have[5,12]

$$||\mathbf{b}||_2 = \sqrt{(\mathbf{\Sigma}_{i=1}^{n}\beta_i^2)} = ||\mathbf{U}\mathbf{b}||_2 = \{\sum_{j=1}^{n}[f(t_j)]^2\}^{1/2}$$

$$= h^{-0.5}\{\sum_{j=1}^{n} h[f(t_j)]^2\}^{1/2} \simeq h^{-0.5}\{\int_0^{2\pi} f^2(t)dt\}^{1/2} = O(h^{-0.5}). \tag{3.14}$$

From (2.6), (3.11), (3.13) and (3.14), we obtain the bound of effective condition number immediately described in the following theorem.

Theorem 3.4. Let $\mathbf{A}_h$ be the discrete real symmetric matrix $\mathbf{A}_h$ in (3.5) by the quadrature rule (3.3). Then the effective condition number has the bound

$$\text{Cond\_eff} = \frac{||\mathbf{b}||}{\lambda_n ||\mathbf{x}||} = O(h^{-1}). \tag{3.15}$$

4 Stability analysis for closed smooth curve Γ

4.1 Description of numerical methods

By the layer potential theory, Dirichlet's problems of Laplace's equation:

$$\left\{\begin{array}{c} \Delta u = 0, \text{ in } \Omega, \\ u = f, \text{ on } \Gamma = \partial\Omega, \end{array}\right. \tag{4.1}$$

are converted into the first kind BIEs[2,23]

$$-\frac{1}{2\pi}\int_{\Gamma} v(x)\ln|x-y|ds_x = f(y), \ y\in\Gamma, \tag{4.2}$$

where $\Omega \subset R^2$ is a bounded domain with a closed smooth edge Γ, and $|x-y| = \{(x_1-y_1)^2+(x_2-y_2)^2\}^{1/2}$. In (4.2) the unknown function $v(x) = \frac{\partial u(x)}{\partial \nu^-} - \frac{\partial u(x)}{\partial \nu^+}$, where ν is a unit outward normal at a point $x \in \Gamma$. From the known results[2,23,27], when the logarithmic capacity (transfinite diameter) $C_\Gamma \neq 1$, there exists a unique solution of (4.2). As soon as $v(x)$ is solved from (4.2), the solutions of (4.1) at interior or exterior points can be calculated by

$$u(y) = -\frac{1}{2\pi}\int_{\Gamma} v(x)\ln|x-y|ds_x, \ y\in R^2\backslash\Gamma.$$

Assume that $C_\Gamma \neq 1$ and Γ can be described by the parameter mapping $x(t) = (x_1(t), x_2(t)) \in \tilde{C}^l[0,2\pi] : [0,2\pi) \to \Gamma$ with

$$\check{\mu} \geq |x^{'}(t)|^2 = |x_1^{'}(t)|^2 + |x_2^{'}(t)|^2 \geq \hat{\mu} > 0,$$

where $\check{\mu}$ and $\hat{\mu}$ are two constants. Define the boundary integral operator

$$(\mathbf{K}v)(t) = \int_0^{2\pi} k(t,\tau)v(\tau)d\tau, \ t\in[0,2\pi), \tag{4.3}$$

where $k(t,\tau) = -\frac{1}{2\pi}\ln|x(t)-x(\tau)|$ and $v(t) = v(x(t))|x^{'}(t)|$. Then Eq (4.2) is converted into

$$\mathbf{K}v = \mathbf{A}v + \mathbf{B}v = f, \tag{4.4}$$

where $\mathbf{B} = \mathbf{K} - \mathbf{A}$ and $(\mathbf{B}v)(t) = \int_0^{2\pi} b(t,\tau)v(\tau)d\tau$ with

$$b(t,\tau) = \left\{\begin{array}{l} -\frac{1}{2\pi}\ln|\frac{e^{1/2}(x(t)-x(\tau))}{2\sin((t-\tau)/2)}|, \text{ for } t-\tau \neq 2\pi Z, \\ -\frac{1}{2\pi}\ln|e^{1/2}x'(t)|, \text{ for } t-\tau = 2\pi Z, \end{array}\right.$$

where $Z = \{0, \pm1, \pm2, ...\}$.

Using the trapezoidal or midpoint rule[10], we obtain the Nyström's approximate operator B_h of B

$$(\mathbf{B}_h v)(t) = \sum_{\tau_j\neq t, j=1}^{n} hb(t,\tau_j)v(\tau_j) + \frac{-h}{2\pi}\ln|e^{1/2}x'(t)|v(t). \tag{4.5}$$

Hence we get the approximate equations of (4.4)

$$\mathbf{A}_h v_h(t_i) + \mathbf{B}_h v_h(t_i) = f(t_i), i = 1, ..., n. \tag{4.6}$$

Lemma 4.1[17]**.** Let Γ ($C_\Gamma \neq 1$) be an arbitrarily closed smooth curve. Assume that $v(t)$ is 6 times differentiable on $[0, 2\pi]$. Assume also that $k(t,\tau)v(\tau)$ is periodic with the period 2π, and that they are 6 times differentiable on {-$\infty, \infty\}\backslash\{t + 2\pi m\}_{m=-\infty}^{\infty}$. Then we have the following:

(1) There exists a unique solution in (4.6) and

$$|v_h(t_i) - v(t_i)| = O(h^3), \ i = 1, ..., n.$$

(2) There exist the functions $w_\mu(t) \in \tilde{C}[0, 2\pi]$ ($\mu = 1, 2$) independent of h such that

$$v(t) - v_h(t) = \sum_{\mu=1}^{2} w_\mu(t) h^{2\mu+1} + O(h^6), \ t \in \{t_i\},$$

where $v_h(t)$ and $v(t)$ are the solutions of (4.6) and (4.4) at $t = t_j$, respectively.

Lemma 4.1 implies that for the closed smooth curve Γ with $C_\Gamma \neq 1$, the superconvergence $O(h^6)$ can also be achieved by Richardson's extrapolations.

4.2 Condition number

From (3.2), (3.3), (3.5), (4.4) and (4.5) we obtain that the discrete matrix $\mathbf{A}_h$ and $\mathbf{B}_h$ all are real symmetric matrices. Now we also estimate the upper and the lower bounds of eigenvalues $|\lambda_i|$ ($i = 0, 1, ..., n-1$) of $\mathbf{K}_k = \mathbf{A}_h + \mathbf{B}_h$. We first cite the following known results[12].

Lemma 4.2. (1) Define $|\mathbf{D}| = [|d_{ij}|]$ for any matrix $\mathbf{D} = [d_{ij}]$, then its spectral radius satisfies $\rho(\mathbf{D}) \leq \rho(|\mathbf{D}|)$. (2) If $|\mathbf{D}| \leq \mathbf{C}$, then $\rho(\mathbf{D}) \leq \rho(\mathbf{C})$. (3) If the diagonal matrix $\mathbf{D} \geq 0$ and the matrix $\mathbf{C} \geq 0$ with $\mathrm{Re}\lambda_i(\mathbf{D} - \mathbf{C}) > 0$, and if the matrix $|\mathbf{M}_1| \geq \mathbf{D}$ and $|\mathbf{M}_2| \leq \mathbf{C}$, then we conclude that (a) $\mathbf{M}_1 - \mathbf{M}_2$ is a nonsingular matrix, (b) $|(\mathbf{M}_1 - \mathbf{M}_2)^{-1}| \leq (\mathbf{D} - \mathbf{C})^{-1}$ and (3) $|\det(\mathbf{M}_1 - \mathbf{M}_2)| \geq \det(\mathbf{D} - \mathbf{C})$.

Lemma 4.3. Let Γ ($C_\Gamma \neq 1$) be an arbitrarily closed smooth curve. Assume that $\mathbf{A}_h$ and $\mathbf{B}_h$ are the discrete matrices defined by (3.3) and (4.6), respectively. Then the eigenvalues $|\lambda_i|$ ($i = 0, 1, ..., n-1$) of discrete matrix $\mathbf{K}_h = \mathbf{A}_h + \mathbf{B}_h$ satisfy

$$\check{c} \geq |\lambda_i| \geq \hat{c} h^{-1}, i = 0, 1, ..., n-1, \tag{4.7}$$

where $\check{c}$ and $\hat{c}$ are two positive constants independent of h.

Proof. From (3.5), (4.3), (4.5) and (4.6), discrete matrices $\mathbf{A}_h$ and $\mathbf{B}_h$ are real symmetric matrices, and their diagonal entries are $a_{ii} = a_0 = -\frac{h}{2\pi} \ln |\frac{e^{-1/2} h}{2\pi}|$ and $b_{ii} = -\frac{h}{2\pi} \ln |e^{1/2} x'(t_i)|$, respectively. Two cases are discussed.

Case I. When $1 \geq e^{1/2} \check{\mu} \geq e^{1/2} |x^{'}(t)| \geq e^{1/2} \hat{\mu} > 0$, we choose $\alpha = a_0 + c_0$, where $c_0 = -\frac{h}{2\pi} \ln(e^{1/2} \check{\mu}) \geq 0$. Let $\mathbf{D}$ =diag(α, ..., α) and $\mathbf{C} = [\frac{c_0}{n}]_{i,j=1}^{n}$.

Obviously, the matrix $\mathbf{D}-\mathbf{C}$ is a circular matrix[10]. From the theory of circular matrix, the eigenvalues $\bar{\lambda}_k$ of matrix $\mathbf{D}-\mathbf{C}$ are given by

$$\bar{\lambda}_k=((a_0+c_0)-\frac{c_0}{n})-\sum_{j=1}^{n-1}\frac{c_0}{n}\varepsilon_k^j,\ k=0,...,n-1 \text{ and } \varepsilon_k=\exp(2k\pi i/n),\ i=\sqrt{-1}.$$

Hence, by some manipulations we have

$$\bar{\lambda}_k=\begin{cases}\frac{1}{2n}+\frac{1}{n}\ln n\geq ch, \text{ as } k=0,\\ \frac{1}{2n}+\frac{1}{n}\ln n-\frac{1}{n}\ln(e^{1/2}\check{\mu})\geq ch, \text{ as } k=1,...,n-1,\end{cases}$$

where c is a positive constant number independent of n.

Case II. When $e^{1/2}\check{\mu}\geq e^{1/2}|x'(t)|\geq e^{1/2}\hat{\mu}\geq 1$, we choose $\alpha=a_0-c_0$, where $c_0=\frac{h}{2\pi}\ln(e^{1/2}\check{\mu})\geq 0$. Let $\mathbf{D}$ =diag(α, ..., α) and $\mathbf{C}=[\frac{c_0}{n}]_{i,j=1}^n$. Eigenvalues $\bar{\lambda}_k$ of matrix $\mathbf{D}-\mathbf{C}$ are given by

$$\bar{\lambda}_k=((a_0-c_0)-\frac{c_0}{n})-\sum_{j=1}^{n-1}\frac{c_0}{n}\varepsilon_k^j$$

$$=\begin{cases}\frac{1}{2n}+\frac{1}{n}\ln n-2c_0, \text{ as } k=0,\\ \frac{1}{2n}+\frac{1}{n}\ln n-c_0, \text{ as } k=1,...,n-1,\end{cases}$$

where $k=0,...,n-1$, and $\varepsilon_k=\exp(2k\pi i/n), i=\sqrt{-1}$. Since $\check{\mu}$ is a bounded positive number, there always exists a positive integer number n_0 such that $n\geq n_0\geq(e^{1/2}\check{\mu})^2\geq 1$. Hence, when $n\geq n_0$ we have $\bar{\lambda}_k\geq ch$, where c is a positive constant number independent of n.

Denote $\mathbf{M}_1=\mathbf{K}_h=\mathbf{A}_h+\mathbf{B}_h$ and $\mathbf{M}_2=[0]_{i,j=1}^n$. Obviously, $|\mathbf{M}_1|\geq\mathbf{D}$, $|\mathbf{M}_2|\leq\mathbf{C}$ and $\mathbf{K}_h=\mathbf{M}_1-\mathbf{M}_2$. Since $\mathrm{Re}\,\bar{\lambda}_i(\mathbf{D}-\mathbf{C})=ch$ and Lemma 4.2, $\mathbf{K}_h=\mathbf{M}_1-\mathbf{M}_2$ is invertible and

$$0<|\bar{\lambda}_i(\mathbf{K}_h^{-1})|\leq\rho(\mathbf{K}_h^{-1})\leq\rho(|\mathbf{K}_h^{-1}|)$$
$$\leq\rho((\mathbf{D}-\mathbf{C})^{-1})\leq ch^{-1},$$

i.e.,

$$|\bar{\lambda}_i(\mathbf{K}_h)|\geq(\rho((\mathbf{D}-\mathbf{C})^{-1}))^{-1}\geq ch,\ i=0,...,n-1.$$

This is the lower bound of $|\lambda_i|$ in (4.7).

Next, we derive the upper bound of eigenvalues $|\lambda_i|$ of $\mathbf{K}_h=\mathbf{A}_h+\mathbf{B}_h$. From [12] we have $\rho(\mathbf{K}_h)=\rho(\mathbf{A}_h+\mathbf{B}_h)\leq\rho(\mathbf{A}_h)+\rho(\mathbf{B}_h)$. Moreover, from Lemma 3.2 we obtain $\rho(\mathbf{A}_h)=c_1$, and from [12] we obtain $\rho(\mathbf{B}_h)\leq c_2$, where c_1 and c_2 are two positive constants. Hence, $\rho(\mathbf{K}_h)\leq c_1+c_2$ and the upper bound of $|\lambda_i|$ in (4.7) follows. This completes the proof of Lemma 4.3.

Based on Lemma 4.3, we have the following theorem.

Theorem 4.4. Let Γ ($C_\Gamma\neq 1$) be an arbitrarily closed smooth curve. Assume that $\mathbf{A}_h$ and $\mathbf{B}_h$ are the discrete real symmetric matrices defined by (3.3) and (4.5), respectively. Then the condition number for (4.6) has the bound

$$\text{Cond.}(\mathbf{K}_h)=O(h^{-1}). \tag{4.8}$$

4.3 Effective condition number

Since $\mathbf{K}_h$ is a real symmetric matrix from (3.5), (4.3) and (4.6), there exists the unitary matrix $\mathbf{U}$ such that[5,19]

$$\mathbf{U}\mathbf{K}_h\mathbf{U}^T = \text{diag}(\sigma_1, ..., \sigma_n).$$

From the known results of [2,23,27], $f(t)$ $(t \in [0, 2\pi])$ is smoother than $v(t)$. Hence, we have for (4.6)

$$||\mathbf{b}|| = \{\Sigma_{i-1}^n |f(t_i)|^2\}^{1/2}$$

$$= h^{-0.5}\{\Sigma_{i-1}^n h|f(t_i)|^2\}^{1/2} \simeq h^{-0.5}\{\int_0^{2\pi} f^2(t)dt\}^{1/2} \leq ch^{-0.5}, \tag{4.9}$$

where $\mathbf{b} = (f(t_1), ..., f(t_n))^T$. From (2.6), (3.13), (4.7), (4.9) and Lemma 4.3, we have the following theorem.

Theorem 4.5. Let Γ $(C_\Gamma \neq 1)$ be an arbitrarily closed smooth curve. Assume that $\mathbf{A}_h$ and $\mathbf{B}_h$ are the real symmetric matrices defined by (3.5) and (4.5), respectively. Then the effective condition number for (4.6) has the bound

$$\text{Cond-eff}=\frac{||\mathbf{b}||}{|\lambda_n|||\mathbf{x}||} \leq ch^{-1} = O(h^{-1}), \tag{4.10}$$

where c is a constant number independent of h.

5 Stability analysis for curved polygons Γ or open contours Γ

Let $\Gamma = \cup_{m=1}^d \Gamma_m$ $(d > 1)$ be curved polygons or open contours with $C_\Gamma \neq 1$, and Γ_m be a piecewise smooth curve. Define the boundary integral operators on Γ_m,

$$(\mathbf{K}_{qm}v_m)(y) = -\frac{1}{2\pi}\int_{\Gamma_m} v_m(x)\log|y-x|ds_x, y \in \Gamma_q,\ m, q = 1, ..., d, \tag{5.1}$$

where $v_m(x) = \frac{\partial u_m(x)}{\partial \nu^-} - \frac{\partial u_m(x)}{\partial \nu^+}$. Then Eq (4.1) can be converted into a matrix operator equation

$$\mathbf{K}\mathbf{v} = \mathbf{F}, \tag{5.2}$$

where $\mathbf{K} = [\mathbf{K}_{qm}]_{q,m=1}^d$, $\mathbf{v} = (v_1(x),\ ...,\ v_d(x))^T$ and $\mathbf{F} = (f_1(y),\ ...,\ f_d(y))^T$. Here, let $\mathbf{K}$ be symmetric operators.

5.1 Mechanical Quadrature Methods

Assume that Γ_m can be described by the parameter mapping $x_m(s) = (x_{m1}(s), x_{m2}(s)) : [0, T_m] \to \Gamma_m$ with $\check{\mu}_m \geq |x_m^{'}(s)| = [|x_{m1}^{'}(s)|^2 + |x_{m2}^{'}(s)|^2]^{1/2} \geq \hat{\mu}_m > 0$, where $\check{\mu}$ and $\hat{\mu}$ are two constants, and T_m is the arc length of Γ_m. Using the $\sin^p-$transformation[22]

$$s = T_m\varphi_p(t) : [0, 1] \to [0, T_m],\ p \in N, \tag{5.3}$$

with $\varphi_p(t) = \vartheta_p(t)/\vartheta_p(1)$ and $\vartheta_p(t) = \int_0^t (\sin \pi t)^p dt$, then the integral operators (5.1) can be converted into integral operators on [0,1] as follows. Define

$$(\mathbf{A}_{qq}w_q)(t) = \int_0^1 a_{qq}(t, \tau)w_q(\tau)d\tau,\ t \in [0, 1], \tag{5.4}$$

$$(\mathbf{B}_{qm}w_m)(t) = \int_0^1 b_{qm}(t, \tau)w_m(\tau)d\tau,\ t \in [0, 1], \tag{5.5}$$

where $a_{qq}(t, \tau) = -\frac{1}{2\pi} \ln |2e^{-1/2} \sin \pi(t-\tau)|$, $w_m(t) = v_m(x_m(T_m\varphi_p(t)))|x'_m(T_m \varphi_p(t))|T_m\varphi'_p(t)$ and

$$b_{qm}(t, \tau) = \begin{cases} -\frac{1}{2\pi} \ln |\frac{x_q(t)-x_q(\tau)}{2e^{-1/2} \sin \pi(t-\tau)}|, \text{ for } q = m, \\ -\frac{1}{2\pi} \ln |x_q(t) - x_m(\tau)|, \text{ for } q \neq m. \end{cases}$$

In (5.4) and (5.5), $x_m(t) = (x_{m1}(T_m\varphi_p(t)), x_{m2}(T_m\varphi_p(t)))$ $(m = 1, ..., d)$, and $|x_q(t) - x_m(\tau)| = [(x_{q1}(t) - x_{m1}(\tau))^2 + (x_{q2}(t) - x_{m2}(\tau))^2]^{1/2}$. Hence Eq (5.2) becomes

$$(\mathbf{A} + \mathbf{B})\mathbf{W} = \mathbf{G}, \tag{5.6}$$

where $\mathbf{A}$ =diag$(\mathbf{A}_{11}, ..., \mathbf{A}_{qq})$ and $\mathbf{B} = [\mathbf{B}_{qm}]_{q,m=1}^d$ are symmetric operators, and $\mathbf{W} = (w_1, ..., w_d)^T$ and $\mathbf{G} = (g_1, ..., g_d)^T$ with $g_m(t) = f_m(x_m(t))$.

Let $h_m = 1/n_m$ $(n_m \in N,\ m = 1, ..., d)$ be mesh widths for the nodes, $t_j = \tau_j = (j - 1/2)h_m$ $(j = 1, ..., n_m)$. By the trapezoidal or the midpoint rule[10] we construct the Nyström's approximate operator $\mathbf{B}_{qm}^h$ of $\mathbf{B}_{qm}$. For the weakly singular operators $\mathbf{A}_{mm}$, by the quadrature formula[21] (3.3), we can also construct the Nyström approximate operator $\mathbf{A}_{qq}^h$. Setting $t = t_i$ $(i = 1, ..., n_q)$, we obtain the following approximate equations of (5.6)

$$\mathbf{K}_h\mathbf{W}_h = (\mathbf{A}_h + \mathbf{B}_h)\mathbf{W}_h = \mathbf{G}_h, \tag{5.7}$$

where $\mathbf{W}_h = (w_1^h(t_1), ..., w_1^h(t_{n_1}), ..., w_d^h(t_1), ..., w_d^h(t_{n_d}))^T$, $\mathbf{A}_h$ =diag$(\mathbf{A}_{11}^h, ..., \mathbf{A}_{dd}^h)$, $\mathbf{A}_{qq}^h = [a_{qq}(t_j, \tau_i)]_{j,i=1}^{n_q}$, $\mathbf{B}_h = [B_{qm}^h]_{q,m=1}^d$, $\mathbf{B}_{qm}^h = [b_{qm}(t_j, \tau_i)]_{j,i=1}^{n_q,n_m}$, $\mathbf{G}_h = (g_1(t_1), ..., g_1(t_{n_1}), ..., g_d(t_1), ..., g_d(t_{n_d}))^T$, and

$$a_{qq}(t_j, \tau_i) = \begin{cases} -[h_q \ln |2e^{-1/2} \sin \pi(t_i - \tau_j)|]/(2\pi), \text{ as } i \neq j, \\ -[h_q| \ln |2\pi e^{-1/2}h_q/(2\pi)|]/(2\pi), \text{ as } i = j. \end{cases} \tag{5.8}$$

Obviously, Eq (5.7) is a linear equation system with $n-$unknowns, where $n = n_1 + \cdots + n_d$. We cite the results of [15].

Lemma 5.1[15]**.** Assume that Γ_m $(m = 1, ..., d)$ are smooth curves, $\Gamma = \cup_{m=1}^{d}\Gamma_m$ with $C_\Gamma \neq 1$, and $h_0 = \max_{1\le m\le d} h_m$ is sufficiently small. Also let $u^0 \in C^4(\Gamma)\times C^4(\Gamma)$. Then there exists a unique solution $\mathbf{W}^h$ for (5.7) such that

$$\mathbf{W} - \hat{\mathbf{W}}^h = \mathbf{h}^3\boldsymbol{\phi} + O(h_0^4), \tag{5.9}$$

at node points, where a vector function $\boldsymbol{\phi} = (\phi_1, ..., \phi_d)^T \in (C_0[0,1])^d$ is independent of $\mathbf{h}^3 = (h_1^3, ..., h_d^3)$ and $\mathbf{h} = (h_1, ..., h_d)^T$, and the subspace

$$C_0[0,1] = \{v(t) \in C[0,1] : v(t)/\sin^2(\pi t) \in C[0,1]\}$$

of $C[0,1]$ with the norm $||v||^* = \max_{0\le t\le 1} |v(t)/\sin^2(\pi t)|$.

5.2 Condition number and effective condition number

Under the above assumptions that $\mathbf{A}$ and $\mathbf{B}$ are symmetric operators, we conclude from (5.7) and (5.8) that the matrices $\mathbf{A}_h+\mathbf{B}_h$ also are the real symmetric matrices of $n-$order. Following the above sections, we can derive similarly the bounds of condition number and effective condition number, to obtain the following theorem.

Theorem 5.2. Assume that Γ_m $(m = 1, ..., d)$ is smooth curve and $\Gamma = \cup_{m=1}^{d}\Gamma_m$ with $C_\Gamma \neq 1$. Let $\mathbf{A}_h$ be defined by rules (5.8) and $\mathbf{B}_h$ be defined by the trapezoidal or the midpoint rule[10]. Then the condition number for (5.7) has the bound

$$\text{Cond.}{=}O(h^{-1}),\ h = \min_{1\le m\le d} h_m, \tag{5.10}$$

and effective condition number for (5.7) has the bound

$$\text{Cond\_eff}{=}\frac{||\mathbf{b}||}{\lambda_n||\mathbf{x}||} = O(h^{-1}),\ h = \min_{1\le m\le d} h_m. \tag{5.11}$$

Remark. In Sections 3-5, we derive the condition number

$$\text{Cond.}{=}O(h^{-1}) \tag{5.12}$$

and the effective condition number

$$\text{Cond\_eff}{=}O(h^{-1}), \tag{5.13}$$

to indicate

$$\text{Cond\_eff}{=}O(\text{Cond.}). \tag{5.14}$$

Both (5.12) and (5.13) imply that the numerical stability of the MQMs is excellent for the first kind BIEs, which also agree with [8]. The new stability analysis in this paper enhances the MQMs, whose error analysis has already been explored in [14-17]. When the partition nodes are quasiuniform for smooth solutions, the improvements of Cond_eff in (5.13) from Cond. in (5.12) are insignificant, since the Cond. itself is not large in practical applications, where h is not very small in computation. However, when $C_\Gamma \to 1$ (see [8]) or the local refinements of nodes are used for singularity problems, the values of Cond. may be large. Then the Cond_eff as from (5.14) may be much smaller than those of Cond.

6 Analysis and comparisons for effective condition number and condition number

In Li et al. [18], the effective condition number is applied to the finite difference method for solving Poisson's equation with the mixed type of Dirichlet and Neumann boundary conditions

$$\begin{aligned} &-\Delta u = f, \text{ in } \Omega, \\ &u = g \text{ on } \Gamma_D, \ \frac{\partial u}{\partial \nu} = g^* \text{ on } \Gamma_N, \end{aligned} \tag{6.1}$$

where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$, Ω is a polygon with the boundary $\partial\Omega = \Gamma_D \cup \Gamma_N$, and ν is the outward normal to $\partial\Omega$. When the functions f, g and g^* are smooth enough, the solution u of (6.1) is also smooth enough. The traditional condition number of the difference matrix is well known

$$\text{Cond.} = O(h_{\min}^{-2}), \tag{6.2}$$

where $h_{\min}$ is the miminal meshspacing of the difference grids used. In [18], the bounds of Cond_eff were derived, to give

$$\text{Cond\_eff} \le c\{||f||_{0,\Omega} + h^{-0.5}||g^*||_{0,\Gamma_N} + h^{-0.5}h_{\min}^{-1}||g||_{0,\Gamma_D}\}, \tag{6.3}$$

where $||f||_{0,\Omega}$, $||g^*||_{0,\Gamma_N}$ and $|||g||_{0,\Gamma_D}$ are the Sobolev norms, h is the maximal meshspacing of the difference grids, c is a constant independent of h. Evidently, Cond_eff in (6.3) is smaller than Cond. in (6.2). In particular, when the boundary conditions are homogeneous, i.e., $g = g^* = 0$, we obtained

$$\text{Cond\_eff}=O(1), \tag{6.4}$$

which is significantly smaller than Cond. in (6.2). Numerical experiments were provided in [18], to support (6.3) and (6.4).

In this paper, for numerical BIE of the first kind by MQMs, when $C_\Gamma \neq 1$, Cond_eff and Cond have the same growth rates as $h \to 0$ (see (5.14)). Also from the data in Tables 1-3 given in Section 7, we can see

$$\frac{\text{Cond.}}{\text{Cond\_eff}} = c \in (1.5, 2], \tag{6.5}$$

Hence, the improvements of Cond_eff to Cond. are insignificant for the stability. For numerical partial differential equations (PDE) and BIEs, why are the bounds of Cond_eff over those of Cond. are so different? Below we intend to provide arguments to explain such distinct behaviors of Cond_eff, based on matrix analysis.

6.1 Effective condition number for BIE

Denote the real eigenvalues λ_i of the discrete stiffness matrix $\mathbf{K}_h$ (or $\mathbf{A}_h$) of Sections 3-5 in a descent order,

$$|\lambda_1| > |\lambda_2| \ge ... \ge |\lambda_n| > 0, \tag{6.6}$$

with[1]

$$|\lambda_1| \risingdotseq O(1),\ |\lambda_n| \fallingdotseq O(h^p),\ p \geq 0.$$

Hence, the traditional condition number has the bound

$$\text{Cond.}{=}\frac{|\lambda_1|}{|\lambda_n|} \risingdotseq O(h^p). \tag{6.7}$$

In the paper, there exist the bounds

$$||\mathbf{x}|| \risingdotseq ||\mathbf{b}|| \risingdotseq O(h^{-0.5}). \tag{6.8}$$

We have the following theorem.

Theorem 6.1. Let (6.6) and (6.8) hold. Then there exists the equivalence

$$\text{Cond\_eff} \risingdotseq O(\text{Cond.}). \tag{6.9}$$

Proof. We have from (6.6) and (6.8)

$$\text{Cond\_eff}{=}\frac{||\mathbf{b}||}{|\lambda_n|||\mathbf{x}||} \risingdotseq O(\frac{1}{|\lambda_n|}) \risingdotseq O(\frac{|\lambda_1|}{|\lambda_n|}) \risingdotseq O(\text{Cond.}).$$

This completes the proof of Theorem 6.1.

Denote the eigenpairs $(\lambda_i, \mathbf{u}_i)$ of $\mathbf{K}_h$, and the angles θ_i between $\mathbf{b}$ and $\mathbf{u}_i$ by

$$\cos\theta_i = \cos(\mathbf{b}, \mathbf{u}_i) = \frac{(\mathbf{b}, \mathbf{u}_i)}{||\mathbf{b}||} = \frac{\beta_i}{||\mathbf{b}||}, \tag{6.10}$$

where $\beta_i = (\mathbf{b}, \mathbf{u}_i) = \mathbf{u}_i^T\mathbf{b}$.

Lemma 6.2. Let $(\lambda_i, \mathbf{u}_i)$ be eigenpairs of matrix $\mathbf{K}_h$. Suppose that matrix $\mathbf{K}_h$ is symmetric and nonsingular, and that

$$||\mathbf{x}|| \risingdotseq ||\mathbf{b}||. \tag{6.11}$$

Then there exist the bounds

$$\Sigma_{i=1}^n \cos^2\theta_i = 1, \tag{6.12}$$

and

$$\Sigma_{i=1}^n \frac{\cos^2\theta_i}{\lambda_i^2} \risingdotseq O(1), \tag{6.13}$$

where the angles θ_i are given in (6.10).

Proof. We have $\mathbf{b} = \mathbf{\Sigma}_{i=1}^n \beta_i \mathbf{u}_i$. Then we obtain from (6.10)

$$||\mathbf{b}|| = \sqrt{\mathbf{\Sigma}_{i=1}^n \beta_i^2} = (\sqrt{\mathbf{\Sigma}_{i=1}^n \cos^2\theta_i})||\mathbf{b}||,$$

[1]The notation $a \risingdotseq b$ ($a \risingdotseq O(b)$), $b > 0$, denotes that there exist two constants c_1 and such that $c_1 b \leq |a| \leq c_2 b$, $b > 0$.

to give the first desired result (6.12). Next, from $\mathbf{x} = \mathbf{K}_h^{-1}\mathbf{b}$ we have $\mathbf{x} = \Sigma_{i=1}^n \frac{\beta_i}{\lambda_i}\mathbf{u}_i$. Then there exists an equality,

$$||\mathbf{x}|| = \sqrt{\Sigma_{i=1}^n \frac{\beta_i^2}{\lambda_i^2}} = \sqrt{\Sigma_{i=1}^n \frac{\cos^2\theta_i}{\lambda_i^2}}||\mathbf{b}||.$$

We obtain from (6.11)

$$\Sigma_{i=1}^n \frac{\cos^2\theta_i}{\lambda_i^2} = [\frac{||\mathbf{x}||}{||\mathbf{b}||}]^2 \risingdotseq O(1).$$

This is the second result (6.13), and completes the proof of Lemma 6.2.

From (6.6), we may denote the eigenvalues by

$$|\lambda_i| = c_i h^{p_i} \risingdotseq O(h^{p_i}), \tag{6.14}$$

where c_i are positive constants, and the powers p_i are given by

$$0 = p_1 \leq p_2 < ... \leq p_n = p, \tag{6.15}$$

Lemma 6.3. Let the conditions in Lemma 6.2 and (6.14) hold. Then there exist the bounds

$$|\cos\theta_i| = O(h^{p_i}) = O(|\lambda_i|), \tag{6.16}$$

where θ_i are defined in (6.10).

Proof. We have from Lemma 6.2

$$|\cos\theta_i| = |\lambda_i|\sqrt{\frac{\cos^2\theta_i}{\lambda_i^2}} \leq |\lambda_i|\sqrt{\Sigma_{i=1}^n \frac{\cos^2\theta_i}{\lambda_i^2}} \leq c|\lambda_i| \leq ch^{p_i}, \tag{6.17}$$

to give the desired result (6.16), and complete the proof of Lemma 6.3.

In fact, $\mathbf{u}_1$ in the eigenpair $(\lambda_1, \mathbf{u}_1)$ is the low frequency eigenvector, and $\mathbf{u}_n$ in the eigenpair $(\lambda_n, \mathbf{u}_n)$ is the high frequency eigenvector. The $\mathbf{u}_i$ is said the low frequency eigenvector if its corresponding eigenvalue λ_i satisfying $|\lambda_i| \risingdotseq O(|\lambda_1|) \risingdotseq O(1)$, or the high frequency eigenvector if $|\lambda_i| \risingdotseq O(|\lambda_n|)$. Also $\mathbf{b}$ is rich in $\mathbf{u}_i$ if

$$|\cos\theta_i| = |\cos(\mathbf{b}, \mathbf{u}_i)| \geq c_0 > 0, \tag{6.18}$$

where c_0 is a constant independent of h. Hence, Lemma 6.3 implies that $\mathbf{b}$ *must not* be rich in high frequency eigenvectors, because $|\cos\theta_i| \to 0$ when h is small.

Lemma 6.4. Let the conditions in Lemma 6.2 hold. Then the solution vector $\mathbf{x}$ is rich in a low frequency eigenvector $\mathbf{u}_k$ if and only if $\mathbf{b}$ is rich in $\mathbf{u}_k$.

Proof. First suppose that $\mathbf{b}$ is rich in the low frequency eigenvector $\mathbf{u}_k$ with

$$|\lambda_k| \risingdotseq O(|\lambda_1|) \risingdotseq O(1), \tag{6.19}$$

where k is a small integer. Eq (6.18) gives

$$|\cos(\mathbf{b}, \mathbf{u}_k)| \geq c_0 > 0. \tag{6.20}$$

Since $\mathbf{x} = \mathbf{K}_h^{-1}\mathbf{b}$ and

$$\frac{||\mathbf{b}||}{||\mathbf{x}||} \geq c_1 > 0, \tag{6.21}$$

where c_1 is a constant independent of h, we have

$$|\cos(\mathbf{x}, \mathbf{u}_k)| = \frac{|(\mathbf{x}, \mathbf{u}_k)|}{||\mathbf{x}||} = \frac{|(\mathbf{K}_h^{-1}\mathbf{b}, \mathbf{u}_k)|}{||\mathbf{x}||} = \frac{|(\mathbf{b}, \mathbf{K_h^{-1}}\mathbf{u}_k)|}{||\mathbf{x}||}$$

$$= \frac{|(\mathbf{b}, \mathbf{u}_k)|}{|\lambda_k|||\mathbf{x}||} = \frac{|\cos(\mathbf{b}, \mathbf{u}_k)|}{|\lambda_k|}\frac{||\mathbf{b}||}{||\mathbf{x}||} \geq \frac{c_0 c_1}{|\lambda_k|} = \bar{c}_0 > 0, \tag{6.22}$$

where we have used $|\lambda_k| \risingdotseq O(1)$. This implies that $\mathbf{x}$ is rich in $\mathbf{u}_k$.

On the other hand, suppose that $|\cos(\mathbf{x}, \mathbf{u}_k)| \geq \bar{c}_0 > 0$, we have

$$|\cos(\mathbf{b}, \mathbf{u}_k)| = \frac{|(\mathbf{b}, \mathbf{u}_k)|}{||\mathbf{b}||} = \frac{|(\mathbf{K}_h\mathbf{x}, \mathbf{u}_k)|}{||\mathbf{b}||} = \frac{|(\mathbf{x}, \mathbf{K_h}\mathbf{u}_k)|}{||\mathbf{b}||}$$

$$= \frac{|\lambda_k|||(\mathbf{x}, \mathbf{u}_k)|}{||\mathbf{b}||} = |\lambda_k|||\cos(\mathbf{x}, \mathbf{u}_k)|\frac{||\mathbf{x}||}{||\mathbf{b}||} \geq |\lambda_k|\frac{\bar{c}_0}{c_1} \geq c_0 > 0.$$

This implies that $\mathbf{b}$ is also rich in $\mathbf{u}_k$, and completes the proof of Lemma 6.4.

Theorem 6.5. Let all conditions in Lemma 6.2 hold. Suppose that $\mathbf{x}$ is rich in a low frequency eigenvector $\mathbf{u}_k$. Then there exists a constant c_0 with $0 < c_0 \leq 1$ independent of h such that

$$\text{Cond\_eff} \geq c_0 \text{Cond}. \tag{6.23}$$

Proof. By the assumption, we have

$$|\cos(\mathbf{x}, \mathbf{u}_k)| \geq \bar{c}_0 > 0, \tag{6.24}$$

where $\bar{c}_0$ is a constant independent of h. Denote the solution vector $\mathbf{x} = \mathbf{x}_1 + \mathbf{x}_2$, where $\mathbf{x}_1 = (\mathbf{x}, \mathbf{u}_k)\mathbf{u}_k$. Then we have

$$||\mathbf{x}_1|| = |(\mathbf{x}, \mathbf{u}_k)| = |\cos(\mathbf{x}, \mathbf{u}_k)|||\mathbf{x}|| \geq \bar{c}_0||\mathbf{x}||,$$

to give

$$||\mathbf{x}|| \leq \frac{1}{\bar{c}_0}||\mathbf{x}_1||. \tag{6.25}$$

Since $\mathbf{b} = \mathbf{K_h}\mathbf{x}$, we have $\mathbf{b} = \mathbf{b}_1 + \mathbf{b}_2$, where $\mathbf{b}_1 = \mathbf{K_h}\mathbf{x}_1 = |\lambda_k|\mathbf{x}_1$. Hence, we obtain

$$||\mathbf{x}_1|| = \frac{1}{|\lambda_k|}||\mathbf{b}_1||. \tag{6.26}$$

Combining (6.25) and (6.26) yields

$$||\mathbf{x}|| \leq \frac{1}{\bar{c}_0}\frac{1}{|\lambda_k|}||\mathbf{b}_1||. \tag{6.27}$$

Now we obtain

$$\text{Cond\_eff}{=}\frac{||\mathbf{b}||}{|\lambda_k|||\mathbf{x}||} \geq \bar{c}_0 \frac{|\lambda_k|}{|\lambda_n|}\frac{||\mathbf{b}||}{||\mathbf{b}_1||}. \tag{6.28}$$

Since $||\mathbf{b}|| \geq ||\mathbf{b}_1||$ and $\frac{|\lambda_k|}{|\lambda_1|} \geq \gamma_k > 0$, Eq (6.28) leads to

$$\text{Cond\_eff} \geq \bar{c}_0 \frac{|\lambda_k|}{|\lambda_n|} \geq \bar{c}_0 \gamma_k \frac{|\lambda_1|}{|\lambda_n|} = \bar{c}_0 \gamma_k \text{Cond.}{=}c_0\text{Cond.}$$

where $c_0 = \bar{c}_0 \gamma_k$. This is the desired result (6.23), and completes the proof of Theorem 6.5.

Remark 6.1. Theorems 6.1 and 6.5 can be applied to many numerical methods for BIEs of the first kind in [8] and the second kind in Atkinson and Han [2,3]. Moreover, the conclusions in this subsection are also valid for the boundary element method (BEM).

6.2 Effective condition number for numerical PDEs

Now, we turn to study Cond_eff for numerical PDEs. For (6.1), consider the smooth problem with the smooth solution u. Let the difference grids (x_i, y_j) are quasiuniform. The quasiuniform grids are said if

$$h/\min_{i,j}\{h_i, k_j\} \leq c, \tag{6.29}$$

where $h = \max\{h_i, k_j\}$, $h_i = x_i - x_{i-1}$, $k_j = y_j - y_{j-1}$, and c is a constant independent of h. The discrete difference equations of (6.1) are denoted by the matrix form

$$\bar{\mathbf{K}}_{\mathbf{h}}\mathbf{x} = \mathbf{b}, \tag{6.30}$$

where the matrix $\bar{\mathbf{K}}_{\mathbf{h}}$ is symmetric and positive definite. Also denote $(\lambda_i, \mathbf{u}_i)$ the eigenpairs of matrix $\mathbf{K}_h$, and the eigenvalues are also given in a descent order

$$c_1 h^{-p} = \lambda_1 > \lambda_2 \geq ... \geq \lambda_n = c_0 > 0, \tag{6.31}$$

where $p \geq 2$, c_1 and c_0 are two constants independent of h. For the homogeneous boundary conditions (i.e., $g = g^* = 0$), the following relations are satisfied:

$$||\mathbf{x}|| \risingdotseq ||\mathbf{b}|| \risingdotseq O(h^{-1}). \tag{6.32}$$

We can derive the following Theorems by following the proof of Theorems 6.1 and 6.5.

Theorem 6.6. Let (6.31) and (6.32) be given. Then Eq (6.4) holds.

Theorem 6.7. Let (6.31) and (6.32) be given. Suppose that $\mathbf{x}$ is rich in a low frequency eigenvector $\mathbf{u}_k$.Then Eq (6.4) holds.

From Theorems 6.1, 6.5-6.7, the effective condition number may be significantly smaller than Cond. only for numerical PDEs, but not for numerical

BIEs. The intrinsic behaviors of Cond_eff result from the follows. The differential operator $-\Delta$ is unbounded so that the eigenvalues of its discrete matrix have the bounds

$$\lambda_n \risingdotseq O(1), \lambda_1 \risingdotseq O(h^{-p}), p \geq 2. \tag{6.33}$$

On the other hand, the integral compact operator is bounded so that the eigenvalues of its discrete matrix have the following different bounds

$$|\lambda_1| \risingdotseq O(1), |\lambda_n| \risingdotseq O(h^{-p}), p \geq 0.$$

Hence, we conclude that the effective condition number is important only for numerical PDEs.

For numerical BIEs, the traditional Cond.=$O(h^{-1})$ is not large, and the Cond_eff has a little helpful for better stability. This fact also displays that the Cond_eff is really *effective* when the traditional Cond. is large. This is just the worth place where the effective condition number works for.

7 Numerical Experiments

We carry out three experiments by MQMs and h^3-Richardson's extrapolation or splitting extrapolation methods (SEM), and verify the error and the stability analysis made in the above sections.

Example 1. Let Γ be a circle with radius $e^{-1/2}$. Based on [2,23,27], $C_\Gamma = e^{-1/2} \neq 1$. Consider the typical BIE of the first kind

$$-\int_{-\pi}^{\pi} \ln|2e^{-1/2}\sin\frac{t-\tau}{2}|w(\tau)d\tau = \frac{\pi}{2}\cos 2t, \tag{7.1}$$

where $w(t) = \cos 2t$ is the true solution. In Table 1.1, we list the errors $e_n = \max_{1\leq i\leq n}|w(t_i) - w^h(t_i)|$, $e_n^E = \max_{1\leq i\leq n}|w(t_i) - w^E(t_i)|$, and values of Cond and Cond-eff, where $w^E(t_i) = (8w^{h/2}(t_i) - w^h(t_i))/7$ and e_n^E denote the extrapolation values and the extrapolation errors, respectively.

Table 1.1 The errors e_n and e_n^E, Cond. and Cond-eff for (7.1).

n	e_n	e_n^E	$\|\lambda_1\|$	$\|\lambda_n\|$	cond	cond-eff
2^3	3.820E-2		3.156	1.088	2.899	1.500
2^4	4.737E-3	4.361E-5	3.143	5.444E-1	5.774	2.899
2^5	5.885E-4	4.086E-6	3.141	2.722E-1	1.154E+1	5.7742
2^6	7.342E-5	1.695E-7	3.141	1.361E-1	2.308E+1	1.154E+1
2^7	9.172E-6	5.955E-9	3.141	6.805E-2	4.616E+1	2.308E+1
2^8	1.146E-6	1.964E-10	3.141	3.402E-2	9.233E+1	4.616E+1
2^9	1.432E-7	6.293E-12	3.141	1.701E-2	1.846E+2	9.233E+1
2^{10}	1.791E-8	2.978E-14	3.141	8.506E-3	3.693E+2	1.846E+2

Table 1.2. the values of λ_k, β_k and θ_k at $n = 32$.

k	λ_k	β_k	θ_k	k	λ_k	β_k	θ_k
1	3.141	-1.001E-15	-1.570	17	0.408	-8.081E-16	-1.570
2	3.141	-2.053E-15	-1.570	18	0.369	6.993E-16	1.570
3	3.141	3.162E-15	1.570	19	0.369	-2.224E-16	-1.570
4	1.571	-6.249	-0.102	20	0.339	-1.055E-15	-1.570
5	1.571	-6.456E-1	1.467	21	0.339	3.333E-16	1.570
6	1.049	-2.155E-15	-1.570	22	0.316	-7.770E-16	-1.570
7	1.049	-1.836E-15	-1.570	23	0.316	-4.167E-17	-1.570
8	0.789	-1.199E-15	-1.570	24	0.299	2.677E-16	1.570
9	0.789	3.882E-15	1.570	25	0.299	-4.991E-15	-1.570
10	0.634	-3.711E-16	-1.570	26	0.287	-4.814E-16	-1.570
11	0.634	1.119E-16	1.570	27	0.287	1.118E-16	1.570
12	0.532	1.944E-15	1.570	28	0.278	8.322E-16	1.570
13	0.532	3.536E-16	1.570	29	0.278	2.778E-16	1.570
14	0.460	1.117E-15	1.570	30	0.273	-3.600E-16	-1.570
15	0.460	1.278E-15	1.570	31	0.273	-5.557E-17	-1.570
16	0.408	-1.668E-16	-1.570	32	0.272	-9.434E-16	-1.570

Example 2[17,21]**.** Let Γ be $x(t) = c_0(e^{\sqrt{-1}t} + c_1 e^{-\sqrt{-1}t})$, $t \in [0, 2\pi]$, which is an elliptic curve, where $c_0 = 50$ and $c_1 = 0.5$. Since $C_\Gamma \neq 1$, the boundary integral equation

$$\int_0^{2\pi} \ln|x(t) - x(\tau)| w(\tau) d\tau = 2\pi \ln|x(t)| \tag{7.2}$$

has the unique solution

$$w(t) = 1 + 4\sum_{k=1}^{\infty} (-1)^k \frac{c_1^k}{1 + c_1^{2k}} \cos(2kt).$$

The computed results are listed in Table 2.1

Table 2.1 The errors e_n and e_n^E, Cond. and Cond-eff for (7.2).

n	e_n	e_n^E	$\|\lambda_1\|$	$\|\lambda_n\|$	Cond	Cond-eff
2^4	2.477E-3		2.458E+1	5.474E-1	4.489E+1	2.929E+1
2^5	2.441E-4	7.481E-5	2.458E+1	2.722E-1	9.030E+1	5.879E+1
2^6	3.045E-5	7.296E-8	2.458E+1	1.361E-1	1.806E+2	1.173E+2
2^7	3.810E-6	4.061E-9	2.458E+1	6.805E-2	3.612E+2	2.346E+2
2^8	4.765E-7	1.529E-10	2.458E+1	3.402E-2	7.224E+2	4.693E+2
2^9	5.956E-8	5.048E-12	2.458E+1	1.701E-2	1.444E+3	9.387E+2
2^{10}	7.445E-9	1.593E-13	2.458E+1	8.506E-3	2.889E+3	1.877E+3

Table 2.2. the values of λ_k, β_k and θ_k at $n = 32$

k	λ_k	β_k	θ_k	k	λ_k	β_k	θ_k
1	24.58	1.534E+2	0.000	17	0.406	-2.365E-14	-1.570
2	4.712	-2.931E-14	-1.570	18	0.369	-8.881E-15	-1.570
3	1.964	-1.964E-14	-1.570	19	0.368	1.159E-14	1.570
4	1.571	-2.086E-14	-1.570	20	0.339	8.881E-15	1.570
5	1.180	4.440E-15	1.570	21	0.339	2.471E-14	1.570
6	1.179	-9.014E-15	-1.570	22	0.316	-8.881E-15	-1.570
7	0.918	-1.536E-14	-1.570	23	0.316	1.366E-14	1.570
8	0.838	8.881E-14	-1.570	24	0.299	-1.332E-14	-1.570
9	0.740	1.831E-14	1.570	25	0.299	1.233E-15	1.570
10	0.653	6.217E-15	1.570	26	0.287	3.908E-14	1.570
11	0.614	1.222E-14	1.570	27	0.287	-1.411E-14	-1.570
12	0.540	0.000	1.570	28	0.278	0.000	1.570
13	0.523	-2.735E-14	-1.570	29	0.278	6.920E-15	1.570
14	0.464	-3.730E-14	-1.570	30	0.273	-8.881E-15	-1.570
15	0.457	-3.193E-14	-1.570	31	0.273	2.418E-14	1.570
16	0.409	-1.687E-14	-1.570	32	0.272	-5.329E-15	-1.570

Example 3[24]**.** Let Γ be an open contour of length 2, in the form of a right-angled wedge:

$$\Gamma = \{(x_1, 0) : 0 \leq x_1 \leq 1\} \cup \{(0, x_2) : 0 \leq x_2 \leq 1\}.$$

The integral equation is chosen as

$$-\int_\Gamma \ln|y - x| v(x) ds_x = 1, \text{ for } (y_1, y_2) \in \Gamma. \tag{7.3}$$

We compute the numerical solution of

$$u(y) = -\int_\Gamma \ln|y - x| v(x) ds_x$$

at $(0.5, 0.5)$, whose true value $u(0.5, 0.5)$ takes 0.621455343.

From [24], although the exact solution $v(x)$ is expected to have a $O(|x - x_0|^{-\frac{1}{3}})$ singularity at the right-angled corner, the dominant singularities in $v(x)$ occur at the two ends, with $O(|x - x_0|^{-\frac{1}{2}})$. Based on [15,16], using $\varphi_6(t)$ in the periodical transformation (5.3), we obtain the numerical results at $Q = (0.5, 0.5)$ by MQMs and list Cond. and Cond-eff in Tables 3.1. Let n_m $(m = 1, 2)$ be the number of uniform partition on $[0, 1]$ corresponding to the mth edge Γ_m of Γ. Based on (5.9), we can obtain the splitting extrapilation errors $e^E(Q) = |u_E(Q) - u(Q)|$, where

$$u_E(Q) = \frac{8}{7}[\sum_{m=1}^{d} u_{h^{(m)}}(Q) - (d - \frac{7}{8}) u_{h^{(0)}}(Q)], \, d = 2,$$

is the splitting extrapilation values. The errors $|u^h(Q) - u(Q)|$ and the splitting extrapilation errors $e^E(Q)$ are also listed in Table 3.1, where $(n_1, n_2) = (8, 8)$ and $(16, 16)$.

Table 3.1. The errors, Cond. and Cond_eff for (7.3).

(n_1, n_2)	$\|u^h - u\|$	$\|\lambda_1\|$	$\|\lambda_n\|$	Cond.	Cond-eff
(4,4)	4.413E-2	0.104	4.287	40.865	12.042
(8,4)	2.166E-2	0.113	4.147	36.420	11.979
(4,8)	2.166E-2	0.113	4.147	36.420	12.049
e^E	7.229E-3				
(8,8)	1.738E-3	0.055	4.374	78.474	22.395
(16,8)	9.452E-4	0.056	4.312	76.007	23.820
(8,16)	9.452E-4	0.056	4.312	76.007	23.856
e^E	7.495E-5				
(16,16)	1.383E-4	0.028	4.378	154.828	44.046
(32,16)	7.805E-5	0.028	4.357	153.680	47.609
(16,32)	7.805E-5	0.028	4.357	153.680	47.618
e^E	5.184E-7				
(32,32)	1.725E-5	0.014	4.375	308.348	87.754
(64,32)	9.703E-6	0.014	4.369	308.234	95.208
(32,64)	9.703E-6	0.014	4.369	308.234	95.210
e^E	1.350E-9				

Table 3.2. the values of λ_k, β_k and $\cos\theta_k$ at $(8, 8)$ in Table 3.1.

k	λ_k	β_k	$\|\cos\theta_k\|$	k	λ_k	β_k	$\|\cos\theta_k\|$
1	4.374	3.006	0.752	9	0.280	-3.608E-16	6.123E-17
2	2.309	1.110E-16	6.123E-17	10	0.225	-0.021	5.334E-3
3	2.076	2.542	0.636	11	0.218	8.326E-17	6.123E-17
4	0.941	0.653	0.163	12	0.187	0.014	3.496E-3
5	0.551	1.110E-16	6.123E-17	13	0.184	4.718E-16	6.123E-17
6	0.532	0.258	0.065	14	0.162	5.153E-3	1.288E-3
7	0.396	2.775E-16	6.123E-17	15	0.090	-2.775E-16	6.123E-17
8	0.308	0.078	0.019	16	0.055	3.665E-17	6.123E-17

Now, let us examine the numerical results in Tables 1-2. We can see numerically,

$$\frac{e|_{n=2^{m+1}}}{e|_{n=2^m}} \approx 2^3, \ \frac{e^E|_{n=2^{m+1}}}{e^E|_{n=2^m}} \to 2^5 \tag{7.4}$$

to indicate the empirical convergence rate $O(h^3)$, and the first extrapilation convergence rate $O(h^5)$. From Tables 1-3, we can also see

$$|\lambda_1| \approx C, \text{ and } |\lambda_n| = O(h^{-1}). \tag{7.5}$$

Eq (7.5) coincide with the theoretical estimates of $|\lambda_1|$ and $|\lambda_n|$ given in Sections 3-5 perfectly. Next, from Tables 1-3 we have

$$\frac{\text{Cond}|_{n=2^{m+1}}}{\text{Cond}|_{n=2^m}} \approx 2 \text{ and } \frac{\text{Cond-eff}|_{n=2^{m+1}}}{\text{Cond-eff}|_{n=2^m}} \approx 2 \ (m = 3, ..., 9), \tag{7.6}$$

and

$$\frac{\text{Cond}|_{(2^{m+1},2^{m+1})}}{\text{Cond}|_{(2^m,2^m)}} \approx 2 \text{ and } \frac{\text{Cond-eff}|_{(2^{m+1},2^{m+1})}}{\text{Cond-eff}|_{(2^m,2^m)}} \approx 2\ (m=2,3,4), \tag{7.7}$$

to indicate that Cond=$O(h^{-1})$ and Cond-eff=$O(h^{-1})$, which are consistent with Theorems 3.3-3.4, 4.4-4.5, and 5.2. Moreover, from Table 3.1 we have

$$\frac{|u^h-u|_{(4,4)}}{|u^h-u|_{(8,8)}} = 25.39, \frac{|u^h-u|_{(8,8)}}{|u^h-u|_{(16,16)}} = 12.56 \text{ and } \frac{|u^h-u|_{(16,16)}}{|u^h-u|_{(32,32)}} = 8.01, \tag{7.8}$$

Hence, the SEMs can provide more accurate solutions. Note that from Table 3.1 with the total number $n = \sum_{m=1}^{2} n_m = 32$ and 64, the error of SEMs is $5.184E$-7 and $1.350E-9$, respectively. In contrast, when $n = 256$ the $u^h = 0.62125$ is given in [24] by Galerkin methods, where the approximating space S^h is the piecewise constant space. This fact displays the efficiency of MQMs and SEMs.

Finally, to scrutinize the spectral distribution of vector $\mathbf{b}$ in $\mathbf{u}_k$, we compute all $\beta_k = \mathbf{u}_k^T\mathbf{b}$ and $\theta_k = arc\cos(\beta_k/||\mathbf{b}||)$, and list them in Tables 1.2 and 2.2 in the descending order of $|\lambda_k|$. We can see that $\theta_k \approx \pm\frac{\pi}{2}$ except $\theta_4 = -0.102$ and $\theta_1 = 0.000$ from Tables 1.2 and 2.2, respectively. This fact conincides with Theorem 6.5. Besides in Table 3,2, we list all β_k and cos$\theta_k = \beta_k/||\mathbf{b}||$, to find the dominant distribution with cos$\theta_1 = 0.752$, cos$\theta_3 = 0.636$ and cos$\theta_4 = 0.163$ because $\cos^2\theta_1+\cos^2\theta_3+\cos^2\theta_4 = 0.996569 \approx 1$. Hence, the dominate rich vectors of $\mathbf{b}$ happen just at the low frequency eigenvectors $\mathbf{u}_1$, $\mathbf{u}_3$ and $\mathbf{u}_4$, also to agree with Theorem 6.5.

8 concluding remarks

1. New stability analysis is made for the mechanical quadrature methods (MQMs) for the first kind BIEs in [14-17], based on Cond. and Cond-eff defined in (2.6). The key analysis is the estimates of eigenvalues for the discrete matrices resulting from MQMs. The main results are

$$\text{Cond-eff} \leftrightharpoons \text{Cond.} \leftrightharpoons O(h^{-1}). \tag{8.1}$$

Although the effective condition number is smaller than Cond., the improvements of Cond-eff are insignificant for stability analysis. However, Eqs. (8.1) display an excellent stability for the MQMs. Since the MQMs provide not only the $O(h^3)$ convergence rates but also the excellent stability, the MQMs are more advantageous over the other methods, such as the Galerkin method, the collocation method, and the modified quarature method in [2,6,20,23,27].

2. In Section 6, based on matrix analysis, we prove again that the Cond. and the Cond-eff have the same growth rate for numerical BIEs. Note that the improvements of stability by the Cond-eff are not as significant as those for numerical PDE in [18]. Such intrinsic differences result from the fact that the operators of BIEs and PDEs are bounded and unbounded respectively. Hence, the eigenvalues of their discrete matrices have different bounds in (6.6) and (6.33). Moreover, the analysis of Cond-eff in Section 6.1 is valid for all numerical methods for BIEs of the first kind [8], the second kind [2], and the boundary element methods.

3. Numerical experiments are carried out for the arbitrary boundary Γ with $C_\Gamma \neq 1$ by MQMs and SEM, and the computed results agree with the stability analysis perfectly. The extrapolation and the SEM techniques are applied to the first kind BIEs, to greatly improve the solution accuracy.

4. Finally, let us compare the stability analysis in this paper with that in Christianan and Saranen [8] in more detail. In [8], p.48, the algorithms (2.6) were also proposed, where the local condition number was called. For typical BIE of the first kind in Section 3, they have discussed three methods: (1) The Galerlin method, (2) the collocation method, and (3) the modified quadrature method, to derive the same growth rates as in (5.12) and (5.13) for smooth problems under $C_\Gamma \neq 1$. In this paper, we discuss three types of the first kind BIEs in Sections 3-5; (1) the typical BIE of the first kind, (2) the case of Γ being closed smooth, and (3) the case of Γ being curved polygons and open contours. In Type (3), since there exist corner singularities, the algorithms and their analysis are more challenging. Compared with the modified quadratic method in [8], the MQMs are more advantageous: (1) the high $O(h^3)$ convergence rates, (2) wide applications for three types of the first kind BIEs, and (3) the superconvergence $O(h^6)$ by the Richarson's extrapolation or the splitting extrapolation. Although the above analysis has been made in [14-17], but no stability analysis exists for the MQMs so far. This paper is the first time to explore their stability analyse (8.1), which grants the MQMs an excellent stability.

9 References

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