On the equivalence of the Trefftz method and method of fundamental solutions for Laplace and biharmonic equation

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Outline

- Laplace problem
- Trefftz method and MFS
- Connection between Trefftz method and MFS
- Numerical example
- Conclusion

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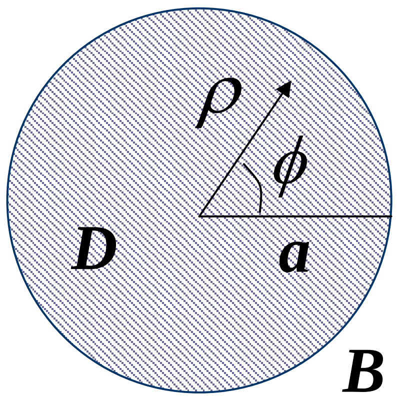
Laplace problem

Two-dimensional Laplace problem with a circular domain

G.E.: $\nabla^2 u(x) = 0, \quad x \in D$

B.C.: $u(x) = \bar{u}, \quad x \in B$



where

∇^2 denotes the Laplacian operator

$u(x)$ is the potential function

ρ is the radius of the field point

ϕ is the angle along the field point

Outline

- Laplace and problems
- Trefftz method and MFS
- Connection between Trefftz method and MFS
- Numerical example
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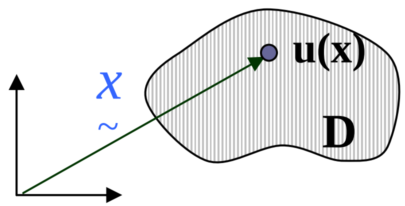
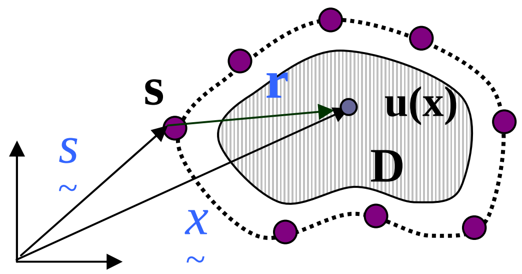
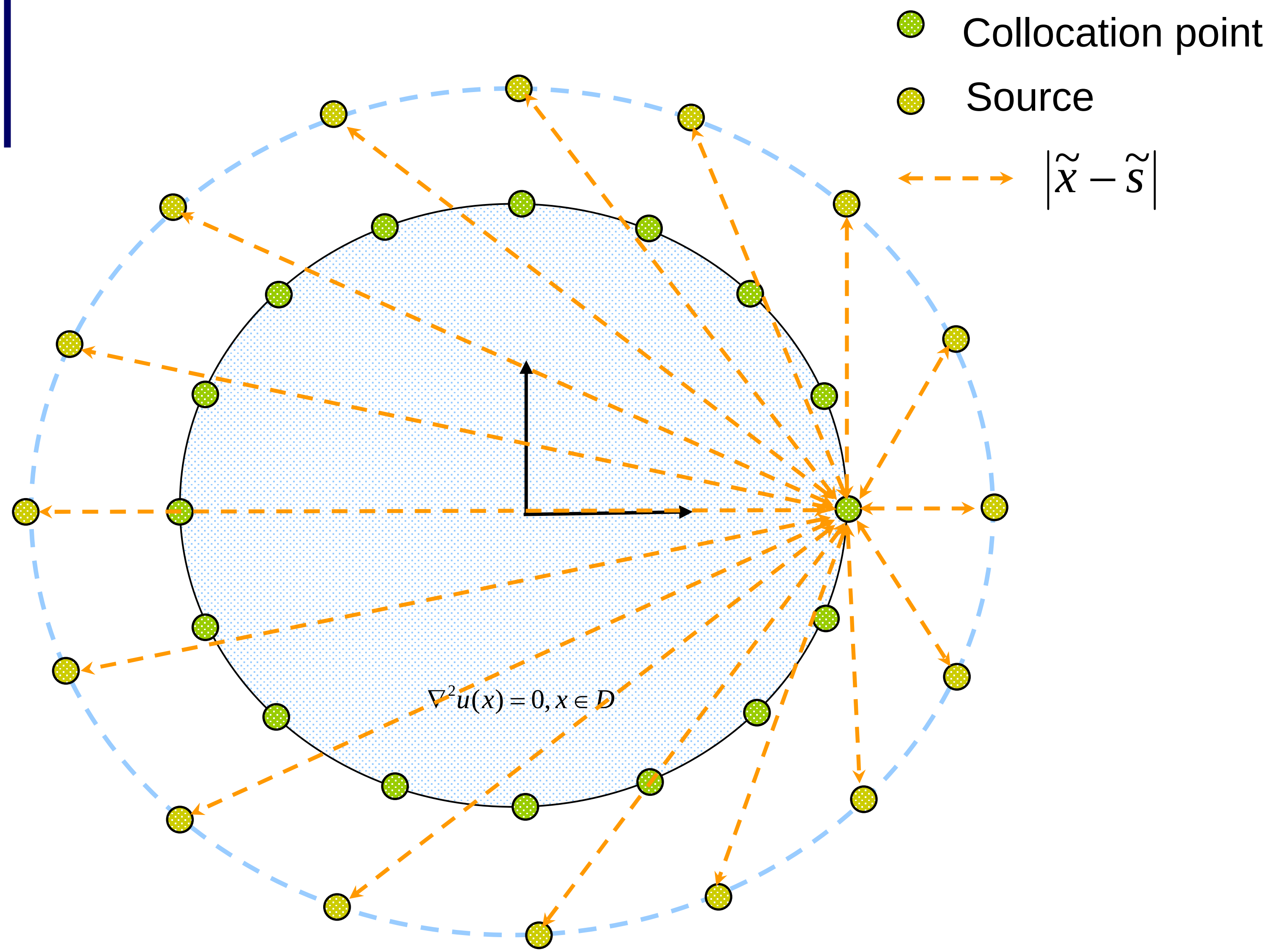
Trefftz method and MFS

Method	Trefftz method	MFS
Definition	$u(x)=\sum_{j=1}^{N_T} c_j u_j(x)$	$u(x)=\sum_{j=1}^{N_M} w_j U(x,s_j)$
		
Base	$u_j(x)$ (T-complete function)	$U(x,s)=\psi(r)$, $r=\lvert x-s\rvert$
G.E.	$\mathcal{L}\,u(x)=0,\quad x\in D$	$\mathcal{L}\,U(x,s)=0,\quad x\in D$ (singularity at s)
Match B.C.	Determine c_j	Determine w_j

N_T is the number of complete functions

N_M is the number of source points in the MFS



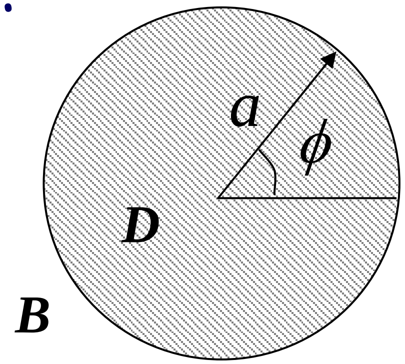
Collocation point
Source
$|\tilde{x}-\tilde{s}|$
$\nabla^2 u(x)=0, x\in D$

Laplace problem

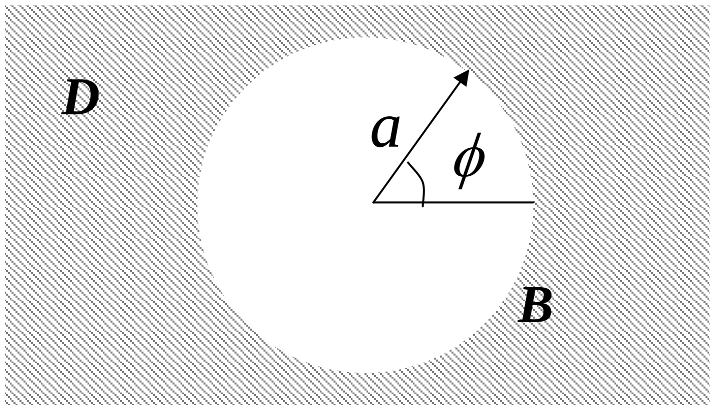
Two-dimensional Laplace problem with a circular domain

G.E.: $\nabla^2 u(x) = 0, \quad x \in D$ **B.C.:** $u(x) = \bar{u}, \quad x \in B$

Interior :



Exterior :



Boundary Condition: $$u(a,\phi) = \bar{a}_0 + \sum_{n=1}^{N} \bar{a}_n \cos(n\phi) + \sum_{n=1}^{N} \bar{b}_n \sin(n\phi)$$

Trefftz method

Representation of the field solution :

$$u(x) = \sum_{j=1}^{2N_T+1} w_j u_j(x)$$

where

$2N_T + 1$ is the number of complete functions

w_j is the unknown coefficient

u_j is the T-complete function which satisfies the Laplace equation

T-complete set function

1 → $\nabla^2 1 = 0$

$\rho^n \cos(n\phi)$ → $\nabla^2 \rho^n \cos(n\phi) = 0$

$\rho^n \sin(n\phi)$ → $\nabla^2 \rho^n \sin(n\phi) = 0$

Laplace Equation

where $$\nabla^2 = \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial \phi^2}$$

Derivation of unknown coefficients (Trefftz)

T-complete set functions : **Interior:** $1,\ \rho^n\cos(n\phi),\ \rho^n\sin(n\phi)$

Exterior: $\ln\rho,\ \rho^{-n}\cos(n\phi),\ \rho^{-n}\sin(n\phi)$

Field solution:

Interior :

$$u^I(a,\phi)=a_0+\sum_{n=1}^{N_T}a_n a^n\cos(n\phi)+\sum_{n=1}^{N_T}b_n a^n\sin(n\phi)$$

Exterior :

$$u^E(a,\phi)=a_0\ln a+\sum_{n=1}^{N_T}a_n a^{-n}\cos(n\phi)+\sum_{n=1}^{N_T}b_n a^{-n}\sin(n\phi)$$

By matching the boundary condition at $\rho=a$

Interior problem:

$$a_0=\bar{a}_0,$$

$$a_n=\frac{\bar{a}_n}{a^n},\qquad n=1,2,\ldots N_T$$

$$b_n=\frac{\bar{b}_n}{a^n}\qquad n=1,2,\ldots N_T$$

Exterior problem:

$$a_0=\frac{1}{\ln a}\bar{a}_0,$$

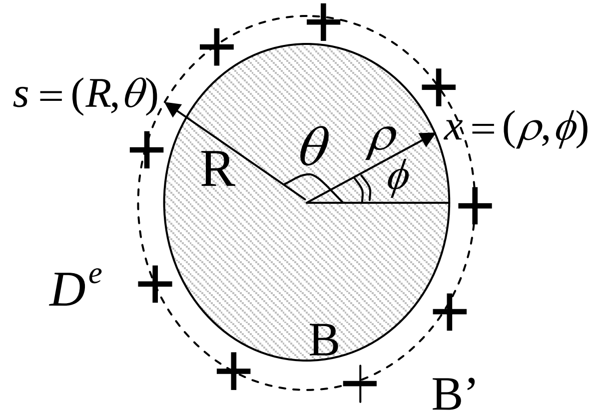
$$a_n=a^n\bar{a}_n,\qquad n=1,2,\ldots N_T$$

$$b_n=a^n\bar{b}_n\qquad n=1,2,\ldots N_T$$

Method of Fundamental Solutions (MFS)

Field solution :

$$u(x) = \sum_{j=1}^{N_M} c_j U(x, s_j), \quad s_j \in D^e$$



where

N_M is the number of source points in the MFS

c_j is the unknown coefficient

$U(x, s_j)$ is the fundamental solution

D^e is the complementary domain

s is the source point

x is the collocation point

Fundamental Solution

$$\nabla_x^2 U(x,s) = 2\pi\delta(x-s)$$

⇓

$$U(x,s) = \ln(r) \qquad r = |\underline{x} - \underline{s}|$$

where

$$\nabla_x^2 = \frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}$$

Derivation of unknown coefficients (MFS)

Degenerate kernel : $U(R,\theta,\rho,\phi)=\ln r=\begin{cases} U^i(R,\theta;\rho,\phi)=\ln(R)-\sum_{n=1}^{\infty}\frac{1}{n}(\frac{\rho}{R})^n\cos(n(\theta-\phi)), & R\geq\rho \\ U^e(R,\theta;\rho,\phi)=\ln(\rho)-\sum_{n=1}^{\infty}\frac{1}{n}(\frac{R}{\rho})^n\cos(n(\theta-\phi)), & R<\rho \end{cases}$

Field solution:

Interior : $u^I(a,\phi)=\sum_{j=1}^{N_M}c_j[\ln(R)-\sum_{n=1}^{\infty}\frac{1}{n}(\frac{a}{R})^n\cos(n(\theta_j-\phi))],\quad 0<\phi<2\pi$

Exterior : $u^E(a,\phi)=\sum_{j=1}^{N_M}c_j[\ln(a)-\sum_{n=1}^{\infty}\frac{1}{n}(\frac{R}{a})^n\cos(n(\theta_j-\phi))],\quad 0<\phi<2\pi$

Interior problem:

$$\bar{a}_0=\sum_{j=1}^{N_M}c_j\ln(R)$$

$$\frac{\bar{a}_n}{\rho^n}=-\sum_{j=1}^{N_M}c_j\frac{1}{n}(\frac{1}{R})^n\cos(n\theta_j),\quad n=1,2,\dots N_M$$

$$\frac{\bar{b}_n}{\rho^n}=-\sum_{j=1}^{N_M}c_j\frac{1}{n}(\frac{1}{R})^n\sin(n\theta_j),\quad n=1,2,\dots N_M$$

Exterior problem:

$$\frac{1}{\ln(a)}\bar{a}_0=\sum_{j=1}^{N_M}c_j$$

$$a^n\bar{a}_n=-\sum_{j=1}^{N_M}c_j\frac{1}{n}(R)^n\cos(n\theta_j),\quad n=1,2,\dots N_M$$

$$a^n\bar{b}_n=-\sum_{j=1}^{N_M}c_j\frac{1}{n}(R)^n\sin(n\theta_j),\quad n=1,2,\dots N_M$$

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Relationship between the two methods

$$\{\underline{w}\} = [K]\{\underline{c}\}$$

Trefftz — MFS

By setting $2N_T + 1 = N_M = 2N + 1$

(Trefftz method) (MFS)

$$\underline{w} = \begin{Bmatrix} a_0 \\ a_1 \\ b_1 \\ a_2 \\ b_2 \\ \vdots \\ a_N \\ b_N \end{Bmatrix}$$

Interior:

$$[K^I] = \begin{bmatrix} \ln R & \ln R & \ln R & \cdots & \ln R \\ \frac{-1}{R}\cos(\theta_1) & \frac{-1}{R}\cos(\theta_2) & \frac{-1}{R}\cos(\theta_3) & \cdots & \frac{-1}{R}\cos(\theta_{2N+1}) \\ \frac{-1}{R}\sin(\theta_1) & \frac{-1}{R}\sin(\theta_2) & \frac{-1}{R}\sin(\theta_3) & \cdots & \frac{-1}{R}\sin(\theta_{2N+1}) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{-1}{N}(\frac{1}{R})^N\cos(N\theta_1) & \frac{-1}{N}(\frac{1}{R})^N\cos(N\theta_2) & \frac{-1}{N}(\frac{1}{R})^N\cos(N\theta_3) & \cdots & \frac{-1}{N}(\frac{1}{R})^N\cos(N\theta_{2N+1}) \\ \frac{-1}{N}(\frac{1}{R})^N\sin(N\theta_1) & \frac{-1}{N}(\frac{1}{R})^N\sin(N\theta_2) & \frac{-1}{N}(\frac{1}{R})^N\sin(N\theta_3) & \cdots & \frac{-1}{N}(\frac{1}{R})^N\sin(N\theta_{2N+1}) \end{bmatrix}$$

Exterior:

$$[K^E] = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ -R\cos(\theta_1) & -R\cos(\theta_2) & -R\cos(\theta_3) & \cdots & -R\cos(\theta_{2N+1}) \\ -R\sin(\theta_1) & -R\sin(\theta_2) & -R\sin(\theta_3) & \cdots & -R\sin(\theta_{2N+1}) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{-1}{N}(R)^N\cos(N\theta_1) & \frac{-1}{N}(R)^N\cos(N\theta_2) & \frac{-1}{N}(R)^N\cos(N\theta_3) & \cdots & \frac{-1}{N}(R)^N\cos(N\theta_{2N+1}) \\ \frac{-1}{N}(R)^N\sin(N\theta_1) & \frac{-1}{N}(R)^N\sin(N\theta_2) & \frac{-1}{N}(R)^N\sin(N\theta_3) & \cdots & \frac{-1}{N}(R)^N\sin(N\theta_{2N+1}) \end{bmatrix}$$

$$\underline{c} = \begin{Bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ \vdots \\ c_{2N} \\ c_{2N+1} \end{Bmatrix}$$

Matrix ($K = T_R T_\theta$)

$$[T_\theta]=\begin{bmatrix} 1 & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & 1 \\ \cos(\theta_1) & \cos(\theta_2) & \cdots & \cdots & \cdots & \cdots & \cdots & \cos(\theta_{2N+1}) \\ \sin(\theta_1) & \sin(\theta_2) & \cdots & \cdots & \cdots & \cdots & \cdots & \sin(\theta_{2N+1}) \\ \cos(2\theta_1) & \cos(2\theta_2) & \cdots & \cdots & \cdots & \cdots & \cdots & \cos(2\theta_{2N+1}) \\ \sin(2\theta_1) & \sin(2\theta_2) & \cdots & \cdots & \cdots & \cdots & \cdots & \sin(2\theta_{2N+1}) \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \cos(N\theta_1) & \cos(N\theta_2) & \cdots & \cdots & \cdots & \cdots & \cdots & \cos(N\theta_{2N+1}) \\ \sin(N\theta_1) & \sin(N\theta_2) & \cdots & \cdots & \cdots & \cdots & \cdots & \sin(N\theta_{2N+1}) \end{bmatrix}_{(2N+1)\times(2N+1)}$$

$$\Rightarrow\quad [T_\theta][T_\theta]^T=\begin{bmatrix} 2N+1 & 0 & \cdots & \cdots & 0 \\ 0 & \frac{2N+1}{2} & \cdots & \cdots & 0 \\ 0 & 0 & \frac{2N+1}{2} & \cdots & \vdots \\ \vdots & \vdots & \cdots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & \frac{2N+1}{2} \end{bmatrix}_{(2N+1)\times(2N+1)}$$

$$\Rightarrow\quad \det[T_\theta]=\frac{(2N+1)^{N+\frac{1}{2}}}{2^N}\neq 0,\quad N\in natural\ \ number$$

Matrix ($K = T_R T_\theta$)

$$[T_R]^I = \begin{bmatrix} \ln(R) & 0 & 0 & 0 & \cdots & \cdots & 0 & 0 \\ 0 & \frac{-1}{R} & 0 & 0 & \cdots & \cdots & 0 & 0 \\ 0 & 0 & \frac{-1}{R} & 0 & \cdots & \cdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \frac{-1}{2}(\frac{1}{R})^2 & \cdots & \cdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \frac{-1}{2}(\frac{1}{R})^2 & \cdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \cdots & \ddots & 0 & 0 \\ 0 & 0 & 0 & \cdots & \cdots & \cdots & \frac{-1}{N}(\frac{1}{R})^N & 0 \\ 0 & 0 & 0 & \cdots & \cdots & \cdots & 0 & \frac{-1}{N}(\frac{1}{R})^N \end{bmatrix}_{(2N+1)\times(2N+1)}$$

$R >> a$, $N \to \infty$ ⟺ **ill-posed problem**

$\ln R = 0$ $(R = 1)$ ⟺ **Degenerate scale problem**

$\frac{\ln(a)}{\ln(a)} \Leftarrow$

$$[T_R]^E = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & \cdots & 0 & 0 \\ 0 & -R & 0 & 0 & \cdots & \cdots & 0 & 0 \\ 0 & 0 & -R & 0 & \cdots & \cdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \frac{-1}{2}(R)^2 & \cdots & \cdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \frac{-1}{2}(R)^2 & \cdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \cdots & \ddots & 0 & 0 \\ 0 & 0 & 0 & \cdots & \cdots & \cdots & \frac{-1}{N}(R)^N & 0 \\ 0 & 0 & 0 & \cdots & \cdots & \cdots & 0 & \frac{-1}{N}(R)^N \end{bmatrix}_{(2N+1)\times(2N+1)}$$

$R << a$, $N \to \infty$ ⟺ **ill-posed problem**

$(a = 1)$ ⟺ **Nonuniqueness**

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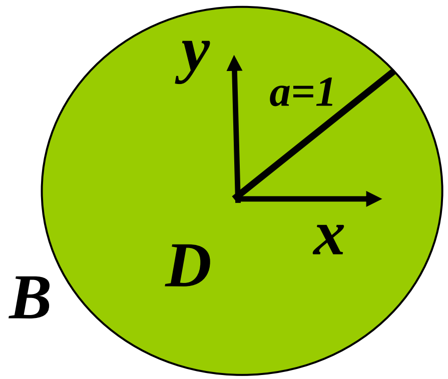
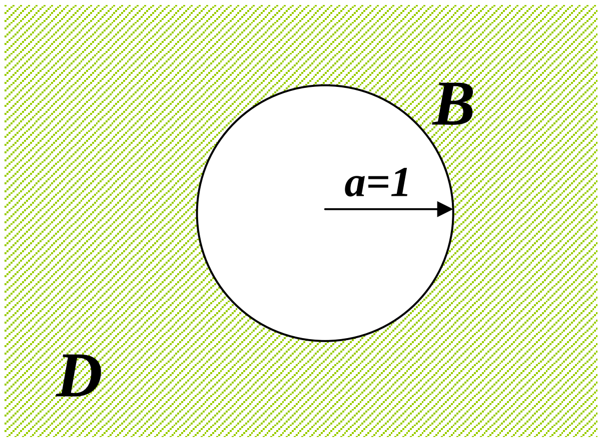
Numerical example 1

$$G.E.: \nabla^2 u(x) = 0$$

$$B.C.: u(x) = \cos(3\theta)$$



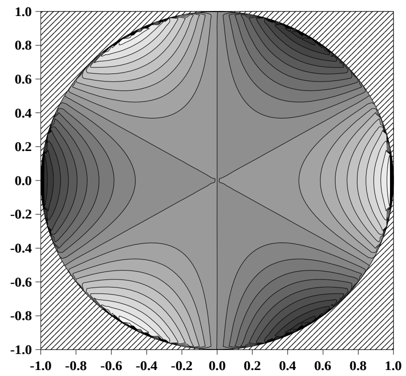
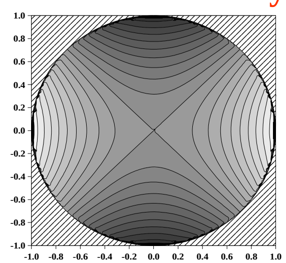
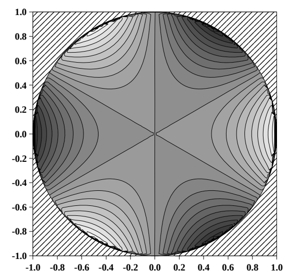
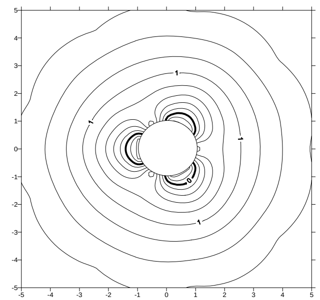
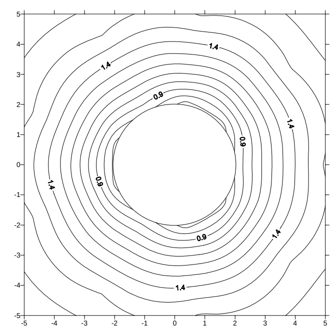
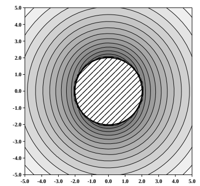
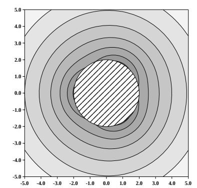


Exact solution: $u(r,\theta) = r^3 \cos(3\theta)$ $u(r,\theta) = c \ln r + \frac{1}{r^3} \cos(3\theta)$

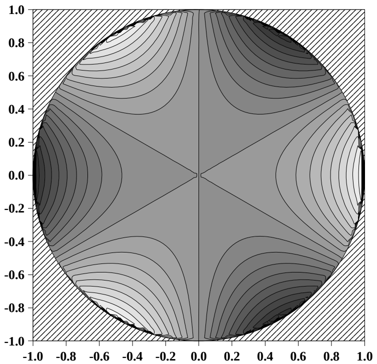
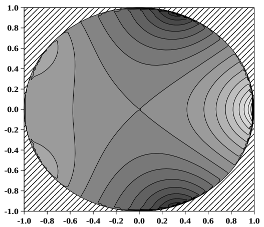
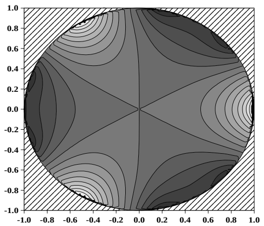
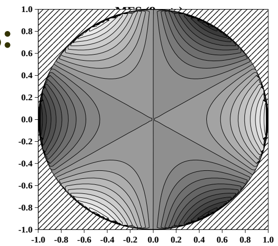
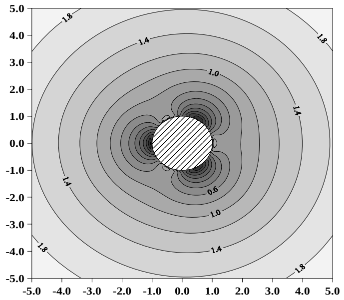
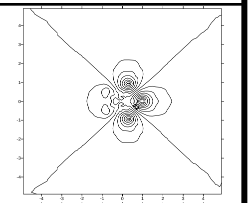
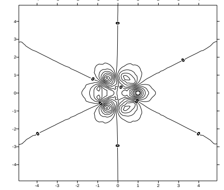
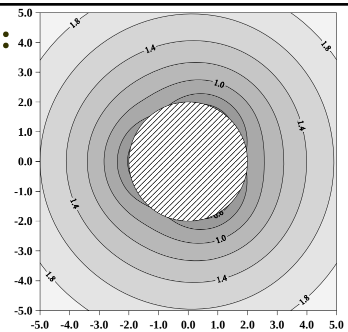
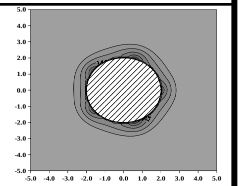
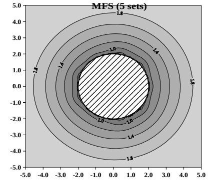
1. **Trefftz method for simply-connected problem**

2. **MFS for simply-connected problem**

Trefftz method

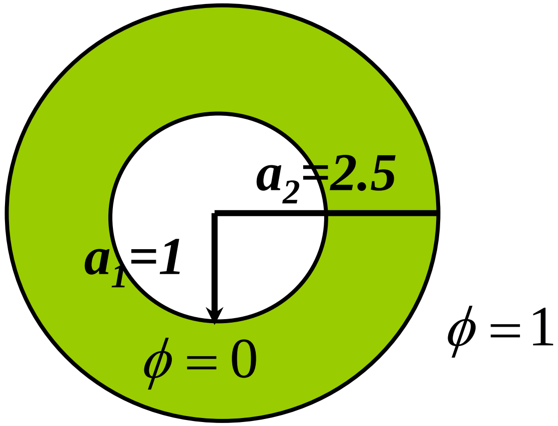
Trefftz method for the simply-connected problem			
Interior problem		Exterior problem	
Exact solution	Numerical solution	Exact solution	Numerical solution
 Exact solution	5 Points: B.C. aliasing base deficiency  Trefftz method (5 sets) 9 Points:  Trefftz method (9 sets)	*a*=1 	5 Points: B.C. aliasing Failure ($\ln \rho$) 9 Points: Failure ($\ln \rho$)
		a=2 	5 Points: B.C. aliasing  Trefftz solution (5 sets) 9 Points:  Trefftz solution (9 sets)

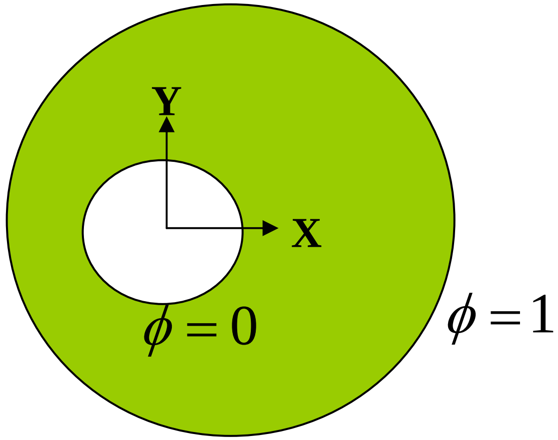
MFS

MFS for simply-connected problem			
Interior problem		Exterior problem	
Exact solution	Numerical solution	Exact solution	Numerical solution
 Exact solution	5 Points: B.C. aliasing  MFS (5 sets) 9 Points:  55 Points:  MFS (55 sets)	*a*=1:  Exact solution (r=1)	5 Points: B.C. aliasing Failure (ln *a*)  9 Points: Failure (ln *a*) 
		a=2:  Exact solution (r=2)	5 Points: B.C. aliasing  MFS (5 sets) 9 Points:  MFS (9 sets)

Numerical example 2

$$G.E.: \nabla^2 \phi = 0$$





Exact solution:

$$u(\rho,\phi)=\frac{\ln\rho}{\ln 2.5}$$

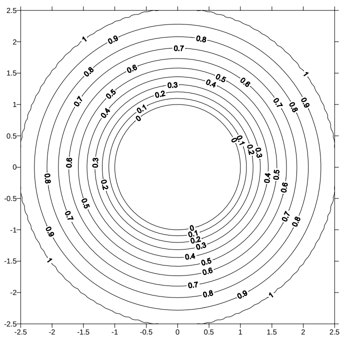
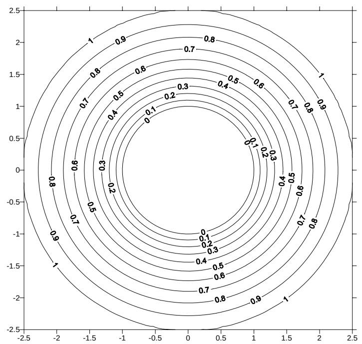
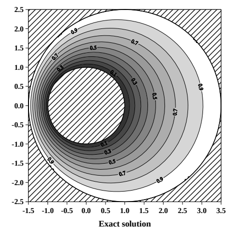
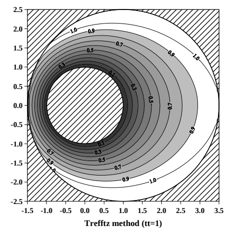
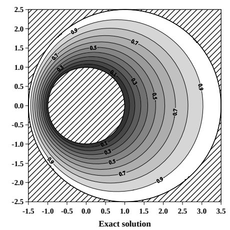
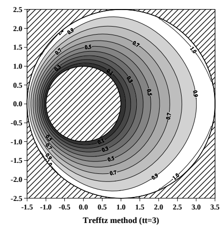
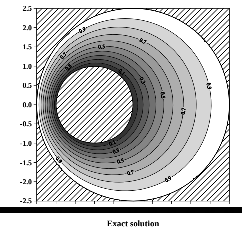
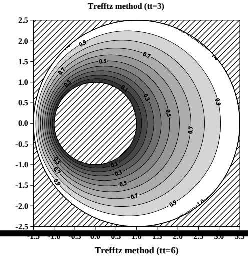
$$u(\rho,\phi)=\frac{1}{2\ln 2}\{\frac{16\rho^2+1+8\rho\cos\phi}{\rho^2+16+8\rho\cos\phi}\}$$

1. **Trefftz method for multiply-connected problem**

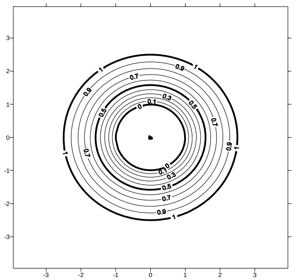
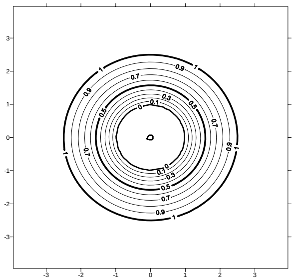
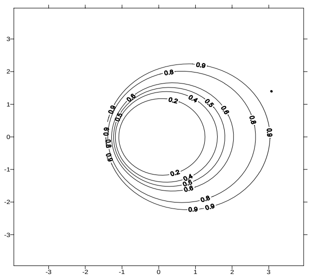
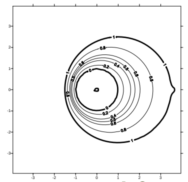
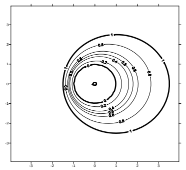
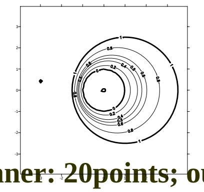
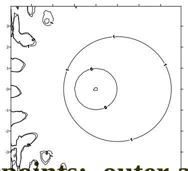
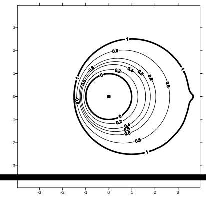
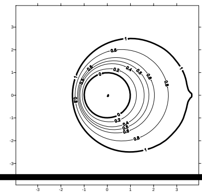
2. **MFS for multiply-connected problem**

Trefftz method

Trefftz method for multiply-connected problem

Concentric circle		Eccentric circle		
Exact solution	Numerical solution	Exact solution		Numerical solution
26 Points  $u(\rho,\phi)=\dfrac{\ln \rho}{\ln 2.5}$	26 Points 		6 Points	
			14 Points	
			26 Points	

MFS

MFS for the multiply-connected problem			
Concentric circle		Eccentric circle	
Exact solution	Numerical solution	Exact solution	Numerical solution
Inner circle: 20 points **outer circle: 60points** $u(\rho,\phi)=\dfrac{\ln\rho}{\ln 2.5}$ 	**Inner circle: a_1=0.9 outer circle :a_2=2.6** **Inner circle: 20 points** **outer circle: 60points** 	**Inner circle: 20 points** **outer circle: 60points** $u(\rho,\phi)=\dfrac{1}{2\ln 2}\times\{\dfrac{16\rho^2+1+8\rho\cos\alpha}{\rho^2+16+8\rho\cos\alpha}\}$ 	**Inner: 20points; outer: 60points; inner a_1=0.9** **outer a_2=2.6**  **outer a_2 =3.0**  **outer a_2 =4.0**  **outer a_2 =10.0**  **Inner: 20points; outer: 60points; outer $a_2$2.6** **inner a_1=0.5**  **inner a_1 =0.3** 

Outline

- Laplace problem
- Trefftz method and MFS
- Connection between Trefftz method and MFS
- Numerical example
- Conclusion

Conclusion

- The proof of the mathematical equivalence between the Trefftz method and MFS for Laplace equation was derived successfully.
- The T-complete set functions in the Trefftz method for interior and exterior problems are imbedded in the degenerate kernels of the fundamental solutions
- The sources of degenerate scale and ill-posed behavior in the MFS are easily found in the present formulation.

The end

Thanks for your kind attention