A Note on the Application of Large Mass and Stiffness Techniques for Multi-support Excitations

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Why the Results Diverge by Using MSC/NASTRAN ?

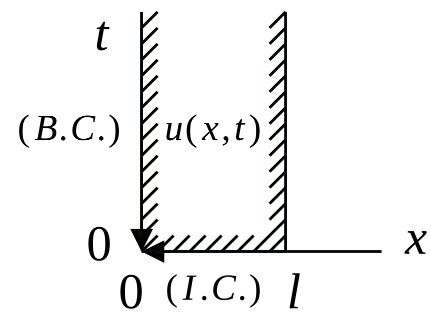
- **Rayleigh Damping Proportional to Mass Only**

$$[C] = \alpha[M] + \beta[K], \ \beta = 0$$

- **Large Stiffness Method Considering the Former Modes**

- **Discontinuity Between Boundary (B.C.) and Initial Conditions(I.C.)**

$$t \to 0 \quad x \to 0$$
$$\dot{u}(0,t) \neq \dot{u}(x,0)$$
$$(B.C.) \quad (I.C.)$$



Sources of Divergence in Modal Dynamics by Crandall

- **S. H. Crandall and A. Yildiz , Transactions of the ASME, 1962**

Beam modal (Euler-Bernoulli, Rayleigh, Timoshenko and shear beam)

Damping mechanism (Rayleigh, viscoelastic, internal, external damping)

Random response (primary and secondary fields)

Transient response is not available

?

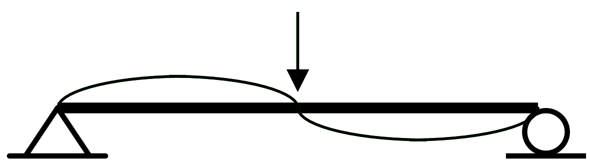
Loading type (boundary force and support excitations are not available)

Sources of Divergence in This Paper

- **Rayleigh damping proportional to mass only for support motion**
- **Series solution without considering the boundary effect for support motion**
- **Large stiffness technique without considering the contributions of high frequency modes**
- **Discontinuity between boundary and initial conditions**

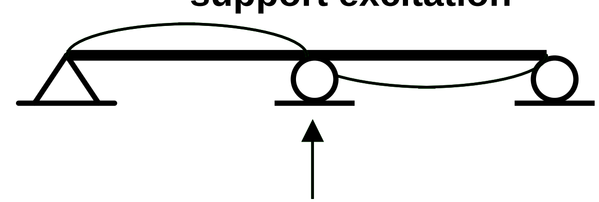
New Concept of Modal Participation Factor(MPF)

- **Modal displacement governs the MPF for body force excitations**
- **Modal displacement governs the MPF for boundary force excitations**
- **Modal reaction governs the MPF for boundary support excitations**
- $$\Gamma_{ij} = \frac{R_{ij}}{(-\omega_i^2)}$$



force excitation

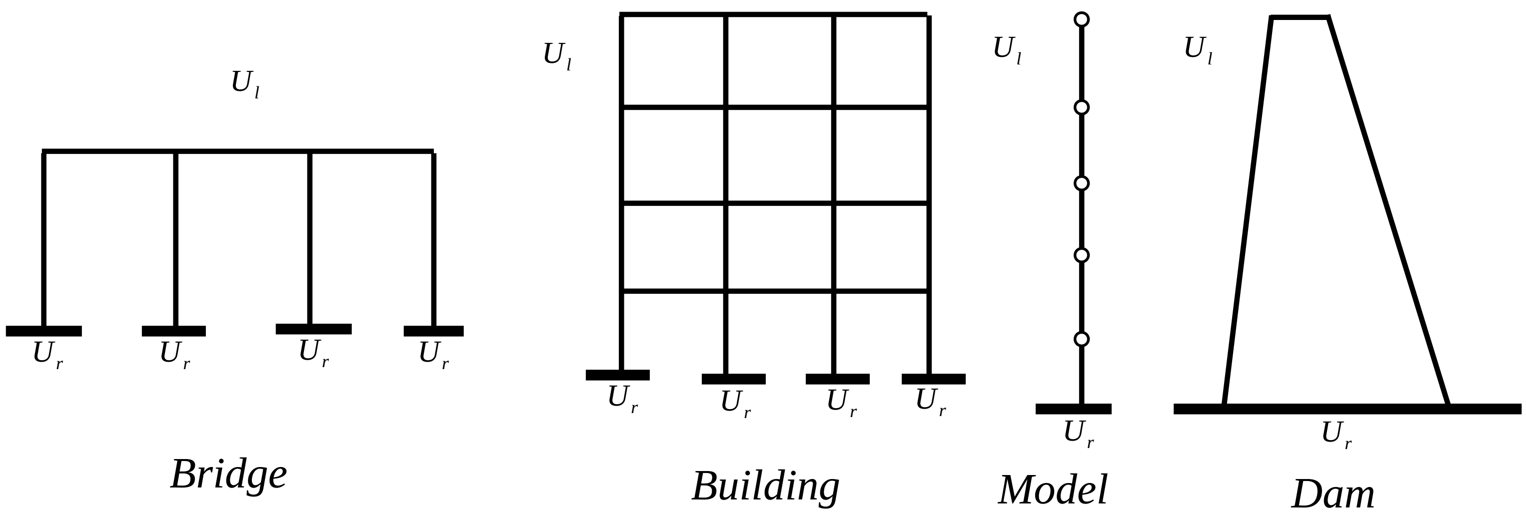
support excitation



Discrete MDOF System

Governing Equation:

$$\begin{bmatrix} M_{ll} & M_{lr} \\ M_{rl} & M_{rr} \end{bmatrix} \begin{Bmatrix} \ddot{U}_l \\ \ddot{U}_r \end{Bmatrix} + \begin{bmatrix} C_{ll} & C_{lr} \\ C_{rl} & C_{rr} \end{bmatrix} \begin{Bmatrix} \dot{U}_l \\ \dot{U}_r \end{Bmatrix} + \begin{bmatrix} K_{ll} & K_{lr} \\ K_{rl} & K_{rr} \end{bmatrix} \begin{Bmatrix} U_l \\ U_r \end{Bmatrix} = \begin{Bmatrix} P_l \\ P_r \end{Bmatrix}$$

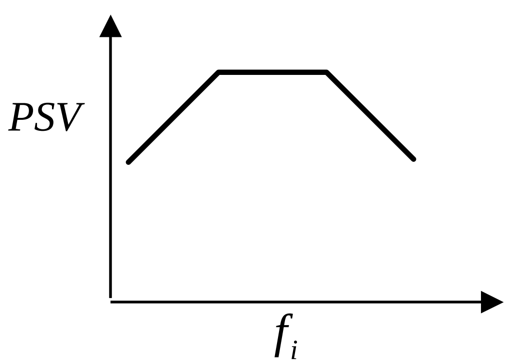


Support Motion Problems

• **Response** ← **Superposition of Modal Response (ABS, CQC, SRSS)**

← **Modal Participation Factor** Γ_i

+ Response Spectrum



Formulations

- **Governing Equation:**

$$\begin{bmatrix} M_{ll} & M_{lr} \\ M_{rl} & M_{rr} \end{bmatrix} \begin{Bmatrix} \ddot{U}_l \\ \ddot{U}_r \end{Bmatrix} + \begin{bmatrix} C_{ll} & C_{lr} \\ C_{rl} & C_{rr} \end{bmatrix} \begin{Bmatrix} \dot{U}_l \\ \dot{U}_r \end{Bmatrix} + \begin{bmatrix} K_{ll} & K_{lr} \\ K_{rl} & K_{rr} \end{bmatrix} \begin{Bmatrix} U_l \\ U_r \end{Bmatrix} = \begin{Bmatrix} P_l \\ P_r \end{Bmatrix}$$

- **Decomposition:**

$$\{U\} = \begin{Bmatrix} U_l \\ U_r \end{Bmatrix} = \{U^s\} + \{U^d\}$$

- **Quasi-static Part:**

$$\{U^s\} = \begin{bmatrix} K_{ll}^{-1} K_{lr} \\ I \end{bmatrix} \{U_r\} = [\ \{G_1\} \cdots \{G_j\} \cdots \{G_{N_r}\}\ ] \begin{Bmatrix} U_{r1}(t) \\ \vdots \\ U_{rj}(t) \\ \vdots \\ U_{rN_r}(t) \end{Bmatrix}$$

Modal Formulations

- **Mode superposition** $$\{U^d\} = [\Phi_{Nl}]\{Q^d\}$$
- **Modal Equation** $$\ddot{Q}_i^d + 2\xi_i\omega_i\dot{Q}_i^d + \omega_i^2 Q_i^d = -\Gamma_{ij}\ddot{U}_{rj}(t)$$
- **Modal Participation Factor (Conventional Method)** $$\Gamma_{ij} = \begin{Bmatrix} \phi_i \\ 0 \end{Bmatrix}^T [M]\{G_j\}$$
- **Modal Participation Factor (Modal Reaction Method)** $$\Gamma_{ij} = \frac{R_{ij}}{(-\omega_i^2)}$$

Methods of Solution for Multi-Support Motions

• Large Mass Technique	(1983)	→	Multi-support Motions	(1992)
• Large Stiffness Technique	(1983)	→	Multi-support Motions	(1992)
• Cesaro Sum	(1890)	→	Base Shear Force	(1992)
• Quasi-static Decomposition	(1950)	→	Discrete System	(1975)
• Stokes' Transformation	(1880)	→	Base Shear Force	(1993)

Methods of Solution for Multi-Support Motions

- **Mode Superposition (Series Solution)**
- **Large Mass Technique (Including Rigid Body Modes)**
- **Large Stiffness Technique (Including High Frequency Modes)**
- **Cesaro sum**
- **Quasi-static Decomposition (Mindlin)**
- **Stokes' Transformation**

Capabilities of Multi-support Excitation in FEM Packages

- **MSC/NASTRAN**

  Large mass technique

  Large stiffness technique

- **ABAQUS**

  Single base excitation(* BASE MOTION), modal reaction available

- **SAP90, ETABS**

  ?

- **MLTDYN(NTUCE)**

  Modal reaction method

Large Stiffness Method and Large Mass Method

Large mass **Large stiffness** **Large mass** **Large stiffness**

Large Stiffness Technique

- **Large Stiffness Technique (Without Including High Frequency Modes)**

  Divergence (slope, moment and shear force)

- **Large Stiffness Technique (Including High Frequency Modes)**

  Boundary effect (moment and shear force)

- **Large Stiffness Ratio** 10^6
- **The Additional High Frequency Modes Display the Boundary Terms**

Large Mass Technique

- **Large Mass Technique (Without Including Rigid Body Modes)**

  Divergence (slope, moment and shear force)

- **Large Mass Technique (Including Rigid Body Modes)**

  Boundary effect (moment and shear force)

- **Large Mass Ratio** 10^6
- **The Rigid Body Modes Display the Boundary Terms**

Results of Large Stiffness Method
Without Considering High Frequency Modes

Displacement

Moment

Slope

Shear force

Results of Large Stiffness Method
Including High Frequency Modes

Displacement

Moment

Slope

Shear force

Rayleigh Damping Model

- **Rayleigh Damping Proportional to Mass Only** $[C] = \alpha[M] + \beta[K],\ \beta = 0$

$\alpha,\ \beta\ \ fixed$

$\xi_1 = 0.05\ \ fixed$

Divergence Due to Mass Proportional Damping

Displacement

Moment

Slope

Shear force

Divergence of Acceleration Profile Due to Discontinuity Between Boundary and Initial Conditions at t=0 second

Displacement (Initial condition)	Velocity (Initial condition)	Acceleration (Divergence)

Displacement and Acceleration Histories at the Middle Point x/l=0.5
Due to Discontinuity Between Boundary and Initial Conditions

Analytical results	Large stiffness method (with high freq. modes)	Large stiffness method (no high freq. modes)
Displacement	**Displacement**	**Displacement**
Acceleration	**Acceleration**	**Acceleration**

Relations of Series Representation, Large Stiffness Technique, Cesaro Sum, Quasi-static Decomposition and Stokes' Transformation

Motivations of Quasi-static Decomposition and Stokes' Transformation

Quasi-static decomposition

$$u(x,t) = \underline{\underline{U(x,t)}} + \sum_{n=0}^{N} q_n(t)\, u_n(x)$$

(Physical meaning)

Differentiation ↓

$$u'(x,t) = U'(x,t) + \sum_{n=0}^{N} q_n(t)\, u_n'(x)$$

Stokes' transformation

$$u(x,t) \cong \sum_{n=0}^{N} \overline{q}_n(t)\, u_n(x)$$

Asymptotic analysis ↓

$$= U(x,t) + \sum_{n=0}^{N} c_n(t)\, u_n(x) + \sum_{n=0}^{N} \overline{q}_n(t)\, u_n(x)$$

Integration ↑

$$u'(x,t) = \sum_{n=0}^{N} \underline{\underline{b_n(t)\, u_n'(x)}} + \sum_{n=0}^{N} \overline{q}_n(t)\, u_n'(x)$$

($\sum b_n(t)\,u_n'(x) \to -\infty$; $\sum \overline{q}_n(t)\,u_n'(x) \to \infty + F.P.$)

(Mathematical way)

$$= U'(x,t) + \sum_{n=0}^{N} c_n(t)\, u_n'(x) + \sum_{n=0}^{N} \overline{q}_n(t)\, u_n'(x)$$

($\sum c_n(t)\,u_n'(x) \to -\infty$; $\sum \overline{q}_n(t)\,u_n'(x) \to \infty + F.P.$)

$$q_n(t) = c_n(t) + \overline{q}_n(t)$$

$$U(x,t) \cong -\sum_{n=0}^{N} c_n(t)\, u_n(x)$$

$$\sum_{n=0}^{N} b_n(t)\, u_n'(x) =$$ **Series representation for distribution on boundary**

Three Analytical Ways and Two Simulation Techniques to Introduce the Quasi-static Part

- **By Solving Boundary Value Problem Directly**

  Quasi-static decomposition method (Mindlin and Goodman)

- **By Integrating the Secondary Field Derived from Stokes' Transformation**

  Boundary terms are available

- **By Adding and Subtracting Technique Using Asymptotic Analysis**

  Series representation (Eringen and Suhubi, Yeh and Liaw)

- **Large Mass Technique (MSC/NASTRAN) --- Rigid Body Modes**
- **Large Stiffness Technique (MSC/NASTRAN) --- High Frequency Modes**

Cesaro Regularization Technique

- **Series Solution(Partial Sum)**

$$
\begin{aligned}
s_0 &= a_0 \\
s_1 &= a_0 + a_1 \\
s_2 &= a_0 + a_1 + a_2 \\
\vdots &= \vdots
\end{aligned}
$$

$$s_{n-1} = a_0 + a_1 + a_2 + \cdots + a_{n-1}$$

$$(\textit{partial sum}) \quad s_n = a_0 + a_1 + a_2 + \cdots + a_{n-1} + a_n \quad (\textit{divergent},\ n \to \infty)$$

$$\frac{s_0 + s_1 + \cdots + s_{n-1} + s_n}{n+1} = a_0 + \frac{n}{n+1} a_1 + \frac{n-1}{n+1} a_2 + \cdots + \frac{2}{n+1} a_{n-1} + \frac{1}{n+1} a_n \quad (\textit{convergent},\ n \to \infty)$$

$$(\textit{Cesaro sum}) \quad S_n = \frac{1}{n+1} \sum_{k=0}^{n} (n-k+1)\ a_k \quad (\textit{moving average})$$

Stokes' Transformation

- **Term by Term Differentiation Is Not Always Legal**
- **Boundary Term Is Present for Some Cases**

$$f'(x) = \frac{d}{dx}\left\{ f(x) \right\} = \frac{d}{dx}\left\{ \sum_{k=0}^{n} c_k\ u_k(x) \right\} = \sum_{k=0}^{n} c_k\ u_k'(x) + \underline{\underline{\sum_{k=0}^{n} b_k\ u_k'(x)}}$$

Boundary term

if $\sum_{k=0}^{n} b_k\ u_k'(x) \neq 0$

- **Term by Term Differentiation Is Legal**

if $\sum_{k=0}^{n} b_k\ u_k'(x) = 0$

The Series Representation Solutions of the Three Analytical Formulations

The Proper Use of Each Method for Different Structures

Conclusions

- **Sources of Divergence Have Been Identified**
- **New Point of View for Modal Participation Factor has been developed**
  - **Physical meaning**
  - **To save CPU time (100 : 1)**
- **New Method for Multi-support Motion --- Stokes' Transformation**
  - **Free from calculating quasi-static solution**
  - **Accelerate convergence rate**
- **Large Stiffness and Large Mass Technique Have Been Tested**
  - **Boundary effect of moment and shear diagrams should be noted**