Mathematical analysis of the true and spurious eigensolutions for free vibration of plate using real-part BEM

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Outlines

1. Introduction
2. Boundary integral equations for plate eigenproblems
3. Mathematical analysis (Continuous system)
4. Mathematical analysis (Discrete system)
5. Conclusions

Introduction

G.E. $$\nabla^4 u(x) = \lambda^4 u(x), x \in \Omega$$

u	lateral displacement	λ	frequency parameter
Ω	domain		

$$\lambda^4 = \frac{\omega^2 \rho h}{D}$$

ω	circular frequency	ρ	surface density
h	plate thickness	D	flexural rigidity

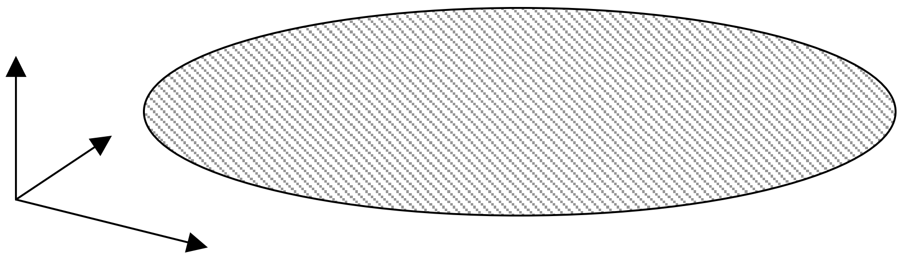
$$D = \frac{Eh^3}{12(1-\nu^2)}$$

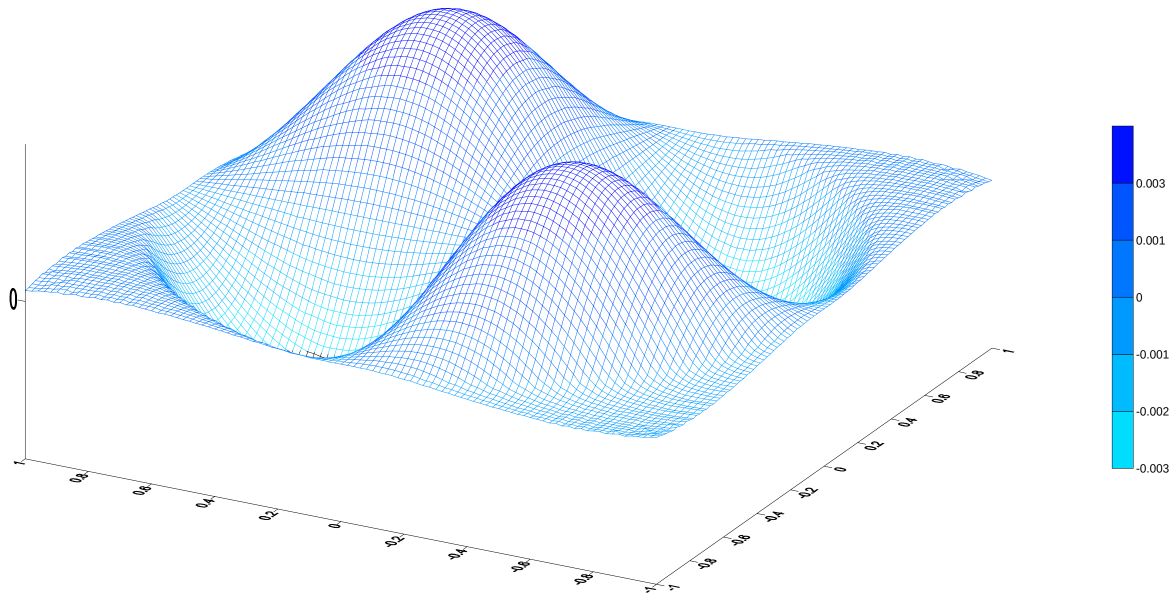
E	Young's modulus	ν	Poisson ratio

Literature review

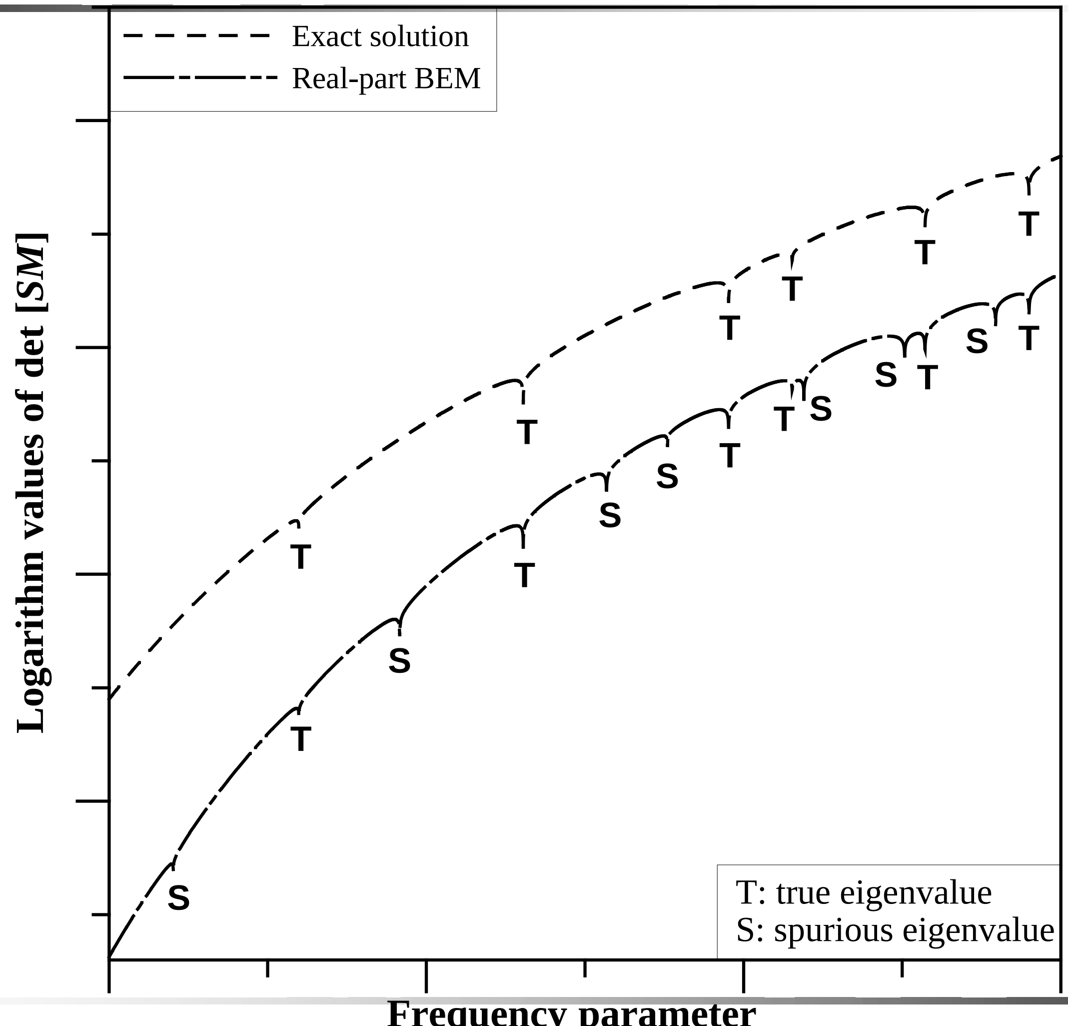
1. Tai and Shaw 1974 (complex-valued BEM)
2. De Mey 1976, Hutchinson and Wong 1979 (real-part kernel)
3. Wong and Hutchinson (real-part direct BEM program)
4. Shaw 1979, Hutchinson 1988, Niwa *et al*. 1982 (real-part kernel)
5. Chen *et al*. (dual formulation, domain partition, SVD updating technique, CHEEF method)

Free vibration of plate





True and spurious eigenvalues



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Boundary integral equations for plate eigenproblems

(1) Displacement

$$u(x) = \int_B \{-U(s,x)v(s) + \Theta(s,x)m(s) - M(s,x)\theta(s) + V(s,x)u(s)\}\, dB(s), \quad x \in \Omega$$

$K_\theta(\cdot)$

(2) Slope

$$\theta(x) = \int_B \{-U_\theta(s,x)v(s) + \Theta_\theta(s,x)m(s) - M_\theta(s,x)\theta(s) + V_\theta(s,x)u(s)\}\, dB(s), \quad x \in \Omega$$

$K_m(\cdot)$

(3) Normal moment

$$m(x) = \int_B \{-U_m(s,x)v(s) + \Theta_m(s,x)m(s) - M_m(s,x)\theta(s) + V_m(s,x)u(s)\}\, dB(s), \quad x \in \Omega$$

$K_v(\cdot)$

(4) Effective shear force

$$v(x) = \int_B \{-U_v(s,x)v(s) + \Theta_v(s,x)m(s) - M_v(s,x)\theta(s) + V_v(s,x)u(s)\}\, dB(s), \quad x \in \Omega$$

Operators

$$K_\theta(\cdot) = \frac{\partial(\cdot)}{\partial n}$$

$$K_m(\cdot) = \nu\nabla^2(\cdot) + (1-\nu)\frac{\partial^2(\cdot)}{\partial n^2}$$

$$K_v(\cdot) = \frac{\partial\nabla^2(\cdot)}{\partial n} + (1-\nu)\frac{\partial}{\partial t}(\frac{\partial^2(\cdot)}{\partial n\partial t})$$

Kernel functions

Fundamental solution

$$\nabla^4 U_c(s,x) - \lambda^4 U_c(s,x) = \delta(x-s)$$

$$U_c(s,x) = \frac{i}{8\lambda^2}(H_0^{(1)}(\lambda r) + H_0^{(2)}(i\lambda r))$$

Kernel functions

$$U(s,x) = \mathrm{Re}[\frac{i}{8\lambda^2}(H_0^{(1)}(\lambda r) + H_0^{(2)}(i\lambda r))]$$

$$\Theta(s,x) = K_\theta(U(s,x))$$

$$M(s,x) = K_m(U(s,x))$$

$$V(s,x) = K_v(U(s,x))$$

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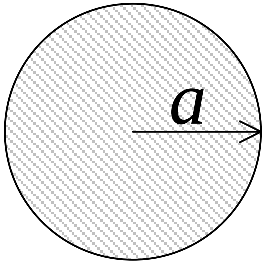
Mathematical analysis (Continuous system)

For clamped circular plate ($u = 0$ and $\theta = 0$)

$$v(s) = a_0 + \sum_{n=1}^{\infty} (a_n \cos(n\bar{\phi}) + b_n \cos(n\bar{\phi}))$$

$$m(s) = p_0 + \sum_{n=1}^{\infty} (p_n \cos(n\bar{\phi}) + q_n \cos(n\bar{\phi}))$$



Null-field integral equations

$$0 = \int_B \{-U(s,x)v(s) + \Theta(s,x)m(s) - M(s,x)\theta(s) + V(s,x)u(s)\}\, dB(s), \quad x \in \Omega^e$$

$$0 = \int_B \{-U_\theta(s,x)v(s) + \Theta_\theta(s,x)m(s) - M_\theta(s,x)\theta(s) + V_\theta(s,x)u(s)\}\, dB(s), \quad x \in \Omega^e$$

Expansion formulae

$$0 = \int_B \{-U(s,x)[a_0 + \sum_{n=1}^{\infty}(a_n \cos(n\bar{\phi}) + b_n \cos(n\bar{\phi}))] + \Theta(s,x)[p_0 + \sum_{n=1}^{\infty}(p_n \cos(n\bar{\phi}) + q_n \cos(n\bar{\phi}))]dB(s)$$

$$0 = \int_B \{-U_\theta(s,x)[a_0 + \sum_{n=1}^{\infty}(a_n \cos(n\bar{\phi}) + b_n \cos(n\bar{\phi}))] + \Theta_\theta(s,x)[p_0 + \sum_{n=1}^{\infty}(p_n \cos(n\bar{\phi}) + q_n \cos(n\bar{\phi}))]dB(s)$$

Degenerate kernel

$$K_0(\lambda r) = \begin{cases} \sum_{m=-\infty}^{\infty} K_m(\lambda\bar{\rho}) I_m(\lambda\rho)\cos(m(\bar{\phi}-\phi)), & \bar{\rho} > \rho \\ \sum_{m=-\infty}^{\infty} K_m(\lambda\rho) I_m(\lambda\bar{\rho})\cos(m(\bar{\phi}-\phi)), & \rho > \bar{\rho} \end{cases}$$

$$Y_0(\lambda r) = \begin{cases} \sum_{m=-\infty}^{\infty} Y_m(\lambda\bar{\rho}) J_m(\lambda\rho)\cos(m(\bar{\phi}-\phi)), & \bar{\rho} > \rho \\ \sum_{m=-\infty}^{\infty} Y_m(\lambda\rho) J_m(\lambda\bar{\rho})\cos(m(\bar{\phi}-\phi)), & \rho > \bar{\rho} \end{cases}$$

Relationship

$$p_n = \frac{1}{\lambda} \frac{Y_n(\lambda a) J_n(\lambda a) - K_n(\lambda a) I_n(\lambda a)}{Y_n(\lambda a) J'_n(\lambda a) - K_n(\lambda a) I'_n(\lambda a)} a_n, \quad N = 1, 2, \ldots,$$

$$q_n = \frac{1}{\lambda} \frac{Y_n(\lambda a) J_n(\lambda a) - K_n(\lambda a) I_n(\lambda a)}{Y_n(\lambda a) J'_n(\lambda a) - K_n(\lambda a) I'_n(\lambda a)} b_n, \quad N = 1, 2, \ldots,$$

$$p_n = \frac{1}{\lambda} \frac{Y'_n(\lambda a) J_n(\lambda a) - K'_n(\lambda a) I_n(\lambda a)}{Y'_n(\lambda a) J'_n(\lambda a) - K'_n(\lambda a) I'_n(\lambda a)} a_n, \quad N = 1, 2, \ldots,$$

$$q_n = \frac{1}{\lambda} \frac{Y_n(\lambda a) J_n(\lambda a) - K_n(\lambda a) I_n(\lambda a)}{Y_n(\lambda a) J'_n(\lambda a) - K_n(\lambda a) I'_n(\lambda a)} b_n, \quad N = 1, 2, \ldots,$$

Eigenequation

$$\frac{1}{\lambda}\frac{Y_n(\lambda a)J_n(\lambda a)-K_n(\lambda a)I_n(\lambda a)}{Y_n(\lambda a)J_n'(\lambda a)-K_n(\lambda a)I_n'(\lambda a)}=\frac{1}{\lambda}\frac{Y_n'(\lambda a)J_n(\lambda a)-K_n'(\lambda a)I_n(\lambda a)}{Y_n'(\lambda a)J_n'(\lambda a)-K_n'(\lambda a)I_n'(\lambda a)}$$

$$[K_{n+1}(\lambda a)Y_n(\lambda a)-K_n(\lambda a)Y_{n+1}(\lambda a)]\{I_{n+1}(\lambda a)J_n(\lambda a)+I_n(\lambda a)J_{n+1}(\lambda a)\}=0$$

Spurious Eigenequation　　　　True Eigenequation

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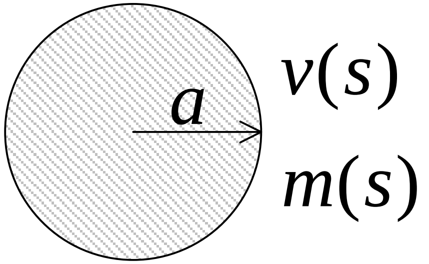
Mathematical analysis (Discrete system)

For clamped circular plate($u=0$ and $\theta=0$)

$$0=[U]\{v\}+[\Theta]\{m\}$$

$$0=[U_\theta]\{v\}+[\Theta_\theta]\{m\}$$



$$[SM]\begin{Bmatrix} v \\ m \end{Bmatrix}=0$$

$$[SM]=\begin{bmatrix} U & \Theta \\ U_\theta & \Theta_\theta \end{bmatrix}_{4N\times 4N}$$

Circulant

$$[U]=\begin{bmatrix} z_0 & z_1 & z_2 & \cdots & z_{2N-1} \\ z_{2N-1} & z_0 & z_1 & \cdots & z_{2N-2} \\ z_{2N-2} & z_{2N-1} & z_0 & \cdots & z_{2N-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ z_1 & z_2 & z_3 & z_{2N-1} & z_0 \end{bmatrix}_{2N\times 2N}$$

$$z_m=\int_{(m-\frac{1}{2})\Delta\bar{\phi}}^{(m+\frac{1}{2})\Delta\bar{\phi}}[-U(a,\bar{\phi},a,\phi)]a\,d\bar{\phi}\approx -U(a,\bar{\phi}_m,a,\phi)a\,\Delta\bar{\phi},$$

$$m=0,1,2,\ldots,2N-1$$

The properties of the circulant

$$\mu_\ell^{[U]} = \sum_{m=0}^{2N-1} z_m \alpha_\ell^m = \sum_{m=0}^{2N-1} z_m e^{i\frac{2\pi m\ell}{2N}},$$

$$\ell = 0, \pm 1, \pm 2, \ldots, \pm(N-1), N$$

$$\alpha_\ell = e^{i\frac{2\pi\ell}{2N}}, \quad \ell = 0, \pm 1, \pm 2, .., \pm N-1, N$$

$$or \quad \ell = 0,1,2,..,2N-1$$

$$\mu_\ell^{[U]} = \lim_{N\to\infty} \sum_{m=0}^{2N-1} \cos(m\ell\Delta\bar{\phi})[-U(a,\bar{\phi}_m,a,0)]a\,\Delta\bar{\phi}$$

$$\approx \int_0^{2\pi} \cos(\ell\bar{\phi})[-U(a,\bar{\phi},a,0)]a\,d\bar{\phi}$$

Eigenvalues of the four matrics

$[U]$ $$\mu_\ell^{[U]} = -\frac{\pi a}{4\lambda^2}[Y_\ell(\lambda a)J_\ell(\lambda a) - K_\ell(\lambda a)I_\ell(\lambda a)]$$

$[\Theta]$ $$\mu_\ell^{[\Theta]} = \frac{\pi a}{4\lambda}[Y_\ell(\lambda a)J'_\ell(\lambda a) - K_\ell(\lambda a)I'_\ell(\lambda a)]$$

$[U_\theta]$ $$\kappa_\ell^{[U]} = -\frac{\pi a}{4\lambda^2}[Y'_\ell(\lambda a)J_\ell(\lambda a) - K'_\ell(\lambda a)I_\ell(\lambda a)]$$

$[\Theta_\theta]$ $$\kappa_\ell^{[\Theta]} = \frac{\pi a}{4\lambda^2}[Y'_\ell(\lambda a)J'_\ell(\lambda a) - K'_\ell(\lambda a)I'_\ell(\lambda a)]$$

$$\ell = 0, \pm 1, \pm 2, \ldots, \pm(N-1), N$$

Eigenvalue decomposition

$$[U]=\Phi\Sigma_U\Phi^{-1}$$

$$=\Phi\begin{bmatrix}\mu_0^{[U]} & 0 & 0 & \cdots & 0 & 0 & 0\\ 0 & \mu_1^{[U]} & 0 & \cdots & 0 & 0 & 0\\ 0 & 0 & \mu_{-1}^{[U]} & \cdots & 0 & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & \mu_{N-1}^{[U]} & 0 & 0\\ 0 & 0 & 0 & \cdots & 0 & \mu_{-(N-1)}^{[U]} & 0\\ 0 & 0 & 0 & \cdots & 0 & 0 & \mu_N^{[U]}\end{bmatrix}\Phi^{-1}$$

$$[\Theta]=\Phi\Sigma_\Theta\Phi^{-1}$$

$$=\Phi\begin{bmatrix}\mu_0^{[\Theta]} & 0 & 0 & \cdots & 0 & 0 & 0\\ 0 & \mu_1^{[\Theta]} & 0 & \cdots & 0 & 0 & 0\\ 0 & 0 & \mu_{-1}^{[\Theta]} & \cdots & 0 & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & \mu_{N-1}^{[\Theta]} & 0 & 0\\ 0 & 0 & 0 & \cdots & 0 & \mu_{-(N-1)}^{[\Theta]} & 0\\ 0 & 0 & 0 & \cdots & 0 & 0 & \mu_N^{[\Theta]}\end{bmatrix}\Phi^{-1}$$

$$[U_\theta]=\Phi\Sigma_{U_\theta}\Phi^{-1}$$

$$=\Phi\begin{bmatrix}\kappa_0^{[U]} & 0 & 0 & \cdots & 0 & 0 & 0\\ 0 & \kappa_1^{[U]} & 0 & \cdots & 0 & 0 & 0\\ 0 & 0 & \kappa_{-1}^{[U]} & \cdots & 0 & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & \kappa_{N-1}^{[U]} & 0 & 0\\ 0 & 0 & 0 & \cdots & 0 & \kappa_{-(N-1)}^{[U]} & 0\\ 0 & 0 & 0 & \cdots & 0 & 0 & \kappa_N^{[U]}\end{bmatrix}\Phi^{-1}$$

$$[\Theta_\theta]=\Phi\Sigma_{\Theta_\theta}\Phi^{-1}$$

$$=\Phi\begin{bmatrix}\kappa_0^{[\Theta]} & 0 & 0 & \cdots & 0 & 0 & 0\\ 0 & \kappa_1^{[\Theta]} & 0 & \cdots & 0 & 0 & 0\\ 0 & 0 & \kappa_{-1}^{[\Theta]} & \cdots & 0 & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & \kappa_{N-1}^{[\Theta]} & 0 & 0\\ 0 & 0 & 0 & \cdots & 0 & \kappa_{-(N-1)}^{[\Theta]} & 0\\ 0 & 0 & 0 & \cdots & 0 & 0 & \kappa_N^{[\Theta]}\end{bmatrix}\Phi^{-1}$$

Determinant

$$[SM] = \begin{bmatrix} \Phi\Sigma_U\Phi^{-1} & \Phi\Sigma_\Theta\Phi^{-1} \\ \Phi\Sigma_{U_\theta}\Phi^{-1} & \Phi\Sigma_{\Theta_\theta}\Phi^{-1} \end{bmatrix}$$

$$= \begin{bmatrix} \Phi & 0 \\ 0 & \Phi \end{bmatrix} \begin{bmatrix} \Sigma_U & \Sigma_\Theta \\ \Sigma_{U_\theta} & \Sigma_{\Theta_\theta} \end{bmatrix} \begin{bmatrix} \Phi & 0 \\ 0 & \Phi \end{bmatrix}^{-1}$$

$$\det[SM] = \det\begin{bmatrix} \Sigma_U & \Sigma_\Theta \\ \Sigma_{U_\theta} & \Sigma_{\Theta_\theta} \end{bmatrix} = \prod_{\ell=-(N-1)}^{N} (\mu_\ell^{[U]}\kappa_\ell^{[\Theta]} - \mu_\ell^{[\Theta]}\kappa_\ell^{[U]})$$

Eigenequation

$$\det[SM]$$

$$= \prod_{\ell=-(N-1)}^{N} \frac{-\pi a^2}{16\lambda^2}\{[Y_\ell(\lambda a)J_\ell(\lambda a)-K_\ell(\lambda a)I_\ell(\lambda a)][Y'_\ell(\lambda a)J'_\ell(\lambda a)-K'_\ell(\lambda a)I'_\ell(\lambda a)]$$

$$-[Y_\ell(\lambda a)J'_\ell(\lambda a)-K_\ell(\lambda a)I'_\ell(\lambda a)][Y'_\ell(\lambda a)J_\ell(\lambda a)-K'_\ell(\lambda a)I_\ell(\lambda a)]\}$$

$$= \prod_{\ell=-(N-1)}^{N} \frac{-\pi a^2}{16\lambda^2}[K_{\ell+1}(\lambda a)Y_\ell(\lambda a)-K_\ell(\lambda a)Y_{\ell+1}(\lambda a)]$$

$$\{I_{\ell+1}(\lambda a)J_\ell(\lambda a)+I_\ell(\lambda a)J_{\ell+1}(\lambda a)\}$$

$$[K_{\ell+1}(\lambda a)Y_\ell(\lambda a)-K_\ell(\lambda a)Y_{\ell+1}(\lambda a)]\{I_{\ell+1}(\lambda a)J_\ell(\lambda a)+I_\ell(\lambda a)J_{\ell+1}(\lambda a)\}=0$$

Spurious Eigenequation   True Eigenequation

Spurious eigenequations using the real-part BEMs

Eqs. number	Spurious eigenequation using the real-part BEM
u, θ (1) and (2)	$K_{\ell+1}Y_{\ell} - K_{\ell}Y_{\ell+1} = 0$
u,*m* (1) and (3)	$(1-\nu)(K_{\ell}Y_{\ell+1} - K_{\ell+1}Y_{\ell}) - 2\lambda\rho K_{\ell}Y_{\ell} = 0$
u,*v* (1) and (4)	$\ell^2(1-\nu)(K_{\ell}Y_{\ell+1} - K_{\ell+1}Y_{\ell}) - 2\lambda\rho K_{\ell}Y_{\ell} + \lambda^2\rho^2(K_{\ell+1}Y_{\ell} + K_{\ell}Y_{\ell+1}) = 0$
θ,*m* (2) and (3)	$\ell^2(1-\nu)(K_{\ell}Y_{\ell+1} - K_{\ell+1}Y_{\ell}) - 2\lambda\rho K_{\ell}Y_{\ell} + \lambda^2\rho^2(K_{\ell+1}Y_{\ell} + K_{\ell}Y_{\ell+1}) = 0$
θ,*v* (2) and (4)	$2\lambda\rho(\ell^2 K_{\ell}Y_{\ell} + \lambda^2\rho^2 K_{\ell+1}Y_{\ell+1}) - 2\lambda^2\rho^2\ell(K_{\ell+1}Y_{\ell} + K_{\ell}Y_{\ell+1})$ $+[2\ell - (3-\nu)\ell^2 - 2\lambda\rho\ell(1-\ell)](K_{\ell}Y_{\ell+1} - K_{\ell+1}Y_{\ell}) = 0$
m,*v* (3) and (4)	$4\ell\lambda\rho(-1+\ell)[1-\ell(1-\nu)-\lambda\rho]K_{\ell}Y_{\ell} + [\ell^4(1-\nu)^2 + \lambda^4\rho^4 - 2\ell(1-\nu)(-1+\lambda\rho)](K_{\ell}Y_{\ell+1} - K_{\ell+1}Y_{\ell})$ $-\ell^2[3-4\nu+\nu^2-2\lambda\rho(1-\nu)](K_{\ell}Y_{\ell+1} - K_{\ell+1}Y_{\ell}) - 2\lambda^2\rho^2(1-\nu)(\ell-\ell^2)(K_{\ell}Y_{\ell+1} + K_{\ell+1}Y_{\ell}) + 2\lambda^3\rho^3(1-\nu)K_{\ell+1}Y_{\ell+1} = 0$

where $\ell = 0, \pm 1, \pm 2, \pm 3, \ldots\ldots$ And true eigenequation is $I_{\ell+1}J_{\ell} + I_{\ell}J_{\ell+1} = 0$

Flow chart for clamped plate using the real-part BEM

G.E. $\nabla^4 u(x) = \lambda^4 u(x), \quad x \in \Omega$

↓

BIE

↓

BEM

↓ B.C. $u = 0 \quad \theta = 0$

$$0 = [U]\{v\} + [\Theta]\{m\}$$
$$0 = [U_\theta]\{v\} + [\Theta_\theta]\{m\}$$

Construction of [SM]

$$[SM] = \begin{bmatrix} U & \Theta \\ U_\theta & \Theta_\theta \end{bmatrix}_{4N\times 4N}$$

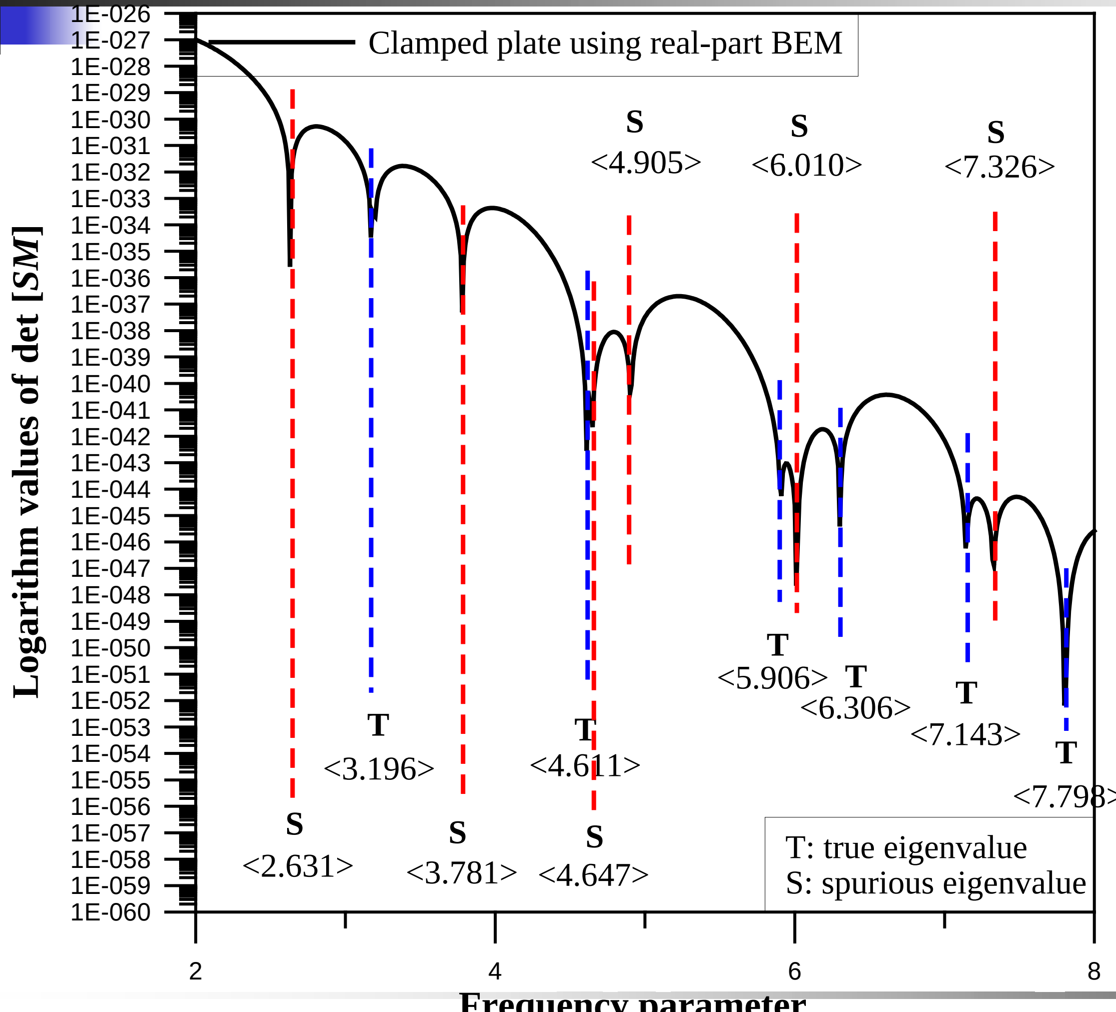
↓

$$\det[SM] = \prod_{\ell=-(N-1)}^{N} (\mu_\ell^{[U]}\kappa_\ell^{[\Theta]} - \mu_\ell^{[\Theta]}\kappa_\ell^{[U]})$$
$$= 0$$

↓

True and spurious eigenvalues

Determinant v.s Frequency parameter



True Eigenequation

$$\{I_{\ell+1}(\lambda a)J_{\ell}(\lambda a)+I_{\ell}(\lambda a)J_{\ell+1}(\lambda a)\}=0$$

Spurious Eigenequation

$$[K_{\ell+1}(\lambda a)Y_{\ell}(\lambda a)-K_{\ell}(\lambda a)Y_{\ell+1}(\lambda a)]=0$$

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Conclusions

1. A real-part formulation has been derived for the eigenproblem of the clamped plate.
2. For a circular plate, the true and spurious eigenvalues and eigenequations were derived analytically in continuous and discrete systems.
3. Since any two equations in the plate formulation (4 equations) can be chosen, 6 options can be considered.
4. The occurrence of spurious eigensolution only depends on the formulation instead of the boundary condition.

The End

Thanks for your kind attention