國 立 臺 灣 海 洋 大 學
河 海 工 程 學 系
碩 士 學 位 論 文

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Mathematical analysis and numerical study for free vibration of plate using BEM
邊界元素法於板自由振動之數學分析與數值研究

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中 華 民 國 九 十 二 年 六 月

誌　謝

從大二“ 工程數學 ”開始，即受到陳正宗 老師在課業上細心的教導，同時在因緣巧合下進入了“ 力學聲響振動研究室 ”打工。大四時的國科會大學生專題計畫又承蒙陳老師的指導與照顧。大學畢業後進入河工所就讀，真正拜得陳老師門下，開始學習如何作研究及撰寫論文並順利畢業，至今算算也五年了，在這五年的時間中，其中的點點滴滴，都令我記憶猶新。

吾師對於學術的執著與熱忱，及對學生負責的態度，是學生一直督促自己學習的原動力。雖然做研究的歷程非常艱辛，但是研究成果完成時所得到的喜悅是無可比擬的。因此本文的完成，首先要感謝恩師諄諄不倦的教誨及指導。再者，感謝陳義麟 博士的共同指導，在求學期間與研究過程中細心的教導及協助，並提供研究經驗與心得來幫助我思考問題。論文口試期間，承蒙台大土木系楊德良 教授、洪宏基教授對本文的細心指教及建議，使本文更為充實及完善，令學生更加瞭解物理觀念及數學問題密不可分的關係。在研究及撰寫本文過程中，感謝實驗室所有的學長、同學、學弟妹，桂鴻、慶鋒、韋誌、銘翰、書睿、立偉、宗衛、應德、清森、雅雯、文成、嘉俊、新閔、…及大學同學、宗佑、怡伶、建中、盈翔、昭安，…謝謝他們的精神鼓勵與實質上的幫忙。並感謝國科會 NSC-91-2211-E-019-009 計畫的獎助金，讓我在做研究過程中不用擔心經濟方面的壓力。

最後，感謝我的父親與母親。一直以來，父母親是我的精神及經濟支柱，並不斷地鼓勵我奮發向上。如今獲得碩士學位，最感謝的是父母親的支持，讓我在這求學的路上，獲得許多知識及技能。「爸，媽！謝謝你們。祝你們天天快樂，永遠健康！」

感謝所有曾幫助，關心以及在乎過我的人，謝謝你們對我的付出與關懷，希望你們能分享我的喜悅，並讓我們永遠牽繫著這份情感。

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Contents

Table captions

Figure captions

Notations

a	radius of the circular plate or radius of the outer circle for annular plate
a_n	Fourier coefficient of boundary density
$a_{i,n}$	Fourier coefficient of boundary density
B	boundary
B_1	outer boundary of an annular domain
B_2	inner boundary of an annular domain
b_n	Fourier coefficient of boundary density
$b_{i,n}$	Fourier coefficient of boundary density
$[C]$	updating matrix
$[C^*]$	updating matrix
D	flexural rigidity
E	Young's modulus
h	plate thickness
$I_n(\cdot)$	the n-th order modified Bessel function of the first kind
$I'_n(\cdot)$	derivative of $I_n(\cdot)$
$J_n(\cdot)$	the n-th order Bessel function of the first kind
$J'_n(\cdot)$	derivative of $J_n(\cdot)$
$K_n(\cdot)$	the n-th order modified Bessel function of the second kind
$K'_n(\cdot)$	derivative of $K_n(\cdot)$
m	normal moment
$M(s,x)$	kernel function
$M_\theta(s,x)$	kernel function
$M_m(s,x)$	kernel function
$M_v(s,x)$	kernel function
$[M]$	influence matrix of the kernel function $M(s,x)$
$[M_\theta]$	influence matrix of the kernel function $M_\theta(s,x)$
$[M_m]$	influence matrix of the kernel function $M_m(s,x)$

$[M_v]$	influence matrix of the kernel function $M_v(s,x)$
n	normal vector
n_s	normal vector at the source point s
n_x	normal vector at the source point x
p_n	Fourier coefficient of boundary density
$p_{i,n}$	Fourier coefficient of boundary density
$P.V.$	principal value
q_n	Fourier coefficient of boundary density
$q_{i,n}$	Fourier coefficient of boundary density
r	distance between the source point s and the field point x, $r \equiv \|s-x\|$
s	position vector of the source point
s_{B1}	source point s locates at B_1
s_{B2}	source point s locates at B_2
$[SM]$	influence matrix
$[SM^c]$	influence matrix for the clamped circular plate
$[SM^s]$	influence matrix for the simply-supported circular plate
$[SM^f]$	influence matrix for the free circular plate
$[SM^{cc}]$	influence matrix for the C-C annular plate
$[SM^{ss}]$	influence matrix for the S-S annular plate
$[SM^{ff}]$	influence matrix for the F-F annular plate
$[SM_1^c]$	influence matrix for the clamped circular plate
$[SM_2^c]$	influence matrix for the clamped circular plate
$[SM_1^{cc}]$	influence matrix for the C-C annular plate
$[SM_2^{cc}]$	influence matrix for the C-C annular plate
t	tangential vector
t_s	tangential vector at the field point s
t_x	tangential vector at the field point x
u	displacement
$U(s,x)$	kernel function

$U_\theta(s,x)$	kernel function
$U_m(s,x)$	kernel function
$U_v(s,x)$	kernel function
$[U]$	influence matrix of the kernel function $U(s,x)$
$[U_\theta]$	influence matrix of the kernel function $U_\theta(s,x)$
$[U_m]$	influence matrix of the kernel function $U_m(s,x)$
$[U_v]$	influence matrix of the kernel function $U_v(s,x)$
v	effective shear force
$V(s,x)$	kernel function
$V_\theta(s,x)$	kernel function
$V_m(s,x)$	kernel function
$V_v(s,x)$	kernel function
$[V]$	influence matrix of the kernel function $V(s,x)$
$[V_\theta]$	influence matrix of the kernel function $V_\theta(s,x)$
$[V_m]$	influence matrix of the kernel function $V_m(s,x)$
$[V_v]$	influence matrix of the kernel function $V_v(s,x)$
x	position vector of the field point
x_{B1}	field point x locates at B_1
x_{B2}	field point x locates at B_2
$Y_n(\cdot)$	the n-th order Bessel function of second kind
$Y_n'(\cdot)$	derivative of $Y_n(\cdot)$
α	coefficient
$\alpha_n^I(\cdot)$	function of $I_n(\cdot)$
$\alpha_n^J(\cdot)$	function of $J_n(\cdot)$
$\alpha_n^K(\cdot)$	function of $K_n(\cdot)$
$\alpha_n^Y(\cdot)$	function of $Y_n(\cdot)$
$\beta_n^I(\cdot)$	function of $I_n(\cdot)$
$\beta_n^J(\cdot)$	function of $J_n(\cdot)$
$\beta_n^K(\cdot)$	function of $K_n(\cdot)$

$\beta_n^Y(\cdot)$	function of $Y_n(\cdot)$
$\gamma_n^I(\cdot)$	function of $I_n(\cdot)$
$\gamma_n^J(\cdot)$	function of $J_n(\cdot)$
$\gamma_n^K(\cdot)$	function of $K_n(\cdot)$
$\gamma_n^Y(\cdot)$	function of $Y_n(\cdot)$
$\mu_\ell^{[U]}$	eigenvalue of the matrix $[U]$
$\mu_\ell^{[\Theta]}$	eigenvalue of the matrix $[\Theta]$
$\mu_\ell^{[M]}$	eigenvalue of the matrix $[M]$
$\mu_\ell^{[V]}$	eigenvalue of the matrix $[V]$
$\kappa_\ell^{[U]}$	eigenvalue of the matrix $[U_\theta]$
$\kappa_\ell^{[\Theta]}$	eigenvalue of the matrix $[\Theta_\theta]$
$\kappa_\ell^{[M]}$	eigenvalue of the matrix $[M_\theta]$
$\kappa_\ell^{[V]}$	eigenvalue of the matrix $[V_\theta]$
$\zeta_\ell^{[U]}$	eigenvalue of the matrix $[U_m]$
$\zeta_\ell^{[\Theta]}$	eigenvalue of the matrix $[\Theta_m]$
$\zeta_\ell^{[M]}$	eigenvalue of the matrix $[M_m]$
$\zeta_\ell^{[V]}$	eigenvalue of the matrix $[V_m]$
$\delta_\ell^{[U]}$	eigenvalue of the matrix $[U_v]$
$\delta_\ell^{[\Theta]}$	eigenvalue of the matrix $[\Theta_v]$
$\delta_\ell^{[M]}$	eigenvalue of the matrix $[M_v]$
$\delta_\ell^{[V]}$	eigenvalue of the matrix $[V_v]$
$\delta(x-s)$	Dirac-Delta function
θ	slope
$\Theta(s,x)$	kernel function
$\Theta(s,x)$	kernel function
$\Theta_\theta(s,x)$	kernel function
$\Theta_m(s,x)$	kernel function
$\Theta_v(s,x)$	kernel function
$[\Theta]$	influence matrix of the kernel function $\Theta(s,x)$

$[\Theta_\theta]$	influence matrix of the kernel function $\Theta_\theta(s,x)$
$[\Theta_m]$	influence matrix of the kernel function $\Theta_m(s,x)$
$[\Theta_v]$	influence matrix of the kernel function $\Theta_v(s,x)$
$\mathcal{K}_\theta(\cdot)$	slope operator
$\mathcal{K}_m(\cdot)$	moment operator
$\mathcal{K}_v(\cdot)$	effective shear force operator
$(\overline{\rho},\overline{\phi})$	polor coordinate of s
(ρ,ϕ)	polor coordinate of x
(ρ_i,ϕ_i)	polor coordinate of the collocation point ($i=1,\ 2,\ 3$)
∇^4	biharmonic operator
Σ	diagonal matrix of SVD
Φ	left unitary matrix of SVD
λ	frequency parameter
ω	circular frequency
Ω	domain
Ω^e	complementary domain
ρ_0	surface density of plate
ν	Poisson ratio

摘要

本論文針對使用邊界元素法求解單連通及多連通板自由振動問題所產生的假根問題作深入的解析與數值研究。文中對於產生假根的機制, 分別以連續系統與離散系統的觀點來探討。在邊界元素法中, 將二維聲場或薄膜振動問題中的奇異及超奇異積分方程式或單層及雙層勢能之 對偶積分方程式拓展至板問題之四條邊界積分方程式, 任取兩條積分方程式聯立即可求解板自由振動之特徵值問題。在連續系統中, 以 Fourier 級數及基本解的退化核函數來推導出圓形板之自由振動眞假特徵方程。在離散系統中, 利用基本解的退化核函數與循環矩陣之特性解析圓形板之自由振動眞假特徵方程。進而得到單連通及多連通板問題中假根的產生機制, 並可得知不論採用任兩條方程式來求解, 都可得到眞正的特徵頻率, 而假根的產生則隨著使用之方程式不同而有所改變。在多連通問題之假根出現的的位置 (特徵值) 即對應內部邊界之内域問題的共振頻率。爲克服假根問題, 本文提出三種解決方法, 分別爲 SVD 補充式、Burton & Miller 法與 CHEEF (CHIEF) 法。最後本文藉由不同的數值算例, 來驗證上述的理論推導。

關鍵字: 邊界元素法, 板自由振動, Fourier 級數, 退化核函數, 循環矩陣, 假根, 奇異值分解法之補充式, Burton & Miller 法, CHEEF 法, CHIEF 法。

Abstract

In this thesis, the spurious eigenequations for the simply and multiply-connected plate eigenproblems are studied in the continuous and discrete systems. Since any two boundary integral equations in the plate formulation (4 equations) can be chosen, 6 (C_2^4) options can be considered instead of only two approaches (single-layer and double-layer methods, singular and hypersingular equations) are adopted for the eigenproblems of the membrane and acoustic problems. The occurring mechanism of the spurious eigenequation for the simply-connected and multiply-connected plate eigenproblems in the real-part, imaginary-part and complex-valued formulations were studied analytically. For the continuous system, degenerate kernels for the fundamental solution and the Fourier series expansion for the boundary density are employed to derive the true and spurious eigenequations analytically for a circular plate. For the discrete system, the degenerate kernels for the fundamental solution and circulants resulting from the circular boundary are employed to determine the spurious eigenequation. True eigenequation depends on the cases while spurious eigenequation is embedded in each formulation for simply-connected and multiply-connected plates. The spurious eigenvalues for the multiply-connected plate eigenproblem is the true eigenvalue of the associated simply-connected problem with the radius b which is the inner boundary of the multiply-connected domain. Also, we provide three methods (SVD updating technique, Burton & Miller method and CHEEF (CHIEF) method) to suppress the occurrence of the spurious eigenvalues for the free vibration of simply and multiply-connected plate eigenproblems. Several examples were demonstrated to check the validity of the formulation.

Keywords: boundary element method, free vibration of plate, Fourier series, degenerate kernel, circulants, spurious eigenvalues, SVD technique of updating term, Burton & Miller method, CHEEF method, CHIEF method.

Chapter 1 Introduction

1.1 Motivation of the research

For the simply-connected problems of interior acoustics or membrane, either the real-part or imaginary-part BEM instead of the complex-valued BEM results in spurious eigenequations. Tai and Shaw [72] first employed BEM to solve membrane vibration using a complex-valued kernel. De Mey [36, 37], Hutchinson and Wong [45] employed only the real-part kernel to solve the membrane and plate vibrations, respectively, free of the complex-valued computation in sacrifice of occurrence of spurious eigenequations. Kamiya *et al.* [52, 53] and Yeih *et al.* [78] independently linked the relation of multiply reciprocity method (MRM) and real-part BEM. Wong and Hutchinson [47] have presented a direct BEM for plate vibration involving displacement, slope, moment and shear force. They were able to obtain eigenvalues for the clamped plates by employing only the real-part BEM with obvious computational gains. However, this saving leads to the spurious eigenvalues in addition to the true ones for free vibration analysis. One has to investigate the mode shapes in order to identify and reject the spurious ones. Shaw [69] commented that only the real-part approach was incorrect since the eigenequation must satisfy the real-part and imaginary-part equations at the same time. Hutchinson [46] replied that the claim of incorrectness was perhaps a little strong since the real-part BEM does not miss any true eigenvalue although the solution is contaminated by spurious ones according to his numerical experience. However, no proof was provided. Precently, Kuo *et al.* and Chen *et al.* have proved the fact through a circular case for the real-part and imaginary-part BEMs, respectively. If we usually need to look for the eigenmode as well as eigenvalue, the sorting for the spurious eigenequations pay a small price by identifying the mode shapes. Chen *et al.* [18] commented that the spurious modes can be reasonable which may mislead the judgement of the true and spurious ones, since the true and spurious modes may have the same nodal line in case of different eigenvalues. This is the reason why Chen and his coworkers have developed many systematic techniques, *e.g.,* dual formulation [18], domain partition [9], SVD updating technique [17], CHEEF method [10], for sorting out the true and the spurious eigenvalues. Niwa *et al.* [62] also stated that "One must take care to use the complete Green's function for outgoing waves, as attempts to use just the real (singular) or imaginary (regular) part separately will not provide the complete spectrum". As quoted from the reply of Hutchinson [46], this comment is not correct since the real-part BEM does not lose any true eigenvalue. The reason is that the real-part and imaginary-part kernels satisfy the Hilbert transform pair. They are not fully independent. To use both parts, real and imaginary kernels may be not economical. Complete eigenspectrum is imbedded in either real or imaginary-part kernel. The Hilbert transform is the constraint in the frequency domain corresponding to the casual effect in the time-domain fundamental solution. The physical meaning of the real-part kernel is the standing wave [39]. Tai and Shaw [72] claimed that spurious eigenvalues are not present if the complex-valued kernel is employed for the eigenproblem. However, it is true only for the case of problem with a simply-connected domain. For multiply-connected problems, spurious eigenequation still occurs even though the complex-valued BEM is utilized. This finding and the spurious eigenvalues have been verified in the membrane and acoustic problems [21, 31]. The spurious eigenvalues occurs in two aspects as shown in Figure 1-1: one is for the simply-connected eigenproblem by using the real-part or imaginary-part BEM; the other is for the multiply-connected eigenproblem even though the complex-valued BEM is utilized.

In this thesis, the spurious eigenequation for the simply and multiply-connected plate eigenproblems will be studied. Since any two boundary integral equations in the plate formulation (4 equations) can be chosen, 6 (C_2^4) options can be considered instead of only two approaches (single and double layer method, or singular and hypersingular equations) are adopted for the eigenproblems of the membrane and acoustic problems. The occurring mechanism of the spurious eigenequation for the simply and multiply-connected plate eigenproblems in each formulation will be studied analytically in the continuous and discrete systems. Also, we will provide three methods (SVD updating technique, the Burton & Miller method and the CHEEF (Combined Helmholtz Exterior integral Equation Formulation) or CHIEF (Combined Helmholtz Interior integral Equation Formulation) method to suppress the occurrence of the spurious eigenvalues for the free vibration of simply and multiply-connected plate problems.

1.2 Organization of the thesis

The frame of this thesis is shown in Figure 1-1. Two kinds of the eigenvalue problems, simply and multiply-connected cases are both considered. The occurring mechanism and its treatment for the spurious eigenvalues are covered.

In the Chapter 2, the eigenproblem for the simply-connected plate is solved by using the boundary element method. The true and spurious eigenequations for the simply-connected plate eigenproblem is derived by using the real and imaginary-part BEMs. Since any two boundary integral equations in the plate formulation (4 equations) can be chosen, 6 (C_2^4) options can be considered. The occurring mechanism of the spurious eigenequation for the plate eigenproblem in each formulation is studied analytically in the continuous and discrete systems. Three types of plates subject to clamped, simply-supported and free boundary conditions are illustrated to check the validity of the present formulations. Four alternatives (SVD updating technique, the Burton & Miller method, the complex-valued BEM and the CHEEF method) are adopted to suppress the occurrence of the spurious eigenvalues for the simply-connected problem in the Chapter 3. A clamped circular plate case is demonstrated analytically in the continuous and discrete systems. In the Chapter 4, the eigenproblem for the multiply-connected plate is solved by using the complex-valued BEM. Since any two boundary integral equations in the plate formulation (4 equations) can be chosen, 6 (C_2^4) options can be considered. The occurring mechanism of the spurious eigenequation for the multiply-connected plate eigenproblem in each formulation is studied analytically in the continuous and discrete systems. Three types of plates subject to C-C, S-S and F-F (C, S and F mean clamped, simply-supported and free boundary conditions, the first and second indices denote the outer and inner boundaries, respectively) are demonstrated analytically in the continuous and discrete systems. Nine numerical examples of plates subject to C-C, C-S, C-F, S-C, S-S, S-F, F-C, F-S and F-F are illustrated to check the validity of the present formulations. In the Chapter 5, three alternatives (SVD updating technique, the Burton & Miller method and the CHIEF method) are adopted to suppress the occurrence of the spurious eigenvalues for the multiply-connected problem when the complex-valued BEM is used. One clamped-clamped annular plate is demonstrated analytically in the discrete systems. Finally, we draw out some important conclusions items by item and reveal some further topics in the Chapter 6.

Chapter 2 Boundary element method for the free vibration of simply-connected plate

Summary

In this chapter, the eigenproblem for the simply-connected plate is solved by using the boundary element method. The true and spurious eigenequations for the plate eigenproblem are derived by using the real and imaginary-part BEMs. Since any two boundary integral equations in the plate formulation (4 equations) can be chosen, $6\ (C_2^4)$ options can be considered. The occurring mechanism of the spurious eigenequation in the plate eigenproblem in each formulation is studied analytically in the continuous and discrete systems. For the continuous system, degenerate kernels for the fundamental solution and the Fourier series expansion for the boundary densities are employed to derive the true and spurious eigenequations analytically for a circular plate. For the discrete system, the degenerate kernels for the fundamental solution and circulants resulting from the circular boundary are employed to determine the spurious eigenequation. Three types of plates subject to clamped, simply-supported and free boundary conditions are illustrated to check the validity of the present formulations.

2-1 Boundary integral equations for plate eigenproblems

The governing equation for free flexural vibration of a uniform thin plate is written as follows:

$$\nabla^4 u(x) = \lambda^4 u(x), \quad x \in \Omega, \tag{2-1}$$

where u is the lateral displacement, $\lambda^4 = \frac{\omega^2 \rho_0 h}{D}$, λ is the frequency parameter, ω is the circular frequency, ρ_0 is the surface density, D is the flexural rigidity expressed as $D = \frac{Eh^3}{12(1-\nu^2)}$ in terms of Young's modulus E, the Poisson ratio ν and the plate thickness h, and Ω is the domain of the thin plate. The integral equations for the domain point can be derived from the

Rayleigh-Green identity [54] as follows:

$$\begin{aligned} u(x) = &- \int_B U(s,x)v(s)\,dB(s) + \int_B \Theta(s,x)m(s)\,dB(s) \\ &- \int_B M(s,x)\theta(s)\,dB(s) + \int_B V(s,x)u(s)\,dB(s), \quad x \in \Omega, \end{aligned} \tag{2-2}$$

$$\begin{aligned} \theta(x) = &- \int_B U_\theta(s,x)v(s)\,dB(s) + \int_B \Theta_\theta(s,x)m(s)\,dB(s) \\ &- \int_B M_\theta(s,x)\theta(s)\,dB(s) + \int_B V_\theta(s,x)u(s)\,dB(s), \quad x \in \Omega, \end{aligned} \tag{2-3}$$

$$\begin{aligned} m(x) = &- \int_B U_m(s,x)v(s)\,dB(s) + \int_B \Theta_m(s,x)m(s)\,dB(s) \\ &- \int_B M_m(s,x)\theta(s)\,dB(s) + \int_B V_m(s,x)u(s)\,dB(s), \quad x \in \Omega, \end{aligned} \tag{2-4}$$

$$\begin{aligned} v(x) = &- \int_B U_v(s,x)v(s)\,dB(s) + \int_B \Theta_v(s,x)m(s)\,dB(s) \\ &- \int_B M_v(s,x)\theta(s)\,dB(s) + \int_B V_v(s,x)u(s)\,dB(s), \quad x \in \Omega, \end{aligned} \tag{2-5}$$

where B is the boundary, u, θ, m and v mean the displacement, slope, normal moment, effective shear force, s and x are the source and field points, respectively, U, Θ, M and V kernel functions will be elaborated on later. By moving the field point to the boundary, the Eqs.(2-2)-(2-5) reduce to

$$\begin{aligned} \alpha\, u(x) = &- P.V. \int_B U(s,x)v(s)\,dB(s) + P.V. \int_B \Theta(s,x)m(s)\,dB(s) \\ &- P.V. \int_B M(s,x)\theta(s)\,dB(s) + P.V. \int_B V(s,x)u(s)\,dB(s),\ x \in B, \end{aligned} \tag{2-6}$$

$$\begin{aligned} \alpha\, \theta(x) = &- P.V. \int_B U_\theta(s,x)v(s)\,dB(s) + P.V. \int_B \Theta_\theta(s,x)m(s)\,dB(s), \\ &- P.V. \int_B M_\theta(s,x)\theta(s)\,dB(s) + P.V. \int_B V_\theta(s,x)u(s)\,dB(s),\ x \in B, \end{aligned} \tag{2-7}$$

$$\begin{aligned} \alpha\, m(x) = &- P.V. \int_B U_m(s,x)v(s)\,dB(s) + P.V. \int_B \Theta_m(s,x)m(s)\,dB(s) \\ &- P.V. \int_B M_m(s,x)\theta(s)\,dB(s) + P.V. \int_B V_m(s,x)u(s)\,dB(s),\ x \in B, \end{aligned} \tag{2-8}$$

$$\begin{aligned} \alpha\, v(x) = &- P.V. \int_B U_v(s,x)v(s)\,dB(s) + P.V. \int_B \Theta_v(s,x)m(s)\,dB(s) \\ &- P.V. \int_B M_v(s,x)\theta(s)\,dB(s) + P.V. \int_B V_v(s,x)u(s)\,dB(s),\ x \in B, \end{aligned} \tag{2-9}$$

where $P.V.$ denotes the principal value, and $\alpha = \frac{1}{2}$ for a smooth boundary point. The kernel function $U(s,x)$ is the fundamental solution $U_c(s,x)$ which satisfies

$$\nabla^4 U_c(s,x) - \lambda^4 U_c(s,x) = -\delta(x-s). \tag{2-10}$$

where $\delta(x-s)$ is the Dirac-Delta function. Considering the two singular solutions ($Y_0(\lambda r)$ and $K_0(\lambda r)$, which are the zeroth-order of second kind Bessel and modified Bessel functions, respectively) [47] and two regular solutions ($J_0(\lambda r)$ and $I_0(\lambda r)$, which are the zeroth-order of first kind Bessel and modified Bessel functions, respectively) in the fundamental solution, we have

$$U_c(s,x) = \frac{1}{8\lambda^2}[(Y_0(\lambda r) + iJ_0(\lambda r)) + \frac{2}{\pi}(K_0(\lambda r) + iI_0(\lambda r))] \tag{2-11}$$

where $r \equiv |s-x|$ and $i^2 = -1$. The other three kernels, $\Theta(s,x)$, $M(s,x)$ and $V(s,x)$, are defined as follows:

$$\Theta(s,x) = \mathcal{K}_\theta(U(s,x)), \tag{2-12}$$

$$M(s,x) = \mathcal{K}_m(U(s,x)), \tag{2-13}$$

$$V(s,x) = \mathcal{K}_v(U(s,x)), \tag{2-14}$$

where $\mathcal{K}_\theta(\cdot)$, $\mathcal{K}_m(\cdot)$ and $\mathcal{K}_v(\cdot)$ mean the operators defined by

$$\mathcal{K}_\theta(\cdot) \equiv \frac{\partial(\cdot)}{\partial n}, \tag{2-15}$$

$$\mathcal{K}_m(\cdot) \equiv \nu\nabla^2(\cdot) + (1-\nu)\frac{\partial^2(\cdot)}{\partial n^2}, \tag{2-16}$$

$$\mathcal{K}_v(\cdot) \equiv \frac{\partial\nabla^2(\cdot)}{\partial n} + (1-\nu)\frac{\partial}{\partial t}[(\frac{\partial^2(\cdot)}{\partial n\partial t})], \tag{2-17}$$

where n and t are the normal vector and tangential vector, respectively. The operators $\mathcal{K}_\theta$, $\mathcal{K}_m$ and $\mathcal{K}_v$ can be applied to U, Θ, M and V kernels. The kernel functions can be expressed as:

$$\Theta(s,x) = \mathcal{K}_\theta(U(s,x)) = \frac{\partial U(s,x)}{\partial n_s}, \tag{2-18}$$

$$M(s,x) = \mathcal{K}_m(U(s,x)) = \nu\nabla_s^2 U(s,x) + (1-\nu)\frac{\partial^2 U(s,x)}{\partial n_s^2}, \tag{2-19}$$

$$V(s,x) = \mathcal{K}_v(U(s,x)) = \frac{\partial\nabla_s^2 U(s,x)}{\partial n_s} + (1-\nu)\frac{\partial}{\partial t_s}[(\frac{\partial^2 U(s,x)}{\partial n_s\partial t_s})]. \tag{2-20}$$

The displacement, slope, normal moment and effective shear force are derived by

$$\theta(x) = \mathcal{K}_\theta(u(x)), \tag{2-21}$$

$$m(x) = \mathcal{K}_m(u(x)), \tag{2-22}$$

$$v(x) = \mathcal{K}_v(u(x)). \tag{2-23}$$

Once the field point x locates outside the domain, the null-field BIEs of the direct method in the Eqs.(2-6)-(2-9) yield

$$\begin{aligned} 0 = &-\int_B U(s,x)v(s)\,dB(s) + \int_B \Theta(s,x)m(s)\,dB(s) \\ &-\int_B M(s,x)\theta(s)\,dB(s) + \int_B V(s,x)u(s)\,dB(s),\ x \in \Omega^e, \end{aligned} \tag{2-24}$$

$$\begin{aligned} 0 = &-\int_B U_\theta(s,x)v(s)\,dB(s) + \int_B \Theta_\theta(s,x)m(s)\,dB(s) \\ &-\int_B M_\theta(s,x)\theta(s)\,dB(s) + \int_B V_\theta(s,x)u(s)\,dB(s),\ x \in \Omega^e, \end{aligned} \tag{2-25}$$

$$\begin{aligned} 0 = &-\int_B U_m(s,x)v(s)\,dB(s) + \int_B \Theta_m(s,x)m(s)\,dB(s) \\ &-\int_B M_m(s,x)\theta(s)\,dB(s) + \int_B V_m(s,x)u(s)\,dB(s),\ x \in \Omega^e, \end{aligned} \tag{2-26}$$

$$\begin{aligned} 0 = &-\int_B U_v(s,x)v(s)\,dB(s) + \int_B \Theta_v(s,x)m(s)\,dB(s) \\ &-\int_B M_v(s,x)\theta(s)\,dB(s) + \int_B V_v(s,x)u(s)\,dB(s),\ x \in \Omega^e, \end{aligned} \tag{2-27}$$

where Ω^e is the complementary domain. Note that the null-field BIEs are not singular, since x and s never coincide. When the boundary is discretized into $2N$ constant elements, the linear algebraic equations of the Eqs.(2-24)-(2-27) by moving the field point x close to the boundary B can be obtained as follows:

$$0 = [U]\{v\} + [\Theta]\{m\} + [M]\{\theta\} + [V]\{u\}, \tag{2-28}$$

$$0 = [U_\theta]\{v\} + [\Theta_\theta]\{m\} + [M_\theta]\{\theta\} + [V_\theta]\{u\}, \tag{2-29}$$

$$0 = [U_m]\{v\} + [\Theta_m]\{m\} + [M_m]\{\theta\} + [V_m]\{u\}, \tag{2-30}$$

$$0 = [U_v]\{v\} + [\Theta_v]\{m\} + [M_v]\{\theta\} + [V_v]\{u\}, \tag{2-31}$$

where $[U]$, $[\Theta]$, $[M]$, $[V]$, $[U_\theta]$, $[\Theta_\theta]$, $[M_\theta]$, $[V_\theta]$, $[U_m]$, $[\Theta_m]$, $[M_m]$, $[V_m]$, $[U_v]$, $[\Theta_v]$, $[M_v]$ and $[V_v]$ are the sixteen influence matrices with a dimension of $2N$ by $2N$, $\{u\}$, $\{\theta\}$, $\{m\}$ and

$\{v\}$ are the vectors of boundary data with a dimension of $2N$ by one. The real or imaginary-part BEM for solving the eigenfrequencies of plates is proposed for saving half effort in computation instead of using the complex-valued BEM. Imaginary-part BEM can also avoid the singular integrals. By employing the real or imaginary-part fundamental solution, the spurious eigenequations in conjunction with the true eigenequation are obtained for the free vibration of plate. To verify this finding, the real-part and imaginary-part BEMs for solving the eigenproblem of simply-connected plate are demonstrated analytically for the continuous and discrete systems in the following subsections.

2-2 Mathematical analysis using the real-part BEM

For the real-part BEM, the kernel function $U(s,x)$ is the real-part of the fundamental solution

$$U(s,x) = Re[U_c(s,x)] = \frac{1}{8\lambda^2}[Y_0(\lambda r) + \frac{2}{\pi}K_0(\lambda r)]. \tag{2-32}$$

In order to obtain the true and spurious eigenequations for plate vibration in the real-part BEM, the degenerate kernel is adopted to analytically derive the true and spurious eigenequations in the continuous and discrete systems of a circular plate. For the continuous system, the spurious eigenequation is derived by using the degenerate kernel and Fourier series. For the discrete system, mathematical analysis for the spurious eigenequation is done by using the degenerate kernel and circulants. Three cases (clamped, simply-supported and free plates) are demonstrated analytically in the continuous and the discrete systems, respectively, in the following subsections.

2-2-1 Continuous system

Case 1. Clamped circular plate

For the clamped circular plate ($u = 0$ and $\theta = 0$) with a radius a, we can obtain the eigenequation in the continuous formulation. The moment and shear force, $m(s)$ and $v(s)$ along the circular boundary, can be expanded into Fourier series by

$$m(s)=\sum_{n=0}^{\infty}(p_n^c cos(n\overline{\phi})+q_n^c sin(n\overline{\phi})),\quad s\in B, \tag{2-33}$$

$$v(s)=\sum_{n=0}^{\infty}(a_n^c cos(n\overline{\phi})+b_n^c sin(n\overline{\phi})),\quad s\in B, \tag{2-34}$$

where the superscript " c " denotes the clamped case, $\overline{\phi}$ is the angle on the circular boundary, a_n^c, b_n^c, p_n^c and q_n^c are the undetermined Fourier coefficients. Substituting the Eqs.(2-33) and (2-34) into the Eqs.(2-24) and (2-25), we have

$$\begin{aligned}0=&-\int_0^{2\pi}U(s,x)[\sum_{n=0}^{\infty}(a_n^c cos(n\overline{\phi})+b_n^c sin(n\overline{\phi}))]\,dB\\&+\int_0^{2\pi}\Theta(s,x)[\sum_{n=0}^{\infty}(p_n^c cos(n\overline{\phi})+q_n^c sin(n\overline{\phi}))]\,dB,\ x\ \in\ B,\end{aligned} \tag{2-35}$$

$$\begin{aligned}0=&-\int_0^{2\pi}U_\theta(s,x)[\sum_{n=0}^{\infty}(a_n^c cos(n\overline{\phi})+b_n^c sin(n\overline{\phi}))]\,dB\\&+\int_0^{2\pi}\Theta_\theta(s,x)[\sum_{n=0}^{\infty}(p_n^c cos(n\overline{\phi})+q_n^c sin(n\overline{\phi}))]\,dB,\ x\ \in\ B.\end{aligned} \tag{2-36}$$

The kernel functions, $U(s,x)$, $\Theta(s,x)$, $U_\theta(s,x)$ and $\Theta_\theta(s,x)$, can be expanded by using the expansion formulae,

$$Y_0(\lambda r)=\begin{cases}Y_0^i(\lambda r)=\sum_{m=-\infty}^{\infty}Y_m(\lambda\overline{\rho})J_m(\lambda\rho)\cos(m(\overline{\phi}-\phi)),\ \ \overline{\rho}>\rho,\\ Y_0^e(\lambda r)=\sum_{m=-\infty}^{\infty}Y_m(\lambda\rho)J_m(\lambda\overline{\rho})\cos(m(\overline{\phi}-\phi)),\ \ \rho>\overline{\rho},\end{cases} \tag{2-37}$$

$$K_0(\lambda r)=\begin{cases}K_0^i(\lambda r)=\sum_{m=-\infty}^{\infty}K_m(\lambda\overline{\rho})I_m(\lambda\rho)\cos(m(\overline{\phi}-\phi)),\ \ \overline{\rho}>\rho,\\ K_0^e(\lambda r)=\sum_{m=-\infty}^{\infty}K_m(\lambda\rho)I_m(\lambda\overline{\rho})\cos(m(\overline{\phi}-\phi)),\ \ \rho>\overline{\rho},\end{cases} \tag{2-38}$$

where J_m and I_m denote the first kind of the *mth*-order Bessel and modified Bessel functions, Y_m and K_m denote the second kind of the *mth*-order Bessel and modified Bessel functions. The superscripts " i " and " e " denote the interior point ($\overline{\rho}>\rho$) and the exterior point ($\overline{\rho}<\rho$), $s=(\overline{\rho},\overline{\phi})$ and $x=(\rho,\phi)$ are the polar coordinates of s and x, respectively. In this case, $\overline{\rho}=\rho=a$ and $dB(s)=a\,d\overline{\phi}$ for the circular plate with a radius a. Similarly, the other kernels can also be expanded into degenerate forms. By using the degenerate kernels (all the complex-valued degenerate kernels can be found in the Appendix 1) into the Eq.(2-35) and by employing the orthogonality condition of the Fourier series, the Fourier coefficients a_n^c, b_n^c, p_n^c and q_n^c satisfy

$$p_n^c = \frac{1}{\lambda}\frac{[Y_n(\lambda a)J_n(\lambda a) + \frac{2}{\pi}K_n(\lambda a)I_n(\lambda a)]}{[Y_n(\lambda a)J_n'(\lambda a) + \frac{2}{\pi}K_n(\lambda a)I_n'(\lambda a)]}a_n^c, \qquad n = 0, 1, 2, \ldots, \tag{2-39}$$

$$q_n^c = \frac{1}{\lambda}\frac{[Y_n(\lambda a)J_n(\lambda a) + \frac{2}{\pi}K_n(\lambda a)I_n(\lambda a)]}{[Y_n(\lambda a)J_n'(\lambda a) + \frac{2}{\pi}K_n(\lambda a)I_n'(\lambda a)]}b_n^c, \qquad n = 0, 1, 2, \ldots. \tag{2-40}$$

Similarly, the Eq.(2-36) yields,

$$p_n^c = \frac{1}{\lambda}\frac{[Y_n'(\lambda a)J_n(\lambda a) + \frac{2}{\pi}K_n'(\lambda a)I_n(\lambda a)]}{[Y_n'(\lambda a)J_n'(\lambda a) + \frac{2}{\pi}K_n'(\lambda a)I_n'(\lambda a)]}a_n^c, \qquad n = 0, 1, 2, \ldots, \tag{2-41}$$

$$q_n^c = \frac{1}{\lambda}\frac{[Y_n'(\lambda a)J_n(\lambda a) + \frac{2}{\pi}K_n'(\lambda a)I_n(\lambda a)]}{[Y_n'(\lambda a)J_n'(\lambda a) + \frac{2}{\pi}K_n'(\lambda a)I_n'(\lambda a)]}b_n^c, \qquad n = 0, 1, 2, \ldots. \tag{2-42}$$

To seek nontrivial data for the generalized coefficients of a_n^c, p_n^c, b_n^c and q_n^c, we can obtain the eigenequation by using either the Eqs.(2-39) and (2-41) or the Eqs.(2-40) and (2-42)

$$\frac{Y_n(\lambda a)J_n(\lambda a) + \frac{2}{\pi}K_n(\lambda a)I_n(\lambda a)}{Y_n(\lambda a)J_n'(\lambda a) + \frac{2}{\pi}K_n(\lambda a)I_n'(\lambda a)} = \frac{Y_n'(\lambda a)J_n(\lambda a) + \frac{2}{\pi}K_n'(\lambda a)I_n(\lambda a)}{Y_n'(\lambda a)J_n'(\lambda a) + \frac{2}{\pi}K_n'(\lambda a)I_n'(\lambda a)} \tag{2-43}$$

After recollecting the terms by using the recurrence relations of the Bessel function in the Appendix 2, the Eq.(2-43) can be simplified to

$$[K_{n+1}(\lambda a)Y_n(\lambda a) - Y_{n+1}(\lambda a)K_n(\lambda a)]\{I_{n+1}(\lambda a)J_n(\lambda a) + J_{n+1}(\lambda a)I_n(\lambda a)\} = 0 \tag{2-44}$$

The former part in the Eq.(2-44) inside the middle bracket is the spurious eigenequation while the latter part inside the big bracket is found to be the true eigenequation after comparing with the exact eigenequation [58]. All the eigenequations for the simply-connected circular plate in Leissa's book can be found for comparison in the Appendix 3.

Case 2. Simply-supported circular plate

For the simply-supported circular plate ($u = 0$ and $m = 0$) with a radius a, we can obtain the eigenequation in the continuous formulation. Similarly, the moment and shear force, $\theta(s)$ and $v(s)$ along the circular boundary, can be expanded into Fourier series by

$$\theta(s) = \sum_{n=0}^{\infty}(p_n^s cos(n\overline{\phi}) + q_n^s sin(n\overline{\phi})), \quad s \in B, \tag{2-45}$$

$$v(s) = \sum_{n=0}^{\infty}(a_n^s cos(n\overline{\phi}) + b_n^s sin(n\overline{\phi})), \quad s \in B, \tag{2-46}$$

where the superscript " s " denotes the simply-supported case, $\overline{\phi}$ is the angle on the circular boundary, a_n^s, b_n^s, p_n^s and q_n^s are the undetermined Fourier coefficients. Substituting the Eqs.(2-45) and (2-46) and using the degenerate kernels of $U(s,x)$, $M(s,x)$, $U_\theta(s,x)$ and $M_\theta(s,x)$ into the Eq.(2-24), we have

$$p_n^s = -\frac{[Y_n(\lambda a)J_n(\lambda a) + \frac{2}{\pi}K_n(\lambda a)I_n(\lambda a)]}{[Y_n(\lambda a)\alpha_n^J(\lambda a) + \frac{2}{\pi}K_n(\lambda a)\alpha_n^I(\lambda a)]}a_n^s, \qquad n = 0, 1, 2, \ldots, \tag{2-47}$$

$$q_n^s = -\frac{[Y_n(\lambda a)J_n(\lambda a) + \frac{2}{\pi}K_n(\lambda a)I_n(\lambda a)]}{[Y_n(\lambda a)\alpha_n^J(\lambda a) + \frac{2}{\pi}K_n(\lambda a)\alpha_n^I(\lambda a)]}b_n^s, \qquad n = 0, 1, 2, \ldots. \tag{2-48}$$

where

$$\alpha_n^J(\lambda a) = \lambda^2 J_n''(\lambda a) + \nu[\frac{1}{a}\lambda J_n'(\lambda a) - (\frac{n}{a})^2 J_n(\lambda a)], \tag{2-49}$$

$$\alpha_n^I(\lambda a) = \lambda^2 I_n''(\lambda a) + \nu[\frac{1}{a}\lambda I_n'(\lambda a) - (\frac{n}{a})^2 I_n(\lambda a)]. \tag{2-50}$$

Similarly, the Eq.(2-25) yields,

$$p_n^s = -\frac{[Y_n'(\lambda a)J_n(\lambda a) + \frac{2}{\pi}K_n'(\lambda a)I_n(\lambda a)]}{[Y_n'(\lambda a)\alpha_n^J(\lambda a) + \frac{2}{\pi}K_n'(\lambda a)\alpha_n^I(\lambda a)]}a_n^s, \qquad n = 0, 1, 2, \ldots, \tag{2-51}$$

$$q_n^s = -\frac{[Y_n'(\lambda a)J_n(\lambda a) + \frac{2}{\pi}K_n'(\lambda a)I_n(\lambda a)]}{[Y_n'(\lambda a)\alpha_n^J(\lambda a) + \frac{2}{\pi}K_n'(\lambda a)\alpha_n^I(\lambda a)]}b_n^s, \qquad n = 0, 1, 2, \ldots. \tag{2-52}$$

To seek nontrivial data for the generalized coefficients of a_n^s, p_n^s, b_n^s and q_n^s, we can obtain the eigenequations as

$$\frac{Y_n(\lambda a)J_n(\lambda a) + \frac{2}{\pi}K_n(\lambda a)I_n(\lambda a)}{Y_n(\lambda a)\alpha_n^J(\lambda a) + \frac{2}{\pi}K_n(\lambda a)\alpha_n^I(\lambda a)} = \frac{Y_n'(\lambda a)J_n(\lambda a) + \frac{2}{\pi}K_n'(\lambda a)I_n(\lambda a)}{Y_n'(\lambda a)\alpha_n^J(\lambda a) + \frac{2}{\pi}K_n'(\lambda a)\alpha_n^I(\lambda a)} \tag{2-53}$$

After recollecting the terms by using the recurrence relations of the Bessel function in the Appendix 2, the Eq.(2-53) can be simplified to

$$\begin{aligned} &[K_{n+1}(\lambda a)Y_n(\lambda a) - Y_{n+1}(\lambda a)K_n(\lambda a)] \\ &\{(1-\nu)I_n(\lambda a)J_{n+1}(\lambda a) + I_{n+1}(\lambda a)J_n(\lambda a) - 2\lambda a I_n(\lambda a)J_n(\lambda a)\} = 0 \end{aligned} \tag{2-54}$$

The former part in the Eq.(2-54) inside the middle bracket is the spurious eigenequation while the latter part inside the big bracket is found to be the true eigenequation after comparing with the exact eigenequation [58].

Case 3. Free circular plate

For the free circular plate ($m = 0$ and $v = 0$) with a radius a, we can obtain the eigenequation in the continuous formulation. Similarly, the displacement and slope, $u(s)$ and $\theta(s)$ along the circular boundary, can be expanded into Fourier series by

$$u(s) = \sum_{n=0}^{\infty}(p_n^f cos(n\overline{\phi}) + q_n^f sin(n\overline{\phi})), \quad s \in B, \tag{2-55}$$

$$\theta(s) = \sum_{n=0}^{\infty}(a_n^f cos(n\overline{\phi}) + b_n^f sin(n\overline{\phi})), \quad s \in B, \tag{2-56}$$

where the superscript " f " denotes the free case, $\overline{\phi}$ is the angle on the circular boundary, a_n^f, b_n^f, p_n^f and q_n^f are the undetermined Fourier coefficients. Substituting the Eqs.(2-55) and (2-56) and using the degenerate kernels of $M(s,x)$, $V(s,x)$, $M_\theta(s,x)$ and $V_\theta(s,x)$ into the Eq.(2-24), we have

$$p_n^f = -\frac{[Y_n(\lambda a)\alpha_n^J(\lambda a) + \frac{2}{\pi}K_n(\lambda a)\alpha_n^I(\lambda a)]}{[Y_n(\lambda a)\beta_n^J(\lambda a) + \frac{2}{\pi}K_n(\lambda a)\beta_n^I(\lambda a) + \frac{1-\nu}{a}[Y_n(\lambda a)\gamma_n^J(\lambda a) + \frac{2}{\pi}K_n(\lambda a)\gamma_n^I(\lambda a)]]}a_n^f, \quad n = 0, 1, 2, \ldots, \tag{2-57}$$

$$q_n^f = -\frac{[Y_n(\lambda a)\alpha_n^J(\lambda a) + \frac{2}{\pi}K_n(\lambda a)\alpha_n^I(\lambda a)]}{[Y_n(\lambda a)\beta_n^J(\lambda a) + \frac{2}{\pi}K_n(\lambda a)\beta_n^I(\lambda a) + \frac{1-\nu}{a}[Y_n(\lambda a)\gamma_n^J(\lambda a) + \frac{2}{\pi}K_n(\lambda a)\gamma_n^I(\lambda a)]]}b_n^f, \quad n = 0, 1, 2, \ldots. \tag{2-58}$$

where

$$\beta_n^J(\lambda a) = \lambda^3 J_n'''(\lambda a) + \nu[\frac{1}{a}\lambda^2 J_n''(\lambda a) - (\frac{n}{a})^2\lambda J_n'(\lambda a) - \frac{1}{a^2}\lambda J_n'(\lambda a) + (\frac{2n^2}{a^3})J_n(\lambda a)], \tag{2-59}$$

$$\beta_n^I(\lambda a) = \lambda^3 I_n'''(\lambda a) + \nu[\frac{1}{a}\lambda^2 I_n''(\lambda a) - (\frac{n}{a})^2\lambda I_n'(\lambda a) - \frac{1}{a^2}\lambda I_n'(\lambda a) + (\frac{2n^2}{a^3})I_n(\lambda a)], \tag{2-60}$$

$$\gamma_n^J(\lambda a) = -n^2[\frac{1}{a^2}J_n(\lambda a) + \frac{\lambda}{a}J'(\lambda a)], \tag{2-61}$$

$$\gamma_n^I(\lambda a) = -n^2[\frac{1}{a^2}I_n(\lambda a) + \frac{\lambda}{a}I'(\lambda a)]. \tag{2-62}$$

Similarly, the Eq.(2-25) yields,

$$p_n^f = -\frac{[Y_n'(\lambda a)J_n(\lambda a) + \frac{2}{\pi}K_n'(\lambda a)I_n(\lambda a)]}{[Y_n'(\lambda a)\beta_n^J(\lambda a) + \frac{2}{\pi}K_n'(\lambda a)\beta_n^I(\lambda a) + \frac{1-\nu}{a}[Y_n'(\lambda a)\gamma_n^J(\lambda a) + \frac{2}{\pi}K_n'(\lambda a)\gamma_n^I(\lambda a)]]}a_n^f, \quad n = 0, 1, 2, \ldots, \tag{2-63}$$

$$q_n^f = -\frac{[Y_n'(\lambda a)J_n(\lambda a) + \frac{2}{\pi}K_n'(\lambda a)I_n(\lambda a)]}{[Y_n'(\lambda a)\beta_n^J(\lambda a) + \frac{2}{\pi}K_n'(\lambda a)\beta_n^I(\lambda a) + \frac{1-\nu}{a}[Y_n'(\lambda a)\gamma_n^J(\lambda a) + \frac{2}{\pi}K_n'(\lambda a)\gamma_n^I(\lambda a)]]}b_n^f, \quad n = 0, 1, 2, \ldots. \tag{2-64}$$

To seek nontrivial data for the generalized coefficients of a_n^f, p_n^f, b_n^f and q_n^f, we can obtain the eigenequation

$$\begin{aligned} &\frac{[Y_n(\lambda a)\alpha_n^J(\lambda a) + \frac{2}{\pi}K_n(\lambda a)\alpha_n^I(\lambda a)]}{[Y_n(\lambda a)\beta_n^J(\lambda a) + \frac{2}{\pi}K_n(\lambda a)\beta_n^I(\lambda a) + \frac{1-\nu}{a}[Y_n(\lambda a)\gamma_n^J(\lambda a) + \frac{2}{\pi}K_n(\lambda a)\gamma_n^I(\lambda a)]]} \\ =\ &\frac{[Y_n'(\lambda a)J_n(\lambda a) + \frac{2}{\pi}K_n'(\lambda a)I_n(\lambda a)]}{[Y_n'(\lambda a)\beta_n^J(\lambda a) + \frac{2}{\pi}K_n'(\lambda a)\beta_n^I(\lambda a) + \frac{1-\nu}{a}[Y_n'(\lambda a)\gamma_n^J(\lambda a) + \frac{2}{\pi}K_n'(\lambda a)\gamma_n^I(\lambda a)]]} \end{aligned} \tag{2-65}$$

After recollecting the terms by using the recurrence relations of the Bessel function in the Appendix 2, the Eq.(2-65) can be simplified to

$$\begin{aligned} &[K_{n+1}(\lambda a)Y_n(\lambda a) - Y_{n+1}(\lambda a)K_n(\lambda a)] \\ &\{\lambda a(1-\nu)[-4n^2(n-1)I_n(\lambda a)J_n(\lambda a) - 2\lambda^2a^2I_{n+1}(\lambda a)J_{n+1}(\lambda a)] \\ &+ 2n\lambda^2a^2(1-\nu)(1-n)(I_{n+1}(\lambda a)J_n(\lambda a) - I_n(\lambda a)J_{n+1}(\lambda a)) \\ &+ [n^2(1-\nu)^2(n^2-1) + \lambda^4a^4](I_{n+1}(\lambda a)J_n(\lambda a) + I_n(\lambda a)J_{n+1}(\lambda a))\} = 0 \end{aligned} \tag{2-66}$$

The former part in the Eq.(2-66) inside the middle bracket is the spurious eigenequation which also appears in the clamped and simply-supported cases as shown in the Eqs.(2-44), (2-54) and (2-66). It indicates that the spurious eigenequations of the Eqs.(2-44), (2-54) and (2-66) are the same since the same formulation (null-field formulation of the Eqs.(2-24) and (2-25)) is used. This reveals that spurious eigenequation depends on the formulation instead of the specified boundary condition. It is noted that the true eigenequation of free plate does not agree with that of the Leissa's result [58]. However, the same true eigenvalues are obtained numerically between the present and Leissa's results. After finding the eigenvalues according to the Leissa's eigenequation, the eigenvalues are not consistent to the data in his book. After careful check, the eigenequation in the Leissa's book was a misprint where the index in the numerator of the right hand side of the equation should be J instead of I [42]. The eigenequation in the bigger bracket of the Eq.(2-66) can be simplified to be equivalent to the Leissa's result as shown in the Table 2-1.

2-2-2 Discrete system

Case 1. Clamped circular plate

For the clamped circular plate ($u = 0$ and $\theta = 0$) with a radius a, the Eqs.(2-28) and (2-29) can be rewritten as

$$\{0\} = [U]\{v\} + [\Theta]\{m\}, \tag{2-67}$$

$$\{0\} = [U_\theta]\{v\} + [\Theta_\theta]\{m\}, \tag{2-68}$$

By assembling the Eqs.(2-67) and (2-68) together, we have

$$[SM^c]\left\{\begin{array}{c} v \\ m \end{array}\right\} = \{0\}, \tag{2-69}$$

where the superscript " c " denotes the clamped case and

$$[SM^c] = \left[\begin{array}{cc} U & \Theta \\ U_\theta & \Theta_\theta \end{array}\right]_{4N\times 4N}. \tag{2-70}$$

For the existence of nontrivial solution of $\left\{\begin{array}{c} v \\ m \end{array}\right\}$, the determinant of the matrix versus eigenvalue must be zero, i.e.,

$$det[SM^c] = 0. \tag{2-71}$$

Since the rotation symmetry is preserved for a circular boundary, the influence matrices for the discrete system are found to be the circulants with the following forms into the Eq.(2-67), we have

$$[U] = \begin{bmatrix} z_0 & z_1 & z_2 & \cdots & z_{2N-1} \\ z_{2N-1} & z_0 & z_1 & \cdots & z_{2N-2} \\ z_{2N-2} & z_{2N-1} & z_0 & \cdots & z_{2N-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ z_1 & z_2 & z_3 & \cdots & z_0 \end{bmatrix}_{2N\times 2N} \tag{2-72}$$

The coefficients of each element can be obtained by using degenerate kernel

$$z_m = \int_{(m-\frac{1}{2})\Delta\overline{\phi}}^{(m+\frac{1}{2})\Delta\overline{\phi}} [-U(a,\overline{\phi};a,\phi)]\, ad\overline{\phi} \approx [-U(a,\overline{\phi}_m;a,\phi)]\, a\Delta\overline{\phi}, \quad m = 0,1,2,\cdots,2N-1 \tag{2-73}$$

where $\Delta\overline{\phi} = \frac{2\pi}{2N}$, $\overline{\phi}_m = m\Delta\overline{\phi}$. By introducing the following bases for the circulants, $[I]$, $([C_{2N}])^1$, $([C_{2N}])^2$, $\cdots$, $([C_{2N}])^{2N-1}$, we can expand matrix $[U]$ into

$$[U] = z_0[I] + z_1([C_{2N}])^1 + z_2([C_{2N}])^2 + \cdots + z_{2N-1}([C_{2N}])^{2N-1}, \tag{2-74}$$

where

$$[C_{2N}] = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 0 \end{bmatrix}_{2N\times 2N} . \tag{2-75}$$

Based on the similar properties for the matrices of $[U]$ and $[C_{2N}]$, we have

$$\mu_\ell^{[U]} = z_0 + z_1\alpha_\ell + z_2\alpha_\ell^2 + \cdots + z_{2N-1}\alpha_\ell^{2N-1}, \quad \ell = 0,1,2,\cdots,2N-1. \tag{2-76}$$

where $\mu_\ell^{[U]}$ and α_ℓ are the eigenvalues for $[U]$ and $[C_{2N}]$, respectively. It is easily found that the eigenvalues for the circulants $[C_{2N}]$, are the roots for $\alpha^{2N} = 1$ as shown below:

$$\alpha_\ell = e^{i\frac{2\pi\ell}{2N}}, \quad \ell = 0, \pm1, \pm2, \cdots, \pm(N-1), N \text{ or } \ell = 0,1,2,\cdots,2N-1. \tag{2-77}$$

The eigenvector for the circulant $[C_{2N}]$ is

$$\{\phi_\ell\} = \begin{Bmatrix} 1 \\ \alpha_\ell \\ \alpha_\ell^2 \\ \vdots \\ \alpha_\ell^{2N-1} \end{Bmatrix}_{2N\times 1} . \tag{2-78}$$

Substituting the Eq.(2-77) into the Eq.(2-76), we have

$$\mu_\ell^{[U]} = \sum_{m=0}^{2N-1} z_m \alpha_\ell^m = \sum_{m=0}^{2N-1} z_m e^{i\frac{2\pi}{2N}m\ell}, \quad \ell = 0, \pm 1, \pm 2, \cdots, \pm(N-1), N. \tag{2-79}$$

According to the definition for z_m in the Eq.(2-73), we have

$$z_m = z_{2N-m}, \quad m = 0, 1, 2, \cdots, 2N-1. \tag{2-80}$$

Substitution of the Eq.(2-80) into the Eq.(2-79) yields

$$\mu_\ell^{[U]} = z_0 + (-1)^\ell z_N + \sum_{m=1}^{N-1} (\alpha_\ell^m + \alpha_\ell^{2N-m}) z_m = \sum_{m=0}^{2N-1} \cos(m\ell\Delta\overline{\phi}) z_m. \tag{2-81}$$

Substituting the Eq.(2-73) into the Eq.(2-81) for $\phi = 0$ without loss of generality, the Reimann sum of infinite terms reduces to the following integral

$$\mu_\ell^{[U]} = \lim_{N\to\infty} \sum_{m=0}^{2N-1} \cos(m\ell\Delta\overline{\phi})[-U(a, \overline{\phi}_m; a, 0)] \approx \int_0^{2\pi} \cos(\ell\overline{\phi})[-U(a, \overline{\phi}_m; a, 0)]\, a d\overline{\phi}, \tag{2-82}$$

By using the degenerate kernel for $U(s, x)$ and the orthogonal conditions, the Eq.(2-82) reduces to

$$\mu_\ell^{[U]} = -\frac{\pi a}{4\lambda^2}[Y_\ell(\lambda a) J_\ell(\lambda a) + \frac{2}{\pi} K_\ell(\lambda a) I_\ell(\lambda a)], \quad \ell = 0, \pm 1, \pm 2, \cdots, \pm(N-1), N. \tag{2-83}$$

Similarly, we have

$$\mu_\ell^{[\Theta]} = \frac{\pi a}{4\lambda}[Y_\ell(\lambda a) J'_\ell(\lambda a) + \frac{2}{\pi} K_\ell(\lambda a) I'_\ell(\lambda a)], \quad \ell = 0, \pm 1, \pm 2, \cdots, \pm(N-1), N. \tag{2-84}$$

$$\kappa_\ell^{[U]} = -\frac{\pi a}{4\lambda}[Y'_\ell(\lambda a) J_\ell(\lambda a) + \frac{2}{\pi} K'_\ell(\lambda a) I_\ell(\lambda a)], \quad \ell = 0, \pm 1, \pm 2, \cdots, \pm(N-1), N. \tag{2-85}$$

$$\kappa_\ell^{[\Theta]} = \frac{\pi a}{4}[Y'_\ell(\lambda a) J'_\ell(\lambda a) + \frac{2}{\pi} K'_\ell(\lambda a) I'_\ell(\lambda a)], \quad \ell = 0, \pm 1, \pm 2, \cdots, \pm(N-1), N. \tag{2-86}$$

where $\mu_\ell^{[\Theta]}$, $\kappa_\ell^{[U]}$ and $\kappa_\ell^{[\Theta]}$ are the eigenvalues of $[\Theta]$, $[U_\theta]$ and $[\Theta_\theta]$ matrices, respectively. Since the four matrices $[U]$, $[\Theta]$, $[U_\theta]$ and $[\Theta_\theta]$ are all symmetric circulants, they can be expressed by

$$[U] = \Phi \Sigma_U \Phi^{-1} = \Phi \begin{bmatrix} \mu_0^{[U]} & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \mu_1^{[U]} & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \mu_{-1}^{[U]} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \mu_{(N-1)}^{[U]} & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & \mu_{-(N-1)}^{[U]} & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & \mu_N^{[U]} \end{bmatrix}_{2N \times 2N} \Phi^{-1}, \tag{2-87}$$

$$[\Theta] = \Phi \Sigma_\Theta \Phi^{-1} = \Phi \begin{bmatrix} \mu_0^{[\Theta]} & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \mu_1^{[\Theta]} & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \mu_{-1}^{[\Theta]} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \mu_{(N-1)}^{[\Theta]} & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & \mu_{-(N-1)}^{[\Theta]} & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & \mu_N^{[\Theta]} \end{bmatrix}_{2N \times 2N} \Phi^{-1}, \tag{2-88}$$

$$[U_\theta] = \Phi \Sigma_{U_\theta} \Phi^{-1} = \Phi \begin{bmatrix} \kappa_0^{[U]} & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \kappa_1^{[U]} & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \kappa_{-1}^{[U]} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \kappa_{(N-1)}^{[U]} & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & \kappa_{-(N-1)}^{[U]} & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & \kappa_N^{[U]} \end{bmatrix}_{2N \times 2N} \Phi^{-1}, \tag{2-89}$$

$$
[\Theta_\theta] = \Phi \Sigma_{\Theta_\theta} \Phi^{-1} = \Phi \begin{bmatrix} \kappa_0^{[\Theta]} & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \kappa_1^{[\Theta]} & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \kappa_{-1}^{[\Theta]} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \kappa_{(N-1)}^{[\Theta]} & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & \kappa_{-(N-1)}^{[\Theta]} & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & \kappa_N^{[\Theta]} \end{bmatrix}_{2N \times 2N} \Phi^{-1}, \tag{2-90}
$$

where

$$
\Phi = \frac{1}{\sqrt{2N}} \begin{bmatrix} 1 & 1 & 0 & \cdots & 1 & 0 & 1 \\ 1 & \cos(\frac{2\pi}{2N}) & \sin(\frac{2\pi}{2N}) & \cdots & \cos(\frac{2\pi(N-1)}{2N}) & \sin(\frac{2\pi(N-1)}{2N}) & \cos(\frac{2\pi N}{2N}) \\ 1 & \cos(\frac{4\pi}{2N}) & \sin(\frac{4\pi}{2N}) & \cdots & \cos(\frac{4\pi(N-1)}{2N}) & \sin(\frac{4\pi(N-1)}{2N}) & \cos(\frac{4\pi N}{2N}) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & \cos(\frac{2\pi(2N-2)}{2N}) & \sin(\frac{2\pi(2N-2)}{2N}) & \cdots & \cos(\frac{\pi(4N-4)(N-1)}{2N}) & \sin(\frac{\pi(4N-4)(N-1)}{2N}) & \cos(\frac{\pi(4N-4)(N)}{2N}) \\ 1 & \cos(\frac{2\pi(2N-1)}{2N}) & \sin(\frac{2\pi(2N-1)}{2N}) & \cdots & \cos(\frac{\pi(4N-2)(N-1)}{2N}) & \sin(\frac{\pi(4N-2)(N-1)}{2N}) & \cos(\frac{\pi(4N-2)(N)}{2N}) \end{bmatrix}_{2N \times 2N}. \tag{2-91}
$$

By employing the Eqs.(2-87)-(2-90) for the Eq.(2-70), we have

$$
[SM^c] = \begin{bmatrix} \Phi \Sigma_U \Phi^{-1} & \Phi \Sigma_\Theta \Phi^{-1} \\ \Phi \Sigma_{U_\theta} \Phi^{-1} & \Phi \Sigma_{\Theta_\theta} \Phi^{-1} \end{bmatrix}_{4N \times 4N}, \tag{2-92}
$$

the Eq.(2-92) can be reformulated into

$$
[SM^c] = \begin{bmatrix} \Phi & 0 \\ 0 & \Phi \end{bmatrix} \begin{bmatrix} \Sigma_U & \Sigma_\Theta \\ \Sigma_{U_\theta} & \Sigma_{\Theta_\theta} \end{bmatrix} \begin{bmatrix} \Phi & 0 \\ 0 & \Phi \end{bmatrix}^{-1}. \tag{2-93}
$$

By using the property of the determinant in the Appendix 4, the determinant of $[SM^c]_{4N \times 4N}$ is

$$
det[SM^c] = det \begin{bmatrix} \Sigma_U & \Sigma_\Theta \\ \Sigma_{U_\theta} & \Sigma_{\Theta_\theta} \end{bmatrix} = \prod_{\ell=-(N-1)}^{N} (\mu_\ell^{[U]} \kappa_\ell^{[\Theta]} - \mu_\ell^{[\Theta]} \kappa_\ell^{[U]}), \tag{2-94}
$$

since Φ is orthogonal. By employing the Eqs.(2-83)-(2-86) for the Eq.(2-94), we have

$$
\begin{aligned}
det[SM^c] = \prod_{\ell=-(N-1)}^{N} & \frac{\pi^2 a^2}{16\lambda^2} \\
& \{[Y_\ell(\lambda a) J_\ell(\lambda a) + \frac{2}{\pi} K_\ell(\lambda a) I_\ell(\lambda a)][Y_\ell'(\lambda a) J_\ell'(\lambda a) + \frac{2}{\pi} K_\ell'(\lambda a) I_\ell'(\lambda a)] \\
& - [Y_\ell(\lambda a) J_\ell'(\lambda a) + \frac{2}{\pi} K_\ell(\lambda a) I_\ell'(\lambda a)][Y_\ell'(\lambda a) J_\ell(\lambda a) + \frac{2}{\pi} K_\ell'(\lambda a) I_\ell(\lambda a)]\}
\end{aligned} \tag{2-95}
$$

By using the recurrence relations of the Bessel function in the Appendix 2, the Eq.(2-95) can be simplified into

$$det[SM^c] = \prod_{\ell=-(N-1)}^{N} \frac{\pi a^2}{8\lambda^2}[K_{\ell+1}(\lambda a)Y_\ell(\lambda a) - Y_{\ell+1}(\lambda a)K_\ell(\lambda a)] \\ \{I_{\ell+1}(\lambda a)J_\ell(\lambda a) + J_{\ell+1}(\lambda a)I_\ell(\lambda a)\} = 0 \tag{2-96}$$

Zero determinant in the Eq.(2-96) implies that the eigenequation is

$$[K_{\ell+1}(\lambda a)Y_\ell(\lambda a) - Y_{\ell+1}(\lambda a)K_\ell(\lambda a)]\{I_{\ell+1}(\lambda a)J_\ell(\lambda a) + J_{\ell+1}(\lambda a)I_\ell(\lambda a)\} = 0, \\ \ell = 0, \pm 1, \pm 2, \cdots, \pm(N-1), N. \tag{2-97}$$

After comparing with the analytical solution for the clamped circular plate [58], the true eigenequation for the continuous system can be obtained by approaching N in the the discrete system to infinity. The former part in the Eq.(2-97) inside the middle bracket is the spurious eigenequation while the latter part inside the big bracket is found to be the true eigenequation. The result of the Eq.(2-97) in the discrete system matches well with the Eq.(2-44) in the continuous system.

Case 2. Simply-supported circular plate

For the simply-supported circular plate ($u = 0$ and $m = 0$) with a radius a, we have

$$[SM^s] = \begin{bmatrix} U & M \\ U_\theta & M_\theta \end{bmatrix}_{4N\times 4N}, \tag{2-98}$$

where the superscript " s " denotes the simply-supported case. Since the rotation symmetry is preserved for a circular boundary, the eigenvalues of the influence matrices for the discrete system can be found by using the circulants as shown below:

$$\mu_\ell^{[M]} = -\frac{\pi a}{4\lambda^2}[Y_\ell(\lambda a)\alpha_\ell^J(\lambda a) + \frac{2}{\pi}K_\ell(\lambda a)\alpha_\ell^I(\lambda a)], \quad \ell = 0, \pm 1, \pm 2, \cdots, \pm(N-1), N. \tag{2-99}$$

$$\kappa_\ell^{[M]} = -\frac{\pi a}{4\lambda}[Y_\ell'(\lambda a)\alpha_\ell^J(\lambda a) + \frac{2}{\pi}K_\ell'(\lambda a)\alpha_\ell^I(\lambda a)], \quad \ell = 0, \pm 1, \pm 2, \cdots, \pm(N-1), N. \tag{2-100}$$

where $\mu_\ell^{[M]}$ and $\kappa_\ell^{[M]}$ are the eigenvalues of $[M]$ and $[M_\theta]$ matrices, respectively. Since the two matrices $[M]$ and $[M_\theta]$ are all symmetric circulants, they can be expressed by

$$[M] = \Phi\Sigma_M\Phi^T, \tag{2-101}$$

$$[M_\theta] = \Phi\Sigma_{M_\theta}\Phi^T. \tag{2-102}$$

By employing the Eqs.(2-87), (2-89), (2-101) and (2-102) for the Eq.(2-98), we have

$$[SM^s] = \begin{bmatrix} \Phi & 0 \\ 0 & \Phi \end{bmatrix} \begin{bmatrix} \Sigma_U & \Sigma_M \\ \Sigma_{U_\theta} & \Sigma_{M_\theta} \end{bmatrix} \begin{bmatrix} \Phi & 0 \\ 0 & \Phi \end{bmatrix}^T . \tag{2-103}$$

By using the property of the determinant in the Appendix 4, the determinant of $[SM^s]_{4N\times 4N}$ is

$$det[SM^s] = det \begin{bmatrix} \Sigma_U & \Sigma_M \\ \Sigma_{U_\theta} & \Sigma_{M_\theta} \end{bmatrix} = \prod_{\ell=-(N-1)}^{N} (\mu_\ell^{[U]}\kappa_\ell^{[M]} - \mu_\ell^{[M]}\kappa_\ell^{[U]}), \tag{2-104}$$

since Φ is orthogonal. By employing the Eqs.(2-83), (2-85), (2-99) and (2-100) for the Eq.(2-104), we have

$$\begin{aligned} det[SM^s] = & \prod_{\ell=-(N-1)}^{N} \frac{\pi^2 a^2}{16\lambda^3} \\ & \{[Y_\ell(\lambda a)J_\ell(\lambda a) + \frac{2}{\pi}K_\ell(\lambda a)I_\ell(\lambda a)][Y_\ell'(\lambda a)\alpha_\ell^J(\lambda a) + \frac{2}{\pi}K_\ell'(\lambda a)\alpha_\ell^I(\lambda a)] \\ & - [Y_\ell(\lambda a)\alpha_\ell^J(\lambda a) + \frac{2}{\pi}K_\ell(\lambda a)\alpha_\ell^I(\lambda a)][Y_\ell'(\lambda a)J_\ell(\lambda a) + \frac{2}{\pi}K_\ell'(\lambda a)I_\ell(\lambda a)]\} \end{aligned} \tag{2-105}$$

By using the recurrence relations of the Bessel function in the Appendix 2, the Eq.(2-105) can be simplified into

$$\begin{aligned} det[SM^s] = & \prod_{\ell=-(N-1)}^{N} \frac{\pi a}{8\lambda^2}[K_{\ell+1}(\lambda a)Y_\ell(\lambda a) - Y_{\ell+1}(\lambda a)K_\ell(\lambda a)] \\ & \{(1-\nu)I_\ell(\lambda a)J_{n+1}(\lambda a) + I_{n+1}(\lambda a)J_\ell(\lambda a) - 2\lambda a I_\ell(\lambda a)J_\ell(\lambda a)\} = 0 \end{aligned} \tag{2-106}$$

Zero determinant in the Eq.(2-106) implies that the eigenequation is

$$\begin{aligned} & [K_{\ell+1}(\lambda a)Y_\ell(\lambda a) - Y_{\ell+1}(\lambda a)K_\ell(\lambda a)] \\ & \{(1-\nu)I_\ell(\lambda a)J_{n+1}(\lambda a) + I_{n+1}(\lambda a)J_\ell(\lambda a) - 2\lambda a I_\ell(\lambda a)J_\ell(\lambda a)\} = 0, \\ & \qquad \ell = 0, \pm 1, \pm 2, \cdots, \pm(N-1), N. \end{aligned} \tag{2-107}$$

After comparing with the analytical solution for the simply-supported circular plate [58], the true eigenequation for the continuous system can be obtained by approaching N in the discrete system to infinity. The former part in the Eq.(2-107) inside the middle bracket is the spurious eigenequation while the latter part inside the big bracket is found to be the true eigenequation. The result of the Eq.(2-107) in the discrete system match well with the Eq.(2-54) in the continuous system.

Case 3. Free circular plate

For the free circular plate ($m = 0$ and $v = 0$) with a radius a, we have

$$[SM^f] = \begin{bmatrix} M & V \\ M_\theta & V_\theta \end{bmatrix}_{4N\times 4N}, \tag{2-108}$$

where the superscript " f " denotes the free case. Since the rotation symmetry is preserved for a circular boundary, the eigenvalues of the influence matrices for the discrete system can be found by using the circulants as shown below:

$$\begin{aligned} \mu_\ell^{[V]} &= -\frac{\pi a}{4\lambda^2}[Y_\ell(\lambda a)\beta_\ell^J(\lambda a) + \frac{2}{\pi}K_\ell(\lambda a)\beta_\ell^I(\lambda a) \\ &+ \frac{1-\nu}{a}[Y_\ell(\lambda a)\gamma_\ell^J(\lambda a) + \frac{2}{\pi}K_\ell(\lambda a)\gamma_n^I(\lambda a)], \quad \ell = 0, \pm 1, \pm 2, \cdots, \pm(N-1), N. \end{aligned} \tag{2-109}$$

$$\begin{aligned} \kappa_\ell^{[V]} &= -\frac{\pi a}{4\lambda}[Y_\ell'(\lambda a)\beta_\ell^J(\lambda a) + \frac{2}{\pi}K_\ell'(\lambda a)\beta_\ell^I(\lambda a) \\ &+ \frac{1-\nu}{a}[Y_\ell'(\lambda a)\gamma_\ell^J(\lambda a) + \frac{2}{\pi}K_\ell'(\lambda a)\gamma_n^I(\lambda a)], \quad \ell = 0, \pm 1, \pm 2, \cdots, \pm(N-1), N. \end{aligned} \tag{2-110}$$

where $\mu_\ell^{[V]}$ and $\kappa_\ell^{[V]}$ are the eigenvalues of $[V]$ and $[V_\theta]$ matrices, respectively. Since the two matrices $[V]$ and $[V_\theta]$ are all symmetric circulants, they can be expressed by

$$[V] = \Phi\Sigma_V\Phi^T, \tag{2-111}$$

$$[V_\theta] = \Phi\Sigma_{V_\theta}\Phi^T, \tag{2-112}$$

By employing the Eqs.(2-101), (2-102), (2-111) and (2-112) for the Eq.(2-108), we have

$$[SM^f] = \begin{bmatrix} \Phi & 0 \\ 0 & \Phi \end{bmatrix} \begin{bmatrix} \Sigma_M & \Sigma_V \\ \Sigma_{M_\theta} & \Sigma_{V_\theta} \end{bmatrix} \begin{bmatrix} \Phi & 0 \\ 0 & \Phi \end{bmatrix}^T. \tag{2-113}$$

By using the property of the determinant in the Appendix 4, the determinant of $[SM^f]_{4N\times 4N}$ is

$$det[SM^f] = det \begin{bmatrix} \Sigma_U & \Sigma_M \\ \Sigma_{U_\theta} & \Sigma_{M_\theta} \end{bmatrix} = \prod_{\ell=-(N-1)}^{N} (\mu_\ell^{[M]}\kappa_\ell^{[V]} - \mu_\ell^{[V]}\kappa_\ell^{[M]}), \tag{2-114}$$

since Φ is orthogonal. By employing the Eqs.(2-99), (2-100), (2-109) and (2-110) for the Eq.(2-114), we have

$$det[SM^f] = \prod_{\ell=-(N-1)}^{N} \frac{\pi^2 a^2}{16\lambda^3}$$
$$\{[Y_\ell(\lambda a)J_\ell(\lambda a) + \frac{2}{\pi}K_\ell(\lambda a)I_\ell(\lambda a)][Y'_\ell(\lambda a)\alpha_\ell^J(\lambda a) + \frac{2}{\pi}K'_\ell(\lambda a)\alpha_\ell^I(\lambda a)]$$
$$- [Y_\ell(\lambda a)\alpha_\ell^J(\lambda a) + \frac{2}{\pi}K_\ell(\lambda a)\alpha_\ell^I(\lambda a)][Y'_\ell(\lambda a)J_\ell(\lambda a) + \frac{2}{\pi}K'_\ell(\lambda a)I_\ell(\lambda a)]\}. \tag{2-115}$$

By using the recurrence relations of the Bessel function in the Appendix 2, the Eq.(2-115) can be simplified into

$$det[SM^f] = \prod_{\ell=-(N-1)}^{N} \frac{\pi a}{8\lambda^2}$$
$$[K_{\ell+1}(\lambda a)Y_\ell(\lambda a) - Y_{\ell+1}(\lambda a)K_\ell(\lambda a)]$$
$$\{\lambda a(1-\nu)[-4\ell^2(\ell-1)]I_\ell(\lambda a)J_\ell(\lambda a) - 2\lambda^2 a^2 I_{\ell+1}(\lambda a)J_{\ell+1}(\lambda a)]$$
$$+ 2\ell\lambda^2 a^2(1-\nu)(1-\ell)(I_{\ell+1}(\lambda a)J_\ell(\lambda a) - I_\ell(\lambda a)J_{\ell+1}(\lambda a))$$
$$+ [\ell^2(1-\nu)^2(\ell^2-1) + \lambda^4 a^4](I_{\ell+1}(\lambda a)J_\ell(\lambda a) + I_\ell(\lambda a)J_{\ell+1}(\lambda a))\} = 0. \tag{2-116}$$

Zero determinant in the Eq.(2-116) implies that the eigenequation is

$$\begin{aligned}
&[K_{\ell+1}(\lambda a)Y_\ell(\lambda a) - Y_{\ell+1}(\lambda a)K_\ell(\lambda a)] \\
&\{\lambda a(1-\nu)[-4\ell^2(\ell-1)I_\ell(\lambda a)J_\ell(\lambda a) - 2\lambda^2 a^2 I_{\ell+1}(\lambda a)J_{\ell+1}(\lambda a)] \\
&+ 2\ell\lambda^2 a^2(1-\nu)(1-\ell)(I_{\ell+1}(\lambda a)J_\ell(\lambda a) - I_\ell(\lambda a)J_{\ell+1}(\lambda a)) \\
&+ [\ell^2(1-\nu)^2(\ell^2-1) + \lambda^4 a^4](I_{\ell+1}(\lambda a)J_\ell(\lambda a) + I_\ell(\lambda a)J_{\ell+1}(\lambda a))\} = 0 \\
&\ell = 0, \pm 1, \pm 2, \cdots, \pm(N-1), N.
\end{aligned} \tag{2-117}$$

After comparing with the analytical solution for the free circular plate [58], the true eigenequation for the continuous system can be obtained by approaching N in the discrete system to infinity. The former part in the Eq.(2-117) inside the middle bracket is the spurious eigenequation while the latter part inside the big bracket is the true eigenequation. The result of the

Eq.(2-117) in the discrete system match well with the Eq.(2-66) in the continuous system. After comparing the Eq.(2-97) with the Eqs.(2-107) and (2-117), the same spurious eigenequation ($[K_{\ell+1}(\lambda a)Y_{\ell}(\lambda a) - Y_{\ell+1}(\lambda a)K_{\ell}(\lambda a)] = 0$) is simultaneously embedded in the u, θ formulation no matter what the boundary condition is.

Since any two equations in the plate formulation (the Eqs.(2-28)-(2-31)) can be chosen, $6(C_2^4)$ options of the formulation can be considered. If we choose different combinations of the formulae for any one of the the clamped, simply-supported or free circular plate cases, we can obtain the same true eigenequation but different spurious eigenequations. At the same time, the clamped, simply-supported and free circular plates result in the same spurious eigenequation, once the same formulation is chosen. The occurrence of spurious eigenequation only depends on the formulation instead of the specified boundary condition. True eigenequation depends on the specified boundary condition instead of the formulation. All the resluts are summarized in the Table 2-2.

2-3 Mathematical analysis using the imaginary-part BEM

For the imaginary-part BEM, the kernel function $U(s,x)$ is the imaginary-part of the fundamental solution

$$U(s,x) = Im[U_c(s,x)] = \frac{1}{8\lambda^2}[J_0(\lambda r) + \frac{2}{\pi} I_0(\lambda r)]. \tag{2-118}$$

In order to obtain the true and spurious eigenequations for plate vibration using the imaginary-part BEM, the degenerate kernel is adopted to analytically derive the true and spurious eigenequation in the continuous and discrete systems of a circular plate. For the continuous system, the spurious eigenequation is derived by using the degenerate kernel and Fourier series. For the discrete system, mathematical analysis for the spurious eigenequation is done by using the degenerate kernel and circulants. Three cases (clamped, simply-supported and free plate) are demonstrated analytically in the continuous and the discrete systems, respectively, in the following subsections.

2-3-1 Continuous system

Case 1. Clamped circular plate

For the clamped circular plate ($u = 0$ and $\theta = 0$) with a radius a, we can obtain the eigenequation in the continuous formulation. The moment and shear force, $m(s)$ and $v(s)$ along the circular boundary, can be expanded into Fourier series by

$$m(s) = \sum_{n=0}^{\infty}(p_n^c cos(n\overline{\phi}) + q_n^c sin(n\overline{\phi})), \quad s \in B, \tag{2-119}$$

$$v(s) = \sum_{n=0}^{\infty}(a_n^c cos(n\overline{\phi}) + b_n^c sin(n\overline{\phi})), \quad s \in B, \tag{2-120}$$

where the superscript " c " denotes the clamped case, $\overline{\phi}$ is the angle on the circular boundary, a_n^c, b_n^c, p_n^c and q_n^c are the undetermined Fourier coefficients. Substituting the Eqs.(2-119) and (2-120) into the Eqs.(2-24) and (2-25), we have

$$\begin{aligned} 0 = &- \int_0^{2\pi} U(s,x)[\sum_{n=0}^{\infty}(a_n^c cos(n\overline{\phi}) + b_n^c sin(n\overline{\phi}))]\, dB \\ &+ \int_0^{2\pi} \Theta(s,x)[\sum_{n=0}^{\infty}(p_n^c cos(n\overline{\phi}) + q_n^c sin(n\overline{\phi}))]\, dB,\ x \in B, \end{aligned} \tag{2-121}$$

$$\begin{aligned} 0 = &- \int_0^{2\pi} U_\theta(s,x)[\sum_{n=0}^{\infty}(a_n^c cos(n\overline{\phi}) + b_n^c sin(n\overline{\phi}))]\, dB \\ &+ \int_0^{2\pi} \Theta_\theta(s,x)[\sum_{n=0}^{\infty}(p_n^c cos(n\overline{\phi}) + q_n^c sin(n\overline{\phi}))]\, dB,\ x \in B. \end{aligned} \tag{2-122}$$

The kernel functions, $U(s,x)$, $\Theta(s,x)$, $U_\theta(s,x)$ and $\Theta_\theta(s,x)$, can be expanded by using the expansion formulae,

$$J_0(\lambda r) = \begin{cases} J_0^i(\lambda r) = \sum_{m=-\infty}^{\infty} J_m(\lambda\overline{\rho})J_m(\lambda\rho)\cos(m(\overline{\phi}-\phi)), & \overline{\rho} > \rho, \\ J_0^e(\lambda r) = \sum_{m=-\infty}^{\infty} J_m(\lambda\rho)J_m(\lambda\overline{\rho})\cos(m(\overline{\phi}-\phi)), & \rho > \overline{\rho}, \end{cases} \tag{2-123}$$

$$I_0(\lambda r) = \begin{cases} I_0^i(\lambda r) = \sum_{m=-\infty}^{\infty} (-1)^m I_m(\lambda\overline{\rho})I_m(\lambda\rho)\cos(m(\overline{\phi}-\phi)), & \overline{\rho} > \rho, \\ I_0^e(\lambda r) = \sum_{m=-\infty}^{\infty} (-1)^m I_m(\lambda\rho)I_m(\lambda\overline{\rho})\cos(m(\overline{\phi}-\phi)), & \rho > \overline{\rho}, \end{cases} \tag{2-124}$$

Similarly, the other kernels can also be expanded into degenerate forms. By using the degenerate kernels into the Eq.(2-121) and by employing the orthogonality condition of the Fourier series, the Fourier coefficients a_n^c, b_n^c, p_n^c and q_n^c satisfy

$$p_n^c = \frac{1}{\lambda}\frac{[J_n(\lambda a)J_n(\lambda a) + \frac{2}{\pi}(-1)^n I_n(\lambda a)I_n(\lambda a)]}{[J_n(\lambda a)J_n'(\lambda a) + \frac{2}{\pi}(-1)^n I_n(\lambda a)I_n'(\lambda a)]}a_n^c, \qquad n = 0, 1, 2, \ldots, \tag{2-125}$$

$$q_n^c = \frac{1}{\lambda}\frac{[J_n(\lambda a)J_n(\lambda a) + \frac{2}{\pi}(-1)^n I_n(\lambda a)I_n(\lambda a)]}{[J_n(\lambda a)J_n'(\lambda a) + \frac{2}{\pi}(-1)^n I_n(\lambda a)I_n'(\lambda a)]}b_n^c, \qquad n = 0, 1, 2, \ldots. \tag{2-126}$$

Similarly, the Eq.(2-122) yields,

$$p_n^c = \frac{1}{\lambda}\frac{[J_n'(\lambda a)J_n(\lambda a) + \frac{2}{\pi}(-1)^n I_n'(\lambda a)I_n(\lambda a)]}{[J_n'(\lambda a)J_n'(\lambda a) + \frac{2}{\pi}(-1)^n I_n'(\lambda a)I_n'(\lambda a)]}a_n^c, \qquad n = 0, 1, 2, \ldots, \tag{2-127}$$

$$q_n^c = \frac{1}{\lambda}\frac{[J_n'(\lambda a)J_n(\lambda a) + \frac{2}{\pi}(-1)^n I_n'(\lambda a)I_n(\lambda a)]}{[J_n'(\lambda a)J_n'(\lambda a) + \frac{2}{\pi}(-1)^n I_n'(\lambda a)I_n'(\lambda a)]}b_n^c, \qquad n = 0, 1, 2, \ldots. \tag{2-128}$$

To seek nontrivial data for the generalized coefficients of a_n^c, p_n^c, b_n^c and q_n^c, we can obtain the eigenequation by using either the Eqs.(2-125) and (2-127) or the Eqs.(2-126) and (2-128)

$$\frac{J_n(\lambda a)J_n(\lambda a) + \frac{2}{\pi}(-1)^n I_n(\lambda a)I_n(\lambda a)}{J_n(\lambda a)J_n'(\lambda a) + \frac{2}{\pi}(-1)^n I_n(\lambda a)I_n'(\lambda a)} = \frac{J_n'(\lambda a)J_n(\lambda a) + \frac{2}{\pi}(-1)^n I_n'(\lambda a)I_n(\lambda a)}{J_n'(\lambda a)J_n'(\lambda a) + \frac{2}{\pi}(-1)^n I_n'(\lambda a)I_n'(\lambda a)} \tag{2-129}$$

After recollecting the terms by using the recurrence relations of the Bessel function in the Appendix 2, the Eq.(2-129) can be simplified to

$$[I_{n+1}(\lambda a)J_n(\lambda a) + J_{n+1}(\lambda a)I_n(\lambda a)]\{I_{n+1}(\lambda a)J_n(\lambda a) + J_{n+1}(\lambda a)I_n(\lambda a)\} = 0 \tag{2-130}$$

The former part in the Eq.(2-130) inside the middle bracket is the spurious eigenequation while the latter part inside the big bracket is found to be the true eigenequation after comparing with the exact eigenequation [58]. The true eigenequation in the Eq.(2-130) by using the imaginary-part BEM is the same as the former one in the Eq.(2-44) by using the real-part method. In this case, it is interesting to find that the true and spurious eigenequation are the same. We can also comment that no spurious eigenvalue occurs although the spurious multiplicity appears. This case is similar to the membrane case with eigenequation $J_n(\lambda a)J_n(\lambda a) = 0$ as shown in Table 2-3 when the imaginary-part BEM is employed to solve the Dirichlet problem [12].

Case 2. Simply-supported circular plate

For the simply-supported circular plate ($u = 0$ and $m = 0$) with a radius a, we can obtain the eigenequation in the continuous formulation. Similarly, the moment and shear force, $\theta(s)$ and $v(s)$ along the circular boundary, can be expanded into Fourier series by

$$\theta(s) = \sum_{n=0}^{\infty}(p_n^s cos(n\overline{\phi}) + q_n^s sin(n\overline{\phi})), \quad s \in B, \tag{2-131}$$

$$v(s) = \sum_{n=0}^{\infty}(a_n^s cos(n\overline{\phi}) + b_n^s sin(n\overline{\phi})), \quad s \in B, \tag{2-132}$$

where the superscript " s " denotes the simply-supported case, $\overline{\phi}$ is the angle on the circular boundary, a_n^s, b_n^s, p_n^s and q_n^s are the undetermined Fourier coefficients. Substituting the Eqs.(2-131) and (2-132) and using the degenerate kernels of $U(s,x)$, $M(s,x)$, $U_\theta(s,x)$ and $M_\theta(s,x)$ into the Eq.(2-24), we have

$$p_n^s = -\frac{[J_n(\lambda a)J_n(\lambda a) + \frac{2}{\pi}(-1)_n^n(\lambda a)I_n(\lambda a)]}{[J_n(\lambda a)\alpha_n^J(\lambda a) + \frac{2}{\pi}(-1)^n I_n(\lambda a)\alpha_n^I(\lambda a)]}a_n^s, \qquad n = 0, 1, 2, \ldots, \tag{2-133}$$

$$q_n^s = -\frac{[J_n(\lambda a)J_n(\lambda a) + \frac{2}{\pi}(-1)^n I_n(\lambda a)I_n(\lambda a)]}{[J_n(\lambda a)\alpha_n^J(\lambda a) + \frac{2}{\pi}(-1)^n I_n(\lambda a)\alpha_n^I(\lambda a)]}b_n^s, \qquad n = 0, 1, 2, \ldots. \tag{2-134}$$

Similarly, the Eq.(2-25) yields,

$$p_n^s = -\frac{[J_n'(\lambda a)J_n(\lambda a) + \frac{2}{\pi}(-1)^n I_n'(\lambda a)I_n(\lambda a)]}{[J_n'(\lambda a)\alpha_n^J(\lambda a) + \frac{2}{\pi}(-1)^n I_n'(\lambda a)\alpha_n^I(\lambda a)]}a_n^s, \qquad n = 0, 1, 2, \ldots, \tag{2-135}$$

$$q_n^s = -\frac{[J_n'(\lambda a)J_n(\lambda a) + \frac{2}{\pi}(-1)^n I_n'(\lambda a)I_n(\lambda a)]}{[J_n'(\lambda a)\alpha_n^J(\lambda a) + \frac{2}{\pi}(-1)^n I_n'(\lambda a)\alpha_n^I(\lambda a)]}b_n^s, \qquad n = 0, 1, 2, \ldots. \tag{2-136}$$

To seek nontrivial data for the generalized coefficients of a_n^s, p_n^s, b_n^s and q_n^s, we can obtain the eigenequations as

$$\frac{J_n(\lambda a)J_n(\lambda a) + \frac{2}{\pi}(-1)^n I_n(\lambda a)I_n(\lambda a)}{J_n(\lambda a)\alpha_n^J(\lambda a) + \frac{2}{\pi}(-1)^n I_n(\lambda a)\alpha_n^I(\lambda a)} = \frac{J_n'(\lambda a)J_n(\lambda a) + \frac{2}{\pi}(-1)^n I_n'(\lambda a)I_n(\lambda a)}{J_n'(\lambda a)\alpha_n^J(\lambda a) + \frac{2}{\pi}(-1)^n I_n'(\lambda a)\alpha_n^I(\lambda a)} \tag{2-137}$$

After recollecting the terms by using the recurrence relations of the Bessel function in the Appendix 2, the Eq.(2-137) can be simplified to

$$\begin{aligned}&[I_{n+1}(\lambda a)J_n(\lambda a) + J_{n+1}(\lambda a)I_n(\lambda a)] \\ &\{(1-\nu)I_n(\lambda a)J_{n+1}(\lambda a) + I_{n+1}(\lambda a)J_n(\lambda a) - 2\lambda a I_n(\lambda a)J_n(\lambda a)\} = 0\end{aligned} \tag{2-138}$$

The former part in the Eq.(2-138) inside the middle bracket is the spurious eigenequation while the latter part inside the big bracket is found to be the true eigenequation after comparing with the exact eigenequation [58]. The true eigenequation in the Eq.(2-138) by using the imaginary-part BEM is the same as the former one in the Eq.(2-54) by using the real-part method.

Case 3. Free circular plate

For the free circular plate ($m = 0$ and $v = 0$) with a radius a, we can obtain the eigenequation in the continuous formulation. Similarly, the displacement and slope, $u(s)$ and $\theta(s)$ along the circular boundary, can be expanded into Fourier series by

$$u(s) = \sum_{n=0}^{\infty}(p_n^f cos(n\overline{\phi}) + q_n^f sin(n\overline{\phi})), \quad s \in B, \tag{2-139}$$

$$\theta(s) = \sum_{n=0}^{\infty}(a_n^f cos(n\overline{\phi}) + b_n^f sin(n\overline{\phi})), \quad s \in B, \tag{2-140}$$

where the superscript " f " denotes the free case, $\overline{\phi}$ is the angle on the circular boundary, a_n^f, b_n^f, p_n^f and q_n^f are the undetermined Fourier coefficients. Substituting the Eqs.(2-139) and (2-140) and using the degenerate kernels of $M(s,x)$, $V(s,x)$, $M_\theta(s,x)$ and $V_\theta(s,x)$ into the Eq.(2-24), we have

$$p_n^f = -\frac{[J_n(\lambda a)\alpha_n^J(\lambda a) + \frac{2}{\pi}(-1)^n I_n(\lambda a)\alpha_n^I(\lambda a)]}{[J_n(\lambda a)\beta_n^J(\lambda a) + \frac{2}{\pi}(-1)^n I_n(\lambda a)\beta_n^I(\lambda a) + \frac{1-\nu}{a}[J_n(\lambda a)\gamma_n^J(\lambda a) + \frac{2}{\pi}(-1)^n I_n(\lambda a)\gamma_n^I(\lambda a)]]} a_n^f, \quad n = 0, 1, 2, \ldots, \tag{2-141}$$

$$q_n^f = -\frac{[J_n(\lambda a)\alpha_n^J(\lambda a) + \frac{2}{\pi}(-1)^n I_n(\lambda a)\alpha_n^I(\lambda a)]}{[J_n(\lambda a)\beta_n^J(\lambda a) + \frac{2}{\pi}(-1)^n I_n(\lambda a)\beta_n^I(\lambda a) + \frac{1-\nu}{a}[J_n(\lambda a)\gamma_n^J(\lambda a) + \frac{2}{\pi}(-1)^n I_n(\lambda a)\gamma_n^I(\lambda a)]]} b_n^f, \quad n = 0, 1, 2, \ldots. \tag{2-142}$$

Similarly, the Eq.(2-25) yields,

$$p_n^f = -\frac{[J_n'(\lambda a)J_n(\lambda a) + \frac{2}{\pi}(-1)^n I_n'(\lambda a)I_n(\lambda a)]}{[J_n'(\lambda a)\beta_n^J(\lambda a) + \frac{2}{\pi}(-1)^n I_n'(\lambda a)\beta_n^I(\lambda a) + \frac{1-\nu}{a}[J_n'(\lambda a)\gamma_n^J(\lambda a) + \frac{2}{\pi}(-1)^n I_n'(\lambda a)\gamma_n^I(\lambda a)]]} a_n^f, \quad n = 0, 1, 2, \ldots, \tag{2-143}$$

$$q_n^f = -\frac{[J_n'(\lambda a)J_n(\lambda a) + \frac{2}{\pi}(-1)^n I_n'(\lambda a)I_n(\lambda a)]}{[J_n'(\lambda a)\beta_n^J(\lambda a) + \frac{2}{\pi}(-1)^n I_n'(\lambda a)\beta_n^I(\lambda a) + \frac{1-\nu}{a}[J_n'(\lambda a)\gamma_n^J(\lambda a) + \frac{2}{\pi}(-1)^n I_n'(\lambda a)\gamma_n^I(\lambda a)]]} b_n^f,$$
$$n = 0, 1, 2, \ldots. \quad (2\text{-}144)$$

To seek nontrivial data for the generalized coefficients of a_n^f, p_n^f, b_n^f and q_n^f, we can obtain the eigenequation

$$\frac{[J_n(\lambda a)\alpha_n^J(\lambda a) + \frac{2}{\pi}(-1)^n I_n(\lambda a)\alpha_n^I(\lambda a)]}{[J_n(\lambda a)\beta_n^J(\lambda a) + \frac{2}{\pi}(-1)^n I_n(\lambda a)\beta_n^I(\lambda a) + \frac{1-\nu}{a}[J_n(\lambda a)\gamma_n^J(\lambda a) + \frac{2}{\pi}(-1)^n I_n(\lambda a)\gamma_n^I(\lambda a)]]}$$
$$= \frac{[J_n'(\lambda a)J_n(\lambda a) + \frac{2}{\pi}(-1)^n I_n'(\lambda a)I_n(\lambda a)]}{[J_n'(\lambda a)\beta_n^J(\lambda a) + \frac{2}{\pi}(-1)^n I_n'(\lambda a)\beta_n^I(\lambda a) + \frac{1-\nu}{a}[J_n'(\lambda a)\gamma_n^J(\lambda a) + \frac{2}{\pi}(-1)^n I_n'(\lambda a)\gamma_n^I(\lambda a)]]} \quad (2\text{-}145)$$

After recollecting the terms by using the recurrence relations of the Bessel function in the Appendix 2, the Eq.(2-145) can be simplified to

$$\begin{aligned} &[I_{n+1}(\lambda a)J_n(\lambda a) + J_{n+1}(\lambda a)I_n(\lambda a)] \\ &\{\lambda a(1-\nu)[-4n^2(n-1)I_n(\lambda a)J_n(\lambda a) - 2\lambda^2 a^2 I_{n+1}(\lambda a)J_{n+1}(\lambda a)] \\ &+ 2n\lambda^2 a^2(1-\nu)(1-n)(I_{n+1}(\lambda a)J_n(\lambda a) - I_n(\lambda a)J_{n+1}(\lambda a)) \\ &+ [n^2(1-\nu)^2(n^2-1) + \lambda^4 a^4](I_{n+1}(\lambda a)J_n(\lambda a) + I_n(\lambda a)J_{n+1}(\lambda a))\} = 0 \end{aligned} \quad (2\text{-}146)$$

The former part in the Eq.(2-146) inside the middle bracket is the spurious eigenequation which also appears in the clamped and simply-supported cases. It indicates that the spurious eigenequations of the Eqs.(2-130), (2-138) and (2-146) are the same since the same formulation (null-field formulation of the Eqs.(2-24) and (2-25)) is used. This indicates that spurious eigenequation depends on the formulation instead of the specified boundary condition. The true eigenequation in the Eq.(2-146) by using the imaginary-part BEM agrees well with the former one in the Eq.(2-66) by using the real-part BEM.

2-3-2 Discrete system

Case 1. Clamped circular plate

For the clamped circular plate ($u = 0$ and $\theta = 0$) with a radius a, we have

$$[SM^c] = \begin{bmatrix} U & \Theta \\ U_\theta & \Theta_\theta \end{bmatrix}_{4N\times 4N}. \quad (2\text{-}147)$$

where the superscript " c " denotes the clamped case. Since the rotation symmetry is preserved for a circular boundary, the eigenvalues of the influence matrices for the discrete system can be found by using the circulants as shown below:

$$\mu_\ell^{[U]} = -\frac{\pi a}{4\lambda^2}[J_\ell(\lambda a)J_\ell(\lambda a) + \frac{2}{\pi}(-1)^\ell I_\ell(\lambda a)I_\ell(\lambda a)], \quad \ell = 0, \pm1, \pm2, \cdots, \pm(N-1), N. \quad (2\text{-}148)$$

$$\mu_\ell^{[\Theta]} = \frac{\pi a}{4\lambda}[J_\ell(\lambda a)J'_\ell(\lambda a) + \frac{2}{\pi}(-1)^\ell I_\ell(\lambda a)I'_\ell(\lambda a)], \qquad \ell = 0, \pm1, \pm2, \cdots, \pm(N-1), N. \quad (2\text{-}149)$$

$$\kappa_\ell^{[U]} = -\frac{\pi a}{4\lambda}[J'_\ell(\lambda a)J_\ell(\lambda a) + \frac{2}{\pi}(-1)^\ell I'_\ell(\lambda a)I_\ell(\lambda a)], \quad \ell = 0, \pm1, \pm2, \cdots, \pm(N-1), N. \quad (2\text{-}150)$$

$$\kappa_\ell^{[\Theta]} = \frac{\pi a}{4}[J'_\ell(\lambda a)J'_\ell(\lambda a) + \frac{2}{\pi}(-1)^\ell I'_\ell(\lambda a)I'_\ell(\lambda a)], \qquad \ell = 0, \pm1, \pm2, \cdots, \pm(N-1), N. \quad (2\text{-}151)$$

where $\mu_\ell^{[\Theta]}$, $\kappa_\ell^{[U]}$ and $\kappa_\ell^{[\Theta]}$ are the eigenvalues of $[\Theta]$, $[U_\theta]$ and $[\Theta_\theta]$ matrices, respectively. Since the four matrices $[U]$, $[\Theta]$, $[U_\theta]$ and $[\Theta_\theta]$ are all symmetric circulants, they can be expressed by

$$[U] = \Phi\Sigma_U\Phi^T \quad (2\text{-}152)$$

$$[U_\theta] = \Phi\Sigma_{U_\theta}\Phi^T \quad (2\text{-}153)$$

$$[\Theta_\theta] = \Phi\Sigma_{\Theta_\theta}\Phi^T \quad (2\text{-}154)$$

By employing the Eqs.(2-152)-(2-154) for the Eq.(2-147), we have

$$[SM^c] = \begin{bmatrix} \Phi & 0 \\ 0 & \Phi \end{bmatrix} \begin{bmatrix} \Sigma_U & \Sigma_\Theta \\ \Sigma_{U_\theta} & \Sigma_{\Theta_\theta} \end{bmatrix} \begin{bmatrix} \Phi & 0 \\ 0 & \Phi \end{bmatrix}^T. \quad (2\text{-}155)$$

By using the property of the determinant in the Appendix 4, the determinant of $[SM^c]_{4N\times 4N}$ is

$$det[SM^c] = det\begin{bmatrix} \Sigma_U & \Sigma_\Theta \\ \Sigma_{U_\theta} & \Sigma_{\Theta_\theta} \end{bmatrix} = \prod_{\ell=-(N-1)}^{N} (\mu_\ell^{[U]}\kappa_\ell^{[\Theta]} - \mu_\ell^{[\Theta]}\kappa_\ell^{[U]}), \tag{2-156}$$

since Φ is orthogonal. By employing the Eqs.(2-148)-(2-151) for the Eq.(2-156), we have

$$\begin{aligned} det[SM^c] = \prod_{\ell=-(N-1)}^{N} \frac{\pi^2 a^2}{16\lambda^2} \\ \{[J_\ell(\lambda a)J_\ell(\lambda a) + \frac{2}{\pi}(-1)^\ell I_\ell(\lambda a)I_\ell(\lambda a)][J'_\ell(\lambda a)J'_\ell(\lambda a) + \frac{2}{\pi}(-1)^\ell I'_\ell(\lambda a)I'_\ell(\lambda a)] \\ - [J_\ell(\lambda a)J'_\ell(\lambda a) + \frac{2}{\pi}(-1)^\ell I_\ell(\lambda a)I'_\ell(\lambda a)][J'_\ell(\lambda a)J_\ell(\lambda a) + \frac{2}{\pi}(-1)^\ell I'_\ell(\lambda a)I_\ell(\lambda a)]\} \end{aligned} \tag{2-157}$$

By using the recurrence relations of the Bessel function in the Appendix 2, the Eq.(2-157) can be simplified into

$$\begin{aligned} det[SM^c] = \prod_{\ell=-(N-1)}^{N} \frac{\pi a^2}{8\lambda^2}[I_{\ell+1}(\lambda a)J_\ell(\lambda a) + J_{\ell+1}(\lambda a)I_\ell(\lambda a)] \\ \{I_{\ell+1}(\lambda a)J_\ell(\lambda a) + J_{\ell+1}(\lambda a)I_\ell(\lambda a)\} = 0 \end{aligned} \tag{2-158}$$

Zero determinant in the Eq.(2-158) implies that the eigenequation is

$$\begin{aligned} [I_{\ell+1}(\lambda a)J_\ell(\lambda a) + J_{\ell+1}(\lambda a)I_\ell(\lambda a)]\{I_{\ell+1}(\lambda a)J_\ell(\lambda a) + J_{\ell+1}(\lambda a)I_\ell(\lambda a)\} = 0, \\ \ell = 0, \pm 1, \pm 2, \cdots, \pm(N-1), N. \end{aligned} \tag{2-159}$$

After comparing with the analytical solution for the clamped circular plate [58], the true eigenequation for the continuous system can be obtained by approaching N in the the discrete system to infinity. The former part in the Eq.(2-159) inside the middle bracket is the spurious eigenequation while the latter part inside the big bracket is found to be the true eigenequation. The result of the Eq.(2-159) in the discrete system matches well with the Eq.(2-130) in the continuous system.

Case 2. Simply-supported circular plate

For the simply-supported circular plate ($u = 0$ and $m = 0$) with a radius a, we have

$$[SM^s] = \begin{bmatrix} U & M \\ U_\theta & M_\theta \end{bmatrix}_{4N\times 4N}, \tag{2-160}$$

where the superscript " s " denotes the simply-supported case. Since the rotation symmetry is preserved for a circular boundary, the eigenvalues of the influence matrices for the discrete system can be found by using the circulants as shown below:

$$\mu_\ell^{[M]} = -\frac{\pi a}{4\lambda^2}\left[J_\ell(\lambda a)\alpha_\ell^J(\lambda a) + \frac{2}{\pi}(-1)^\ell I_\ell(\lambda a)\alpha_\ell^I(\lambda a)\right], \quad \ell = 0, \pm 1, \pm 2, \cdots, \pm(N-1), N. \tag{2-161}$$

$$\kappa_\ell^{[M]} = -\frac{\pi a}{4\lambda}\left[J'_\ell(\lambda a)\alpha_\ell^J(\lambda a) + \frac{2}{\pi}(-1)^\ell I'_\ell(\lambda a)\alpha_\ell^I(\lambda a)\right], \quad \ell = 0, \pm 1, \pm 2, \cdots, \pm(N-1), N. \tag{2-162}$$

where $\mu_\ell^{[M]}$ and $\kappa_\ell^{[M]}$ are the eigenvalues of $[M]$ and $[M_\theta]$ matrices, respectively. Since the two matrices $[M]$ and $[M_\theta]$ are all symmetric circulants, they can be expressed by

$$[M] = \Phi \Sigma_M \Phi^T, \tag{2-163}$$

$$[M_\theta] = \Phi \Sigma_{M_\theta} \Phi^T. \tag{2-164}$$

By employing the Eqs.(2-152), (2-153), (2-163) and (2-164) for the Eq.(2-160), we have

$$[SM^s] = \begin{bmatrix} \Phi & 0 \\ 0 & \Phi \end{bmatrix} \begin{bmatrix} \Sigma_U & \Sigma_M \\ \Sigma_{U_\theta} & \Sigma_{M_\theta} \end{bmatrix} \begin{bmatrix} \Phi & 0 \\ 0 & \Phi \end{bmatrix}^T. \tag{2-165}$$

By using the property of the determinant in the Appendix 4, the determinant of $[SM^s]_{4N\times 4N}$ is

$$det[SM^s] = det \begin{bmatrix} \Sigma_U & \Sigma_M \\ \Sigma_{U_\theta} & \Sigma_{M_\theta} \end{bmatrix} = \prod_{\ell=-(N-1)}^{N} (\mu_\ell^{[U]}\kappa_\ell^{[M]} - \mu_\ell^{[M]}\kappa_\ell^{[U]}), \tag{2-166}$$

since Φ is orthogonal. By employing the Eqs.(2-148), (2-150), (2-161) and (2-162) for the Eq.(2-166), we have

$$\begin{aligned} det[SM^s] = \prod_{\ell=-(N-1)}^{N} \frac{\pi^2 a^2}{16\lambda^3} \\ &\{[J_\ell(\lambda a)J_\ell(\lambda a) + \frac{2}{\pi}(-1)^\ell I_\ell(\lambda a)I_\ell(\lambda a)][J'_\ell(\lambda a)\alpha_\ell^J(\lambda a) + \frac{2}{\pi}(-1)^\ell I'_\ell(\lambda a)\alpha_\ell^I(\lambda a)] \\ &- [J_\ell(\lambda a)\alpha_\ell^J(\lambda a) + \frac{2}{\pi}(-1)^\ell I_\ell(\lambda a)\alpha_\ell^I(\lambda a)][J'_\ell(\lambda a)J_\ell(\lambda a) + \frac{2}{\pi}(-1)^\ell I'_\ell(\lambda a)I_\ell(\lambda a)]\} \end{aligned} \tag{2-167}$$

By using the recurrence relations of the Bessel function in the Appendix 2, the Eq.(2-167) can be simplified into

$$det[SM^s] = \prod_{\ell=-(N-1)}^{N} \frac{\pi a}{8\lambda^2}[I_{\ell+1}(\lambda a)J_\ell(\lambda a) + J_{\ell+1}(\lambda a)I_\ell(\lambda a)] \\ \{(1-\nu)I_\ell(\lambda a)J_{n+1}(\lambda a) + I_{n+1}(\lambda a)J_\ell(\lambda a) - 2\lambda a I_\ell(\lambda a)J_\ell(\lambda a)\} = 0 \tag{2-168}$$

Zero determinant in the Eq.(2-168) implies that the eigenequation is

$$[I_{\ell+1}(\lambda a)J_\ell(\lambda a) + J_{\ell+1}(\lambda a)I_\ell(\lambda a)] \\ \{(1-\nu)I_\ell(\lambda a)J_{n+1}(\lambda a) + I_{n+1}(\lambda a)J_\ell(\lambda a) - 2\lambda a I_\ell(\lambda a)J_\ell(\lambda a)\} = 0, \\ \ell = 0, \pm 1, \pm 2, \cdots, \pm(N-1), N. \tag{2-169}$$

After comparing with the analytical solution for the simply-supported circular plate [58], the true eigenequation for the continuous system can be obtained by approaching N in the discrete system to infinity. The former part in the Eq.(2-169) inside the middle bracket is the spurious eigenequation while the latter part inside the big bracket is found to be the true eigenequation. The result of the Eq.(2-169) in the discrete system match well with the Eq.(2-138) in the continuous system.

Case 3. Free circular plate

For the free circular plate ($m = 0$ and $v = 0$) with a radius a, we have

$$[SM^f] = \begin{bmatrix} M & V \\ M_\theta & V_\theta \end{bmatrix}_{4N\times 4N}, \tag{2-170}$$

where the superscript " f " denotes the free case. Since the rotation symmetry is preserved for a circular boundary, the eigenvalues of the influence matrices for the discrete system can be found by using the circulants as shown below:

$$\mu_\ell^{[V]} = -\frac{\pi a}{4\lambda^2}[J_\ell(\lambda a)\beta_\ell^J(\lambda a) + \frac{2}{\pi}(-1)^\ell I_\ell(\lambda a)\beta_\ell^I(\lambda a) \\ + \frac{1-\nu}{a}[J_\ell(\lambda a)\gamma_\ell^J(\lambda a) + \frac{2}{\pi}(-1)^\ell I_\ell(\lambda a)\gamma_n^I(\lambda a)], \quad \ell = 0, \pm 1, \pm 2, \cdots, \pm(N-1), N. \tag{2-171}$$

$$\kappa_\ell^{[V]} = -\frac{\pi a}{4\lambda}[J'_\ell(\lambda a)\beta_\ell^J(\lambda a) + \frac{2}{\pi}(-1)^\ell I'_\ell(\lambda a)\beta_\ell^I(\lambda a)$$
$$+ \frac{1-\nu}{a}[J'_\ell(\lambda a)\gamma_\ell^J(\lambda a) + \frac{2}{\pi}(-1)^\ell I'_\ell(\lambda a)\gamma_n^I(\lambda a)], \quad \ell = 0, \pm 1, \pm 2, \cdots, \pm(N-1), N. \tag{2-172}$$

where $\mu_\ell^{[V]}$ and $\kappa_\ell^{[V]}$ are the eigenvalues of $[V]$ and $[V_\theta]$ matrices, respectively. Since the two matrices $[V]$ and $[V_\theta]$ are all symmetric circulants, they can be expressed by

$$[V] = \Phi\Sigma_V\Phi^T, \tag{2-173}$$

$$[V_\theta] = \Phi\Sigma_{V_\theta}\Phi^T, \tag{2-174}$$

By employing the Eqs.(2-163), (2-164), (2-173) and (2-174) for the Eq.(2-170), we have

$$[SM^f] = \begin{bmatrix} \Phi & 0 \\ 0 & \Phi \end{bmatrix} \begin{bmatrix} \Sigma_M & \Sigma_V \\ \Sigma_{M_\theta} & \Sigma_{V_\theta} \end{bmatrix} \begin{bmatrix} \Phi & 0 \\ 0 & \Phi \end{bmatrix}^T. \tag{2-175}$$

By using the property of the determinant in the Appendix 4, the determinant of $[SM^f]_{4N\times 4N}$ is

$$det[SM^f] = det \begin{bmatrix} \Sigma_U & \Sigma_M \\ \Sigma_{U_\theta} & \Sigma_{M_\theta} \end{bmatrix} = \prod_{\ell=-(N-1)}^{N} (\mu_\ell^{[M]}\kappa_\ell^{[V]} - \mu_\ell^{[V]}\kappa_\ell^{[M]}), \tag{2-176}$$

since Φ is orthogonal. By employing the Eqs.(2-161), (2-162), (2-171) and (2-172) for the Eq.(2-176), we have

$$det[SM^f] = \prod_{\ell=-(N-1)}^{N} \frac{\pi^2 a^2}{16\lambda^3}$$
$$\{[J_\ell(\lambda a)J_\ell(\lambda a) + \frac{2}{\pi}(-1)^\ell I_\ell(\lambda a)I_\ell(\lambda a)][J'_\ell(\lambda a)\alpha_\ell^J(\lambda a) + \frac{2}{\pi}(-1)^\ell I'_\ell(\lambda a)\alpha_\ell^I(\lambda a)]$$
$$- [J_\ell(\lambda a)\alpha_\ell^J(\lambda a) + \frac{2}{\pi}(-1)^\ell I_\ell(\lambda a)\alpha_\ell^I(\lambda a)][J'_\ell(\lambda a)J_\ell(\lambda a) + \frac{2}{\pi}(-1)^\ell I'_\ell(\lambda a)I_\ell(\lambda a)]\}. \tag{2-177}$$

By using the recurrence relations of the Bessel function in the Appendix 2, the Eq.(2-177) can be simplified into

$$
\begin{aligned}
det[SM^f] = \prod_{\ell=-(N-1)}^{N} & \frac{\pi a}{8\lambda^2} \\
& [I_{\ell+1}(\lambda a)J_\ell(\lambda a) + J_{\ell+1}(\lambda a)I_\ell(\lambda a)] \\
& \{\lambda a(1-\nu)[-4\ell^2(\ell-1)]I_\ell(\lambda a)J_\ell(\lambda a) - 2\lambda^2 a^2 I_{\ell+1}(\lambda a)J_{\ell+1}(\lambda a)] \\
& + 2\ell\lambda^2 a^2(1-\nu)(1-\ell)(I_{\ell+1}(\lambda a)J_\ell(\lambda a) - I_\ell(\lambda a)J_{\ell+1}(\lambda a)) \\
& + [\ell^2(1-\nu)^2(\ell^2-1) + \lambda^4 a^4](I_{\ell+1}(\lambda a)J_\ell(\lambda a) + I_\ell(\lambda a)J_{\ell+1}(\lambda a))\} = 0.
\end{aligned}
\tag{2-178}
$$

Zero determinant in the Eq.(2-178) implies that the eigenequation is

$$
\begin{aligned}
& [I_{\ell+1}(\lambda a)J_\ell(\lambda a) + J_{\ell+1}(\lambda a)I_\ell(\lambda a)] \\
& \{\lambda a(1-\nu)[-4\ell^2(\ell-1)I_\ell(\lambda a)J_\ell(\lambda a) - 2\lambda^2 a^2 I_{\ell+1}(\lambda a)J_{\ell+1}(\lambda a)] \\
& + 2\ell\lambda^2 a^2(1-\nu)(1-\ell)(I_{\ell+1}(\lambda a)J_\ell(\lambda a) - I_\ell(\lambda a)J_{\ell+1}(\lambda a)) \\
& + [\ell^2(1-\nu)^2(\ell^2-1) + \lambda^4 a^4](I_{\ell+1}(\lambda a)J_\ell(\lambda a) + I_\ell(\lambda a)J_{\ell+1}(\lambda a))\} = 0 \\
& \ell = 0, \pm 1, \pm 2, \cdots, \pm(N-1), N.
\end{aligned}
\tag{2-179}
$$

After comparing with the analytical solution for the free circular plate [58], the true eigenequation for the continuous system can be obtained by approaching N in the discrete system to infinity. The former part in the Eq.(2-179) inside the middle bracket is the spurious eigenequation while the latter part inside the big bracket is the true eigenequation. The result of the Eq.(2-179) in the discrete system match well with the Eq.(2-146) in the continuous system. After comparing the Eq.(2-159) with the Eqs.(2-169) and (2-179), the same spurious eigenequation ($[I_{\ell+1}(\lambda a)J_\ell(\lambda a) + J_{\ell+1}(\lambda a)I_\ell(\lambda a)] = 0$) is simultaneously embedded in the u, θ formulation no matter what the boundary condition is.

Since any two equations in the plate formulation (the Eqs.(2-28)-(2-31)) can be chosen, $6(C_2^4)$ options of the formulation can be considered. If we choose different combinations of the formulae for any one of the the clamped, simply-supported or free circular plate cases, we can obtain the same true eigenequation but different spurious eigenequations. At the same time, the clamped, simply-supported and free circular plates result in the same spurious eigenequation, once the same formulation is chosen. The occurrence of spurious eigenequation only depends on the formulation instead of the specified boundary condition. True eigenequation depends on the specified boundary condition instead of the formulation. All the resluts are summarized in the Table 2-4.

2-4 Numerical results and discussions

Circular plate (clamped, simply-supported and free (C, S and F) boundary conditions)

A circular plate with a radius of one meter $(a = 1\,m)$ and the Poisson ratio $\nu = 0.33$ are considered. The boundary is discretized into ten constant elements. Since any two equations in the plate formulation (the Eqs.(2-28)-(2-31)) can be chosen, $6(C_2^4)$ options of the formulation can be considered. By using the real-part and imaginary-part BEMs, the numerical results are shown below:

Real-part BEM:

Based on the six real-part formulations, the determinants of $[SM]$ versus frequency parameter λ for the clamped, simply-supported and free circular plates are shown in Figures 2-1 $\sim$ 2-3, respectively. In each figure, we find that the true eigenvalues depends on the specified boundary condition (C, S and F) instead of the six formulations (a, b, c, d, e and f). The spurious eigenvalues are embedded in each formulation as shown in Figures 2-1 $\sim$ 2-3, which satisfy the spurious eigenequations in the Table 2-2. For example, if we use the u, θ formulation to solve the circular plates subject to different boundary conditions (Figures 2-1.(a), 2-2.(a) and 2-3.(a)), the spurious eigenvalues appear at the positions which satisfy the spurious eigenequation $[K_{\ell+1}(\lambda a)Y_\ell(\lambda a) - Y_{\ell+1}(\lambda a)K_\ell(\lambda a)] = 0$ as embedded in Eqs.(2-44), (2-54), (2-66), (2-97), (2-107) and (2-117). In order to distinguish the spurious eigenvalues, Figures 2-4.(a)-(f) show the determinant of $[SM]$ versus λ using the formualtion (*e.g.* u, θ and u, m formulation) to solve the the circular plates subject to different boundary conditions. It is found that any one of the clamped, simply-supported and free cases results in the same spurious eigenvalues, once the formulation (a, b and c - u, θ formulation; d, e and f - u, m formulation) is employed. The numerical results reconfirm that the occurrence of spurious eigenvalues only depends on the formulation instead of the specified boundary condition. All the spurious eigenequations in the real-part BEM are summarized in the the Table 2-2.

Imaginary-part BEM:

Based on the six imaginary-part formulations, the determinant of $[SM]$ versus frequency parameter λ for the clamped, simply-supported and free circular plates are shown in Figures 2-5 $\sim$ 2-7, respectively. In each figure, we find that the true eigenvalues depends on the specified boundary condition (C, S and F) instead of the six formulations (a, b, c, d, e and f). The spurious eigenvalues are embedded in each formulation as shown in Figures 2-5 $\sim$ 2-7, which satisfy the spurious eigenequations in the Table 2-4. For example, if we use the u, θ formulation to solve the circular plates subject to different boundary conditions (Figures 2-5.(a), 2-6.(a) and 2-7.(a)), the spurious eigenvalues appear at the positions which satisfy the spurious eigenequation $[I_{\ell+1}(\lambda a)J_{\ell}(\lambda a)+J_{\ell+1}(\lambda a)I_{\ell}(\lambda a)]=0$ as embedded in Eqs.(2-130), (2-138), (2-146), (2-159), (2-169) and (2-179). In order to distinguish the spurious eigenvalues, Figures 2-8.(a)-(f) show the determinant of $[SM]$ versus λ using the formualtion (*e.g.* u, θ and u, m formulation) to solve the circular plates subject to different boundary conditions. It is found that any one of the clamped, simply-supported and free cases results in the same spurious eigenvalues, once we use the formulation (a, b and c - u, θ formulation; d, e and f - u, m formulation). The numerical results reconfirm that the occurrence of spurious eigenvalues only depends on the formulation instead of the specified boundary condition. All the spurious eigenequations in the imaginary-part BEM are summarized in the Table 2-4.

The numerical results of the true eigenvalues agree well with the data in Leissa [58] by using the real and imaginary-part BEMs as shown in the Tables 2-5, 2-6 and 2-7 for the clamped, simply-supported and free circular plates, respectively.

2-5 Concluding remarks

The real and imaginary-part BEM formulations have been derived for the free vibration of plates. For a circular plate, the true and spurious eigenequations were derived analytically by using the degenerate kernel, Fourier series and circulants in the continuous and discrete systems. The eigenvalues were determined numerically. Since any two equations in the plate formulation (4 equations) can be chosen, C_2^4 (6) options can be considered. The occurrence of spurious eigenequation only depends on the formulation instead of the specified boundary condition, while the true eigenequation is independent of the formulation and is relevant to the specified boundary condition. All the results are shown in the Tables 2-2 and 2-4. Three cases were demonstrated analytically and numerically to see the validity of the present method. Although the circular case lacks generality, it leads significant insight into the occurring mechanism of true and spurious eigenequation. Here, the proof is only limited to the circular case, it is a great help to the researchers who may require analytical explanation to understand why the spurious eigenequation occurs. The same algorithm in the discrete system can be applied to solve arbitrary-shaped plate numerically without any difficulty. Nevertheless, mathematical derivation in the continuous and discrete systems can not be done analytically. How to avoid the occurrence of the spurious eigenvalues will be addressed in the next chapter.

Chapter 3 Treatment of the spurious eigenvalues for simply-connected eigenproblems

Summary

In this chapter, four alternatives (SVD updating technique, the Burton & Miller method, the complex-valued BEM and the CHEEF method) are adopted to suppress the occurrence of the spurious eigenvalues for the simply-connected eigenproblem of plate in the real and imaginary-part BEMs. A clamped case is demonstrated analytically in the continuous and discrete systems.

3-1 SVD updating technique

3-1-1 Continuous system

A conventional approach to detect the nonunique solution is the criterion of satisfying all the Eqs.(2-28)-(2-31) at the same time in the real-part BEM. For the clamped plate ($u = 0$ and $\theta = 0$), the moment and shear force ($m(s)$ and $v(s)$) along the circular boundary, can be expanded into Fourier series as shown in the Eqs.(2-33) and (2-34) in the Chapter 2. By substituting the degenerate kernels into the Eqs.(2-24)-(2-25) and by employing the orthogonality condition of the Fourier series, the Fourier coefficients a_n^c, b_n^c, p_n^c and q_n^c satisfy the Eqs.(2-39)-(2-42) as shown in the Chapter 2. If we employ the Eq.(2-26) to sovle the same eigenproblem, we obtain the Fourier coefficients a_n^c, b_n^c, p_n^c and q_n^c satisfying

$$p_n^c = \frac{[\alpha_n^Y(\lambda a)J_n(\lambda a) + \frac{2}{\pi}\alpha_n^K(\lambda a)I_n(\lambda a)]}{[\alpha_n^Y(\lambda a)J_n'(\lambda a) + \frac{2}{\pi}\alpha_n^K(\lambda a)I_n'(\lambda a)]}a_n^c, \qquad n = 0, 1, 2, \ldots, \tag{3-1}$$

$$q_n^c = \frac{[\alpha_n^Y(\lambda a)J_n(\lambda a) + \frac{2}{\pi}\alpha_n^K(\lambda a)I_n(\lambda a)]}{[\alpha_n^Y(\lambda a)J_n'(\lambda a) + \frac{2}{\pi}\alpha_n^K(\lambda a)I_n'(\lambda a)]}b_n^c, \qquad n = 0, 1, 2, \ldots. \tag{3-2}$$

Similarly, the Eq.(2-27) yields,

$$p_n^c = \frac{[\beta_n^Y(\lambda a)J_n(\lambda a) + \frac{2}{\pi}\beta_n^K(\lambda a)(\lambda a)I_n(\lambda a) + \frac{1-\nu}{a}[\gamma_n^Y(\lambda a)J_n(\lambda a) + \frac{2}{\pi}\gamma_n^K(\lambda a)I_n(\lambda a)]]}{[\beta_n^Y(\lambda a)J_n'(\lambda a) + \frac{2}{\pi}\beta_n^K(\lambda a)(\lambda a)I_n'(\lambda a) + \frac{1-\nu}{a}[\gamma_n^Y(\lambda a)J_n'(\lambda a) + \frac{2}{\pi}\gamma_n^K(\lambda a)I_n'(\lambda a)]]}a_n^c, \quad n = 0, 1, 2, \ldots, \tag{3-3}$$

$$q_n^c = \frac{[\beta_n^Y(\lambda a)J_n(\lambda a) + \frac{2}{\pi}\beta_n^K(\lambda a)(\lambda a)I_n(\lambda a) + \frac{1-\nu}{a}[\gamma_n^Y(\lambda a)J_n(\lambda a) + \frac{2}{\pi}\gamma_n^K(\lambda a)I_n(\lambda a)]]}{[\beta_n^Y(\lambda a)J_n'(\lambda a) + \frac{2}{\pi}\beta_n^K(\lambda a)(\lambda a)I_n'(\lambda a) + \frac{1-\nu}{a}[\gamma_n^Y(\lambda a)J_n'(\lambda a) + \frac{2}{\pi}\gamma_n^K(\lambda a)I_n'(\lambda a)]]} b_n^c, \quad n = 0, 1, 2, \ldots. \tag{3-4}$$

When the eigenvalues satisfy the true eigenequation of

$$\{I_{n+1}(\lambda a)J_n(\lambda a) + J_{n+1}(\lambda a)I_n(\lambda a)\} = 0, \quad n = 0, \pm 1, \pm 2, \cdots, \pm(N-1), N, \tag{3-5}$$

the Eqs.(2-39), (2-41), (3-1) and (3-3) can be simplified to

$$p_n^c = \frac{I(\lambda a)}{\lambda I_n'(\lambda a)} a_n^c, \; n = 0, 1, 2, \ldots. \tag{3-6}$$

In this case, we obtain the nontrivial data of the true boundary mode in the column vector form by employing the Eq.(3-6) as shown below:

$$\begin{Bmatrix} a_0^c \\ p_0^c \\ a_1^c \\ b_1^c \\ p_1^c \\ q_1^c \\ \vdots \\ a_n^c \\ b_n^c \\ p_n^c \\ q_n^c \\ \vdots \\ a_{2N}^c \\ b_{2N}^c \\ p_{2N}^c \\ q_{2N}^c \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ I_n(\lambda a))/\lambda I_n'(\lambda a) \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} a_n^c + \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ I_n(\lambda a))/\lambda I_n'(\lambda a) \\ \vdots \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} b_n^c, \tag{3-7}$$

where a_n^c and b_n^c are arbitrary. The column vector of the true boundary modes are the same by using any one of the Eqs.(2-39), (2-41), (3-1) and (3-3). In case of the spurious eigenvalue, the Eqs.(2-39), (2-41), (3-1) and (3-3) can not obtain the commom term. After recollecting any two terms of the Eqs.(2-39), (2-41), (3-1) and (3-3) by using the recurrence relations of the Bessel function in the Appendix 2, it is found that all the results can be simplified to six different spurious eigenequations as shown in the Table 2-2. The same true eigenequation is commonly imbedded in the six real-part formulations. The only possibility to seek nontrivial data for the generalized coefficients is only the common one (true eigenequation in the Eq.(3-5)) to be zero.

This indicates that only the true eigenequation of the clamped circular plate is sorted out since the true eigenequation is simultaneously embedded in the six real-part formulations. The result matches well with the Eqs.(2-44) and (2-97) in the continuous and discrete systems, respectively. Since we solve the same problem by using the imaginary-part BEM, only the true eigenequation of the clamped circular plate is sorted out for the same reason that the true eigenequation is simultaneously embedded in the six imaginary-part formulations. This is the mathematical meaning of the SVD technique of updating term in the continuous system. We will apply the SVD updating technique in the discrete system.

3-1-2 Discrete system

In the discrete system, the approach to detect the spurious eigensolution is the criterion of satisfying all the Eqs.(2-28)-(2-31) at the same time in the real-part BEM. For the clamped plate ($u = 0$ and $\theta = 0$), the Eqs.(2-28)-(2-31) reduce to

$$0 = [U]\{v\} + [\Theta]\{m\}, \tag{3-8}$$

$$0 = [U_\theta]\{v\} + [\Theta_\theta]\{m\}, \tag{3-9}$$

$$0 = [U_m]\{v\} + [\Theta_m]\{m\}, \tag{3-10}$$

$$0 = [U_v]\{v\} + [\Theta_v]\{m\}. \tag{3-11}$$

After rearranging the terms, the Eqs.(3-8) and (3-9) can be assembled to

$$[SM_1^c]\left\{\begin{array}{c} v \\ m \end{array}\right\} = \{0\}, \tag{3-12}$$

where

$$[SM_1^c] = \left[\begin{array}{cc} U & \Theta \\ U_\theta & \Theta_\theta \end{array}\right]. \tag{3-13}$$

Similarly, the Eqs.(3-10) and (3-11) yield

$$[SM_2^c]\left\{\begin{array}{c} v \\ m \end{array}\right\} = \{0\}, \tag{3-14}$$

where

$$[SM_2^c] = \begin{bmatrix} U_m & \Theta_m \\ U_v & \Theta_v \end{bmatrix}. \tag{3-15}$$

Since the real-part BEM misses the imaginary-part information, we can reconstruct the independent equation by differentiation. To obtain an overdetermined system, we can combine the Eqs.(3-12) and (3-14) by using the SVD technique of updating term as shown below:

$$[C]\left\{\begin{array}{c} v \\ m \end{array}\right\} = \{0\}, \tag{3-16}$$

where

$$[C] = \begin{bmatrix} SM_1^c \\ SM_2^c \end{bmatrix}_{8N\times 4N}. \tag{3-17}$$

Since the eigenequation is nontrivial, the rank of the matrix $[C]$ must be smaller than $4N$, the $4N$ singular values for the matrix $[C]$ must have at least one zero value. The explicit form for the matrix $[C]$ can be decomposed into

$$[C] = \begin{bmatrix} \Phi & 0 & 0 & 0 \\ 0 & \Phi & 0 & 0 \\ 0 & 0 & \Phi & 0 \\ 0 & 0 & 0 & \Phi \end{bmatrix}_{8N\times 8N} \begin{bmatrix} \Sigma_U & \Sigma_\Theta \\ \Sigma_{U_\theta} & \Sigma_{\Theta_\theta} \\ \Sigma_{U_m} & \Sigma_{\Theta_m} \\ \Sigma_{U_v} & \Sigma_{\Theta_v} \end{bmatrix}_{8N\times 4N} \begin{bmatrix} \Phi^T & 0 \\ 0 & \Phi^T \end{bmatrix}_{4N\times 4N}. \tag{3-18}$$

Based on the equivalence between the SVD technique and the least-squares method in mathematical essence, the least squares form leads to

$$[C]^T[C] = \begin{bmatrix} \Phi & 0 \\ 0 & \Phi \end{bmatrix}_{4N\times 4N} [D]_{4N\times 4N} \begin{bmatrix} \Phi & 0 \\ 0 & \Phi \end{bmatrix}^T_{4N\times 4N} \tag{3-19}$$

where

$$[D] = \begin{bmatrix} \Sigma_U & \Sigma_{U_\theta} & \Sigma_{U_m} & \Sigma_{U_v} \\ \Sigma_\Theta & \Sigma_{\Theta_\theta} & \Sigma_{\Theta_m} & \Sigma_{\Theta_v} \end{bmatrix}_{4N\times 8N} \begin{bmatrix} \Sigma_U & \Sigma_\Theta \\ \Sigma_{U_\theta} & \Sigma_{\Theta_\theta} \\ \Sigma_{U_m} & \Sigma_{\Theta_m} \\ \Sigma_{U_v} & \Sigma_{\Theta_v} \end{bmatrix}_{8N\times 4N} \tag{3-20}$$

If the determinant of the matrix $[C]^T[C]$ is zero, we can obtain the nontrivial solution. Since Φ is orthogonal, the determinant of the matrix $[C]^T[C]$ is equal to the determinant of the matrix $[D]$. By calculating the determinant of the matrix $[D]$, we have

$$\begin{aligned} det[D] = &\prod_{\ell=-(N-1)}^{N} \\ &[(\mu_\ell^{[U]}\kappa_\ell^{[\Theta]} - \mu_\ell^{[\Theta]}\kappa_\ell^{[U]})^2 + (\mu_\ell^{[U]}\zeta_\ell^{[\Theta]} - \mu_\ell^{[\Theta]}\zeta_\ell^{[U]})^2 + (\mu_\ell^{[U]}\delta_\ell^{[\Theta]} - \mu_\ell^{[\Theta]}\delta_\ell^{[U]})^2 \\ &+ (\kappa_\ell^{[U]}\zeta_\ell^{[\Theta]} - \kappa_\ell^{[\Theta]}\zeta_\ell^{[U]})^2 + (\kappa_\ell^{[U]}\delta_\ell^{[\Theta]} - \kappa_\ell^{[\Theta]}\delta_\ell^{[U]})^2 + (\zeta_\ell^{[U]}\delta_\ell^{[\Theta]} - \zeta_\ell^{[\Theta]}\delta_\ell^{[U]})^2], \end{aligned} \tag{3-21}$$

where $\zeta_\ell^{[U]}$, $\zeta_\ell^{[\Theta]}$, $\delta_\ell^{[\Theta]}$ and $\delta_\ell^{[U]}$ are the eigenvalues of the matrics $[U_m]$, $[\Theta_m]$, $[U_v]$ and $[\Theta_v]$, respectively. The only possibility for the zero determinant of the matrix $[D]$ occurs when the six terms, $(\mu_\ell^{[U]}\kappa_\ell^{[\Theta]} - \mu_\ell^{[\Theta]}\kappa_\ell^{[U]})$, $(\mu_\ell^{[U]}\zeta_\ell^{[\Theta]} - \mu_\ell^{[\Theta]}\zeta_\ell^{[U]})$, $(\mu_\ell^{[U]}\delta_\ell^{[\Theta]} - \mu_\ell^{[\Theta]}\delta_\ell^{[U]})$, $(\kappa_\ell^{[U]}\zeta_\ell^{[\Theta]} - \kappa_\ell^{[\Theta]}\zeta_\ell^{[U]})$, $(\kappa_\ell^{[U]}\delta_\ell^{[\Theta]} - \kappa_\ell^{[\Theta]}\delta_\ell^{[U]})$ and $(\zeta_\ell^{[U]}\delta_\ell^{[\Theta]} - \zeta_\ell^{[\Theta]}\delta_\ell^{[U]})$ are all zeros at the same time for the same ℓ. Here we can find that the six terms exactly result in the six different spurious eigenequations as shown in the Table 2-2, and the same true eigenequation is commonly imbedded in the six real-part formulations. The only possibility for the zero determinant of the matrix $[D]$ is the common term (true eigenequation in the Eq.(3-5)) to be zero.

This indicates that only the true eigenequation of the clamped circular plate is sorted out in the SVD updating matrix since the true eigenequation is simultaneously embedded in the six real-part formulations (the Eq.(3-21)). The result matches well with the Eqs.(2-44) and (2-97) in the continuous and discrete systems, respectively. Since we solve the same problem by using the imaginary-part BEM, only the true eigenequation of the clamped circular plate is sorted out for the same reason that the true eigenequation is simultaneously embedded in the six imaginary-part formulations.

3-2 Burton & Miller method and the complex-valued BEM

In the exterior acoustics of Helmholtz equation by using the dual BEM, Burton & Miller utilized the product of hypersingular equation with an imaginary constant and added the singular equation to deal with the fictitious-frequency problem which results in a non-uniqueness solution. We will extend this concept to suppress the appearence of the spurious eigenequation of simply-connected plate in the real-part or imaginary-part BEM.

3-2-1 Continuous system

For the clamped circular plate with a radius a, combination of the Eqs.(2-24) and (2-26) with an imaginary number by using the real-part BEM yields

$$0 = -\int_B [U(s,x) + iU_m(s,x)]v(s)\,dB(s) + \int_B [\Theta(s,x) + i\Theta_m(s,x)]m(s)\,dB(s) \quad \text{(3-22)}$$

Similarly, the Eqs.(2-25) and (2-27) yield,

$$0 = -\int_B [U_\theta(s,x) + iU_v(s,x)]v(s)\,dB(s) + \int_B [\Theta_\theta(s,x) + i\Theta_v(s,x)]m(s)\,dB(s) \quad \text{(3-23)}$$

The moment and shear force, $m(s)$ and $v(s)$ along the circular boundary, can be expanded into Fourier series as shown in the Eqs.(2-33) and (2-34) in the Chapter 2. By using the degenerate kernels into the Eq.(3-22) and by employing the orthogonality condition of the Fourier series, the Fourier coefficients a_n^c, b_n^c, p_n^c and q_n^c satisfy

$$p_n^c = \frac{[Y_n(\lambda a)J_n(\lambda a) + \frac{2}{\pi}K_n(\lambda a)I_n(\lambda a)] + i[\alpha_n^Y(\lambda a)J_n(\lambda a) + \frac{2}{\pi}\alpha_n^K(\lambda a)I_n(\lambda a)]}{[Y_n(\lambda a)J_n'(\lambda a) + \frac{2}{\pi}K_n(\lambda a)I_n'(\lambda a)] + i[\alpha_n^Y(\lambda a)J_n'(\lambda a) + \frac{2}{\pi}\alpha_n^K(\lambda a)I_n'(\lambda a)]} a_n^c,$$
$$n = 0, 1, 2, \ldots, \quad \text{(3-24)}$$

$$q_n^c = \frac{[Y_n(\lambda a)J_n(\lambda a) + \frac{2}{\pi}K_n(\lambda a)I_n(\lambda a)] + i[\alpha_n^Y(\lambda a)J_n(\lambda a) + \frac{2}{\pi}\alpha_n^K(\lambda a)I_n(\lambda a)]}{[Y_n(\lambda a)J_n'(\lambda a) + \frac{2}{\pi}K_n(\lambda a)I_n'(\lambda a)] + i[\alpha_n^Y(\lambda a)J_n'(\lambda a) + \frac{2}{\pi}\alpha_n^K(\lambda a)I_n'(\lambda a)]} b_n^c,$$
$$n = 0, 1, 2, \ldots. \quad \text{(3-25)}$$

Similarly, the Eq.(3-23) yields,

$$p_n^c = \frac{\lambda[Y_n'(\lambda a)J_n(\lambda a) + \frac{2}{\pi}K_n'(\lambda a)I_n(\lambda a)]}{\lambda[Y_n'(\lambda a)J_n'(\lambda a) + \frac{2}{\pi}K_n'(\lambda a)I_n'(\lambda a)]}$$
$$\frac{+i[\beta_n^Y(\lambda a)J_n(\lambda a) + \frac{2}{\pi}\beta_n^K(\lambda a)I_n(\lambda a) + \frac{1-\nu}{a}[\gamma_n^Y(\lambda a)J_n(\lambda a) + \frac{2}{\pi}\gamma_n^K(\lambda a)(\lambda a)I_n(\lambda a)]]}{+i[\beta_n^Y(\lambda a)J_n'(\lambda a) + \frac{2}{\pi}\beta_n^K(\lambda a)I_n'(\lambda a) + \frac{1-\nu}{a}[\gamma_n^Y(\lambda a)J_n'(\lambda a) + \frac{2}{\pi}\gamma_n^K(\lambda a)I_n'(\lambda a)]]} a_n^c,$$
$$n = 0, 1, 2, \ldots, \quad \text{(3-26)}$$

$$
\begin{aligned}
q_n^c =& \frac{\lambda[Y_n'(\lambda a)J_n(\lambda a)+\frac{2}{\pi}K_n'(\lambda a)I_n(\lambda a)]}{\lambda[Y_n'(\lambda a)J_n'(\lambda a)+\frac{2}{\pi}K_n'(\lambda a)I_n'(\lambda a)]} \\
& \frac{+i[\beta_n^Y(\lambda a)J_n(\lambda a)+\frac{2}{\pi}\beta_n^K(\lambda a)I_n(\lambda a)+\frac{1-\nu}{a}[\gamma_n^Y(\lambda a)J_n(\lambda a)+\frac{2}{\pi}\gamma_n^K(\lambda a)(\lambda a)I_n(\lambda a)]]}{+i[\beta_n^Y(\lambda a)J_n'(\lambda a)+\frac{2}{\pi}\beta_n^K(\lambda a)I_n'(\lambda a)+\frac{1-\nu}{a}[\gamma_n^Y(\lambda a)J_n'(\lambda a)+\frac{2}{\pi}\gamma_n^K(\lambda a)I_n'(\lambda a)]]} b_n^c, \\
& n=0,1,2,\ldots.
\end{aligned}
\tag{3-27}
$$

To seek nontrivial data for the generalized coefficients of a_n^c, p_n^c, b_n^c and q_n^c, we can obtain the eigenequation by using either the Eqs.(3-24) and (3-26) or the Eqs.(3-25) and (3-27)

$$
\begin{aligned}
& \frac{\lambda[Y_n'(\lambda a)J_n(\lambda a)+\frac{2}{\pi}K_n'(\lambda a)I_n(\lambda a)]}{\lambda[Y_n'(\lambda a)J_n'(\lambda a)+\frac{2}{\pi}K_n'(\lambda a)I_n'(\lambda a)]} \\
& \frac{+i[\beta_n^Y(\lambda a)J_n(\lambda a)+\frac{2}{\pi}\beta_n^K(\lambda a)I_n(\lambda a)+\frac{1-\nu}{a}[\gamma_n^Y(\lambda a)J_n(\lambda a)+\frac{2}{\pi}\gamma_n^K(\lambda a)(\lambda a)I_n(\lambda a)]]}{+i[\beta_n^Y(\lambda a)J_n'(\lambda a)+\frac{2}{\pi}\beta_n^K(\lambda a)I_n'(\lambda a)+\frac{1-\nu}{a}[\gamma_n^Y(\lambda a)J_n'(\lambda a)+\frac{2}{\pi}\gamma_n^K(\lambda a)I_n'(\lambda a)]]} \\
=& \frac{\lambda[Y_n'(\lambda a)J_n(\lambda a)+\frac{2}{\pi}K_n'(\lambda a)I_n(\lambda a)]}{\lambda[Y_n'(\lambda a)J_n'(\lambda a)+\frac{2}{\pi}K_n'(\lambda a)I_n'(\lambda a)]} \\
& \frac{+i[\beta_n^Y(\lambda a)J_n(\lambda a)+\frac{2}{\pi}\beta_n^K(\lambda a)I_n(\lambda a)+\frac{1-\nu}{a}[\gamma_n^Y(\lambda a)J_n(\lambda a)+\frac{2}{\pi}\gamma_n^K(\lambda a)(\lambda a)I_n(\lambda a)]]}{+i[\beta_n^Y(\lambda a)J_n'(\lambda a)+\frac{2}{\pi}\beta_n^K(\lambda a)I_n'(\lambda a)+\frac{1-\nu}{a}[\gamma_n^Y(\lambda a)J_n'(\lambda a)+\frac{2}{\pi}\gamma_n^K(\lambda a)I_n'(\lambda a)]]}
\end{aligned}
\tag{3-28}
$$

After recollecting the terms by using the recurrence relations of the Bessel function in the Appendix 2, the Eq.(3-28) can be simplified to

$$
[A(\lambda)+iB(\lambda)]\{I_{n+1}(\lambda a)J_n(\lambda a)+J_{n+1}(\lambda a)I_n(\lambda a)\}=0 \tag{3-29}
$$

Since the term $[A(\lambda)+iB(\lambda)]$ is never zero for any λ, we can obtain the true eigenvalues by using the real-part BEM in conjunction with the Burton & Miller concept. Nevertheless, if we combine the u, θ and m, v formulations or u, v and θ, m formulations, the method fails. The reason is that the u, v and θ, m formulation have the same spurious eigenequation. It occurs that $A(\lambda)$ ($B(\lambda)$) may always be zero for any λ, this results in the spurious eigenvalue since the other coefficient ($B(\lambda)$ ($A(\lambda)$)) may be zero. Only the combination of u, m and θ, v real-part formulation can obtain the true eigenvalues. All the explicit forms of the $[A(\lambda)+iB(\lambda)]$ are shown in the Table 3-1 by using the real-part BEM. Similarly, we can use the imaginary-part BEM in conjunction with the Burton & Miller concept to solve the same problem. Only the combination of u, m and θ, v imaginary-part formulations can obtain the true eigenvalues. All the results of the $[A(\lambda) + iB(\lambda)]$ are shown in the Table 3-2 by using the imaginary-part BEM.

Since the real (resp. imaginary)-part BEM misses imaginary (resp. real)-part information, we can reconstruct the independent equation by adding the other real (resp. imaginary)-part BEM multiplied by an imaginary unit. By employing the Burton & Miller concept to combine the real and imaginary-part for the same formulation (*e. g.* u, θ formulae), the complex-valued BEM can be treated as a special case of Burton & Miller method. This indicates that Burton & Miller method and the complex-valued BEM are mathematical equivalent if we choose the same formulation (*e. g.* u, θ formulae). In this case, we construct the real-part formulation (u, θ formulae) and combine with m, v formulae by multiplying an imaginary unit.

Now, we consider the complex-valued BEM to solve the same problem by using

$$U(s,x) = U_c(s,x) = \frac{1}{8\lambda^2}[(Y_0(\lambda r) + iJ_0(\lambda r)) + \frac{2}{\pi}(K_0(\lambda r) + iI_0(\lambda r))] \tag{3-30}$$

By using the degenerate kernels into the Eq.(3-22) and by employing the orthogonality condition of the Fourier series for the clamped case, the Fourier coefficients a_n^c, b_n^c, p_n^c and q_n^c satisfy

$$p_n^c = \frac{1}{\lambda}\frac{[Y_n(\lambda a)J_n(\lambda a) + \frac{2}{\pi}K_n(\lambda a)I_n(\lambda a)] + i[J_n(\lambda a)J_n(\lambda a) + \frac{2}{\pi}I_n(\lambda a)I_n(\lambda a)]}{[Y_n(\lambda a)J_n'(\lambda a) + \frac{2}{\pi}K_n(\lambda a)I_n'(\lambda a)] + i[J_n(\lambda a)J_n'(\lambda a) + \frac{2}{\pi}I_n(\lambda a)I_n'(\lambda a)]}a_n^c, \quad n = 0, 1, 2, \ldots, \tag{3-31}$$

$$q_n^c = \frac{1}{\lambda}\frac{[Y_n(\lambda a)J_n(\lambda a) + \frac{2}{\pi}K_n(\lambda a)I_n(\lambda a)] + i[J_n(\lambda a)J_n(\lambda a) + \frac{2}{\pi}I_n(\lambda a)I_n(\lambda a)]}{[Y_n(\lambda a)J_n'(\lambda a) + \frac{2}{\pi}K_n(\lambda a)I_n'(\lambda a)] + i[J_n(\lambda a)J_n'(\lambda a) + \frac{2}{\pi}I_n(\lambda a)I_n'(\lambda a)]}b_n^c, \quad n = 0, 1, 2, \ldots. \tag{3-32}$$

Similarly, the Eq.(3-23) yields,

$$p_n^c = \frac{1}{\lambda}\frac{[Y_n'(\lambda a)J_n(\lambda a)+\frac{2}{\pi}K_n'(\lambda a)I_n(\lambda a)]+i[J_n(\lambda a)J_n(\lambda a)+\frac{2}{\pi}I_n(\lambda a)I_n(\lambda a)]}{[Y_n'(\lambda a)J_n'(\lambda a)+\frac{2}{\pi}K_n'(\lambda a)I_n'(\lambda a)]+i[J_n(\lambda a)J_n'(\lambda a)+\frac{2}{\pi}I_n(\lambda a)I_n'(\lambda a)]}a_n^c,$$
$$n=0,1,2,\ldots, \tag{3-33}$$

$$q_n^c = \frac{1}{\lambda}\frac{[Y_n'(\lambda a)J_n(\lambda a)+\frac{2}{\pi}K_n'(\lambda a)I_n(\lambda a)]+i[J_n(\lambda a)J_n(\lambda a)+\frac{2}{\pi}I_n(\lambda a)I_n(\lambda a)]}{[Y_n'(\lambda a)J_n'(\lambda a)+\frac{2}{\pi}K_n'(\lambda a)I_n'(\lambda a)]+i[J_n(\lambda a)J_n'(\lambda a)+\frac{2}{\pi}I_n(\lambda a)I_n'(\lambda a)]}b_n^c,$$
$$n=0,1,2,\ldots. \tag{3-34}$$

To seek nontrivial data for the generalized coefficients of a_n^c, p_n^c, b_n^c and q_n^c, we can obtain the eigenequation by using either the Eqs.(3-24) and (3-33) or the Eqs.(3-32) and (3-34)

$$\begin{aligned}&\frac{[Y_n(\lambda a)J_n(\lambda a)+\frac{2}{\pi}K_n(\lambda a)I_n(\lambda a)]+i[J_n(\lambda a)J_n(\lambda a)+\frac{2}{\pi}I_n(\lambda a)I_n(\lambda a)]}{[Y_n(\lambda a)J_n'(\lambda a)+\frac{2}{\pi}K_n(\lambda a)I_n'(\lambda a)]+i[J_n(\lambda a)J_n'(\lambda a)+\frac{2}{\pi}I_n(\lambda a)I_n'(\lambda a)]}\\ =&\frac{[Y_n'(\lambda a)J_n(\lambda a)+\frac{2}{\pi}K_n'(\lambda a)I_n(\lambda a)]+i[J_n(\lambda a)J_n(\lambda a)+\frac{2}{\pi}I_n(\lambda a)I_n(\lambda a)]}{[Y_n'(\lambda a)J_n'(\lambda a)+\frac{2}{\pi}K_n'(\lambda a)I_n'(\lambda a)]+i[J_n(\lambda a)J_n'(\lambda a)+\frac{2}{\pi}I_n(\lambda a)I_n'(\lambda a)]}\end{aligned} \tag{3-35}$$

After recollecting the terms by using the recurrence relations of the Bessel function in the Appendix 2, the Eq.(3-28) can be simplified to

$$[A(\lambda)+iB(\lambda)]\{I_{n+1}(\lambda a)J_n(\lambda a)+J_{n+1}(\lambda a)I_n(\lambda a)\}=0 \tag{3-36}$$

Since the term $[A(\lambda)+iB(\lambda)]$ is never zero for any λ, we can obtain the true eigenvalues by using the complex-valued BEM. All the explicit forms of the $[A(\lambda)+iB(\lambda)]$ are shown in the Table 3-3 by using the complex-valued BEM.

3-2-2 Discrete system

By combining the Eqs.(3-12) and (3-14) with an imaginary number in the real-part BEM, we have

$$[[SM_1^c]+i[SM_2^c]]\left\{\begin{array}{c} v \\ m \end{array}\right\}=0. \tag{3-37}$$

The determinant of the $[SM_1^c] + i[SM_2^c]$ is obtained by using the circulant as

$$det[[SM_1^c] + i[SM_2^c]] = \prod_{\ell=-(N-1)}^{N} [A(\lambda) + iB(\lambda)]\{I_{\ell+1}(\lambda a)J_\ell(\lambda a) + J_{\ell+1}(\lambda a)I_\ell(\lambda a)\} \tag{3-38}$$

Since the term $[A(\lambda) + iB(\lambda)]$ is never zero for any λ, we can obtain the true eigenvalues by using the real-part BEM in conjunction with the Burton & Miller concept. Nevertheless, if we combine the u, θ and m, v formulations or u, v and θ, m formulations, the method fails. The reason is that the u, v and θ, m formulation have the same spurious eigenequation. It occurs that $A(\lambda)$ ($B(\lambda)$) may always be zero for any λ, this results in the spurious eigenvalue since the other coefficient ($B(\lambda)$ ($A(\lambda)$)) may be zero. Only the combination of u, m and θ, v real-part formulation can obtain the true eigenvalues. All the explicit forms of the $[A(\lambda) + iB(\lambda)]$ are shown in the Table 3-1 for the real-part BEM. Similarly, we can use the imaginary-part BEM in conjunction with the Burton & Miller concept to solve the same problem. Only the combination of u, m and θ, v imaginary-part formulation can obtain the true eigenvalues. All the explicit forms of the $[A(\lambda) + iB(\lambda)]$ are shown in the Table 3-2 for the imaginary-part BEM.

Since the real (resp. imaginary)-part BEM misses the imaginary (resp. real)-part information, we can reconstruct the independent equation by adding the other real (resp. imaginary)-part BEM with multiplication an imaginary unit. By employing the Burton & Miller concept to combine the real and imaginary-part for the same formulation (*e. g.* u, θ formulae), the complex-valued BEM can be treated as a special case of Burton & Miller method. This indicates that Burton & Miller method and the complex-valued BEM are mathematical equivalent if we choose the same formulation (*e. g.* u, θ formulae). In this case, we construct one real-part formulation (u, θ formulae) and combine with m, v formulae by multiplying an imaginary number.

Now, we consider the complex-valued BEM to solve the same problem using

$$U(s,x) = U_c(s,x) = \frac{1}{8\lambda^2}[(Y_0(\lambda r) + iJ_0(\lambda r)) + \frac{2}{\pi}(K_0(\lambda r) + iI_0(\lambda r))] \tag{3-39}$$

For the clamped circular plate ($u=0$ and $\theta=0$), we have

$$[SM]\left\{\begin{array}{c} v \\ m \end{array}\right\} = 0. \tag{3-40}$$

We can also obtain the determinant of the $[SM]$ by using the circulant as

$$det[SM] = \prod_{\ell=-(N-1)}^{N}\{I_{\ell+1}(\lambda a)J_\ell(\lambda a) + J_{\ell+1}(\lambda a)I_\ell(\lambda a)\}[A+iB] \tag{3-41}$$

Since the term $[A(\lambda)+iB(\lambda)]$ is never zero for any λ, we can obtain the true eigenvalues by using the complex-valued BEM. All the explicit forms of the $[A(\lambda)+iB(\lambda)]$ are shown in the Table 3-3 for the complex-valued BEM.

3-3 CHEEF method

3-3-1 Continuous system

By substituting the degenerate kernels in the real-part BEM of the Eqs.(2-37) and (2-38) for the interior point $(0<\rho<a)$ and the relationship between the Fourier coefficients of the Eq.(2-39) into u formula in the Eq.(2-6), we have

$$\begin{aligned} u(\rho,\phi) =&\{\frac{J_n(\lambda\rho)[Y_n(\lambda a)(Y_n(\lambda a)J_{n+1}(\lambda a)-Y_{n+1}(\lambda a)J_n(\lambda a))-\frac{2}{\pi}K_n(\lambda a)(I_{n+1}(\lambda a)Y_n(\lambda a)+Y_{n+1}(\lambda a)I_n(\lambda a))]}{8\lambda^2[Y_n(\lambda a)J_n'(\lambda a)+\frac{2}{\pi}K_n(\lambda a)I_n'(\lambda a)]} \\ &+\frac{I_n(\lambda\rho)[Y_n(\lambda a)(J_{n+1}(\lambda a)K_n(\lambda a)-K_{n+1}(\lambda a)J_n(\lambda a))-\frac{2}{\pi}K_n(\lambda a)(\frac{2}{\pi}I_{n+1}(\lambda a)K_n(\lambda a)+\frac{2}{\pi}K_{n+1}(\lambda a)I_n(\lambda a))]}{8\lambda^2[Y_n(\lambda a)J_n'(\lambda\rho)+\frac{2}{\pi}K_n(\lambda a)I_n'(\lambda\rho)]}\} \\ &(p_n^c cos(n\phi)+q_n^c sin(n\phi)),\ 0<\rho<a,\ 0\le\phi<2\pi. \end{aligned} \tag{3-42}$$

For the exterior point $(a<\rho)$, the null-field integral equation yields

$$\begin{aligned} 0 =&\frac{[J_n(\lambda a)I_{n+1}(\lambda a)+I_n(\lambda a)J_{n+1}(\lambda a)][\frac{2}{\pi}Y_n(\lambda a)K_n(\lambda\rho)-\frac{2}{\pi}K_n(\lambda a)Y_n(\lambda\rho)]}{8\lambda^2[Y_n(\lambda a)J_n'(\lambda a)+\frac{2}{\pi}K_n(\lambda a)I_n'(\lambda a)]} \\ &(p_n^c cos(n\phi)+q_n^c sin(n\phi)),\ a<\rho,\ 0\le\phi<2\pi. \end{aligned} \tag{3-43}$$

When the true eigenvalue λ satisfies the Eq.(3-5), we can find that the field of the interior $(\rho<a)$ and exterior $(a<\rho)$ points are the nontrivial solution and null-field, respectively. It is found that the solution of the exterior point $(a<\rho)$ is not a null-field for the spurious eigenvalue. This provides us a clue to filter out the spurious eigenvalue. By this way, the exterior point can be chosen to obtain an independent constraint in the null-field equation in order to suppress the occurrence of the spurious eigenvalues for the simply-connected plate eigenproblem. In another words, the Eq.(3-43) can be a discriminant for the spurious eigenvalue once the null-field equation is not satisfied.

In this section, we employ the CHEEF method to deal with the problem of spurious eigenvalues. Firstly, we choose a CHEEF point (ρ_1, ϕ_1) outside the domain $(a < \rho_1,\, 0 < \phi_1 < 2\pi)$. By substituting the CHEEF point into the Eq.(3-43), we have

$$0 = \frac{[J_n(\lambda a)I_{n+1}(\lambda a) + I_n(\lambda a)J_{n+1}(\lambda a)][\frac{2}{\pi}Y_n(\lambda a)K_{n+1}(\lambda\rho_1) - \frac{2}{\pi}K_n(\lambda a)Y_{n+1}(\lambda\rho_1)]}{8\lambda^2[Y_n(\lambda a)J_n'(\lambda a) + \frac{2}{\pi}K_n(\lambda a)I_n'(\lambda a)]} (p_n^c cos(n\phi_1) + q_n^c sin(n\phi_1)) \tag{3-44}$$

Comparing the Eq.(3-44) with the true eigenequation in the Eq.(3-5), the Eq.(3-44) shows the consistency of the null-field when $a < \rho_1$. For the spurious eigenvalue, the Eq.(3-44) can not be satisfied once $[Y_n(\lambda a)K_n(\lambda\rho_1) - K_n(\lambda a)Y_n(\lambda\rho_1)] \neq 0$. By this way, the Eq.(3-44) can provide the independent constraint to detect the spurious eigenequation $[Y_n(\lambda a)K_{n+1}(\lambda a) - K_n(\lambda a)Y_{n+1}(\lambda a)] = 0$ if $[Y_n(\lambda a)K_n(\lambda\rho_1) - K_n(\lambda a)Y_n(\lambda\rho_1)] \neq 0$. Because one added point supplies at most one constraint, an additional point is required for the eigenvalues of multiplicity two, in order to obtain sufficient constraints. Therefore, we add another point (ρ_2, ϕ_2) in the complementary domain $(a < \rho_1,\, 0 < \phi_1 < 2\pi)$. Substituting the field point (ρ_2, ϕ_2) into the Eq.(3-43), we have

$$0 = \frac{[J_n(\lambda a)I_{n+1}(\lambda a) + I_n(\lambda a)J_{n+1}(\lambda a)][\frac{2}{\pi}Y_n(\lambda a)K_n(\lambda\rho_2) - \frac{2}{\pi}K_n(\lambda a)Y_n(\lambda\rho_2)]}{8\lambda^2[Y_n(\lambda a)J_n'(\lambda a) + \frac{2}{\pi}K_n(\lambda a)I_n'(\lambda a)]} (p_n^c cos(n\phi_2) + q_n^c sin(n\phi_2)) \tag{3-45}$$

To seek nontrivial data for the generalized coefficients of p_n^c and q_n^c by using the Eqs.(3-44) and (3-45), we have

$$[W] = \begin{bmatrix} \{\frac{2}{\pi}Y_n(\lambda a)K_n(\lambda\rho_1) - \frac{2}{\pi}K_n(\lambda a)Y_n(\lambda\rho_1)\}cos(n\phi_1) & \{\frac{2}{\pi}Y_n(\lambda a)K_n(\lambda\rho_1) - \frac{2}{\pi}K_n(\lambda a)Y_n(\lambda\rho_1)\}sin(n\phi_1) \\ \{\frac{2}{\pi}Y_n(\lambda a)K_n(\lambda\rho_2) - \frac{2}{\pi}K_n(\lambda a)Y_n(\lambda\rho_2)\}cos(n\phi_2) & \{\frac{2}{\pi}Y_n(\lambda a)K_n(\lambda\rho_2) - \frac{2}{\pi}K_n(\lambda a)Y_n(\lambda\rho_2)\}sin(n\phi_2) \end{bmatrix} \tag{3-46}$$

If the determinant of the matrix $[W]$ is zero, the coefficients p_n^c and q_n^c can be arbitrary; *i.e.*, the Eqs.(3-44) and (3-45) do not provide two independent constraints. In this case, the intersection angle $(\phi_1 - \phi_2)$ between the two selected points satisfying

$$\sin n(\phi_1 - \phi_2) = 0, \text{ or } \phi_1 - \phi_2 = \frac{\pi}{n}, \ n = 1, 2, 3, \cdots, \tag{3-47}$$

makes the two equations dependent. Besides, the CHEEF points (ρ_1, ϕ_1) and (ρ_2, ϕ_2) should not satisfy $[Y_n(\lambda a)K_n(\lambda\rho_1) - K_n(\lambda a)Y_n(\lambda\rho_1)] = 0$ and $[Y_n(\lambda a)K_n(\lambda\rho_2) - K_n(\lambda a)Y_n(\lambda\rho_2)] = 0$. Therefore, we must avoid this point in order to effectively filter out the spurious eigenvalues of multiplicity two.

Here, we extend the concept of the CHEEF method from interior acoustics to the plate vibration. The details for the CHEEF method will be elaborated on the discrete system in real computations.

3-3-2 Discrete system

Consider the eigenproblem for the clamped plate, the Eqs.(2-24) and (2-25) can be rewritten as

$$\begin{bmatrix} U & \Theta \\ U_\theta & \Theta_\theta \end{bmatrix}_{4N\times 4N} \begin{Bmatrix} v \\ m \end{Bmatrix}_{4N\times 1} = \{0\}_{4N\times 1}, \tag{3-48}$$

By moving the field point x to be the CHEEF point outside the domain, we have

$$\begin{bmatrix} U^C & \Theta^C \\ U_\theta^C & \Theta_\theta^C \end{bmatrix}_{2N_C\times 4N} \begin{Bmatrix} v \\ m \end{Bmatrix}_{4N\times 1} = \{0\}_{4N\times 1}, \tag{3-49}$$

where the superscript C denotes the CHEEF point in the null-field equation and the subscript N_c (≥ 1) indicates the number of additional CHEEF points. Combining the Eqs.(3-48) and (3-49) together to obtain the overdetermined system, we have

$$[C^*] \begin{Bmatrix} v \\ m \end{Bmatrix}_{4N\times 1} = \{0\}_{(4N+2N_C)\times 1}, \tag{3-50}$$

where

$$[C^*] = \begin{bmatrix} U & \Theta \\ U_\theta & \Theta_\theta \\ \hline U^C & \Theta^C \\ U_\theta^C & \Theta_\theta^C \end{bmatrix}_{(4N+2N_C)\times 4N} \tag{3-51}$$

Therefore, an overdetermined system is obtained to ensure a unique solution. According to the successful experience of CHEEF technique for interior acoustics, we can overcome the spurious-eigenvalue problem for the simply-connected plate eigenproblem by using the real or imaginary BEM. Also, the optimum number of the CHEEF points and their appropriate positions will be addressed in the numerical results.

The concept of the CHEEF method and SVD technique of updating term are the same in constructing an overdetermined system to solve a unique solution. From the computation point of view, CHEEF method used the minimum number of dimension than that of SVD technique of updating term although it may take risk for the failure CHEEF points. In another words, it works to overcome the spurious eigenproblems by using only real-part or imaginary-part formulation, if we provide enough constraints by adding the CHEEF points. Because the CHEEF method has used the Hilbert-transform relation between the real-part and imaginary-part kernels. Therefore, the conventional method (complex-valued BEM) to solve the eigenproblem seems to overlook the Hilbert-transform relation and take too much computation.

3-4 Numerical results and discussions

Circular plate (clamped, simply-supported and free boundary conditions)

A circular plate with a radius of one meter $(a = 1\,m)$ and the Poisson ratio $\nu = 0.33$ are considered. The circular boundary is discretized into ten constant elements. Since any two equations in the plate formulation (the Eqs.(2-28)-(2-31)) can be chosen, $6(C_2^4)$ options of the formulation can be considered. The numerical results of the four methods to suppress the spurious are shown below:

SVD updating technique

Figures 3-1.(a)-(f) show the determinant of the $[C]^T[C]$ versus λ for the clamped, simply-supported and free circular plates using the six real-part formulations in conjunction with the SVD technique of updating term. It is found that the spurious eigenvalues are filtered out and only the true eigenvalues appear as predicted in Eq.(3-21) for the clamped case. Figures 3-2.(a)-(f) show the determinant of the $[C]^T[C]$ versus λ for the clamped, simply-supported and free circular using the six imaginary-part formulations in conjunction with the SVD technique of updating term. Similarly, only the true eigenvalues are obtained without contaimination of the spurious eigenvalues. Good agreement is made.

Burton & Miller method and the complex-valued BEM

Figures 3-3 $\sim$ 3-5 show the determinant of the $[SM]$ versus λ for the clamped, simply-supported and free circular plates using the six real-part formulations in conjunction with Burton & Miller concept. The failure cases are shown in the (a), (c), (d) and (f) cases as predicted in the Table 3-1. Only the combination of u, m and θ, v formulation can obtain the true eigenvalues in Figures 3-3.(b), 3-3.(e), 3-4.(b), 3-4.(e), 3-5.(b) and 3-5.(e) as predicted in the Table 3-1, since $A(\lambda)$ and $B(\lambda)$ can not be zero at the same time. For the case by using the u, θ formualtion in conjunction the m, v formulations after multiplying an imaginary number for solving the circular plates subject to different boundary conditions (3-3.(a), 3-4.(a) and 3-5.(a)), the spurious eigenvalues occur since $A(\lambda)$ may be zero for the spurious eigenvalues and $B(\lambda)$ is always zero in the Table 3-2. Figures 3-6 $\sim$ 3-8 show the determinant of the $[SM]$ versus λ for the clamped, simply-supported and free circular plates using the six imaginary-part formulations in conjunction with the Burton & Miller concept. The failure cases are shown in the (a), (c), (d) and (f) as predicted in the Table 3-2. It must be noted that the spurious eigenvlaues still occur in Figures 3-8(a) and 3-8(f) for the free circular plate by using the imaginary-part BEM in conjunction with the Burton & Miller concept. The spurious eigenequations of u, θ and m, v imaginary formulation are imbedded in the term $A(\lambda)$ by using the combination of u, θ and m, v imaginary formulation, and the spurious eigenequation of m, v imaginary formulation are the same with the true eigenequation of the free circular plate. The true and spurious eigenvalues are very close in the plot range ($0 < \lambda < 8$) by using u, θ and m, v imaginary-part formulations in conjunction with the Burton & Miller concept. Only the combination of u, m and θ, v formulation can obtain the true eigenvalues in the Figures 3-6.(b), 3-6.(e), 3-7.(b), 3-7.(e), 3-8.(b) and 3-8.(e) as predicted in the Table 3-1, since $A(\lambda)$ and $B(\lambda)$ can not be zero at the same time. For the case by using the u, θ formualtion in conjunction the m, v formulations after multiplying an imaginary number for solving the circular plates subject to different boundary conditions (3-6.(a), 3-7.(a) and 3-8.(a)), the spurious eigenvalues occur since $A(\lambda)$ may be zero for the spurious eigenvalues since $B(\lambda)$ is always zero in the Table 3-2. By using the complex-valued BEM, the six C_2^4 formulations can obtain the true eigenvalues. Figures 3-9 $\sim$ 3-11 show the determinant of the $[SM]$ versus λ for the clamped, simply-supported and free circular plates using the six complex-valued BEM. No spurious eigenvalue appears.

CHEEF method

Figures 3-12.(a)-(c) show the minimum singular value σ_1 of the $[C^*]$ versus λ for the clamped circular plate by using the real-part (u, θ) formulations in conjunction with zero (without CHEEF point), one and two CHEEF points, respectively. The first CHEEF point (ρ_1, ϕ_1) locates at $(1.50, \pi/4)$. It is interesting to find that only one CHEEF point can not suppress the appearance of all the spurious eigenvalues (3.78, 4.90 and 6.01) as shown in Figure 3-12.(b). By adding another CHEEF point (ρ_2, ϕ_2) which locates at $(1.45, 29\pi/36)$, and the angle $(\phi_1 - \phi_2)$ between the two selected points is $5\pi/9$, only the true eigenvalues are obtained as shown in Figure 3-12.(c) by using the two valid CHEEF points. Similarly, Figures 3-12.(d)-(f) show the minimum singular value σ_1 of the $[C^*]$ versus λ for the *simply-supported* circular plate by using the real-part (u, θ) formulations in conjunction with zero (without CHEEF point), one and two CHEEF points. The CHEEF points (ρ_1, ϕ_1) and (ρ_2, ϕ_2) locate at $(1.50, \pi/4)$ and $(1.45, 29\pi/36)$, and the angle $(\phi_1 - \phi_2)$ between the two selected points is $5\pi/9$. Good agreement is made by using the CHEEF method. Only the true eigenvalues are obtained.

For clamped, simply-supported and free circular cases, the SVD technique of updating term, the Burton & Miller method (u, m and θ, v formulations), the complex-valued BEM and the CHEEF method can obtain the same true eigenvalues. All the numerical resluts of the eigenvalues agree well with the data in Leissa [58]. From the computation point of view, CHEEF method use the minimum number of matrix dimension although it may take risk for the failure CHEEF points. The internal Hilbert-transform relation between the real and imaginary-part kernels has been imbedded. The complete information (real-part and imaginary-part) in the complex-valued BEM is not fully required after comparing with the real-part (imaginary-part) BEM in conjunction with the CHEEF technique. After obtaining the true eigenvalues, we can also obtain the interior modes for plate vibration. The former six interior modes for the clamped circular plate are shown in Figures 3-13 by using the Eq.(3-42). The numerical results of the former six interior modes for the clamped circular plate are shown in Figures 3-14 and 3-15 by using the real-part and complex-valued BEMs.

3-5 Concluding remarks

Four alternatives (SVD updating technique, the Burton & Miller method, the complex-valued BEM and the CHEEF method) were adopted to suppress the occurrence of the spurious eigenvalues for the clamped plate in the real-part and imaginary-part BEMs. The SVD updating technique was employed to deal with the problem of spurious eigenvalue occurring in the simply-connected plate eigenproblem. Then, the numerical experiments of the clamped, simply-supported and free circular plate problems were performed to demonstrate the validity of the remedies. The role of the Burton & Miller method for spurious-eigenvalue problem instead of fictitious-frequency problem of exterior acoustics of simply-connected plate has also been examined. By choosing the valid CHEEF points, we can suppress the occurrence of the spurious eigenvalues for the clamped plate in the real-part or imaginary-part BEM. For clamped, simply-supported and free circular cases, the SVD technique of updating term, the Burton & Miller method (u, m and θ, v formulations), the complex-valued BEM and the CHEEF method can obtain the true eigenvalues and the results agree well with the data in Leissa [58].

Chapter 4 Boundary element method for the free vibration of multiply-connected plate

Summary

In this chapter, the eigenproblem for the multiply-connected plate is solved by using the boundary element method. The true and spurious eigenequations for the plate eigenproblem are derived by using the complex-valued BEM. Since any two boundary integral equations in the plate formulation (4 equations) can be chosen, $6\ (C_2^4)$ options can be considered. The occurring mechanism of the spurious eigenequation for the plate eigenproblem in each formulation is studied analytically in both the continuous and discrete systems. For the continuous system, degenerate kernels for the fundamental solution and the Fourier series expansion for boundary densities are employed to derive the true and spurious eigenequations analytically for an annular plate. For the discrete system, the degenerate kernels of the fundamental solution and circulants for the influence matrices resulting from the annular boundary are employed to determine the spurious eigenequation. Three types of plates subject to C-C, S-S and F-F (C, S and F mean clamped, simply-supported and free boundary conditions, the first and second indices denote the outer and inner boundaries, respectively) are demonstrated analytically in the continuous and discrete systems. Several examples of plates subject to C-C, C-S, C-F, S-C, S-S, S-F, F-C, F-S and F-F are illustrated to check the validity of the present formulations.

4-1 Mathematical analysis using the complex-valued BEM

In the Chapter 2, either real-part or imaginary-part BEM results in the spurious eigenvalues for the simply-connected domain. Although a complex-valued BEM can avoid the appearence of the spurious eigenvalues for simply-connected domain as shown in the Chapter 3, the spurious eigenvalues may occur for the multiply-connected domain [21, 31]. Here, the kernel function $U(s,x)$ is the complex-valued fundamental solution as shown below:

$$U(s,x) = U_c(s,x) = \frac{1}{8\lambda^2}[(Y_0(\lambda r) + iJ_0(\lambda r)) + \frac{2}{\pi}(K_0(\lambda r) + iI_0(\lambda r))]. \tag{4-1}$$

In order to derive the true and spurious eigenequations for multiply-connected plate using the complex-valued BEM, the degenerate kernel is adopted to analytically derive the true and spurious eigenequations in the continuous and discrete systems of an annular plate. Here, the same boundary conditions on the outer and inner boundaries (C-C, S-S and F-F) are considered. Three cases are demonstrated analytically in the continuous and discrete systems as shown in the following subsections.

4-1-1 Continuous system

Case 1. Annular plate clamped on both the outer and inner boundaries

To consider an annular plate clamped on the the outer circle B_1 ($u_1 = 0$ and $\theta_1 = 0$) and the inner circle B_2 ($u_2 = 0$ and $\theta_2 = 0$), where u_1, θ_1, u_2 and θ_2 are the displacement and slope on the B_1 and B_2, respectively. The radii of the outer and inner circles are a and b, respectively. We can obtain the eigenequation in the continuous formulation. The moment and shear force, $m_1(s)$, $m_2(s)$, $v_1(s)$ and $v_2(s)$ along the circular boundary, can be expanded into Fourier series by

$$m_1(s) = \sum_{n=0}^{\infty}(p_{1,n}^{cc}cos(n\overline{\phi}) + q_{1,n}^{cc}sin(n\overline{\phi})), \quad s \in B_1, \tag{4-2}$$

$$m_2(s) = \sum_{n=0}^{\infty}(p_{2,n}^{cc}cos(n\overline{\phi}) + q_{2,n}^{cc}sin(n\overline{\phi})), \quad s \in B_2, \tag{4-3}$$

$$v_1(s) = \sum_{n=0}^{\infty}(a_{1,n}^{cc}cos(n\overline{\phi}) + b_{1,n}^{cc}sin(n\overline{\phi})), \quad s \in B_1, \tag{4-4}$$

$$v_2(s) = \sum_{n=0}^{\infty}(a_{2,n}^{cc}cos(n\overline{\phi}) + b_{2,n}^{cc}sin(n\overline{\phi})), \quad s \in B_2, \tag{4-5}$$

where the superscript " cc " denotes the clamped-clamped case, $\overline{\phi}$ is the angle on the circular boundary, $a_{i,n}^{cc}$, $b_{i,n}^{cc}$, $p_{i,n}^{cc}$ and $q_{i,n}^{cc}$ ($i = 1, 2$) are the undetermined Fourier coefficients on B_i ($i = 1, 2$). When the field point locates on B_1, substitution of the Eqs.(4-2) - (4-5) into the

Eqs.(2-24) and (2-25) yields

$$\begin{aligned}0 =& -\int_{B_1} U(s,x)[\sum_{n=0}^{\infty}(a_{1,n}^{cc}cos(n\overline{\phi}) + b_{1,n}^{cc}sin(n\overline{\phi}))]\,dB(s) \\ & -\int_{B_2} U(s,x)[\sum_{n=0}^{\infty}(a_{2,n}^{cc}cos(n\overline{\phi}) + b_{2,n}^{cc}sin(n\overline{\phi}))]\,dB(s) \\ & +\int_{B_1} \Theta(s,x)[\sum_{n=0}^{\infty}(p_{1,n}^{cc}cos(n\overline{\phi}) + q_{1,n}^{cc}sin(n\overline{\phi}))]\,dB(s) \\ & +\int_{B_2} \Theta(s,x)[\sum_{n=0}^{\infty}(p_{2,n}^{cc}cos(n\overline{\phi}) + q_{2,n}^{cc}sin(n\overline{\phi}))]\,dB(s),\ x\ \in\ B_1,\end{aligned} \tag{4-6}$$

$$\begin{aligned}0 =& -\int_{B_1} U_\theta(s,x)[\sum_{n=0}^{\infty}(a_{1,n}^{cc}cos(n\overline{\phi}) + b_{1,n}^{cc}sin(n\overline{\phi}))]\,dB(s) \\ & -\int_{B_2} U_\theta(s,x)[\sum_{n=0}^{\infty}(a_{2,n}^{cc}cos(n\overline{\phi}) + b_{2,n}^{cc}sin(n\overline{\phi}))]\,dB(s) \\ & +\int_{B_1} \Theta_\theta(s,x)[\sum_{n=0}^{\infty}(p_{1,n}^{cc}cos(n\overline{\phi}) + q_{1,n}^{cc}sin(n\overline{\phi}))]\,dB(s) \\ & +\int_{B_2} \Theta_\theta(s,x)[\sum_{n=0}^{\infty}(p_{2,n}^{cc}cos(n\overline{\phi}) + q_{2,n}^{cc}sin(n\overline{\phi}))]\,dB(s),\ x\ \in\ B_1,\end{aligned} \tag{4-7}$$

When the field point locates on B_2, substitution of the Eqs.(4-2) - (4-5) into the Eqs.(2-24) and (2-25) yields

$$\begin{aligned}0 =& -\int_{B_1} U(s,x)[\sum_{n=0}^{\infty}(a_{1,n}^{cc}cos(n\overline{\phi}) + b_{1,n}^{cc}sin(n\overline{\phi}))]\,dB(s) \\ & -\int_{B_2} U(s,x)[\sum_{n=0}^{\infty}(a_{2,n}^{cc}cos(n\overline{\phi}) + b_{2,n}^{cc}sin(n\overline{\phi}))]\,dB(s) \\ & +\int_{B_1} \Theta(s,x)[\sum_{n=0}^{\infty}(p_{1,n}^{cc}cos(n\overline{\phi}) + q_{1,n}^{cc}sin(n\overline{\phi}))]\,dB(s) \\ & +\int_{B_2} \Theta(s,x)[\sum_{n=0}^{\infty}(p_{2,n}^{cc}cos(n\overline{\phi}) + q_{2,n}^{cc}sin(n\overline{\phi}))]\,dB(s),\ x\ \in\ B_2,\end{aligned} \tag{4-8}$$

$$
\begin{aligned}
0 = &- \int_{B_1} U_\theta(s,x)[\sum_{n=0}^{\infty}(a_{1,n}^{cc}cos(n\overline{\phi}) + b_{1,n}^{cc}sin(n\overline{\phi}))]\, dB(s) \\
&- \int_{B_2} U_\theta(s,x)[\sum_{n=0}^{\infty}(a_{2,n}^{cc}cos(n\overline{\phi}) + b_{2,n}^{cc}sin(n\overline{\phi}))]\, dB(s) \\
&+ \int_{B_1} \Theta_\theta(s,x)[\sum_{n=0}^{\infty}(p_{1,n}^{cc}cos(n\overline{\phi}) + q_{1,n}^{cc}sin(n\overline{\phi}))]\, dB(s) \\
&+ \int_{B_2} \Theta_\theta(s,x)[\sum_{n=0}^{\infty}(p_{2,n}^{cc}cos(n\overline{\phi}) + q_{2,n}^{cc}sin(n\overline{\phi}))]\, dB(s),\ x \in B_2,
\end{aligned}
\tag{4-9}
$$

By using the degenerate kernels in the Appendix 1 into the Eqs.(4-6)-(4-9) and by employing the orthogonality condition of the Fourier series, the Fourier coefficients $a_{i,n}^{cc}$ and $p_{i,n}^{cc}$ ($i = 1,2$) satisfy

$$
[TM_n^{cc}]_{4\times4} \left\{ \begin{matrix} a_{1,n}^{cc} \\ a_{2,n}^{cc} \\ p_{1,n}^{cc} \\ p_{2,n}^{cc} \end{matrix} \right\}_{4\times1} = \left\{ \begin{matrix} 0 \\ 0 \\ 0 \\ 0 \end{matrix} \right\}_{4\times1}
\tag{4-10}
$$

where

$$
[TM_n^{cc}] =
\begin{bmatrix}
-\int_{B_1} U(s_{B1},x_{B1})cos(n\phi)\,dB(s) & -\int_{B_2} U(s_{B2},x_{B1})cos(n\phi)\,dB(s) & \int_{B_1} \Theta(s_{B1},x_{B1})cos(n\phi)\,dB(s) & \int_{B_2} \Theta(s_{B2},x_{B1})cos(n\phi)\,dB(s) \\
-\int_{B_1} U(s_{B1},x_{B2})cos(n\phi)\,dB(s) & -\int_{B_2} U(s_{B2},x_{B2})cos(n\phi)\,dB(s) & \int_{B_1} \Theta(s_{B1},x_{B2})cos(n\phi)\,dB(s) & \int_{B_2} \Theta(s_{B2},x_{B2})cos(n\phi)\,dB(s) \\
-\int_{B_1} U_\theta(s_{B1},x_{B1})cos(n\phi)\,dB(s) & -\int_{B_2} U_\theta(s_{B2},x_{B1})cos(n\phi)\,dB(s) & \int_{B_1} \Theta_\theta(s_{B1},x_{B1})cos(n\phi)\,dB(s) & \int_{B_2} \Theta_\theta(s_{B2},x_{B1})cos(n\phi)\,dB(s) \\
-\int_{B_1} U_\theta(s_{B1},x_{B2})cos(n\phi)\,dB(s) & -\int_{B_2} U_\theta(s_{B2},x_{B2})cos(n\phi)\,dB(s) & \int_{B_1} \Theta_\theta(s_{B1},x_{B2})cos(n\phi)\,dB(s) & \int_{B_2} \Theta_\theta(s_{B2},x_{B2})cos(n\phi)\,dB(s)
\end{bmatrix}
\tag{4-11}
$$

Also, the coefficients of $b_{i,n}^{cc}$ and $q_{i,n}^{cc}$ ($i = 1,2$) have the same relationship in the matrix form. For the existence of nontrivial solution for the generalized coefficients of $a_{i,n}^{cc}$, $p_{i,n}^{cc}$, $b_{i,n}^{cc}$ and $q_{i,n}^{cc}$ ($i = 1,2$), the determinant of the matrix versus the eigenvalue must be zero, i.e.,

$$
det[TM_n^{cc}] = 0. \tag{4-12}
$$

By using the properties of the determinants in the Appendix 5, we can simplify the Eq.(4-11) to

$$
det[TM_n^{cc}] = det([S_n^{u\theta}][T_n^{cc}]) \tag{4-13}
$$

where

$$[S_n^{u\theta}]_{4\times4} = \begin{bmatrix} (Y_n(\lambda a)+iJ_n(\lambda a)) & 0 & (K_n(\lambda a)+iI_n(\lambda a)) & 0 \\ iJ_n(\lambda b) & J_n(\lambda b) & iI_n(\lambda b) & I_n(\lambda b) \\ (Y_n'(\lambda a)+iJ_n'(\lambda a)) & 0 & (K_n'(\lambda a)+iI_n'(\lambda a)) & 0 \\ iJ_n'(\lambda b) & J_n'(\lambda b) & iI_n'(\lambda b) & I_n'(\lambda b) \end{bmatrix}_{4\times4} \tag{4-14}$$

and

$$[T_n^{cc}]_{4\times4} = \begin{bmatrix} J_n(\lambda a) & J_n(\lambda b) & J_n'(\lambda a) & J_n'(\lambda b) \\ Y_n(\lambda a) & Y_n(\lambda b) & Y_n'(\lambda a) & Y_n'(\lambda b) \\ I_n(\lambda a) & K_n(\lambda b) & I_n'(\lambda a) & I_n'(\lambda b) \\ K_n(\lambda a) & I_n(\lambda b) & K_n'(\lambda a) & K_n'(\lambda b) \end{bmatrix}_{4\times4} \tag{4-15}$$

It is noted that the matrix $[T_n^{cc}]$ denotes the matrix of true eigenequation for the C-C case and the matrix $[S_n^{u\theta}]$ denotes the matrix of spurious eigenequation in the u, θ formulation. Zero determinant in the Eq.(4-11) implies that the eigenequation is

$$det([S_n^{u\theta}][T_n^{cc}]) = 0, \quad n = 0, \pm1, \pm2, \cdots, \pm(N-1), N. \tag{4-16}$$

After comparing with the analytical solution for the annular plate [58], the former matrix $[S_n^{u\theta}]$ in the Eq.(4-16) results in the spurious eigenequation while the latter matrix $[T_n^{cc}]$ results in the true eigenequation. The spurious eigenequation in Eq.(4-14) will be elaborated on later.

Case 2. Annular plate simply-supported on both the outer and inner boundaries

To consider an annular plate simply-supported on both the outer circle B_1 ($u_1 = 0$ and $m_1 = 0$) and the inner circle B_2 ($u_2 = 0$ and $m_2 = 0$), where u_1, m_1, u_2 and m_2 are the displacement and moment on the B_1 and B_2, respectively. The radii of the outer and inner circles are a and b, respectively. We can obtain the eigenequation in the continuous formulation. The slope and shear force, $\theta_1(s)$, $\theta_2(s)$, $v_1(s)$ and $v_2(s)$ along the circular boundary, can be expanded into Fourier series by

$$\theta_1(s) = \sum_{n=0}^{\infty}(p_{1,n}^{ss}cos(n\overline{\phi}) + q_{1,n}^{ss}sin(n\overline{\phi})), \quad s \in B_1, \tag{4-17}$$

$$\theta_2(s) = \sum_{n=0}^{\infty}(p_{2,n}^{ss}cos(n\overline{\phi}) + q_{2,n}^{ss}sin(n\overline{\phi})), \quad s \in B_2, \tag{4-18}$$

$$v_1(s) = \sum_{n=0}^{\infty}(a_{1,n}^{ss}cos(n\overline{\phi}) + b_{1,n}^{ss}sin(n\overline{\phi})), \quad s \in B_1, \tag{4-19}$$

$$v_2(s) = \sum_{n=0}^{\infty}(a_{2,n}^{ss}cos(n\overline{\phi}) + b_{2,n}^{ss}sin(n\overline{\phi})), \quad s \in B_2, \tag{4-20}$$

where the superscript " ss " denotes the simply-supported-simply-supported case, $\overline{\phi}$ is the angle on the circular boundary, $a_{i,n}^{ss}$, $b_{i,n}^{ss}$, $p_{i,n}^{ss}$ and $q_{i,n}^{ss}$ $(i = 1, 2)$ are the undetermined Fourier coefficients on B_i $(i = 1, 2)$. When the field point locates on B_1, substitution of the Eqs.(4-17) - (4-20) into the Eqs.(2-24) and (2-25) yields

$$\begin{aligned} 0 = &- \int_{B_1} U(s,x)[\sum_{n=0}^{\infty}(a_{1,n}^{ss}cos(n\overline{\phi}) + b_{1,n}^{ss}sin(n\overline{\phi}))]\, dB(s) \\ &- \int_{B_2} U(s,x)[\sum_{n=0}^{\infty}(a_{2,n}^{ss}cos(n\overline{\phi}) + b_{2,n}^{ss}sin(n\overline{\phi}))]\, dB(s) \\ &- \int_{B_1} M(s,x)[\sum_{n=0}^{\infty}(p_{1,n}^{ss}cos(n\overline{\phi}) + q_{1,n}^{ss}sin(n\overline{\phi}))]\, dB(s) \\ &- \int_{B_2} M(s,x)[\sum_{n=0}^{\infty}(p_{2,n}^{ss}cos(n\overline{\phi}) + q_{2,n}^{ss}sin(n\overline{\phi}))]\, dB(s),\ x \in B_1, \end{aligned} \tag{4-21}$$

$$\begin{aligned} 0 = &- \int_{B_1} U_\theta(s,x)[\sum_{n=0}^{\infty}(a_{1,n}^{ss}cos(n\overline{\phi}) + b_{1,n}^{ss}sin(n\overline{\phi}))]\, dB(s) \\ &- \int_{B_2} U_\theta(s,x)[\sum_{n=0}^{\infty}(a_{2,n}^{ss}cos(n\overline{\phi}) + b_{2,n}^{ss}sin(n\overline{\phi}))]\, dB(s) \\ &- \int_{B_1} M_\theta(s,x)[\sum_{n=0}^{\infty}(p_{1,n}^{ss}cos(n\overline{\phi}) + q_{1,n}^{ss}sin(n\overline{\phi}))]\, dB(s) \\ &- \int_{B_2} M_\theta(s,x)[\sum_{n=0}^{\infty}(p_{2,n}^{ss}cos(n\overline{\phi}) + q_{2,n}^{ss}sin(n\overline{\phi}))]\, dB(s),\ x \in B_1, \end{aligned} \tag{4-22}$$

When the field point locates on B_2, substitution of the Eqs.(4-17) - (4-20) into the Eqs.(2-24) and (2-25) yields

$$
\begin{aligned}
0 = &- \int_{B_1} U(s,x)[\sum_{n=0}^{\infty}(a_{1,n}^{ss}cos(n\overline{\phi}) + b_{1,n}^{ss}sin(n\overline{\phi}))]\, dB(s) \\
&- \int_{B_2} U(s,x)[\sum_{n=0}^{\infty}(a_{2,n}^{ss}cos(n\overline{\phi}) + b_{2,n}^{ss}sin(n\overline{\phi}))]\, dB(s) \\
&- \int_{B_1} M(s,x)[\sum_{n=0}^{\infty}(p_{1,n}^{ss}cos(n\overline{\phi}) + q_{1,n}^{ss}sin(n\overline{\phi}))]\, dB(s) \\
&- \int_{B_2} M(s,x)[\sum_{n=0}^{\infty}(p_{2,n}^{ss}cos(n\overline{\phi}) + q_{2,n}^{ss}sin(n\overline{\phi}))]\, dB(s),\ x \in\ B_2,
\end{aligned}
\tag{4-23}
$$

$$
\begin{aligned}
0 = &- \int_{B_1} U_\theta(s,x)[\sum_{n=0}^{\infty}(a_{1,n}^{ss}cos(n\overline{\phi}) + b_{1,n}^{ss}sin(n\overline{\phi}))]\, dB(s) \\
&- \int_{B_2} U_\theta(s,x)[\sum_{n=0}^{\infty}(a_{2,n}^{ss}cos(n\overline{\phi}) + b_{2,n}^{ss}sin(n\overline{\phi}))]\, dB(s) \\
&- \int_{B_1} M_\theta(s,x)[\sum_{n=0}^{\infty}(p_{1,n}^{ss}cos(n\overline{\phi}) + q_{1,n}^{ss}sin(n\overline{\phi}))]\, dB(s) \\
&- \int_{B_2} M_\theta(s,x)[\sum_{n=0}^{\infty}(p_{2,n}^{ss}cos(n\overline{\phi}) + q_{2,n}^{ss}sin(n\overline{\phi}))]\, dB(s),\ x \in\ B_2,
\end{aligned}
\tag{4-24}
$$

By using the degenerate kernels in the Appendix 1 into the Eqs.(4-21)-(4-24) and by employing the orthogonality condition of the Fourier series, the Fourier coefficients $a_{i,n}^{ss}$ and $p_{i,n}^{ss}$ $(i = 1, 2)$ satisfy

$$
[TM_n^{ss}]_{4\times 4} \left\{ \begin{array}{c} a_{1,n}^{ss} \\ a_{2,n}^{ss} \\ p_{1,n}^{ss} \\ p_{2,n}^{ss} \end{array} \right\}_{4\times 1} = \left\{ \begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \end{array} \right\}_{4\times 1}
\tag{4-25}
$$

where

$$
[TM_n^{ss}] = \left[\begin{array}{cccc}
\int_{B_1} U(s_{B1},x_{B1})cos(n\phi)\, dB(s) & \int_{B_2} U(s_{B2},x_{B1})cos(n\phi)\, dB(s) & \int_{B_1} M(s_{B1},x_{B1})cos(n\phi)\, dB(s) & \int_{B_2} M(s_{B2},x_{B1})cos(n\phi)\, dB(s) \\
\int_{B_1} U(s_{B1},x_{B2})cos(n\phi)\, dB(s) & \int_{B_2} U(s_{B2},x_{B2})cos(n\phi)\, dB(s) & \int_{B_1} M(s_{B1},x_{B2})cos(n\phi)\, dB(s) & \int_{B_2} M(s_{B2},x_{B2})cos(n\phi)\, dB(s) \\
\int_{B_1} U_\theta(s_{B1},x_{B1})cos(n\phi)\, dB(s) & \int_{B_2} U_\theta(s_{B2},x_{B1})cos(n\phi)\, dB(s) & \int_{B_1} M_\theta(s_{B1},x_{B1})cos(n\phi)\, dB(s) & \int_{B_2} M_\theta(s_{B2},x_{B1})cos(n\phi)\, dB(s) \\
\int_{B_1} U_\theta(s_{B1},x_{B2})cos(n\phi)\, dB(s) & \int_{B_2} U_\theta(s_{B2},x_{B2})cos(n\phi)\, dB(s) & \int_{B_1} M_\theta(s_{B1},x_{B2})cos(n\phi)\, dB(s) & \int_{B_2} M_\theta(s_{B2},x_{B2})cos(n\phi)\, dB(s)
\end{array} \right]
\tag{4-26}
$$

Also, The coefficients of $b_{i,n}^{ss}$ and $q_{i,n}^{ss}$ $(i = 1, 2)$ have the same relationship in the matrix form. For the existence of nontrivial solution for the generalized coefficients of $a_{i,n}^{ss}$, $p_{i,n}^{ss}$, $b_{i,n}^{ss}$ and

$q_{i,n}^{ss}$ $(i = 1, 2)$, the determinant of the matrix versus the eigenvalue must be zero, i.e.,

$$det[TM_n^{ss}] = 0. \tag{4-27}$$

By using the properties of the determinants in the Appendix 5, we can simplify the Eq.(4-26) to

$$det[TM_n^{ss}] = det([S_n^{u\theta}][T_n^{ss}]) \tag{4-28}$$

where

$$[T_n^{ss}]_{4\times 4} = \begin{bmatrix} J_n(\lambda a) & J_n(\lambda b) & \alpha_n^J(\lambda a) & \alpha_n^J(\lambda b) \\ Y_n(\lambda a) & Y_n(\lambda b) & \alpha_n^Y(\lambda a) & \alpha_n^Y(\lambda b) \\ I_n(\lambda a) & K_n(\lambda b) & \alpha_n^I(\lambda a) & \alpha_n^I(\lambda b) \\ K_n(\lambda a) & I_n(\lambda b) & \alpha_n^K(\lambda a) & \alpha_n^K(\lambda b) \end{bmatrix}_{4\times 4} \tag{4-29}$$

It is noted that the matrix $[T_n^{ss}]$ denotes the matrix of true eigenequation for the S-S case. Zero determinant in the Eq.(4-26) implies that the eigenequation is

$$det([S_n^{u\theta}][T_n^{ss}]) = 0, \quad n = 0, \pm 1, \pm 2, \cdots, \pm(N-1), N. \tag{4-30}$$

After comparing with the analytical solution for the annular plate [58], the former matrix $[S_n^{u\theta}]$ in the Eq.(4-30) is the same as Eq.(4-14) which results in the spurious eigenequation while the latter matrix $[T_n^{ss}]$ results in the true eigenequation.

Case 3. Annular plate free on both the outer and inner boundaries

To consider an annular plate free on both the outer circle B_1 ($m_1 = 0$ and $v_1 = 0$) and the inner circle B_2 ($m_2 = 0$ and $v_2 = 0$), where m_1, v_1, m_2 and v_2 are the moment and shear force on the B_1 and B_2, respectively. The radii of the outer and inner circles are a and b, respectively. We can obtain the eigenequation in the continuous formulation. The displacement and slope, $u_1(s)$, $u_2(s)$, $\theta_1(s)$ and $\theta_2(s)$ along the circular boundary, can be expanded into Fourier series by

$$u_1(s) = \sum_{n=0}^{\infty}(p_{1,n}^{ff}cos(n\overline{\phi}) + q_{1,n}^{ff}sin(n\overline{\phi})), \quad s \in B_1, \tag{4-31}$$

$$u_2(s) = \sum_{n=0}^{\infty}(p_{2,n}^{ff}cos(n\overline{\phi}) + q_{2,n}^{ff}sin(n\overline{\phi})), \quad s \in B_2, \tag{4-32}$$

$$\theta_1(s) = \sum_{n=0}^{\infty}(a_{1,n}^{ff}cos(n\overline{\phi}) + b_{1,n}^{ff}sin(n\overline{\phi})), \quad s \in B_1, \tag{4-33}$$

$$\theta_2(s) = \sum_{n=0}^{\infty}(a_{2,n}^{ff}cos(n\overline{\phi}) + b_{2,n}^{ff}sin(n\overline{\phi})), \quad s \in B_2, \tag{4-34}$$

where the superscript “ ff ” denotes the free-free case, $\overline{\phi}$ is the angle on the circular boundary, $a_{i,n}^{ff}$, $b_{i,n}^{ff}$, $p_{i,n}^{ff}$ and $q_{i,n}^{ff}$ ($i = 1, 2$) are the undetermined Fourier coefficients on B_i ($i = 1, 2$). When the field point locates on B_1, substitution of the Eqs.(4-31) - (4-34) into the Eqs.(2-24) and (2-25) yields

$$\begin{aligned}
0 = &-\int_{B_1} M(s,x)[\sum_{n=0}^{\infty}(a_{1,n}^{ff}cos(n\overline{\phi}) + b_{1,n}^{ff}sin(n\overline{\phi}))]\, dB(s) \\
&-\int_{B_2} M(s,x)[\sum_{n=0}^{\infty}(a_{2,n}^{ff}cos(n\overline{\phi}) + b_{2,n}^{ff}sin(n\overline{\phi}))]\, dB(s) \\
&+\int_{B_1} V(s,x)[\sum_{n=0}^{\infty}(p_{1,n}^{ff}cos(n\overline{\phi}) + q_{1,n}^{ff}sin(n\overline{\phi}))]\, dB(s) \\
&+\int_{B_2} V(s,x)[\sum_{n=0}^{\infty}(p_{2,n}^{ff}cos(n\overline{\phi}) + q_{2,n}^{ff}sin(n\overline{\phi}))]\, dB(s),\ x \in B_1,
\end{aligned} \tag{4-35}$$

$$\begin{aligned}
0 = &-\int_{B_1} M_\theta(s,x)[\sum_{n=0}^{\infty}(a_{1,n}^{ff}cos(n\overline{\phi}) + b_{1,n}^{ff}sin(n\overline{\phi}))]\, dB(s) \\
&-\int_{B_2} M_\theta(s,x)[\sum_{n=0}^{\infty}(a_{2,n}^{ff}cos(n\overline{\phi}) + b_{2,n}^{ff}sin(n\overline{\phi}))]\, dB(s) \\
&+\int_{B_1} V_\theta(s,x)[\sum_{n=0}^{\infty}(p_{1,n}^{ff}cos(n\overline{\phi}) + q_{1,n}^{ff}sin(n\overline{\phi}))]\, dB(s) \\
&+\int_{B_2} V_\theta(s,x)[\sum_{n=0}^{\infty}(p_{2,n}^{ff}cos(n\overline{\phi}) + q_{2,n}^{ff}sin(n\overline{\phi}))]\, dB(s),\ x \in B_1,
\end{aligned} \tag{4-36}$$

When the field point locates on B_2, substitution of the Eqs.(4-31) - (4-34) into the Eqs.(2-24) and (2-25) yields

$$
\begin{aligned}
0 = &-\int_{B_1} M(s,x)[\sum_{n=0}^{\infty}(a_{1,n}^{ff}cos(n\overline{\phi}) + b_{1,n}^{ff}sin(n\overline{\phi}))]\,dB(s) \\
&-\int_{B_2} M(s,x)[\sum_{n=0}^{\infty}(a_{2,n}^{ff}cos(n\overline{\phi}) + b_{2,n}^{ff}sin(n\overline{\phi}))]\,dB(s) \\
&+\int_{B_1} V(s,x)[\sum_{n=0}^{\infty}(p_{1,n}^{ff}cos(n\overline{\phi}) + q_{1,n}^{ff}sin(n\overline{\phi}))]\,dB(s) \\
&+\int_{B_2} V(s,x)[\sum_{n=0}^{\infty}(p_{2,n}^{ff}cos(n\overline{\phi}) + q_{2,n}^{ff}sin(n\overline{\phi}))]\,dB(s),\ x\ \in\ B_2,
\end{aligned}
\tag{4-37}
$$

$$
\begin{aligned}
0 = &-\int_{B_1} M_\theta(s,x)[\sum_{n=0}^{\infty}(a_{1,n}^{ff}cos(n\overline{\phi}) + b_{1,n}^{ff}sin(n\overline{\phi}))]\,dB(s) \\
&-\int_{B_2} M_\theta(s,x)[\sum_{n=0}^{\infty}(a_{2,n}^{ff}cos(n\overline{\phi}) + b_{2,n}^{ff}sin(n\overline{\phi}))]\,dB(s) \\
&+\int_{B_1} V_\theta(s,x)[\sum_{n=0}^{\infty}(p_{1,n}^{ff}cos(n\overline{\phi}) + q_{1,n}^{ff}sin(n\overline{\phi}))]\,dB(s) \\
&+\int_{B_2} V_\theta(s,x)[\sum_{n=0}^{\infty}(p_{2,n}^{ff}cos(n\overline{\phi}) + q_{2,n}^{ff}sin(n\overline{\phi}))]\,dB(s),\ x\ \in\ B_2,
\end{aligned}
\tag{4-38}
$$

By using the degenerate kernels in the Appendix 1 into the Eqs.(4-35)-(4-38) and by employing the orthogonality condition of the Fourier series, the Fourier coefficients $a_{i,n}^{ff}$ and $p_{i,n}^{ff}$ $(i = 1, 2)$ satisfy

$$
[TM_n^{ff}]_{4\times 4}\left\{\begin{array}{c} a_{1,n}^{ff} \\ a_{2,n}^{ff} \\ p_{1,n}^{ff} \\ p_{2,n}^{ff} \end{array}\right\}_{4\times 1} = \left\{\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \end{array}\right\}_{4\times 1}
\tag{4-39}
$$

where

$$
[TM_n^{ff}] =
\left[\begin{array}{cccc}
-\int_{B_1} M(s_{B1},x_{B1})cos(n\phi)\,dB(s) & -\int_{B_2} M(s_{B2},x_{B1})cos(n\phi)\,dB(s) & \int_{B_1} V(s_{B1},x_{B1})cos(n\phi)\,dB(s) & \int_{B_2} V(s_{B2},x_{B1})cos(n\phi)\,dB(s) \\
-\int_{B_1} M(s_{B1},x_{B2})cos(n\phi)\,dB(s) & -\int_{B_2} M(s_{B2},x_{B2})cos(n\phi)\,dB(s) & \int_{B_1} V(s_{B1},x_{B2})cos(n\phi)\,dB(s) & \int_{B_2} V(s_{B2},x_{B2})cos(n\phi)\,dB(s) \\
-\int_{B_1} M_\theta(s_{B1},x_{B1})cos(n\phi)\,dB(s) & -\int_{B_2} M_\theta(s_{B2},x_{B1})cos(n\phi)\,dB(s) & \int_{B_1} V_\theta(s_{B1},x_{B1})cos(n\phi)\,dB(s) & \int_{B_2} V_\theta(s_{B2},x_{B1})cos(n\phi)\,dB(s) \\
-\int_{B_1} M_\theta(s_{B1},x_{B2})cos(n\phi)\,dB(s) & -\int_{B_2} M_\theta(s_{B2},x_{B2})cos(n\phi)\,dB(s) & \int_{B_1} V_\theta(s_{B1},x_{B2})cos(n\phi)\,dB(s) & \int_{B_2} V_\theta(s_{B2},x_{B2})cos(n\phi)\,dB(s)
\end{array}\right]
\tag{4-40}
$$

Also, the coefficients of $b_{i,n}^{ff}$ and $q_{i,n}^{ff}$ $(i = 1, 2)$, have the same relationship in the matrix form. For the existence of nontrivial solution for the generalized coefficients of $a_{i,n}^{ff}$, $p_{i,n}^{ff}$, $b_{i,n}^{ff}$ and

$q_{i,n}^{ff}$ $(i = 1, 2)$, the determinant of the matrix versus the eigenvalue must be zero, i.e.,

$$det[TM_n^{ff}] = 0. \tag{4-41}$$

By using the properties of the determinants in the Appendix 5, we can simplify the Eq.(4-40) to

$$det[TM_n^{ff}] = det([S_n^{u\theta}][T_n^{ff}]) \tag{4-42}$$

where

$$[T_n^{ff}] = \begin{bmatrix} \alpha_n^J(\lambda a) & \alpha_n^J(\lambda b) & \beta_n^J(\lambda a) + \frac{(1-\nu)}{a}\gamma_n^J(\lambda a) & \beta_n^J(\lambda b) + \frac{(1-\nu)}{b}\gamma_n^J(\lambda b) \\ \alpha_n^Y(\lambda a) & \alpha_n^Y(\lambda b) & \beta_n^Y(\lambda a) + \frac{(1-\nu)}{a}\gamma_n^Y(\lambda a) & \beta_n^Y(\lambda b) + \frac{(1-\nu)}{b}\gamma_n^Y(\lambda b) \\ \alpha_n^I(\lambda a) & \alpha_n^I(\lambda b) & \beta_n^I(\lambda a) + \frac{(1-\nu)}{a}\gamma_n^I(\lambda a) & \beta_n^I(\lambda b) + \frac{(1-\nu)}{b}\gamma_n^I(\lambda b) \\ \alpha_n^K(\lambda a) & \alpha_n^K(\lambda b) & \beta_n^K(\lambda a) + \frac{(1-\nu)}{a}\gamma_n^K(\lambda a) & \beta_n^K(\lambda b) + \frac{(1-\nu)}{b}\gamma_n^K(\lambda b) \end{bmatrix} \tag{4-43}$$

It is noted that the matrix $[T_n^{ff}]$ denotes the matrix of true eigenequation for the F-F case. Zero determinant in the Eq.(4-40) implies that the eigenequation is

$$det([S_n^{u\theta}][T_n^{ff}]) = 0, \quad n = 0, \pm 1, \pm 2, \cdots, \pm(N-1), N. \tag{4-44}$$

After comparing with the analytical solution for the annular plate [58], the former matrix $[S_n^{u\theta}]$ in the Eq.(4-44) is the same as Eq.(4-14) which results in the spurious eigenequation while the latter matrix $[T_n^{ff}]$ results in the true eigenequation.

4-1-2 Discrete system

Case 1. Annular plate clamped on both the outer and inner boundaries

To consider an annular plate clamped on the outer circle B_1 ($u_1 = 0$ and $\theta_1 = 0$) and the inner circle B_2 ($u_2 = 0$ and $\theta_2 = 0$), where u_1, θ_1, u_2 and θ_2 are the displacements and slopes on the B_1 and B_2, respectively. The radii of the outer and inner circles are a and b, respectively. When the outer and inner boundaries are both discretized into $2N$ constant elements, respectively, the Eq.(2-28) by using the complex-valued BEM can be rewritten as

$$\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} U11 & U12 \\ U21 & U22 \end{bmatrix} \begin{Bmatrix} v_1 \\ v_2 \end{Bmatrix} + \begin{bmatrix} \Theta 11 & \Theta 12 \\ \Theta 21 & \Theta 22 \end{bmatrix} \begin{Bmatrix} m_1 \\ m_2 \end{Bmatrix} \tag{4-45}$$

where m_1, v_1, m_2 and v_2 are the column vectors of the normal moment and effective shear force on B_1 and B_2 with a dimension $2N \times 1$, the matrices $[Uij]$ and $[\Theta ij]$ mean the influence matrices of U and Θ kernels which is obtained by collocating the field and source points on B_i and B_j with a dimension $2N \times 2N$, respectively. Similarly, the Eq.(2-29) can be rewritten as

$$\left\{ \begin{array}{c} 0 \\ 0 \end{array} \right\} = \left[\begin{array}{cc} U11_\theta & U12_\theta \\ U21_\theta & U22_\theta \end{array} \right] \left\{ \begin{array}{c} v_1 \\ v_2 \end{array} \right\} + \left[\begin{array}{cc} \Theta11_\theta & \Theta12_\theta \\ \Theta21_\theta & \Theta22_\theta \end{array} \right] \left\{ \begin{array}{c} m_1 \\ m_2 \end{array} \right\} \tag{4-46}$$

where the matrices $[Uij_\theta]$ and $[\Theta ij_\theta]$ mean the influence matrices of the U_θ and Θ_θ kernels which is obtained by the field and source points locating on B_i and B_j with a dimension $2N \times 2N$, respectively. By assembling the Eqs.(4-45) and (4-46) together, we have

$$[SM^{cc}] \left\{ \begin{array}{c} v_1 \\ v_2 \\ m_1 \\ m_1 \end{array} \right\} = \{0\}, \tag{4-47}$$

where the superscript " cc " denotes the clamped-clamped case and

$$[SM^{cc}] = \left[\begin{array}{cccc} U11 & U12 & \Theta11 & \Theta12 \\ U21 & U22 & \Theta21 & \Theta22 \\ U11_\theta & U11_\theta & \Theta11_\theta & \Theta12_\theta \\ U21_\theta & U22_\theta & \Theta21_\theta & \Theta22_\theta \end{array} \right]_{8N \times 8N} . \tag{4-48}$$

For the existence of nontrivial solution, the determinant of the matrix versus eigenvalue must be zero, i.e.,

$$det[SM^{cc}] = 0. \tag{4-49}$$

Since the rotation symmetry is preserved for a circular annular boundary, the influence matrices for the discrete system are found to be the circulants. We can obtain the influence matrices ($[U11]$, $[U12]$, $[\Theta11]$, $[\Theta12]$, $[U21]$, $[U22]$, $[\Theta21]$, $[\Theta22]$, $[U11_\theta]$, $[U12_\theta]$, $[\Theta11_\theta]$, $[\Theta12_\theta]$, $[U21_\theta]$, $[U22_\theta]$, $[\Theta21_\theta]$ and $[\Theta22_\theta]$) which are all symmetric circulants. The eigenvalues of the influence matrices for the discrete system are

$$\mu_\ell^{[U11]} = -\frac{\pi a}{4\lambda^2}[Y_\ell(\lambda a)J_\ell(\lambda a) + \frac{2}{\pi}K_\ell(\lambda a)I_\ell(\lambda a)] + i[J_\ell(\lambda a)J_\ell(\lambda a) + \frac{2}{\pi}(-1)^\ell I_\ell(\lambda a)I_\ell(\lambda a)],\ \ell = 0, \pm1, \cdots, \pm(N-1), N. \quad (4\text{-}50)$$

$$\mu_\ell^{[U12]} = -\frac{\pi b}{4\lambda^2}[Y_\ell(\lambda a)J_\ell(\lambda b) + \frac{2}{\pi}K_\ell(\lambda a)I_\ell(\lambda b)] + i[J_\ell(\lambda a)J_\ell(\lambda b) + \frac{2}{\pi}(-1)^\ell I_\ell(\lambda a)I_\ell(\lambda b)],\ \ell = 0, \pm1, \cdots, \pm(N-1), N. \quad (4\text{-}51)$$

$$\mu_\ell^{[\Theta11]} = \frac{\pi a}{4\lambda}[Y_\ell(\lambda a)J'_\ell(\lambda a) + \frac{2}{\pi}K_\ell(\lambda a)I'_\ell(\lambda a)] + i[J_\ell(\lambda a)J'_\ell(\lambda a) + \frac{2}{\pi}(-1)^\ell I_\ell(\lambda a)I'_\ell(\lambda a)],\ \ell = 0, \pm1, \cdots, \pm(N-1), N. \quad (4\text{-}52)$$

$$\mu_\ell^{[\Theta12]} = \frac{\pi b}{4\lambda}[Y_\ell(\lambda a)J'_\ell(\lambda b) + \frac{2}{\pi}K_\ell(\lambda a)I'_\ell(\lambda b)] + i[J_\ell(\lambda a)J'_\ell(\lambda b) + \frac{2}{\pi}(-1)^\ell I_\ell(\lambda a)I'_\ell(\lambda b)],\ \ell = 0, \pm1, \cdots, \pm(N-1), N. \quad (4\text{-}53)$$

$$\mu_\ell^{[U21]} = -\frac{\pi a}{4\lambda^2}[Y_\ell(\lambda a)J_\ell(\lambda b) + \frac{2}{\pi}K_\ell(\lambda a)I_\ell(\lambda b)] + i[J_\ell(\lambda a)J_\ell(\lambda b) + \frac{2}{\pi}(-1)^\ell I_\ell(\lambda a)I_\ell(\lambda b)],\ \ell = 0, \pm1, \cdots, \pm(N-1), N. \quad (4\text{-}54)$$

$$\mu_\ell^{[U22]} = -\frac{\pi b}{4\lambda^2}[Y_\ell(\lambda b)J_\ell(\lambda b) + \frac{2}{\pi}K_\ell(\lambda b)I_\ell(\lambda b)] + i[J_\ell(\lambda b)J_\ell(\lambda b) + \frac{2}{\pi}(-1)^\ell I_\ell(\lambda b)I_\ell(\lambda b)],\ \ell = 0, \pm1, \cdots, \pm(N-1), N. \quad (4\text{-}55)$$

$$\mu_\ell^{[\Theta21]} = \frac{\pi a}{4\lambda}[Y'_\ell(\lambda a)J_\ell(\lambda b) + \frac{2}{\pi}K'_\ell(\lambda a)I_\ell(\lambda b)] + i[J'_\ell(\lambda a)J_\ell(\lambda b) + \frac{2}{\pi}(-1)^\ell I'_\ell(\lambda a)I_\ell(\lambda b)],\ \ell = 0, \pm1, \cdots, \pm(N-1), N. \quad (4\text{-}56)$$

$$\mu_\ell^{[\Theta22]} = \frac{\pi b}{4\lambda}[Y'_\ell(\lambda b)J_\ell(\lambda b) + \frac{2}{\pi}K'_\ell(\lambda b)I_\ell(\lambda b)] + i[J'_\ell(\lambda b)J_\ell(\lambda b) + \frac{2}{\pi}(-1)^\ell I'_\ell(\lambda b)I_\ell(\lambda b)],\ \ell = 0, \pm1, \cdots, \pm(N-1), N. \quad (4\text{-}57)$$

$$\kappa_\ell^{[U11]} = -\frac{\pi a}{4\lambda}[Y'_\ell(\lambda a)J_\ell(\lambda a) + \frac{2}{\pi}K'_\ell(\lambda a)I_\ell(\lambda a)] + i[J'_\ell(\lambda a)J_\ell(\lambda a) + \frac{2}{\pi}(-1)^\ell I'_\ell(\lambda a)I_\ell(\lambda a)],\ \ell = 0, \pm1, \cdots, \pm(N-1), N. \quad (4\text{-}58)$$

$$\kappa_\ell^{[U12]} = -\frac{\pi b}{4\lambda}[Y'_\ell(\lambda a)J_\ell(\lambda b) + \frac{2}{\pi}K'_\ell(\lambda a)I_\ell(\lambda b)] + i[J'_\ell(\lambda a)J_\ell(\lambda b) + \frac{2}{\pi}(-1)^\ell I'_\ell(\lambda a)I_\ell(\lambda b)],\ \ell = 0, \pm1, \cdots, \pm(N-1), N. \quad (4\text{-}59)$$

$$\begin{aligned}\kappa_\ell^{[\Theta 11]} =& \frac{\pi a}{4}[Y'_\ell(\lambda a)J'_\ell(\lambda a) + \frac{2}{\pi}K'_\ell(\lambda a)I'_\ell(\lambda a)] \\ &+ i[J'_\ell(\lambda a)J'_\ell(\lambda a) + \frac{2}{\pi}(-1)^\ell I'_\ell(\lambda a)I'_\ell(\lambda a)],\ \ell = 0, \pm 1, \cdots, \pm(N-1), N.\end{aligned} \tag{4-60}$$

$$\begin{aligned}\kappa_\ell^{[\Theta 12]} =& \frac{\pi b}{4}[Y'_\ell(\lambda a)J'_\ell(\lambda b) + \frac{2}{\pi}K'_\ell(\lambda a)I'_\ell(\lambda b)] \\ &+ i[J'_\ell(\lambda a)J'_\ell(\lambda b) + \frac{2}{\pi}(-1)^\ell I'_\ell(\lambda a)I'_\ell(\lambda b)],\ \ell = 0, \pm 1, \cdots, \pm(N-1), N.\end{aligned} \tag{4-61}$$

$$\begin{aligned}\kappa_\ell^{[U21]} =& -\frac{\pi a}{4\lambda}[Y_\ell(\lambda a)J'_\ell(\lambda b) + \frac{2}{\pi}K_\ell(\lambda a)I'_\ell(\lambda b)] \\ &+ i[J_\ell(\lambda a)J'_\ell(\lambda b) + \frac{2}{\pi}(-1)^\ell I_\ell(\lambda a)I'_\ell(\lambda b)],\ \ell = 0, \pm 1, \cdots, \pm(N-1), N.\end{aligned} \tag{4-62}$$

$$\begin{aligned}\kappa_\ell^{[U22]} =& -\frac{\pi b}{4\lambda}[Y_\ell(\lambda b)J'_\ell(\lambda b) + \frac{2}{\pi}K_\ell(\lambda b)I'_\ell(\lambda b)] \\ &+ i[J_\ell(\lambda b)J'_\ell(\lambda b) + \frac{2}{\pi}(-1)^\ell I_\ell(\lambda b)I'_\ell(\lambda b)],\ \ell = 0, \pm 1, \cdots, \pm(N-1), N.\end{aligned} \tag{4-63}$$

$$\begin{aligned}\kappa_\ell^{[\Theta 21]} =& \frac{\pi a}{4}[Y'_\ell(\lambda a)J'_\ell(\lambda b) + \frac{2}{\pi}K'_\ell(\lambda a)I'_\ell(\lambda b)] \\ &+ i[J'_\ell(\lambda a)J'_\ell(\lambda b) + \frac{2}{\pi}(-1)^\ell I'_\ell(\lambda a)I'_\ell(\lambda b)],\ \ell = 0, \pm 1, \cdots, \pm(N-1), N.\end{aligned} \tag{4-64}$$

$$\begin{aligned}\kappa_\ell^{[\Theta 22]} =& \frac{\pi b}{4}[Y'_\ell(\lambda b)J'_\ell(\lambda b) + \frac{2}{\pi}K'_\ell(\lambda b)I'_\ell(\lambda b)] \\ &+ i[J'_\ell(\lambda b)J'_\ell(\lambda b) + \frac{2}{\pi}(-1)^\ell I'_\ell(\lambda b)I'_\ell(\lambda b)],\ \ell = 0, \pm 1, \cdots, \pm(N-1), N.\end{aligned} \tag{4-65}$$

where $\mu_\ell^{[U11]}$, $\mu_\ell^{[U12]}$, $\mu_\ell^{[\Theta 11]}$, $\mu_\ell^{[\Theta 12]}$, $\mu_\ell^{[U21]}$, $\mu_\ell^{[U22]}$, $\mu_\ell^{[\Theta 21]}$, $\mu_\ell^{[\Theta 22]}$, $\kappa_\ell^{[U11]}$, $\kappa_\ell^{[U12]}$, $\kappa_\ell^{[\Theta 11]}$, $\kappa_\ell^{[\Theta 12]}$, $\kappa_\ell^{[U21]}$, $\kappa_\ell^{[U22]}$, $\kappa_\ell^{[\Theta 21]}$ and $\kappa_\ell^{[\Theta 22]}$ are the eigenvalues of the matrices $[U11]$, $[U12]$, $[\Theta 11]$, $[\Theta 12]$, $[U21]$, $[U22]$, $[\Theta 21]$, $[\Theta 22]$, $[U11_\theta]$, $[U12_\theta]$, $[\Theta 11_\theta]$, $[\Theta 12_\theta]$, $[U21_\theta]$, $[U22_\theta]$, $[\Theta 21_\theta]$ and $[\Theta 22_\theta]$, respectively. Since the matrices are all symmetric circulants, the matrix $[SM^{cc}]$ in the Eq.(4-48) can be decomposed into

$$[SM^{cc}] = \begin{bmatrix} \Phi & 0 & 0 & 0 \\ 0 & \Phi & 0 & 0 \\ 0 & 0 & \Phi & 0 \\ 0 & 0 & 0 & \Phi \end{bmatrix} \begin{bmatrix} \Sigma_{U11} & \Sigma_{U12} & \Sigma_{\Theta 11} & \Sigma_{\Theta 12} \\ \Sigma_{U21} & \Sigma_{U22} & \Sigma_{\Theta 21} & \Sigma_{\Theta 22} \\ \Sigma_{U11_\theta} & \Sigma_{U12_\theta} & \Sigma_{\Theta 11_\theta} & \Sigma_{\Theta 12_\theta} \\ \Sigma_{U21_\theta} & \Sigma_{U22_\theta} & \Sigma_{\Theta 21_\theta} & \Sigma_{\Theta 22_\theta} \end{bmatrix} \begin{bmatrix} \Phi & 0 & 0 & 0 \\ 0 & \Phi & 0 & 0 \\ 0 & 0 & \Phi & 0 \\ 0 & 0 & 0 & \Phi \end{bmatrix}^T \tag{4-66}$$

Since Φ is orthogonal, the determinant of $[SM^{cc}]_{8N\times 8N}$ is

$$det[SM^{cc}] = det\begin{bmatrix} \Sigma_{U11} & \Sigma_{U12} & \Sigma_{\Theta 11} & \Sigma_{\Theta 12} \\ \Sigma_{U21} & \Sigma_{U22} & \Sigma_{\Theta 21} & \Sigma_{\Theta 22} \\ \Sigma_{U11_\theta} & \Sigma_{U12_\theta} & \Sigma_{\Theta 11_\theta} & \Sigma_{\Theta 12_\theta} \\ \Sigma_{U21_\theta} & \Sigma_{U22_\theta} & \Sigma_{\Theta 21_\theta} & \Sigma_{\Theta 22_\theta} \end{bmatrix}_{8N\times 8N} \tag{4-67}$$

By extending the relationship in the Appendix 4 and employing the properties of the determinants in the Appendix 5, we can simplify the Eq.(4-67) to

$$det[SM^{cc}] = \prod_{\ell=-(N-1)}^{N} det\begin{bmatrix} \mu_\ell^{[U11]} & \mu_\ell^{[U12]} & \mu_\ell^{[\Theta 11]} & \mu_\ell^{[\Theta 12]} \\ \mu_\ell^{[U21]} & \mu_\ell^{[U22]} & \mu_\ell^{[\Theta 21]} & \mu_\ell^{[\Theta 22]} \\ \kappa_\ell^{[U11]} & \kappa_\ell^{[U12]} & \kappa_\ell^{[\Theta 11]} & \kappa_\ell^{[\Theta 12]} \\ \kappa_\ell^{[U21]} & \kappa_\ell^{[U22]} & \kappa_\ell^{[\Theta 21]} & \kappa_\ell^{[\Theta 22]} \end{bmatrix}_{4\times 4} \tag{4-68}$$

By employing the Eqs.(4-50)-(4-63) for the Eq.(4-68), decomposition of the matrice yields

$$det[SM^{cc}] = \prod_{\ell=-(N-1)}^{N} det([S_\ell^{u\theta}][T_\ell^{cc}]) \tag{4-69}$$

where

$$[S_\ell^{u\theta}]_{4\times 4} = \begin{bmatrix} (Y_\ell(\lambda a) + iJ_\ell(\lambda a)) & 0 & (K_\ell(\lambda a) + iI_\ell(\lambda a)) & 0 \\ iJ_\ell(\lambda b) & J_\ell(\lambda b) & iI_\ell(\lambda b) & I_\ell(\lambda b) \\ (Y'_\ell(\lambda a) + iJ'_\ell(\lambda a)) & 0 & (K'_\ell(\lambda a) + iI'_\ell(\lambda a)) & 0 \\ iJ'_\ell(\lambda b) & J'_\ell(\lambda b) & iI'_\ell(\lambda b) & I'_\ell(\lambda b) \end{bmatrix}_{4\times 4} \tag{4-70}$$

and

$$[T_\ell^{cc}]_{4\times 4} = \begin{bmatrix} J_\ell(\lambda a) & J_\ell(\lambda b) & J'_\ell(\lambda a) & J'_\ell(\lambda b) \\ Y_\ell(\lambda a) & Y_\ell(\lambda b) & Y'_\ell(\lambda a) & Y'_\ell(\lambda b) \\ I_\ell(\lambda a) & K_\ell(\lambda b) & I'_\ell(\lambda a) & I'_\ell(\lambda b) \\ K_\ell(\lambda a) & I_\ell(\lambda b) & K'_\ell(\lambda a) & K'_\ell(\lambda b) \end{bmatrix}_{4\times 4} \tag{4-71}$$

It is noted that the matrix $[T_\ell^{cc}]$ denotes the matrix of true eigenequation for the C-C case and the matrix $[S_\ell^{u\theta}]$ denotes the matrix of spurious eigenequation in the u, θ formulation. Zero determinant in the Eq.(4-69) implies that the eigenequation is

$$det([S_\ell^{u\theta}][T_\ell^{cc}]) = 0, \quad \ell = 0, \pm 1, \pm 2, \cdots, \pm(N-1), N. \tag{4-72}$$

After comparing with the analytical solution for the annular plate [58], the former matrix $[S_\ell^{u\theta}]$ in the Eq.(4-72) results in the spurious eigenequation while the latter matrix $[T_\ell^{cc}]$ results in the true eigenequation. The results of the Eq.(4-71) in the discrete system match well with the former one in the continuous system.

Case 2. Annular plate simply-supported on both the outer and inner boundaries

To consider an annular plate simply-supported on both the outer circle B_1 ($u_1 = 0$ and $m_1 = 0$) and the inner circle B_2 ($u_2 = 0$ and $m_2 = 0$), where m_1 and m_2 are the normal moments on B_1 and B_2, respectively. When the outer and inner boundaries are both discretized into $2N$ constant elements, respectively. By using the complex-valued BEM, we have

$$[SM^{ss}] = \begin{bmatrix} U11 & U12 & M11 & M12 \\ U21 & U22 & M21 & M22 \\ U11_\theta & U11_\theta & M11_\theta & M12_\theta \\ U21_\theta & U22_\theta & M21_\theta & M22_\theta \end{bmatrix}_{8N\times 8N} . \tag{4-73}$$

where the superscript " ss " denotes the simply-supported-simply-supported case. Since the rotation symmetry is preserved for a circular annular boundary, the influence matrices for the discrete system are found to be the circulants. We can obtain the influence matrices ($[M11]$, $[M12]$, $[M21]$ and $[M22]$) which are all symmetric circulants. The eigenvalues of the influence matrices for the discrete system are

$$\begin{aligned} \mu_\ell^{[M11]} =& -\frac{\pi a}{4\lambda^2}[Y_\ell(\lambda a)\alpha_\ell^J(\lambda a) + \frac{2}{\pi}K_\ell(\lambda a)\alpha_\ell^I(\lambda a)] \\ &+ i[J_\ell(\lambda a)\alpha_\ell^J(\lambda a) + \frac{2}{\pi}(-1)^\ell I_\ell(\lambda a)\alpha_\ell^I(\lambda a)],\ \ell = 0, \pm 1, \cdots, \pm(N-1), N. \end{aligned} \tag{4-74}$$

$$\begin{aligned} \mu_\ell^{[M12]} =& -\frac{\pi b}{4\lambda^2}[Y_\ell(\lambda a)\alpha_\ell^J(\lambda b) + \frac{2}{\pi}K_\ell(\lambda a)\alpha_\ell^I(\lambda b)] \\ &+ i[J_\ell(\lambda a)\alpha_\ell^J(\lambda b) + \frac{2}{\pi}(-1)^\ell I_\ell(\lambda a)\alpha_\ell^I(\lambda b)],\ \ell = 0, \pm 1, \cdots, \pm(N-1), N. \end{aligned} \tag{4-75}$$

$$\begin{aligned} \mu_\ell^{[M21]} =& -\frac{\pi a}{4\lambda^2}[\alpha_\ell^Y(\lambda a)J_\ell(\lambda b) + \frac{2}{\pi}\alpha_\ell^K(\lambda a)I_\ell(\lambda b)] \\ &+ i[\alpha_\ell^J(\lambda a)J_\ell(\lambda b) + \frac{2}{\pi}(-1)^\ell \alpha_\ell^I(\lambda a)I_\ell(\lambda b)],\ \ell = 0, \pm 1, \cdots, \pm(N-1), N. \end{aligned} \tag{4-76}$$

$$\begin{aligned} \mu_\ell^{[M22]} =& -\frac{\pi b}{4\lambda^2}[\alpha_\ell^Y(\lambda a)J_\ell(\lambda a) + \frac{2}{\pi}\alpha_\ell^K(\lambda a)I_\ell(\lambda a)] \\ &+ i[\alpha_\ell^J(\lambda a)J_\ell(\lambda a) + \frac{2}{\pi}(-1)^\ell \alpha_\ell^I(\lambda a)I_\ell(\lambda a)],\ \ell = 0, \pm 1, \cdots, \pm(N-1), N. \end{aligned} \tag{4-77}$$

$$\begin{aligned}\kappa_\ell^{[M11]} =& -\frac{\pi a}{4\lambda}[Y_\ell'(\lambda a)\alpha_\ell^J(\lambda a)+\frac{2}{\pi}K_\ell'(\lambda a)\alpha_\ell^I(\lambda a)] \\ &+i[J_\ell'(\lambda a)\alpha_\ell^J(\lambda a)+\frac{2}{\pi}(-1)^\ell I_\ell'(\lambda a)\alpha_\ell^I(\lambda a)],\ \ell=0,\pm1,\cdots,\pm(N-1),N.\end{aligned} \tag{4-78}$$

$$\begin{aligned}\kappa_\ell^{[M12]} =& -\frac{\pi b}{4\lambda}[Y_\ell'(\lambda a)\alpha_\ell^J(\lambda b)+\frac{2}{\pi}K_\ell'(\lambda a)\alpha_\ell^I(\lambda b)] \\ &+i[J_\ell'(\lambda a)\alpha_\ell^J(\lambda b)+\frac{2}{\pi}(-1)^\ell I_\ell'(\lambda a)\alpha_\ell^I(\lambda b)],\ \ell=0,\pm1,\cdots,\pm(N-1),N.\end{aligned} \tag{4-79}$$

$$\begin{aligned}\kappa_\ell^{[M21]} =& -\frac{\pi a}{4\lambda}[\alpha_\ell^Y(\lambda a)J_\ell'(\lambda b)+\frac{2}{\pi}\alpha_\ell^K(\lambda a)I_\ell'(\lambda b)] \\ &+i[\alpha_\ell^J(\lambda a)J_\ell'(\lambda b)+\frac{2}{\pi}(-1)^\ell\alpha_\ell^I(\lambda a)I_\ell'(\lambda b)],\ \ell=0,\pm1,\cdots,\pm(N-1),N.\end{aligned} \tag{4-80}$$

$$\begin{aligned}\kappa_\ell^{[M22]} =& -\frac{\pi b}{4\lambda}[\alpha_\ell^Y(\lambda b)J_\ell'(\lambda b)+\frac{2}{\pi}\alpha_\ell^K(\lambda b)I_\ell'(\lambda b)] \\ &+i[\alpha_\ell^J(\lambda b)J_\ell'(\lambda b)+\frac{2}{\pi}(-1)^\ell\alpha_\ell^I(\lambda b)I_\ell'(\lambda b)],\ \ell=0,\pm1,\cdots,\pm(N-1),N.\end{aligned} \tag{4-81}$$

where $\mu_\ell^{[M11]}$, $\mu_\ell^{[M12]}$, $\mu_\ell^{[M21]}$, $\mu_\ell^{[M22]}$, $\kappa_\ell^{[M11]}$, $\kappa_\ell^{[M12]}$, $\kappa_\ell^{[M21]}$ and $\kappa_\ell^{[M22]}$, are the eigenvalues of the matrices $[M11]$, $[M12]$, $[M21]$, $[M22]$, $[M11_\theta]$, $[M12_\theta]$, $[M21_\theta]$ and $[M22_\theta]$, respectively. Since the matrices are all symmetric circulants, the matrix $[SM^{ss}]$ in the Eq.(4-73) can be decomposed into

$$[SM^{ss}] = \begin{bmatrix}\Phi&0&0&0\\0&\Phi&0&0\\0&0&\Phi&0\\0&0&0&\Phi\end{bmatrix}\begin{bmatrix}\Sigma_{U11}&\Sigma_{U12}&\Sigma_{M11}&\Sigma_{M12}\\\Sigma_{U21}&\Sigma_{U22}&\Sigma_{M21}&\Sigma_{M22}\\\Sigma_{U11_\theta}&\Sigma_{U12_\theta}&\Sigma_{M11_\theta}&\Sigma_{M12_\theta}\\\Sigma_{U21_\theta}&\Sigma_{U22_\theta}&\Sigma_{M21_\theta}&\Sigma_{M22_\theta}\end{bmatrix}\begin{bmatrix}\Phi&0&0&0\\0&\Phi&0&0\\0&0&\Phi&0\\0&0&0&\Phi\end{bmatrix}^T \tag{4-82}$$

Since Φ is orthogonal, the determinant of $[SM^{ss}]_{8N\times8N}$ is

$$det[SM^{ss}]_{8n\times8N} = det\begin{bmatrix}\Sigma_{U11}&\Sigma_{U12}&\Sigma_{\Theta11}&\Sigma_{\Theta12}\\\Sigma_{U21}&\Sigma_{U22}&\Sigma_{\Theta21}&\Sigma_{\Theta22}\\\Sigma_{U11_\theta}&\Sigma_{U12_\theta}&\Sigma_{\Theta11_\theta}&\Sigma_{\Theta12_\theta}\\\Sigma_{U21_\theta}&\Sigma_{U22_\theta}&\Sigma_{\Theta21_\theta}&\Sigma_{\Theta22_\theta}\end{bmatrix}_{8N\times8N} \tag{4-83}$$

By extending the relationship in the Appendix 4 and employing the properties of the determinants in the Appendix 5, we can simplify the Eq.(4-83) to

$$det[SM^{ss}] = \prod_{\ell=-(N-1)}^{N} det \begin{bmatrix} \mu_\ell^{[U11]} & \mu_\ell^{[U12]} & \mu_\ell^{[M11]} & \mu_\ell^{[M12]} \\ \mu_\ell^{[U21]} & \mu_\ell^{[U22]} & \mu_\ell^{[M21]} & \mu_\ell^{[M22]} \\ \kappa_\ell^{[U11]} & \kappa_\ell^{[U12]} & \kappa_\ell^{[M11]} & \kappa_\ell^{[M12]} \\ \kappa_\ell^{[U21]} & \kappa_\ell^{[U22]} & \kappa_\ell^{[M21]} & \kappa_\ell^{[M22]} \end{bmatrix}_{4\times 4} \tag{4-84}$$

By employing the Eqs.(4-74)-(4-81) for the Eq.(4-84), decomposition of the matrice yields

$$det[SM^{ss}] = \prod_{\ell=-(N-1)}^{N} det([S_\ell^{u\theta}][T_\ell^{ss}]) \tag{4-85}$$

where

$$[T_\ell^{ss}]_{4\times 4} = \begin{bmatrix} J_\ell(\lambda a) & J_\ell(\lambda b) & \alpha_\ell^J(\lambda a) & \alpha_\ell^J(\lambda b) \\ Y_\ell(\lambda a) & Y_\ell(\lambda b) & \alpha_\ell^Y(\lambda a) & \alpha_\ell^Y(\lambda b) \\ I_\ell(\lambda a) & K_\ell(\lambda b) & \alpha_\ell^I(\lambda a) & \alpha_\ell^I(\lambda b) \\ K_\ell(\lambda a) & I_\ell(\lambda b) & \alpha_\ell^K(\lambda a) & \alpha_\ell^K(\lambda b) \end{bmatrix}_{4\times 4} \tag{4-86}$$

It is noted that the matrix $[T_\ell^{ss}]$ denotes the matrix of true eigenequation for the simply-supported-simply-supported case. Zero determinant in the Eq.(4-85) implies that the eigenequation is

$$det([S_\ell^{u\theta}][T_\ell^{ss}]) = 0, \quad \ell = 0, \pm 1, \pm 2, \cdots, \pm(N-1), N. \tag{4-87}$$

After comparing with the analytical solution for the annular plate [58], the former matrix $[S_\ell^{u\theta}]$ in the Eq.(4-87) results in the spurious eigenequation while the latter matrix $[T_\ell^{ss}]$ results in the true eigenequation. The results of the Eq.(4-86) in the discrete system match well with the former one in the continuous system.

Case 3. Annular plate free on both the outer and inner boundaries

To consider a circular annular plate free on both the outer circle B_1 ($m_1 = 0$ and $v_1 = 0$) and the inner circle B_2 ($m_2 = 0$ and $v_2 = 0$), where v_1 and v_2 are the effective shear forces on B_1 and B_2, respectively. When the outer and inner boundary are both discretized into $2N$

constant elements, respectively By using the complex-valued BEM, we have

$$[SM^{ff}] = \begin{bmatrix} M11 & M12 & V11 & V12 \\ M21 & M22 & V21 & V22 \\ M11_\theta & M11_\theta & V11_\theta & V12_\theta \\ M21_\theta & M22_\theta & V21_\theta & V22_\theta \end{bmatrix}_{8N\times 8N} . \quad (4\text{-}88)$$

where the superscript “ ff ” denotes the free-free case. Since the rotation symmetry is preserved for a circular annular boundary, the influence matrices for the discrete system are found to be the circulants. We can obtain the influence matrices ($[V11]$, $[V12]$, $[V21]$ and $[V22]$) which are all symmetric circulants. The eigenvalues of the influence matrices for the discrete system are

$$\begin{aligned}\mu_\ell^{[V11]} =& \frac{\pi a}{4\lambda^2}[Y_\ell(\lambda a)(\beta_\ell^J(\lambda a)+\frac{(1-\nu)}{a}\gamma_\ell^J(\lambda a))+\frac{2}{\pi}K_\ell(\lambda a)(\beta_\ell^I(\lambda a)+\frac{(1-\nu)}{a}\gamma_\ell^I(\lambda a))]\\ &+i[J_\ell(\lambda a)(\beta_\ell^J(\lambda a)+\frac{(1-\nu)}{a}\gamma_\ell^J(\lambda a))+\frac{2}{\pi}(-1)^\ell I_\ell(\lambda a)(\beta_\ell^I(\lambda a)+\frac{(1-\nu)}{a}\gamma_\ell^I(\lambda a))],\\ &\ell=0,\pm1,\cdots,\pm(N-1),N.\end{aligned} \quad (4\text{-}89)$$

$$\begin{aligned}\mu_\ell^{[V12]} =& \frac{\pi b}{4\lambda^2}[Y_\ell(\lambda a)(\beta_\ell^J(\lambda b)+\frac{(1-\nu)}{b}\gamma_\ell^J(\lambda b))+\frac{2}{\pi}K_\ell(\lambda a)(\beta_\ell^I(\lambda b)+\frac{(1-\nu)}{b}\gamma_\ell^I(\lambda b))]\\ &+i[J_\ell(\lambda a)(\beta_\ell^J(\lambda b)+\frac{(1-\nu)}{b}\gamma_\ell^J(\lambda b))+\frac{2}{\pi}(-1)^\ell I_\ell(\lambda a)(\beta_\ell^I(\lambda b)+\frac{(1-\nu)}{b}\gamma_\ell^I(\lambda b))],\\ &\ell=0,\pm1,\cdots,\pm(N-1),N.\end{aligned} \quad (4\text{-}90)$$

$$\begin{aligned}\mu_\ell^{[V21]} =& \frac{\pi a}{4\lambda^2}[(\beta_\ell^Y(\lambda a)+\frac{(1-\nu)}{a}\gamma_\ell^Y(\lambda a))J_\ell(\lambda b)+\frac{2}{\pi}(\beta_\ell^K(\lambda a)+\frac{(1-\nu)}{a}\gamma_\ell^K(\lambda a))I_\ell(\lambda b)]\\ &+i[(\beta_\ell^J(\lambda a)+\frac{(1-\nu)}{a}\gamma_\ell^J(\lambda a))J_\ell(\lambda b)+\frac{2}{\pi}(-1)^\ell(\beta_\ell^I(\lambda a)+\frac{(1-\nu)}{a}\gamma_\ell^I(\lambda a))I_\ell(\lambda b)],\\ &\ell=0,\pm1,\cdots,\pm(N-1),N.\end{aligned} \quad (4\text{-}91)$$

$$\begin{aligned}\mu_\ell^{[V22]} =& \frac{\pi b}{4\lambda^2}[(\beta_\ell^Y(\lambda b)+\frac{(1-\nu)}{b}\gamma_\ell^Y(\lambda b))J'_\ell(\lambda b)+\frac{2}{\pi}(\beta_\ell^K(\lambda b)+\frac{(1-\nu)}{b}\gamma_\ell^K(\lambda b))I_\ell(\lambda b)]\\ &+i[(\beta_\ell^J(\lambda b)+\frac{(1-\nu)}{b}\gamma_\ell^J(\lambda b))J'_\ell(\lambda b)+\frac{2}{\pi}(-1)^\ell(\beta_\ell^I(\lambda b)+\frac{(1-\nu)}{b}\gamma_\ell^I(\lambda b))I_\ell(\lambda b)],\\ &\ell=0,\pm1,\cdots,\pm(N-1),N.\end{aligned} \quad (4\text{-}92)$$

$$\begin{aligned}\kappa_\ell^{[V11]} =& \frac{\pi a}{4\lambda}[Y'_\ell(\lambda a)(\beta_\ell^J(\lambda a)+\frac{(1-\nu)}{a}\gamma_\ell^J(\lambda a))+\frac{2}{\pi}K'_\ell(\lambda a)(\beta_\ell^I(\lambda a)+\frac{(1-\nu)}{a}\gamma_\ell^I(\lambda a))]\\ &+i[J'_\ell(\lambda a)(\beta_\ell^J(\lambda a)+\frac{(1-\nu)}{a}\gamma_\ell^J(\lambda a))+\frac{2}{\pi}(-1)^\ell I'_\ell(\lambda a)(\beta_\ell^I(\lambda a)+\frac{(1-\nu)}{a}\gamma_\ell^I(\lambda a))],\\ &\ell=0,\pm1,\cdots,\pm(N-1),N.\end{aligned} \quad (4\text{-}93)$$

$$\begin{aligned}\kappa_\ell^{[V12]} =& \frac{\pi b}{4\lambda}[Y'_\ell(\lambda a)(\beta_\ell^J(\lambda b)+\frac{(1-\nu)}{b}\gamma_\ell^J(\lambda b))+\frac{2}{\pi}K'_\ell(\lambda a)(\beta_\ell^I(\lambda b)+\frac{(1-\nu)}{b}\gamma_\ell^I(\lambda b))]\\ &+i[J'_\ell(\lambda a)(\beta_\ell^J(\lambda b)+\frac{(1-\nu)}{b}\gamma_\ell^J(\lambda b))+\frac{2}{\pi}(-1)^\ell I'_\ell(\lambda a)(\beta_\ell^I(\lambda b)+\frac{(1-\nu)}{b}\gamma_\ell^I(\lambda b))],\\ &\ell=0,\pm1,\cdots,\pm(N-1),N.\end{aligned} \quad (4\text{-}94)$$

$$\kappa_\ell^{[V21]} = -\frac{\pi b}{4\lambda}[(\beta_\ell^Y(\lambda a) + \frac{(1-\nu)}{a}\gamma_\ell^Y(\lambda a))J'_\ell(\lambda b) + \frac{2}{\pi}(\beta_\ell^K(\lambda a) + \frac{(1-\nu)}{a}\gamma_\ell^K(\lambda a))I'_\ell(\lambda b)]$$
$$+ i[(\beta_\ell^J(\lambda a) + \frac{(1-\nu)}{a}\gamma_\ell^J(\lambda a))J'_\ell(\lambda b) + \frac{2}{\pi}(-1)^\ell(\beta_\ell^I(\lambda a) + \frac{(1-\nu)}{a}\gamma_\ell^I(\lambda a))I'_\ell(\lambda b)], \quad \text{(4-95)}$$
$$\ell = 0, \pm 1, \cdots, \pm(N-1), N.$$
$$\kappa_\ell^{[V22]} = \frac{\pi b}{4\lambda}[(\beta_\ell^Y(\lambda b) + \frac{(1-\nu)}{b}\gamma_\ell^Y(\lambda b))J'_\ell(\lambda b) + \frac{2}{\pi}(\beta_\ell^K(\lambda b) + \frac{(1-\nu)}{b}\gamma_\ell^K(\lambda b))I'_\ell(\lambda b)]$$
$$+ i[(\beta_\ell^J(\lambda b) + \frac{(1-\nu)}{b}\gamma_\ell^J(\lambda b))J'_\ell(\lambda b) + \frac{2}{\pi}(-1)^\ell(\beta_\ell^I(\lambda b) + \frac{(1-\nu)}{b}\gamma_\ell^I(\lambda b))I'_\ell(\lambda b)], \quad \text{(4-96)}$$
$$\ell = 0, \pm 1, \pm 2, \cdots, \pm(N-1), N.$$

where $\mu_\ell^{[V11]}$, $\mu_\ell^{[V12]}$, $\mu_\ell^{[V21]}$, $\mu_\ell^{[V22]}$, $\kappa_\ell^{[V11]}$, $\kappa_\ell^{[V12]}$, $\kappa_\ell^{[V21]}$ and $\kappa_\ell^{[V22]}$, are the eigenvalues of the matrices $[V11]$, $[M12]$, $[V21]$, $[V22]$, $[V11_\theta]$, $[V12_\theta]$, $[V21_\theta]$ and $[V22_\theta]$, respectively. Since the matrices are all symmetric circulants, the matrix $[SM^{ff}]$ in the Eq.(4-88) can be decomposed into

$$[SM^{ff}] = \begin{bmatrix} \Phi & 0 & 0 & 0 \\ 0 & \Phi & 0 & 0 \\ 0 & 0 & \Phi & 0 \\ 0 & 0 & 0 & \Phi \end{bmatrix} \begin{bmatrix} \Sigma_{M11} & \Sigma_{M12} & \Sigma_{V11} & \Sigma_{V12} \\ \Sigma_{M21} & \Sigma_{M22} & \Sigma_{V21} & \Sigma_{V22} \\ \Sigma_{M11_\theta} & \Sigma_{M12_\theta} & \Sigma_{V11_\theta} & \Sigma_{V12_\theta} \\ \Sigma_{M21_\theta} & \Sigma_{M22_\theta} & \Sigma_{V21_\theta} & \Sigma_{V22_\theta} \end{bmatrix} \begin{bmatrix} \Phi & 0 & 0 & 0 \\ 0 & \Phi & 0 & 0 \\ 0 & 0 & \Phi & 0 \\ 0 & 0 & 0 & \Phi \end{bmatrix}^T \quad \text{(4-97)}$$

Since Φ is orthogonal, the determinant of $[SM^{ff}]_{8N\times 8N}$ is

$$det[SM^{ff}] = det \begin{bmatrix} \Sigma_{M11} & \Sigma_{M12} & \Sigma_{V11} & \Sigma_{V12} \\ \Sigma_{M21} & \Sigma_{M22} & \Sigma_{V21} & \Sigma_{V22} \\ \Sigma_{M11_\theta} & \Sigma_{M12_\theta} & \Sigma_{V11_\theta} & \Sigma_{V12_\theta} \\ \Sigma_{M21_\theta} & \Sigma_{M22_\theta} & \Sigma_{V21_\theta} & \Sigma_{V22_\theta} \end{bmatrix} \quad \text{(4-98)}$$

By extending the relationship in the Appendix 4 and the employing properties of the determinants in the Appendix 5, we can simplify the Eq.(4-98) to

$$det[SM^{ff}] = \prod_{\ell=-(N-1)}^{N} det \begin{bmatrix} \mu_\ell^{[M11]} & \mu_\ell^{[M12]} & \mu_\ell^{[V11]} & \mu_\ell^{[V12]} \\ \mu_\ell^{[M21]} & \mu_\ell^{[M22]} & \mu_\ell^{[V21]} & \mu_\ell^{[V22]} \\ \kappa_\ell^{[M11]} & \kappa_\ell^{[M12]} & \kappa_\ell^{[V11]} & \kappa_\ell^{[V12]} \\ \kappa_\ell^{[M21]} & \kappa_\ell^{[M22]} & \kappa_\ell^{[V21]} & \kappa_\ell^{[V22]} \end{bmatrix}_{4\times 4} \quad \text{(4-99)}$$

By employing the Eqs.(4-50)-(4-63),for the Eq.(4-99), decomposition of the matrix yields

$$det[SM^{ff}] = \prod_{\ell=-(N-1)}^{N} det([S_\ell^{u\theta}][T_\ell^{ff}]) \tag{4-100}$$

where

$$[T_\ell^{ff}] = \begin{bmatrix} \alpha_\ell^J(\lambda a) & \alpha_\ell^J(\lambda b) & \beta_\ell^J(\lambda a) + \frac{(1-\nu)}{a}\gamma_\ell^J(\lambda a) & \beta_\ell^J(\lambda b) + \frac{(1-\nu)}{b}\gamma_\ell^J(\lambda b) \\ \alpha_\ell^Y(\lambda a) & \alpha_\ell^Y(\lambda b) & \beta_\ell^Y(\lambda a) + \frac{(1-\nu)}{a}\gamma_\ell^Y(\lambda a) & \beta_\ell^Y(\lambda b) + \frac{(1-\nu)}{b}\gamma_\ell^Y(\lambda b) \\ \alpha_\ell^I(\lambda a) & \alpha_\ell^I(\lambda b) & \beta_\ell^I(\lambda a) + \frac{(1-\nu)}{a}\gamma_\ell^I(\lambda a) & \beta_\ell^I(\lambda b) + \frac{(1-\nu)}{b}\gamma_\ell^I(\lambda b) \\ \alpha_\ell^K(\lambda a) & \alpha_\ell^K(\lambda b) & \beta_\ell^K(\lambda a) + \frac{(1-\nu)}{a}\gamma_\ell^K(\lambda a) & \beta_\ell^K(\lambda b) + \frac{(1-\nu)}{b}\gamma_\ell^K(\lambda b) \end{bmatrix} \tag{4-101}$$

It is noted that the matrix $[T_\ell^{ff}]$ denotes the matrix of true eigenequation for the F-F case. Zero determinant in the Eq.(4-100) implies that the eigenequation is

$$det([S_\ell^{u\theta}][T_\ell^{ff}]) = 0, \quad \ell = 0, \pm 1, \pm 2, \cdots, \pm(N-1), N. \tag{4-102}$$

After comparing with the analytical solution for the annular plate [58], the former matrix $[S_\ell^{u\theta}]$ in the Eq.(4-102) results in the spurious eigenequation while the latter matrix $[T_\ell^{ff}]$ results in the true eigenequation. The results of the Eq.(4-101) in the discrete system match well with the former one in the continuous system.

The proof in this chapter can also work well for the different boundary conditions on the outer boundary and inner boundary (C-S, C-F, S-C, S-F, F-C and F-S). All the results for each type of boundary conditions of the annular plate are shown in the Table 4-1.

4-1-3 Study of the spurious eigenequation

After comparing the Eq.(4-72) with the Eqs.(4-87) and (4-102) in the discrete system or the results of the Eqs.(4-16), the Eqs.(4-30) and (4-44) in the continuous system for the annular plate, the same spurious eigenequation ($[S_n^{u\theta}] = 0$) is embedded in the u, θ formulation no matter what the boundary condition is. By using the cofactor of the matrix $[S_n^{u\theta}]$ to simplify the zero determinant of the Eq.(4-70) for the spurious eigenequation, we have

$$det[S_n^{u\theta}]_{4\times4} = det\begin{bmatrix} (Y_n(\lambda a)+iJ_n(\lambda a)) & 0 & (K_n(\lambda a)+iI_n(\lambda a)) & 0 \\ iJ_n(\lambda b) & J_n(\lambda b) & iI_n(\lambda b) & I_n(\lambda b) \\ (Y_n'(\lambda a)+iJ_n'(\lambda a)) & 0 & (K_n'(\lambda a)+iI_n'(\lambda a)) & 0 \\ iJ_n'(\lambda b) & J_n'(\lambda b) & iI_n'(\lambda b) & I_n'(\lambda b) \end{bmatrix}_{4\times4}$$
$$= det([Sa_n^{u\theta}][Sb_n^{u\theta}]) \quad (4\text{-}103)$$

where

$$[Sa_n^{u\theta}] = \begin{bmatrix} (Y_n(\lambda a)+iJ_n(\lambda a)) & (K_n(\lambda a)+iI_n(\lambda a)) \\ (Y_n'(\lambda a)+iJ_n'(\lambda a)) & (K_n'(\lambda a)+iI_n'(\lambda a)) \end{bmatrix}_{2\times2} \quad (4\text{-}104)$$

and

$$[Sb_n^{u\theta}] = \begin{bmatrix} J_n(\lambda b) & I_n(\lambda b) \\ J_n'(\lambda b) & I_n'(\lambda b) \end{bmatrix}_{2\times2} \quad (4\text{-}105)$$

It is found that the determinant of the former matrix $[Sa_n^{u\theta}]$ in the Eq.(4-104) is never zoro. The spurious eigenequation is the zero determinant of the matrix $[Sb_n^{u\theta}]$ in the Eq.(4-105). It is interesting that the zero determinant of the $[Sb_n^{u\theta}]$ in the u, θ formulation results in the true eigenequation of simply-connected clamped plate with a radius b. The spurious eigenvalues parasitizing in the u and θ BEM depend on the radius b which is the inner circle of the annular domain. In fact, the multiply-connected problem can be superimposed by two problems, one is an interior problem with B_2 boundary and the other is an exterior problem with B_1 boundary as shown in Figure 4-1. The source which causes the appearance of the spurious eigenvalues stems from the exterior problem with the inner boundary even though the complex-valued kernels are employed as well as the membrane and acoustics behaves [21, 31].

Since any two equations in the plate formulation (the Eqs.(2-28)-(2-31)) in the Chapter 2 can be chosen, $6(C_2^4)$ options of the formulation can be considered. If we choose different formulae for the annular plate, we can obtain the same true eigenequation but different spurious eigenequations. All the resluts of the spurious eigenequation and the relationship between the simply and multiply-connected plate problems are shown in the Table 4-2(a), 4-2(b) and 4-2(c). At the same time, all the cases of the different B.Cs. result in the same spurious eigenequation, once we use the same formulation. The occurrence of spurious eigenequation only depends on the formulation instead of the specified boundary condition. True eigenequation depends on the specified boundary condition instead of the formulation. All the resluts are summarized in the Table 4-1.

4-2 Numerical results and discussions

An annular plate with the outer radius of one meter $(a = 1\,m)$ and the inner radius of 0.5 meter $(b = 0.5\,m)$ of B_1 and B_2, respectively, and the Poisson ratio $\nu = 1/3$ are considered. The outer and inner boundaries are both discretized into ten constant elements, respectively. Since any two equations in the plate formulation (the Eqs.(2-28)-(2-31)) can be chosen, $6(C_2^4)$ options of the formulation can be considered. Since the outer and inner boundaries can be subject to one of the three BCs (clamped, simply-supported and free boundary conditions), nine cases (C-C, C-S, C-F, S-C, S-S, S-F, F-C, F-S and F-F) can be considered.

Figures 4-2 $\sim$ 4-10 show the determinant of $[SM]$ versus frequency parameter λ for the nine cases of annular plate using the six complex-valued formulations. Both the true and spurious eigenvalues occur simultaneously even though the complex-valued BEM is employed. After comparing with (a), (b), (c), (d), (e) and (f) results for each figure, the same true eigenvalues are obtained no matter what the adopted formulation is, it reconfirms that the true eigenvalues depends on the specified boundary condition instead of the formulation, all the true eigenvalues satisfy the true eigenequation in Table 4-1. We obtained different spurious eigenvalues versus different formulations in Figures 4-2 $\sim$ 4-10. After selecting the formualtion (*e.g.* u, θ formulation), the spurious eigenvalues (6.392, 9.222 and 11.810) occur at the positions which satisfy the spurious eigenequation $[S_n^{u\theta}] = 0$ in Eq.(4-70) as shown in figures 4-2.(a), 4-3.(a), 4-4.(a), 4-5.(a), 4-6.(a), 4-7.(a), 4-8.(a), 4-9.(a) and 4-10.(a). The spurious eigenvalues in the six (C_2^4) different formulations are shown in the (a), (b), (c), (d), (e) and (f) cases, and their corresponding spurious eigenequation are summarized in Table 4-2.(c). In order to distinguish the spurious eigenvalues, Figures 4-11.(a)-(c) and (d)-(f) show the determinant of $[SM]$ versus λ using the same formualtion (a, b and c - u, θ formulation; d, e and f - u, m formulation) to solve the plates subject to different boundary conditions. It is found that any one of the C-C, S-S and F-F cases results in the same spurious eigenvalues, once the same formulation (a, b and c - u, θ formulation; d, e and f - u, m formulation) is employed. The numerical results reconfirm that the occurrence of spurious eigenvalues only depends on the formulation instead of the specified boundary condition.

The spurious eigenequation of multiply-connected eigenproblem by using the u, θ formulation is found to be the true eigenequation of the simply-connected clamped plate with a radius b which is the inner radius of the annular plate. For demonstration, Figure 4-12.(a) shows the determinant of $[SM]$ versus λ using the complex-valued formualtion to solve the simply-connected clamped plate since u, θ formulation is utilized to solve the annular problem. The true eigenvalues (6.392, 9.222 and 11.810) for the clamped circular plate in Figure 4-12.(a) with a radius $b = 0.5m$ also appears in the spurious eigenvalues in Figures 4-2.(a), 4-3.(a), 4-4.(a), 4-5.(a), 4-6.(a), 4-7.(a), 4-8.(a), 4-9.(a) and 4-10.(a) by using the u, θ complex-valued formulation for the annular plate. In another words, the spurious eigenvalues embedded in each (C_2^4) formulation are corresponding to the associated true eigenvalues of the simply-connected plate as shown in Figures 4-12.(a), 4-12.(b), 4-12.(c), 4-12.(d), 4-12.(e) and 4-12.(f).

In general, all the annular cases result in the same spurious eigenvalues, once the formulation is adopted. The occurrence of spurious eigenvalues only depends on the formulation instead of the specified boundary condition. All the numerical data of the true eigenvalues are summarized in the Table 4-3(a)$\sim$(i), and the eigenvalues agree well with the data in Leissa and Laura *et al.* [58, 75, 76]. To detect the sensitivity how the inner radius b affects the true eigenvalue, the results are shown in Table 4-4.(a) $\sim$ (i). Good agreement with Laura *et al.* [75, 76] is made.

It is worth mentioning that we provide the unified form of the true eigenequations for the three cases of annular plates in Table 4-1 instead of the separate form ($n = 0, 1, 2$) as shown in the Appendix 6 [58]. The same true eigenvalues are compared with the Leissa's numerical results. However, the obtained eigenvalues according to the Leissa's eigenequation are not consistent to those in his book. The possible explanation is that the eigenequations in the Leissa's book for some cases were wrongly typed as shown in the Appendix 6.

4-3 Concluding remarks

A complex-valued BEM formulation has been derived for the free vibration of annular plate. The true and spurious eigenequations were derived analytically by using the Fourier series, degenerate kernels and circulants in both the continuous and discrete systems. The eigenvalues were determined numerically. Since either two equations in the plate formulation (4 equations) can be chosen, C_2^4 (6) options can be considered. The occurrence of spurious eigenequation only depends on the formulation instead of the specified boundary condition, while the true eigenequation is independent of the formulation and is relevant to the specified boundary condition. It is interesting that the spurious eigenequation of multiply-connected plate eigenproblem by using the u, θ formulation is found to be the true eigenequation of simply-connected clamped plate with a radius b which is the inner radius of the annular plate. All the results are shown in the Table 4-2(a) $\sim$ (c). Three cases were demonstrated analytically to see the validity of the present method. Several examples of plates subject to C-C, C-S, C-F, S-C, S-S, S-F, F-C, F-S and F-F were illustrated to check the validity of the present formulations. Although the annular case lacks generality, it leads significant insight into the occurring mechanism of true and spurious eigenequation. Although the proof is only limited to the annular case, it is a great help to the researchers who may require analytical explanation for the reason why the spurious eigenevalues appears. The same algorithm in the discrete system can be applied to solve arbitrary-shaped plate numerically without any difficulty. Nevertheless, mathematical derivation in the continuous and discrete systems can not be done analytically.

Chapter 5 Treatment of the spurious eigenvalues for multiply-connected eigenproblems

Summary

In this chapter, three alternatives (SVD updating technique, the Burton & Miller method and the CHIEF method) are adopted to suppress the occurrence of the spurious eigenvalues for the multiply-connected plate eigenproblem. A clamped-clamped annular plate is demonstrated analytically in the discrete systems.

5-1 SVD updating technique

In the discrete system, the approach to detect the nonunique solution is the criterion of satisfying all the Eqs.(2-28)-(2-31) at the same time by using the complex-valued BEM. After rearranging the terms of the Eqs.(3-8) and (3-9) in the Chapter 3, we have

$$[SM_1^{cc}]\begin{Bmatrix} v_1 \\ v_2 \\ m_1 \\ m_2 \end{Bmatrix} = \{0\}, \tag{5-1}$$

where

$$[SM_1^{cc}] = \begin{bmatrix} U11 & U12 & \Theta 11 & \Theta 12 \\ U21 & U22 & \Theta 21 & \Theta 22 \\ U11_\theta & U12_\theta & \Theta 11_\theta & \Theta 12_\theta \\ U21_\theta & U22_\theta & \Theta 21_\theta & \Theta 22_\theta \end{bmatrix}. \tag{5-2}$$

Similarly, the Eqs.(3-10) and (3-11) yield

$$[SM_2^{cc}]\begin{Bmatrix} v_1 \\ v_2 \\ m_1 \\ m_2 \end{Bmatrix} = \{0\}, \tag{5-3}$$

where

$$[SM_2^{cc}] = \begin{bmatrix} U11_m & U12_m & \Theta 11_m & \Theta 12_m \\ U21_m & U22_m & \Theta 21_m & \Theta 22_m \\ U11_v & U12_v & \Theta 11_v & \Theta 12_v \\ U21_v & U22_v & \Theta 21_v & \Theta 22_v \end{bmatrix}. \tag{5-4}$$

For the clamped-clamped case by using the SVD technique of updating term, we have

$$[C] \begin{Bmatrix} v_1 \\ v_2 \\ m_1 \\ m_2 \end{Bmatrix} = \{0\}, \tag{5-5}$$

where

$$[C] = \begin{bmatrix} SM_1^{cc} \\ SM_2^{cc} \end{bmatrix}_{16N\times 8N}. \tag{5-6}$$

Since the eigenequation is nontrivial, the rank of the matrix $[C]$ must be smaller than $8N$, the $8N$ singular values for the matrix $[C]$ must have at least one zero value. The explicit form for the matrix $[C]$ can be decomposed into

$$[C] = \begin{bmatrix} \Phi & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \Phi & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \Phi & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \Phi & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \Phi & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \Phi & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \Phi & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Phi \end{bmatrix}_{16N\times 16N} \tag{5-7}$$

$$\begin{bmatrix} \Sigma_{U11} & \Sigma_{U12} & \Sigma_{\Theta 11} & \Sigma_{\Theta 12} \\ \Sigma_{U21} & \Sigma_{U22} & \Sigma_{\Theta 21} & \Sigma_{\Theta 22} \\ \Sigma_{U11_\theta} & \Sigma_{U12_\theta} & \Sigma_{\Theta 11_\theta} & \Sigma_{\Theta 12_\theta} \\ \Sigma_{U21_\theta} & \Sigma_{U22_\theta} & \Sigma_{\Theta 21_\theta} & \Sigma_{\Theta 22_\theta} \\ \Sigma_{U11_m} & \Sigma_{U12_m} & \Sigma_{\Theta 11_m} & \Sigma_{\Theta 12_m} \\ \Sigma_{U21_m} & \Sigma_{U22_m} & \Sigma_{\Theta 21_m} & \Sigma_{\Theta 22_m} \\ \Sigma_{U11_v} & \Sigma_{U12_v} & \Sigma_{\Theta 11_v} & \Sigma_{\Theta 12_v} \\ \Sigma_{U21_v} & \Sigma_{U22_v} & \Sigma_{\Theta 21_v} & \Sigma_{\Theta 22_v} \end{bmatrix}_{16N\times 8N} \begin{bmatrix} \Phi & 0 & 0 & 0 \\ 0 & \Phi & 0 & 0 \\ 0 & 0 & \Phi & 0 \\ 0 & 0 & 0 & \Phi \end{bmatrix}^T_{8N\times 8N}. \tag{5-8}$$

where the dimension of each submatrix Φ and Σ is $2N$ by $2N$. Based on the equivalence between the SVD technique and the least-squares method in mathematical essence, the least square form leads to

$$[C]^T[C] = \begin{bmatrix} \Phi & 0 & 0 & 0 \\ 0 & \Phi & 0 & 0 \\ 0 & 0 & \Phi & 0 \\ 0 & 0 & 0 & \Phi \end{bmatrix}_{8N\times 8N} [D]_{8N\times 8N} \begin{bmatrix} \Phi & 0 & 0 & 0 \\ 0 & \Phi & 0 & 0 \\ 0 & 0 & \Phi & 0 \\ 0 & 0 & 0 & \Phi \end{bmatrix}^T_{8N\times 8N} \tag{5-9}$$

where

$$[D] = \begin{bmatrix} \Sigma_{U11} & \Sigma_{U21} & \Sigma_{U11_\theta} & \Sigma_{U21_\theta} & \Sigma_{U11_m} & \Sigma_{U21_m} & \Sigma_{U11_v} & \Sigma_{U21_v} \\ \Sigma_{U12} & \Sigma_{U22} & \Sigma_{U12_\theta} & \Sigma_{U22_\theta} & \Sigma_{U12_m} & \Sigma_{U22_m} & \Sigma_{U12_v} & \Sigma_{U22_v} \\ \Sigma_{\Theta 11} & \Sigma_{\Theta 21} & \Sigma_{\Theta 11_\theta} & \Sigma_{\Theta 21_\theta} & \Sigma_{\Theta 11_m} & \Sigma_{\Theta 21_m} & \Sigma_{\Theta 11_v} & \Sigma_{\Theta 21_v} \\ \Sigma_{\Theta 12} & \Sigma_{\Theta 22} & \Sigma_{\Theta 12_\theta} & \Sigma_{\Theta 22_\theta} & \Sigma_{\Theta 12_m} & \Sigma_{\Theta 22_m} & \Sigma_{\Theta 12_v} & \Sigma_{\Theta 22_v} \end{bmatrix}_{8N\times 16N} \tag{5-10}$$

$$\begin{bmatrix} \Sigma_{U11} & \Sigma_{U12} & \Sigma_{\Theta 11} & \Sigma_{\Theta 12} \\ \Sigma_{U21} & \Sigma_{U22} & \Sigma_{\Theta 21} & \Sigma_{\Theta 22} \\ \Sigma_{U11_\theta} & \Sigma_{U12_\theta} & \Sigma_{\Theta 11_\theta} & \Sigma_{\Theta 12_\theta} \\ \Sigma_{U21_\theta} & \Sigma_{U21_\theta} & \Sigma_{\Theta 21_\theta} & \Sigma_{\Theta 22_\theta} \\ \Sigma_{U11_m} & \Sigma_{U12_m} & \Sigma_{\Theta 11_m} & \Sigma_{\Theta 12_m} \\ \Sigma_{U21_m} & \Sigma_{U22_m} & \Sigma_{\Theta 21_m} & \Sigma_{\Theta 22_m} \\ \Sigma_{U11_v} & \Sigma_{U12_v} & \Sigma_{\Theta 11_v} & \Sigma_{\Theta 12_v} \\ \Sigma_{U21_v} & \Sigma_{U22_v} & \Sigma_{\Theta 21_v} & \Sigma_{\Theta 22_v} \end{bmatrix}_{16N\times 8N} \tag{5-11}$$

If the determinant of the matrix $[C]^T[C]$ is zero, we can obtain the nontrivial solution. Since Φ is orthogonal, the determinant of the matrix $[C]^T[C]$ is equal to the determinant of the matrix $[D]$. By calculating the determinant of the matrix $[D]$ and using the the relationship in the Appendix 7, we can find that the determinant of the matrix $[D]$ can be decomposed into the summation of the square determinant in the C_4^8 matrices. The only possibility for the zero determinant of the matrix $[D]$ occurs when the C_4^8 terms are all zeros at the same time for the same ℓ. After careful check for all the matrices, we find that the true eigenequation $[T_\ell^{cc}]$ is simultaneously embedded in the C_4^8 matrices. This indicates that only the true eigenequation of the clamped-clamped circular annular plate is sorted out in the SVD updating matrix since the true eigenequation is simultaneously embedded in the six complex-valued formulations.

The result matches well with the former one of the Chapter 4 in the discrete system, respectively.

5-2 Burton & Miller method

By combining the Eqs.(5-1) and (5-3) with an imaginary number in the complex-valued BEM, we have

$$[[SM_1^{cc}]+i[SM_2^{cc}]]\left\{\begin{array}{c} v_1 \\ v_2 \\ m_1 \\ m_2 \end{array}\right\}=\{0\}. \tag{5-12}$$

The determinant of the $[SM_1^{cc}]+i[SM_2^{cc}]$ is obtained by using the circulant and the decomposition technique as

$$det[[SM_1^{cc}]+i[SM_2^{cc}]]=\prod_{\ell=-(N-1)}^{N} det([[S_\ell^{u\theta}]+i[S_\ell^{mv}]][T_\ell^{cc}]) \tag{5-13}$$

Since the term $[S_\ell^{u\theta}]+i[S_\ell^{mv}]$ is never zero for any λ, we can obtain the true eigenvalues by using the complex-valued BEM with the Burton & Miller concept. Unfortunately, if we combine the u, θ and m, v formulations or u, v and θ, m formulations, the method fails. The reason is that the u, v and θ, m formulation have the same spurious eigenequation. Only the combination of u, m and θ, v complex-valued formulation can obtain the true eigenvalues. All the explicit forms of the $[S_\ell^{u\theta}]+i[S_\ell^{mv}]$ are shown in the Table 5-1 by using the complex-valued BEM. Since any two equation in the complex-valued formulation results in the spurious eigenvalues, we can reconstruct the independent equation by employing the Burton & Miller concept. When we choose the appropriate combination, the Burton & Miller method works well.

5-3 CHEEF or CHIEF method

Consider the eigenproblem for the clamped annular plate, the Eqs.(2-24) and (2-25) can be rewritten as

$$\begin{bmatrix} U11 & U12 & \Theta 11 & \Theta 12 \\ U21 & U22 & \Theta 21 & \Theta 22 \\ U11_\theta & U12_\theta & \Theta 11_\theta & \Theta 12_\theta \\ U21_\theta & U22_\theta & \Theta 21_\theta & \Theta 22_\theta \end{bmatrix} \begin{Bmatrix} v_1 \\ v_2 \\ m_1 \\ m_2 \end{Bmatrix}_{4N\times 1} = \{0\}_{4N\times 1}. \tag{5-14}$$

By adding the point for the null-field equation to solve the multiply-connected plate eigenproblem, we have two choices for the location of the CHIEF point ($\rho < b$) or CHEEF point ($a < \rho$). Because the spurious eigenequation of multiply-connected plate eigenproblem by using the u, θ formulation is the true eigenequation of simply-connected clamped plate with a radius b which is the inner radius of the annular plate. If the CHEEF point locates on the outer the domain ($a < \rho$), the CHEEF method fails. By moving the field point x to be outside the domain ($\rho < b$) for the CHIEF points, we have

$$\begin{bmatrix} UC1 & UC2 & \Theta C1 & \Theta C2 \\ UC1_\theta & UC2_\theta & \Theta C1_\theta & \Theta C2_\theta \end{bmatrix}_{2N_C\times 4N} \begin{Bmatrix} v_1 \\ v_2 \\ m_1 \\ m_2 \end{Bmatrix}_{8N\times 1} = \{0\}_{8N\times 1}, \tag{5-15}$$

where the index C denotes the CHIEF point in the null-field equation and the subscript N_c (≥ 1) indicates the number of additional CHIEF points. The symbols, $UC1$, $UC2$, $\Theta C1$, $\Theta C2$, $UC1_\theta$, $UC2_\theta$, $\Theta C1_\theta$ and $\Theta C2_\theta$ mean the influence row vectors resulted from of the U, Θ , U_θ and Θ_θ kernels which is obtained by collocating the CHIEF point. Combining the Eqs.(5-14) and (5-15) together to obtain the overdetermined system, we have

$$[C^*] \begin{Bmatrix} v_1 \\ v_2 \\ m_1 \\ m_2 \end{Bmatrix}_{8N\times 1} = \{0\}_{(8N+2N_C)\times 1}, \tag{5-16}$$

where

$$[C^*] = \begin{bmatrix} U11 & U12 & \Theta 11 & \Theta 12 \\ U21 & U22 & \Theta 21 & \Theta 22 \\ U11_\theta & U12_\theta & \Theta 11_\theta & \Theta 12_\theta \\ U21_\theta & U22_\theta & \Theta 21_\theta & \Theta 22_\theta \\ \hline UC1 & UC2 & \Theta C1 & \Theta C2 \\ UC1_\theta & UC2_\theta & \Theta C1_\theta & \Theta C2_\theta \end{bmatrix}_{(8N+2N_C)\times 8N} \tag{5-17}$$

Therefore, an overdetermined system is obtained to ensure a unique solution. According to the successful experience of CHEEF technique for simply-connected eigenproblem, we can overcome the spurious eigenvalues problem in the multiply-connected plate eigenproblem by using the same concept. Also, the optimum number of adding CHIEF points and appropriate positions of the CHIEF points will be addressed in the numerical results.

5-4 Numerical results and discussions

Annular plate (C-C, S-S and F-F boundary conditions)

An annular plate with the outer radius of one meter $(a = 1\,m)$ and the inner radius of 0.5 meter $(b = 0.5\,m)$ of B_1 and B_2, respectively, and the Poisson ratio $\nu = 1/3$ are considered. The outer and inner boundaries are both discretized into ten constant elements, respectively. Since any two equations in the plate formulation (the Eqs.(2-28)-(2-31)) can be chosen, $6(C^4_2)$ options of the formulation can be considered. Three cases, C-C, S-S and F-F, by using the three methods are demonstrated.

SVD updating technique

Figures 5-1.(a)-(f) show the determinant of the $[C]^T[C]$ versus λ for the C-C annular plate using the six complex-valued formulations in conjunction with the SVD technique of updating term. Figures 5-2.(a)-(f) show the determinant of the $[C]^T[C]$ versus λ for the S-S annular plate using the six complex-valued formulations in conjunction with the SVD technique of updating term. Figures 5-3.(a)-(f) show the determinant of the $[C]^T[C]$ versus λ for the F-F annular plate using the six complex-valued formulations in conjunction with the SVD technique of updating term. It is found that all the spurious eigenvalues are filtered out and only the true eigenvalues appear.

Burton & Miller method

Figures 5-4.(a)-(f) $\sim$ 5-6.(a)-(f) show the determinant of the $[SM]$ versus λ for the C-C, S-S and F-F annular plates using the six complex-valued formulations in conjunction with Burton & Miller concept. The failure cases are shown in the (a), (c), (d) and (f) figures as predicted in the Table 5-1. Only the combination of u, m and θ, v formulation can obtain the true eigenvalues in Figures 5-4.(b), 5-4.(e), 5-5.(b), 5-5.(e), 5-6.(b) and 5-6.(e) as predicted in the Table 3-1, since $det([Sb_n^{um}] + i[Sb_n^{\theta v}])$ can not be zero. For the case of the u, θ formualtion in conjunction with the m, v formulations by multiplying an imaginary number for solving the annular plates subject to different boundary conditions (5-4.(a), 5-5.(a) and 5-6.(a)), the spurious eigenvalues occur since $det([Sb_n^{u\theta}] + i[Sb_n^{mv}]) = 0$ as predicted in the Table 5-1.

CHIEF method

Figures 5-7.(a)-(c) show the minimum singular value σ_1 of the $[C^*]$ versus λ for the F-F annular plate by using the complex-valued (u, θ) formulations in conjunction with zero (without CHIEF point), one and two CHIEF points. The first CHIEF point (ρ_1, ϕ_1) locates at $(0.285, \pi/4)$. It is interesting to find that one CHIEF point can not suppress the appearance of all the spurious eigenvalues (9.222 and 11.810) as shown in Figure 5-7.(b). By adding another CHIEF point (ρ_2, ϕ_2) which locates at $(0.275, 29\pi/36)$, where the angle $(\phi_1 - \phi_2)$ between the two selected points is $5\pi/9$, only the true eigenvalues are obtained as shown in Figure 5-7.(c) by using the two CHIEF points. Similarly, Figures 5-7.(d)-(f) show the minimum singular value σ_1 of the $[C^*]$ versus λ for the F-F annular plate by using the complex-valued (u, m) formulation in conjunction with zero (without CHIEF point), one and two CHIEF points. The CHIEF points (ρ_1, ϕ_1) and (ρ_2, ϕ_2) locate at $(0.30, \pi/4)$ and $(0.28, 29\pi/36)$, where the angle $(\phi_1 - \phi_2)$ between the two selected points is $5\pi/9$. Good agreement is made by using the CHIEF method. Only the true eigenvalues are obtained.

For the C-C, S-S and F-F annular cases, the SVD technique of updating term, Burton & Miller method (u, m and θ, v formulations) and CHIEF method can obtain the same true eigenvalues. All the resluts of the eigenvalues agree well with the data in Leissa and Laura *et al.* [58, 75, 76]. For the multiply-connected problems, the CHIEF method saves CPU time in constructing the influence matrix in comparison with the SVD updating technique. However, the Burton & Miller method has the minimum dimension of the influence matrix in the SVD computation.

5-5 Concluding remarks

Three alternatives (SVD updating technique, the Burton & Miller method and the CHIEF method) were adopted to suppress the occurrence of the spurious eigenvalues for the C-C, S-S and F-F annular plates in the complex-valued BEM. The SVD updating technique was employed to deal with the problem of spurious eigenvalue occurring in the multiply-connected plate eigenproblem. Then, the numerical experiments of the C-C, S-S and F-F annular problems were performed to demonstrate the validity of the remedies. The role of the Burton & Miller method for spurious eigenvalues has also been examined. By choosing the useful CHIEF points, we can suppress the occurrence of the spurious eigenvalues for the C-C annular plate by using the complex-valued BEM. For the eigenproblems of the multiply-connected plate, the SVD technique of updating term, Burton & Miller method (u, m and θ, v formulations) and CHIEF method can obtain the true eigenvalues and the eigenvalues agree well with the data in Leissa and Laura *et al.* [58, 75, 76]. For the multiply-connected problems, the CHIEF method save CPU time in constructing the influence matrix. However, the Burton & Miller method have the minimum dimension in the SVD computation.

Chapter 6 Conclusions and further research

6-1 Conclusions

In this thesis, we draw out some important conclusions item by item as below:

1. We have verified that the spurious eigenequations depend on the formulation in the real-part and imaginary-part BEMs for the simply-connected plate eigenproblems (three cases). Six (C_2^4) formulations can be chosen. True eigenequation depends on the cases while spurious eigenequation is embedded in each formulation. Both the continuous and the discrete systems support this finding.

2. The spurious and true eigenequations for the simply-connected plate were analytically derived by using the degenerate kernel, Fourier series expansion and circulants. All the spurious eigenequations embedded in the real and imaginary-part BEMs are summarized in the Tables 2-2 and 2-4.

3. By extending the finding of the supurious eigenvalues for the simply-connected plate, we have verified that the spurious eigenequations depend on the formulation by using the complex-valued BEM for the multiply-connected plate eigenproblems (nine cases) in the continuous and discrete systems. In the same way, six (C_2^4) formulations can be chosen.

4. The spurious eigenvalues occurring in the multiply-connected plate eigenproblem is the true eigenvalue of the associated simply-connected problem with the radius b which is the inner boundary of the multiply-connected domain.

5. We provide the general form of the true eigenequation for the nine cases of the eigenequations of annular plate instead of the separate form in the Leissa's book [58]. All the resluts are summarized in the Table 4-1. The spurious and true eigenequations for the annular plate were analytically derived by using the complex-valued BEM in the continuous and discrete systems.

6. Three remedies, the SVD technique of updating term, the Burton & Miller method and the CHEEF (CHIEF) method, were successfully employed to suppress the appearance of the spurious eigenvalues for simply-connected (multiply-connected) plate eigenproblems.

7. By using the real, imaginary-part or complex-valued BEMs in conjuction with the concept of the Burton & Miller method, only the combination of the u, m and θ, v formulations can obtain the true eigenvalues for the simply-connected and multiply-connected plate eigenproblems, others fail.

8. How to select the CHEEF points was addressed. In addition, the criteria of successful CHEEF points must avoid locating at the point which satisfies $[Y_n(\lambda a)K_n(\lambda\rho_1) - K_n(\lambda a)Y_n(\lambda\rho_1)] = 0$, $[Y_n(\lambda a)K_n(\lambda\rho_2) - K_n(\lambda a)Y_n(\lambda\rho_2)] = 0$ or $\sin n(\phi_1 - \phi_2) = 0$ for the simply-connected plate eigenproblems.

9. For the multiply-connected plate eigenproblem, the spurious eigenequation of multiply-connected plate eigenproblem is the true eigenequation of the associated simply-connected plate with a radius b which is the radius of the inner boundary of the annular plate. If the CHIEF point locates on the outer the domain ($a < \rho$), the CHIEF method fails. Also, the positions of the CHIEF points should be chosen carefully.

10. From the computation point of view for the simply-connected plate, CHEEF method used the minimum number of dimension although it may take risk for the failure CHEEF points.

11. For the multiply-connected problems, the CHIEF method save CPU time in constructing the influence matrix. However, the Burton & Miller method has the minimum dimension in the SVD computation.

12. In this thesis, we only considered the eigenproblem of plate vibration. If we considered the eigenproblem of the acoustics or membrane vibration (Helmholtz equation) by using the proposed methods, we can obtain the previous results [12, 35, 43, 57, 59, 60,

80].

6-2 Further research

In this thesis, the simply-connected and multiply-connected plate eigenproblems have been studied by using the boundary element method (real-part, imaginary-part and comlex-valued BEMs). However, there are several works which need further investigation as follows:

1. A general program for solving the arbitrarily-shaped plate eigenproblem needs to be developed. In this thesis, the analytical solutions of the true and spurious eigenequations are only limited to the circular or annular case, it is helpful to the researchers who may require analytical explanation to understand why the spurious eigenequation occurs. The same algorithm in the discrete system can be applied to solve arbitrary-shaped plate numerically without any difficulty.

2. The singularity of the principal value $P.V.$ and the free term in the boundary integral equations needs study in depth. Furthermore, the dual properties of the sixteen kernels can be investigated.

3. Although the three approaches, the SVD technique of updating term, the Burton & Miller method and CHEEF (CHIEF) method were successfully applied to suppress the occurrence of spurious eigenvalues, the other possible tools, GSVD and preconditioner may be another alternatives to consider especially for the ill-posed cases.

4. Here, we only considered the eigenproblem for the free vibration of simply-connected and multiply-connected plates by using the boundary element method, it may be extended to similar problems such as Stokes' flow or bulcking of beam and plate due to the same biharmonic operator.

5. The boundary element method was successfully employed for the dynamic eigenproblem of the plate. If we approach the frequency λ to be zero, it may work for the static problem.

6. Although multiply-connected problems were treated in this thesis, is better to say that only doubly-connected case was solved. A plate with multiple holes, a truly multiply-connected problem, may be tested by using the proposed methods.
7. The extension of the present BEM approach to messless formulation by lumping the strength of sources is also a possible direction for further study.
8. The property of Calderon projector for the integral formulation of biharmonic operator needs further investigation.
9. In the Chapter 3, the real-part BEM conjunction with the CHEEF method can obtain the true eigenvalues for the simply-connected plate eigenproblems. Can the spurious eigenvalue be elliminated by using the imaginary-part BEM in conjunction with the CHEEF method ?
10. For the simply-connected plate eigenproblem by using the imaginary-part BEM, the resulted ill-posed behavior of the influence matrix needs study in depth.

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Appendix 1 Degenerate kernels of the complex-valued BEM

List of the degenerate kernels

$$U^I(\overline{\rho},\overline{\phi},\rho,\phi) = \frac{1}{8\lambda^2}\sum_{m=-\infty}^{\infty}\{[Y_m(\lambda\overline{\rho})J_m(\lambda\rho) + \frac{2}{\pi}K_m(\lambda\overline{\rho})I_m(\lambda\rho)]\}cos(m(\overline{\phi}-\phi))$$
$$+ \frac{i}{8\lambda^2}\sum_{m=-\infty}^{\infty}\{[J_m(\lambda\overline{\rho})J_m(\lambda\rho) + \frac{2}{\pi}(-1)^m I_m(\lambda\overline{\rho})I_m(\lambda\rho)]\}cos(m(\overline{\phi}-\phi)), \qquad \overline{\rho} > \rho,$$

$$U^E(\overline{\rho},\overline{\phi},\rho,\phi) = \frac{1}{8\lambda^2}\sum_{m=-\infty}^{\infty}\{[Y_m(\lambda\rho)J_m(\lambda\overline{\rho}) + \frac{2}{\pi}K_m(\lambda\rho)I_m(\lambda\overline{\rho})]\}cos(m(\overline{\phi}-\phi))$$
$$+ \frac{i}{8\lambda^2}\sum_{m=-\infty}^{\infty}\{[J_m(\lambda\rho)J_m(\lambda\overline{\rho}) + \frac{2}{\pi}(-1)^m I_m(\lambda\rho)I_m(\lambda\overline{\rho})]\}cos(m(\overline{\phi}-\phi)), \qquad \rho > \overline{\rho},$$

$$U^I_\theta(\overline{\rho},\overline{\phi},\rho,\phi) = \frac{1}{8\lambda}\sum_{m=-\infty}^{\infty}[[Y_m(\lambda\overline{\rho})J'_m(\lambda\rho) + \frac{2}{\pi}K_m(\lambda\overline{\rho})I'_m(\lambda\rho)]\}cos(m(\overline{\phi}-\phi))$$
$$+ \frac{i}{8\lambda}\sum_{m=-\infty}^{\infty}\{[J_m(\lambda\overline{\rho})J'_m(\lambda\rho) + \frac{2}{\pi}(-1)^m I_m(\lambda\overline{\rho})I'_m(\lambda\rho)]]cos(m(\overline{\phi}-\phi)), \qquad \overline{\rho} > \rho,$$

$$U^E_\theta(\overline{\rho},\overline{\phi},\rho,\phi) = \frac{1}{8\lambda}\sum_{m=-\infty}^{\infty}[[Y'_m(\lambda\rho)J_m(\lambda\overline{\rho}) + \frac{2}{\pi}K'_m(\lambda\rho)I_m(\lambda\overline{\rho})]\}cos(m(\overline{\phi}-\phi))$$
$$+ \frac{i}{8\lambda}\sum_{m=-\infty}^{\infty}\{[J'_m(\lambda\rho)J_m(\lambda\overline{\rho}) + \frac{2}{\pi}(-1)^m I'_m(\lambda\rho)I_m(\lambda\overline{\rho})]]cos(m(\overline{\phi}-\phi)), \qquad \rho > \overline{\rho},$$

$$U^I_M(\overline{\rho},\overline{\phi},\rho,\phi) = \frac{1}{8\lambda^2}\sum_{m=-\infty}^{\infty}\{[Y_m(\lambda\overline{\rho})\alpha^J_m(\lambda\rho) + \frac{2}{\pi}K_m(\lambda\overline{\rho})\alpha^I_m(\lambda\rho)]\}cos(m(\overline{\phi}-\phi))$$
$$+ \frac{i}{8\lambda^2}\sum_{m=-\infty}^{\infty}\{[J_m(\lambda\overline{\rho})\alpha^J_m(\lambda\rho) + \frac{2}{\pi}(-1)^m I_m(\lambda\overline{\rho})\alpha^I_m(\lambda\rho)]\}cos(m(\overline{\phi}-\phi)), \qquad \overline{\rho} > \rho,$$

$$U^E_M(\overline{\rho},\overline{\phi},\rho,\phi) = \frac{1}{8\lambda^2}\sum_{m=-\infty}^{\infty}\{[\alpha^Y_m(\lambda\rho)J_m(\lambda\overline{\rho}) + \frac{2}{\pi}\alpha^K_m(\lambda\rho)I_m(\lambda\overline{\rho})]\}cos(m(\overline{\phi}-\phi))$$
$$+ \frac{i}{8\lambda^2}\sum_{m=-\infty}^{\infty}\{[\alpha^J_m(\lambda\rho)J_m(\lambda\overline{\rho}) + \frac{2}{\pi}(-1)^m \alpha^I_m(\lambda\rho)I_m(\lambda\overline{\rho})]\}cos(m(\overline{\phi}-\phi)), \qquad \rho > \overline{\rho},$$

$$U^I_V(\overline{\rho},\overline{\phi},\rho,\phi) = \frac{1}{8\lambda^2}\sum_{m=-\infty}^{\infty}\{[Y_m(\lambda\overline{\rho})\beta^J_m(\lambda\rho) + \frac{2}{\pi}K_m(\lambda\overline{\rho})\beta^I_m(\lambda\rho)]$$
$$+ \frac{1-\nu}{\rho}[Y_m(\lambda\overline{\rho})\gamma^J_m(\lambda\rho) + \frac{2}{\pi}K_m(\lambda\overline{\rho})\gamma^I_m(\lambda\rho)]\}cos(m(\overline{\phi}-\phi))$$
$$+ \frac{i}{8\lambda^2}\sum_{m=-\infty}^{\infty}\{[J_m(\lambda\overline{\rho})\beta^J_m(\lambda\rho) + \frac{2}{\pi}(-1)^m I_m(\lambda\overline{\rho})\beta^I_m(\lambda\rho)]$$
$$+ \frac{1-\nu}{\rho}[J_m(\lambda\overline{\rho})\gamma^J_m(\lambda\rho) + \frac{2}{\pi}(-1)^m I_m(\lambda\overline{\rho})\gamma^I_m(\lambda\rho)]\}cos(m(\overline{\phi}-\phi)), \qquad \overline{\rho} > \rho,$$

$$U^E_V(\overline{\rho},\overline{\phi},\rho,\phi) = \frac{1}{8\lambda^2}\sum_{m=-\infty}^{\infty}\{[\beta^Y_m(\lambda\rho)J_m(\lambda\overline{\rho}) + \frac{2}{\pi}\beta^K_m(\lambda\rho)I_m(\lambda\overline{\rho})]$$
$$+ \frac{1-\nu}{\rho}[\gamma^Y_m(\lambda\rho)J_m(\lambda\overline{\rho}) + \frac{2}{\pi}\gamma^K_m(\lambda\rho)I_m(\lambda\overline{\rho})]]\}cos(m(\overline{\phi}-\phi))$$
$$+ \frac{i}{8\lambda^2}\sum_{m=-\infty}^{\infty}\{[\beta^J_m(\lambda\rho)J_m(\lambda\overline{\rho}) + \frac{2}{\pi}(-1)^m \beta^I_m(\lambda\rho)I_m(\lambda\overline{\rho})]$$
$$+ \frac{1-\nu}{\rho}[\gamma^J_m(\lambda\rho)J_m(\lambda\overline{\rho}) + \frac{2}{\pi}(-1)^m \gamma^I_m(\lambda\rho)I_m(\lambda\overline{\rho})]\}cos(m(\overline{\phi}-\phi)), \qquad \rho > \overline{\rho},$$

$$\Theta^I(\overline{\rho},\overline{\phi},\rho,\phi) = \frac{1}{8\lambda}\sum_{m=-\infty}^{\infty}\{[Y_m'(\lambda\overline{\rho})J_m(\lambda\rho) + \frac{2}{\pi}K_m'(\lambda\overline{\rho})I_m(\lambda\rho)]\}cos(m(\overline{\phi}-\phi))$$
$$+ \frac{i}{8\lambda}\sum_{m=-\infty}^{\infty}\{[J_m'(\lambda\overline{\rho})J_m(\lambda\rho) + \frac{2}{\pi}(-1)^m I_m'(\lambda\overline{\rho})I_m(\lambda\rho)]\}cos(m(\overline{\phi}-\phi)), \qquad \overline{\rho} > \rho,$$
$$\Theta^E(\overline{\rho},\overline{\phi},\rho,\phi) = \frac{1}{8\lambda}\sum_{m=-\infty}^{\infty}\{[Y_m(\lambda\rho)J_m'(\lambda\overline{\rho}) + \frac{2}{\pi}K_m(\lambda\rho)I_m'(\lambda\overline{\rho})]\}cos(m(\overline{\phi}-\phi))$$
$$+ \frac{i}{8\lambda}\sum_{m=-\infty}^{\infty}\{[J_m(\lambda\rho)J_m'(\lambda\overline{\rho}) + \frac{2}{\pi}(-1)^m I_m(\lambda\rho)I_m'(\lambda\overline{\rho})]\}cos(m(\overline{\phi}-\phi)), \qquad \rho > \overline{\rho},$$
$$\Theta_\theta^I(\overline{\rho},\overline{\phi},\rho,\phi) = \frac{1}{8}\sum_{m=-\infty}^{\infty}\{[Y_m'(\lambda\overline{\rho})J_m'(\lambda\rho) + \frac{2}{\pi}K_m'(\lambda\overline{\rho})I_m'(\lambda\rho)]\}cos(m(\overline{\phi}-\phi))$$
$$+ \frac{i}{8}\sum_{m=-\infty}^{\infty}\{[J_m'(\lambda\overline{\rho})J_m'(\lambda\rho) + \frac{2}{\pi}(-1)^m I_m'(\lambda\overline{\rho})I_m'(\lambda\rho)]\}cos(m(\overline{\phi}-\phi)), \qquad \overline{\rho} > \rho,$$
$$\Theta_\theta^E(\overline{\rho},\overline{\phi},\rho,\phi) = \frac{1}{8}\sum_{m=-\infty}^{\infty}\{[Y_m'(\lambda\rho)J_m'(\lambda\overline{\rho}) + \frac{2}{\pi}K_m'(\lambda\rho)I_m'(\lambda\overline{\rho})]\}cos(m(\overline{\phi}-\phi))$$
$$+ \frac{i}{8}\sum_{m=-\infty}^{\infty}\{[J_m'(\lambda\rho)J_m'(\lambda\overline{\rho}) + \frac{2}{\pi}(-1)^m I_m'(\lambda\rho)I_m'(\lambda\overline{\rho})]\}cos(m(\overline{\phi}-\phi)), \qquad \rho > \overline{\rho},$$
$$\Theta_M^I(\overline{\rho},\overline{\phi},\rho,\phi) = \frac{1}{8\lambda}\sum_{m=-\infty}^{\infty}\{[Y_m'(\lambda\overline{\rho})\alpha_m^J(\lambda\rho) + \frac{2}{\pi}K_m'(\lambda\overline{\rho})\alpha_m^I(\lambda\rho)]\}cos(m(\overline{\phi}-\phi))$$
$$+ \frac{i}{8\lambda}\sum_{m=-\infty}^{\infty}\{[J_m'(\lambda\overline{\rho})\alpha_m^J(\lambda\rho) + \frac{2}{\pi}(-1)^m I_m'(\lambda\overline{\rho})\alpha_m^I(\lambda\rho)]\}cos(m(\overline{\phi}-\phi)), \qquad \overline{\rho} > \rho,$$
$$\Theta_M^E(\overline{\rho},\overline{\phi},\rho,\phi) = \frac{1}{8\lambda}\sum_{m=-\infty}^{\infty}\{[\alpha_m^Y(\lambda\rho)J_m'(\lambda\overline{\rho}) + \frac{2}{\pi}\alpha_m^K(\lambda\rho)I_m'(\lambda\overline{\rho})]\}cos(m(\overline{\phi}-\phi))$$
$$+ \frac{i}{8\lambda}\sum_{m=-\infty}^{\infty}\{[\alpha_m^J(\lambda\rho)J_m'(\lambda\overline{\rho}) + \frac{2}{\pi}(-1)^m \alpha_m^I(\lambda\rho)I_m'(\lambda\overline{\rho})]\}cos(m(\overline{\phi}-\phi)), \qquad \rho > \overline{\rho},$$
$$\Theta_V^I(\overline{\rho},\overline{\phi},\rho,\phi) = \frac{1}{8\lambda}\sum_{m=-\infty}^{\infty}\{[Y_m'(\lambda\overline{\rho})\beta_m^J(\lambda\rho) + \frac{2}{\pi}K_m'(\lambda\overline{\rho})\beta_m^I(\lambda\rho)]$$
$$+ \frac{1-\nu}{\rho}[Y_m'(\lambda\overline{\rho})\gamma_m^J(\lambda\rho) + \frac{2}{\pi}K_m'(\lambda\overline{\rho})\gamma_m^I(\lambda\rho)]\}cos(m(\overline{\phi}-\phi))$$
$$+ \frac{i}{8\lambda}\sum_{m=-\infty}^{\infty}\{[J_m'(\lambda\overline{\rho})\beta_m^J(\lambda\rho) + \frac{2}{\pi}(-1)^m I_m'(\lambda\overline{\rho})\beta_m^I(\lambda\rho)]$$
$$+ \frac{1-\nu}{\rho}[J_m'(\lambda\overline{\rho})\gamma_m^J(\lambda\rho) + \frac{2}{\pi}(-1)^m I_m'(\lambda\overline{\rho})\gamma_m^I(\lambda\rho)]\}cos(m(\overline{\phi}-\phi)), \qquad \overline{\rho} > \rho,$$
$$\Theta_V^E(\overline{\rho},\overline{\phi},\rho,\phi) = \frac{1}{8\lambda}\sum_{m=-\infty}^{\infty}\{[\beta_m^Y(\lambda\rho)J_m'(\lambda\overline{\rho}) + \frac{2}{\pi}\beta_m^K(\lambda\rho)I_m'(\lambda\overline{\rho})]$$
$$+ \frac{1-\nu}{\rho}[\gamma_m^Y(\lambda\rho)J_m'(\lambda\overline{\rho}) + \frac{2}{\pi}\gamma_m^K(\lambda\rho)I_m'(\lambda\overline{\rho})]\}cos(m(\overline{\phi}-\phi))$$
$$+ \frac{i}{8\lambda}\sum_{m=-\infty}^{\infty}\{[\beta_m^J(\lambda\rho)J_m'(\lambda\overline{\rho}) + \frac{2}{\pi}(-1)^m \beta_m^I(\lambda\rho)I_m'(\lambda\overline{\rho})]$$
$$+ \frac{1-\nu}{\rho}[\gamma_m^J(\lambda\rho)J_m'(\lambda\overline{\rho}) + \frac{2}{\pi}(-1)^m \gamma_m^I(\lambda\rho)I_m'(\lambda\overline{\rho})]\}cos(m(\overline{\phi}-\phi)), \qquad \rho > \overline{\rho},$$

$$M^I(\overline{\rho},\overline{\phi},\rho,\phi) = \frac{1}{8\lambda^2}\sum_{m=-\infty}^{\infty}\{[\alpha_n^Y(\lambda\overline{\rho})J_m(\lambda\rho) + \frac{2}{\pi}\alpha_m^K(\lambda\overline{\rho})I_m(\lambda\rho)\}cos(m(\overline{\phi}-\phi))$$
$$+ \frac{i}{8\lambda^2}\sum_{m=-\infty}^{\infty}\{[\alpha_m^J(\lambda\overline{\rho})J_m(\lambda\rho) + \frac{2}{\pi}(-1)^m\alpha_m^I(\lambda\overline{\rho})I_m(\lambda\rho)]\}cos(m(\overline{\phi}-\phi)), \qquad \overline{\rho} > \rho,$$
$$M^E(\overline{\rho},\overline{\phi},\rho,\phi) = \frac{1}{8\lambda^2}\sum_{m=-\infty}^{\infty}\{[Y_m(\lambda\rho)\alpha_m^J(\lambda\overline{\rho}) + \frac{2}{\pi}K_m(\lambda\rho)\alpha_m^I(\lambda\overline{\rho})]\}cos(m(\overline{\phi}-\phi))$$
$$+ \frac{i}{8\lambda^2}\sum_{m=-\infty}^{\infty}\{J_m(\lambda\rho)\alpha_m^J(\lambda\overline{\rho}) + \frac{2}{\pi}(-1)^m I_m(\lambda\rho)\alpha_m^I(\lambda\overline{\rho})]\}cos(m(\overline{\phi}-\phi)), \qquad \rho > \overline{\rho},$$
$$M_\theta^I(\overline{\rho},\overline{\phi},\rho,\phi) = \frac{1}{8\lambda}\sum_{m=-\infty}^{\infty}\{[\alpha_n^Y(\lambda\overline{\rho})J'_m(\lambda\rho) + \frac{2}{\pi}\alpha_m^K(\lambda\overline{\rho})I'_m(\lambda\rho)]\}cos(m(\overline{\phi}-\phi))$$
$$+ \frac{i}{8\lambda^2}\sum_{m=-\infty}^{\infty}\{[\alpha_m^J(\lambda\overline{\rho})J'_m(\lambda\rho) + \frac{2}{\pi}(-1)^m\alpha_m^I(\lambda\overline{\rho})I'_m(\lambda\rho)]\}cos(m(\overline{\phi}-\phi)), \qquad \overline{\rho} > \rho,$$
$$M_\theta^E(\overline{\rho},\overline{\phi},\rho,\phi) = \frac{1}{8\lambda}\sum_{m=-\infty}^{\infty}\{[Y'_m(\lambda\rho)\alpha_m^J(\lambda\overline{\rho}) + \frac{2}{\pi}K'_m(\lambda\rho)\alpha_m^I(\lambda\overline{\rho})]\}cos(m(\overline{\phi}-\phi))$$
$$+ \frac{i}{8\lambda^2}\sum_{m=-\infty}^{\infty}\{[J'_m(\lambda\rho)\alpha_m^J(\lambda\overline{\rho}) + \frac{2}{\pi}(-1)^m I'_m(\lambda\rho)\alpha_m^I(\lambda\overline{\rho})]\}cos(m(\overline{\phi}-\phi)), \qquad \rho > \overline{\rho},$$
$$M_M^I(\overline{\rho},\overline{\phi},\rho,\phi) = \frac{1}{8\lambda^2}\sum_{m=-\infty}^{\infty}\{[\alpha_n^Y(\lambda\overline{\rho})\alpha_m^J(\lambda\rho) + \frac{2}{\pi}\alpha_m^K(\lambda\overline{\rho})\alpha_m^I(\lambda\rho)]\}cos(m(\overline{\phi}-\phi))$$
$$+ \frac{i}{8\lambda^2}\sum_{m=-\infty}^{\infty}\{[\alpha_m^J(\lambda\overline{\rho})\alpha_m^J(\lambda\rho) + \frac{2}{\pi}(-1)^m\alpha_m^I(\lambda\overline{\rho})\alpha_m^I(\lambda\rho)]\}cos(m(\overline{\phi}-\phi)), \qquad \overline{\rho} > \rho,$$
$$M_M^E(\overline{\rho},\overline{\phi},\rho,\phi) = \frac{1}{8\lambda^2}\sum_{m=-\infty}^{\infty}\{[\alpha_m^Y(\lambda\rho)\alpha_m^J(\lambda\overline{\rho}) + \frac{2}{\pi}\alpha_m^K(\lambda\rho)\alpha_m^I(\lambda\overline{\rho})]\}cos(m(\overline{\phi}-\phi))$$
$$+ \frac{i}{8\lambda^2}\sum_{m=-\infty}^{\infty}\{[\alpha_m^J(\lambda\rho)\alpha_m^J(\lambda\overline{\rho}) + \frac{2}{\pi}(-1)^m\alpha_m^I(\lambda\rho)\alpha_m^I(\lambda\overline{\rho})]\}cos(m(\overline{\phi}-\phi)), \qquad \rho > \overline{\rho},$$
$$M_V^I(\overline{\rho},\overline{\phi},\rho,\phi) = \frac{1}{8\lambda^2}\sum_{m=-\infty}^{\infty}\{[\alpha_n^Y(\lambda\overline{\rho})\beta_m^J(\lambda\rho) + \frac{2}{\pi}\alpha_m^K(\lambda\overline{\rho})\beta_m^I(\lambda\rho)]$$
$$+ \frac{1-\nu}{\rho}[\alpha_n^Y(\lambda\overline{\rho})\gamma_m^J(\lambda\rho) + \frac{2}{\pi}\alpha_m^K(\lambda\overline{\rho})\gamma_m^I(\lambda\rho)]\}cos(m(\overline{\phi}-\phi))$$
$$+ \frac{i}{8\lambda^2}\sum_{m=-\infty}^{\infty}\{[\alpha_m^J(\lambda\overline{\rho})\beta_m^J(\lambda\rho) + \frac{2}{\pi}(-1)^m\alpha_m^I(\lambda\overline{\rho})\beta_m^I(\lambda\rho)]$$
$$+ \frac{1-\nu}{\rho}[\alpha_m^J(\lambda\overline{\rho})\gamma_m^J(\lambda\rho) + \frac{2}{\pi}(-1)^m\alpha_m^I(\lambda\overline{\rho})\gamma_m^I(\lambda\rho)]\}cos(m(\overline{\phi}-\phi)), \qquad \overline{\rho} > \rho,$$
$$M_V^E(\overline{\rho},\overline{\phi},\rho,\phi) = \frac{1}{8\lambda^2}\sum_{m=-\infty}^{\infty}\{[\beta_m^Y(\lambda\rho)\alpha_m^J(\lambda\overline{\rho}) + \frac{2}{\pi}\beta_m^K(\lambda\rho)\alpha_m^I(\lambda\overline{\rho})]$$
$$+ \frac{1-\nu}{\rho}[\gamma_m^Y(\lambda\rho)\alpha_m^J(\lambda\overline{\rho}) + \frac{2}{\pi}\gamma_m^K(\lambda\rho)\alpha_m^I(\lambda\overline{\rho})]\}cos(m(\overline{\phi}-\phi))$$
$$+ \frac{i}{8\lambda^2}\sum_{m=-\infty}^{\infty}\{[\beta_m^J(\lambda\rho)\alpha_m^J(\lambda\overline{\rho}) + \frac{2}{\pi}(-1)^m\beta_m^I(\lambda\rho)\alpha_m^I(\lambda\overline{\rho})]$$
$$+ \frac{1-\nu}{\rho}[\gamma_m^J(\lambda\rho)\alpha_m^J(\lambda\overline{\rho}) + \frac{2}{\pi}(-1)^m\gamma_m^I(\lambda\rho)\alpha_m^I(\lambda\overline{\rho})]\}cos(m(\overline{\phi}-\phi)), \qquad \rho > \overline{\rho},$$

$$V^I(\overline{\rho},\overline{\phi},\rho,\phi) = \frac{1}{8\lambda^2}\sum_{m=-\infty}^{\infty}\{[\beta_m^Y(\lambda\overline{\rho})J_m(\lambda\rho) + \frac{2}{\pi}\beta_m^K(\lambda\overline{\rho})I_m(\lambda\rho)] + i[\beta_m^J(\lambda\overline{\rho})J_m(\lambda\rho) + \frac{2}{\pi}(-1)^m\beta_m^I(\lambda\overline{\rho})I_m(\lambda\rho)]\}cos(m(\overline{\phi}-\phi))$$
$$+ \frac{1-\nu}{\overline{\rho}}\{\gamma_m^Y(\lambda\overline{\rho})J_m(\lambda\rho) + \frac{2}{\pi}\gamma_m^K(\lambda\overline{\rho})I_m(\lambda\rho)] + i[\gamma_m^J(\lambda\overline{\rho})J_m(\lambda\rho) + \frac{2}{\pi}(-1)^m\gamma_m^I(\lambda\overline{\rho})I_m(\lambda\rho)]\}cos(m(\overline{\phi}-\phi)), \qquad \overline{\rho} > \rho,$$
$$V^E(\overline{\rho},\overline{\phi},\rho,\phi) = \frac{1}{8\lambda^2}\sum_{m=-\infty}^{\infty}\{[Y_m(\lambda\rho)\beta_m^J(\lambda\overline{\rho}) + \frac{2}{\pi}K_m(\lambda\rho)\beta_m^I(\lambda\overline{\rho})] + i[J_m(\lambda\rho)\beta_m^J(\lambda\overline{\rho}) + \frac{2}{\pi}(-1)^m I_m(\lambda\rho)]\beta_m^I(\lambda\overline{\rho})\}cos(m(\overline{\phi}-\phi))$$
$$+ \frac{1-\nu}{\overline{\rho}}\{Y_m(\lambda\rho)\gamma_m^J(\lambda\overline{\rho}) + \frac{2}{\pi}K_m(\lambda\rho)\gamma_m^I(\lambda\overline{\rho})] + i[J_m(\lambda\rho)\gamma_m^J(\lambda\overline{\rho}) + \frac{2}{\pi}(-1)^m I_m(\lambda\rho)\gamma_m^I(\lambda\overline{\rho})]\}cos(m(\overline{\phi}-\phi)), \qquad \rho > \overline{\rho},$$

$$V_\theta^I(\overline{\rho},\overline{\phi},\rho,\phi) = \frac{1}{8\lambda}\sum_{m=-\infty}^{\infty}\{[\beta_m^Y(\lambda\overline{\rho})J'_m(\lambda\rho) + \frac{2}{\pi}\beta_m^K(\lambda\overline{\rho})I'_m(\lambda\rho)] + i[\beta_m^J(\lambda\overline{\rho})J'_m(\lambda\rho) + \frac{2}{\pi}(-1)^m\beta_m^I(\lambda\overline{\rho})I'_m(\lambda\rho)]\}cos(m(\overline{\phi}-\phi))$$
$$+ \frac{1-\nu}{\overline{\rho}}\{\gamma_m^Y(\lambda\overline{\rho})J'_m(\lambda\rho) + \frac{2}{\pi}\gamma_m^K(\lambda\overline{\rho})I'_m(\lambda\rho)] + i[\gamma_m^J(\lambda\overline{\rho})J'_m(\lambda\rho) + \frac{2}{\pi}(-1)^m\gamma_m^I(\lambda\overline{\rho})I'_m(\lambda\rho)]\}cos(m(\overline{\phi}-\phi)), \qquad \overline{\rho} > \rho,$$
$$V_\theta^E(\overline{\rho},\overline{\phi},\rho,\phi) = \frac{1}{8\lambda}\sum_{m=-\infty}^{\infty}\{[Y'_m(\lambda\rho)\beta_m^J(\lambda\overline{\rho}) + \frac{2}{\pi}K'_m(\lambda\rho)\beta_m^I(\lambda\overline{\rho})] + i[J'_m(\lambda\rho)\beta_m^J(\lambda\overline{\rho}) + \frac{2}{\pi}(-1)^m I'_m(\lambda\rho)\beta_m^I(\lambda\overline{\rho})]\}cos(m(\overline{\phi}-\phi))$$
$$+ \frac{1-\nu}{\overline{\rho}}\{Y'_m(\lambda\rho)\gamma_m^J(\lambda\overline{\rho}) + \frac{2}{\pi}K'_m(\lambda\rho)\gamma_m^I(\lambda\overline{\rho})] + i[J'_m(\lambda\rho)\gamma_m^J(\lambda\overline{\rho}) + \frac{2}{\pi}(-1)^m I'_m(\lambda\rho)\gamma_m^I(\lambda\overline{\rho})]\}cos(m(\overline{\phi}-\phi)), \qquad \rho > \overline{\rho},$$

$$V_M^I(\overline{\rho},\overline{\phi},\rho,\phi) = \frac{1}{8\lambda^2}\sum_{m=-\infty}^{\infty}\{[\beta_m^Y(\lambda\overline{\rho})\alpha_m^J(\lambda\rho) + \frac{2}{\pi}\beta_m^K(\lambda\overline{\rho})\alpha_m^I(\lambda\rho)] + i[\beta_m^J(\lambda\overline{\rho})\alpha_m^J(\lambda\rho) + \frac{2}{\pi}(-1)^m\beta_m^I(\lambda\overline{\rho})\alpha_m^I(\lambda\rho)]\}cos(m(\overline{\phi}-\phi))$$
$$+ \frac{1-\nu}{\overline{\rho}}\{\gamma_m^Y(\lambda\overline{\rho})\alpha_m^J(\lambda\rho) + \frac{2}{\pi}\gamma_m^K(\lambda\overline{\rho})\alpha_m^I(\lambda\rho)] + i[\gamma_m^J(\lambda\overline{\rho})\alpha_m^J(\lambda\rho) + \frac{2}{\pi}(-1)^m\gamma_m^I(\lambda\overline{\rho})\alpha_m^I(\lambda\rho)]\}cos(m(\overline{\phi}-\phi)), \qquad \overline{\rho} > \rho,$$
$$V_M^E(\overline{\rho},\overline{\phi},\rho,\phi) = \frac{1}{8\lambda^2}\sum_{m=-\infty}^{\infty}\{[\alpha_n^Y(\lambda\rho)\beta_m^J(\lambda\overline{\rho}) + \frac{2}{\pi}\alpha_m^K(\lambda\rho)\beta_m^I(\lambda\overline{\rho})] + i[\beta_m^J(\lambda\rho)\alpha_m^J(\lambda\rho) + \frac{2}{\pi}(-1)^m\alpha_m^I(\lambda\rho)\beta_m^I(\lambda\overline{\rho})]\}cos(m(\overline{\phi}-\phi))$$
$$+ \frac{1-\nu}{\overline{\rho}}\{\alpha_n^Y(\lambda\rho)\gamma_m^J(\lambda\overline{\rho}) + \frac{2}{\pi}\alpha_m^K(\lambda\rho)\gamma_m^I(\lambda\overline{\rho})] + i[\alpha_m^J(\lambda\rho)\gamma_m^J(\lambda\overline{\rho}) + \frac{2}{\pi}(-1)^m\alpha_m^I(\lambda\rho)\gamma_m^I(\lambda\overline{\rho})]\}cos(m(\overline{\phi}-\phi)), \qquad \rho > \overline{\rho},$$

$$V_V^I(\overline{\rho},\overline{\phi},\rho,\phi) = \frac{1}{8\lambda^2}\sum_{m=-\infty}^{\infty}\{[\beta_m^Y(\lambda\overline{\rho})\beta_m^J(\lambda\rho) + \frac{2}{\pi}\beta_m^K(\lambda\overline{\rho})\beta_m^I(\lambda\rho)] + i[\beta_m^J(\lambda\overline{\rho})\beta_m^J(\lambda\rho) + \frac{2}{\pi}(-1)^m\beta_m^I(\lambda\overline{\rho})\beta_m^I(\lambda\rho)]\}cos(m(\overline{\phi}-\phi))$$
$$+ \frac{1-\nu}{\overline{\rho}}\{\gamma_m^Y(\lambda\overline{\rho})\beta_m^J(\lambda\rho) + \frac{2}{\pi}\gamma_m^K(\lambda\overline{\rho})\beta_m^I(\lambda\rho)] + i[\gamma_m^J(\lambda\overline{\rho})\beta_m^J(\lambda\rho) + \frac{2}{\pi}(-1)^m\gamma_m^I(\lambda\overline{\rho})\beta_m^I(\lambda\rho)]\}cos(m(\overline{\phi}-\phi))$$
$$+ \frac{1-\nu}{\rho}\{[\beta_m^Y(\lambda\overline{\rho})\gamma_m^Y(\lambda\rho) + \frac{2}{\pi}\beta_m^K(\lambda\overline{\rho})\gamma_m^I(\lambda\rho)] + i[\beta_m^J(\lambda\overline{\rho})\gamma_m^J(\lambda\rho) + \frac{2}{\pi}(-1)^m\beta_m^I(\lambda\overline{\rho})\gamma_m^I(\lambda\rho)]\}cos(m(\overline{\phi}-\phi))$$
$$+ \frac{(1-\nu)^2}{\overline{\rho}\rho}\{\gamma_m^Y(\lambda\overline{\rho})\gamma_m^J(\lambda\rho) + \frac{2}{\pi}\gamma_m^K(\lambda\overline{\rho})\gamma_m^I(\lambda\rho)] + i[\gamma_m^J(\lambda\overline{\rho})\gamma_m^J(\lambda\rho) + \frac{2}{\pi}(-1)^m\gamma_m^I(\lambda\overline{\rho})\gamma_m^I(\lambda\rho)]\}cos(m(\overline{\phi}-\phi)), \qquad \overline{\rho} > \rho,$$
$$V_V^E(\overline{\rho},\overline{\phi},\rho,\phi) = \frac{1}{8\lambda^2}\sum_{m=-\infty}^{\infty}\{[\beta_m^Y(\lambda\rho)\beta_m^J(\lambda\overline{\rho}) + \frac{2}{\pi}\beta_m^K(\lambda\rho)\beta_m^I(\lambda\overline{\rho})] + i[\beta_m^J(\lambda\rho)\beta_m^J(\lambda\overline{\rho}) + \frac{2}{\pi}(-1)^m\beta_m^I(\lambda\rho)\beta_m^I(\lambda\overline{\rho})]\}cos(m(\overline{\phi}-\phi))$$
$$+ \frac{1-\nu}{\overline{\rho}}\{\beta_m^Y(\lambda\rho)\gamma_m^J(\lambda\overline{\rho}) + \frac{2}{\pi}\beta_m^K(\lambda\rho)\gamma_m^I(\lambda\overline{\rho})] + i[\beta_m^J(\lambda\rho)\gamma_m^J(\lambda\overline{\rho}) + \frac{2}{\pi}(-1)^m\beta_m^I(\lambda\rho)\gamma_m^I(\lambda\overline{\rho})]\}cos(m(\overline{\phi}-\phi))$$
$$+ \frac{1-\nu}{\rho}\{[\gamma_m^Y(\lambda\rho)\beta_m^J(\lambda\overline{\rho}) + \frac{2}{\pi}\gamma_m^K(\lambda\rho)\beta_m^I(\lambda\overline{\rho})] + i[\gamma_m^J(\lambda\rho)\beta_m^J(\lambda\overline{\rho}) + \frac{2}{\pi}(-1)^m\gamma_m^I(\lambda\rho)\beta_m^I(\lambda\overline{\rho})]\}cos(m(\overline{\phi}-\phi))$$
$$+ \frac{(1-\nu)^2}{\overline{\rho}\rho}\{\gamma_m^Y(\lambda\rho)\gamma_m^J(\lambda\overline{\rho}) + \frac{2}{\pi}\gamma_m^K(\lambda\rho)\gamma_m^I(\lambda\overline{\rho})] + i[\gamma_m^J(\lambda\rho)\gamma_m^J(\lambda\overline{\rho}) + \frac{2}{\pi}(-1)^m\gamma_m^I(\lambda\rho)\gamma_m^I(\lambda\overline{\rho})]\}cos(m(\overline{\phi}-\phi)), \qquad \rho > \overline{\rho},$$

List of the coefficients

$$\alpha_n^Y(\lambda\overline{\rho}) = \lambda^2 Y_m''(\lambda\overline{\rho}) + \nu[\frac{1}{\overline{\rho}}\lambda Y_m'(\lambda\overline{\rho}) - (\frac{m}{\overline{\rho}})^2 Y_m(\lambda\overline{\rho})],$$
$$\alpha_m^J(\lambda\overline{\rho}) = \lambda^2 J_m''(\lambda\overline{\rho}) + \nu[\frac{1}{\overline{\rho}}\lambda J_m'(\lambda\overline{\rho}) - (\frac{m}{\overline{\rho}})^2 J_m(\lambda\overline{\rho})],$$
$$\alpha_m^K(\lambda\overline{\rho}) = \lambda^2 K_m''(\lambda\overline{\rho}) + \nu[\frac{1}{\overline{\rho}}\lambda K_m'(\lambda\overline{\rho}) - (\frac{m}{\overline{\rho}})^2 K_m(\lambda\overline{\rho})],$$
$$\alpha_m^I(\lambda\overline{\rho}) = \lambda^2 I_m''(\lambda\overline{\rho}) + \nu[\frac{1}{\overline{\rho}}\lambda I_m'(\lambda\overline{\rho}) - (\frac{m}{\overline{\rho}})^2 I_m(\lambda\overline{\rho})],$$
$$\beta_m^Y(\lambda\overline{\rho}) = \lambda^3 Y_m'''(\lambda\overline{\rho}) + \frac{\lambda^2}{\overline{\rho}} Y_m''(\lambda\overline{\rho}) - (\frac{\lambda}{\overline{\rho}} + \frac{m^2\lambda}{\overline{\rho}^2}) Y_m'(\lambda\overline{\rho}) + \frac{2m^2}{\lambda^3} Y(\lambda\overline{\rho})],$$
$$\beta_m^J(\lambda\overline{\rho}) = \lambda^3 J_m'''(\lambda\overline{\rho}) + \frac{\lambda^2}{\overline{\rho}} J_m''(\lambda\overline{\rho}) - (\frac{\lambda}{\overline{\rho}} + \frac{m^2\lambda}{\overline{\rho}^2}) J_m'(\lambda\overline{\rho}) + \frac{2m^2}{\lambda^3} J(\lambda\overline{\rho})],$$
$$\beta_m^K(\lambda\overline{\rho}) = \lambda^3 K_m'''(\lambda\overline{\rho}) + \frac{\lambda^2}{\overline{\rho}} K_m''(\lambda\overline{\rho}) - (\frac{\lambda}{\overline{\rho}} + \frac{m^2\lambda}{\overline{\rho}^2}) K_m'(\lambda\overline{\rho}) + \frac{2m^2}{\lambda^3} K(\lambda\overline{\rho})],$$
$$\beta_m^I(\lambda\overline{\rho}) = \lambda^3 I_m'''(\lambda\overline{\rho}) + \frac{\lambda^2}{\overline{\rho}} I_m''(\lambda\overline{\rho}) - (\frac{\lambda}{\overline{\rho}} + \frac{m^2\lambda}{\overline{\rho}^2}) I_m'(\lambda\overline{\rho}) + \frac{2m^2}{\lambda^3} I(\lambda\overline{\rho})],$$
$$\gamma_m^Y(\lambda\overline{\rho}) = m^2[\frac{1}{\overline{\rho}^2} Y_m(\lambda\overline{\rho}) - \frac{1}{\overline{\rho}}\lambda Y_m'(\lambda\overline{\rho})],$$
$$\gamma_m^J(\lambda\overline{\rho}) = m^2[\frac{1}{\overline{\rho}^2} J_m(\lambda\overline{\rho}) - \frac{1}{\overline{\rho}}\lambda J_m'(\lambda\overline{\rho})],$$
$$\gamma_m^K(\lambda\overline{\rho}) = m^2[\frac{1}{\overline{\rho}^2} K_m(\lambda\overline{\rho}) - \frac{1}{\overline{\rho}}\lambda K_m'(\lambda\overline{\rho})],$$
$$\gamma_m^I(\lambda\overline{\rho}) = m^2[\frac{1}{\overline{\rho}^2} I_m(\lambda\overline{\rho}) - \frac{1}{\overline{\rho}}\lambda I_m'(\lambda\overline{\rho})],$$

$$\alpha_m^Y(\lambda\rho) = \lambda^2 Y_m''(\lambda\rho) + \nu[\frac{1}{\rho}\lambda Y_m'(\lambda\rho) - (\frac{m}{\rho})^2 Y_m(\lambda\rho)]$$
$$\alpha_m^J(\lambda\rho) = \lambda^2 J_m''(\lambda\rho) + \nu[\frac{1}{\rho}\lambda J_m'(\lambda\rho) - (\frac{m}{\rho})^2 J_m(\lambda\rho)]$$
$$\alpha_m^K(\lambda\rho) = \lambda^2 K_m''(\lambda\rho) + \nu[\frac{1}{\rho}\lambda K_m'(\lambda\rho) - (\frac{m}{\rho})^2 K_m(\lambda\rho)]$$
$$\alpha_m^I(\lambda\rho) = \lambda^2 I_m''(\lambda\rho) + \nu[\frac{1}{\rho}\lambda I_m'(\lambda\rho) - (\frac{m}{\rho})^2 I_m(\lambda\rho)]$$
$$\beta_m^Y(\lambda\rho) = \lambda^3 Y_m'''(\lambda\rho) + \frac{\lambda^2}{\rho} Y_m''(\lambda\rho) - (\frac{\lambda}{\rho} + \frac{m^2\lambda}{\rho^2}) Y_m'(\lambda\rho) + \frac{2m^2}{\lambda^3} Y(\lambda\rho)]$$
$$\beta_m^J(\lambda\rho) = \lambda^3 J_m'''(\lambda\rho) + \frac{\lambda^2}{\rho} J_m''(\lambda\rho) - (\frac{\lambda}{\rho} + \frac{m^2\lambda}{\rho^2}) J_m'(\lambda\rho) + \frac{2m^2}{\lambda^3} J(\lambda\rho)]$$
$$\beta_m^K(\lambda\rho) = \lambda^3 K_m'''(\lambda\rho) + \frac{\lambda^2}{\rho} K_m''(\lambda\rho) - (\frac{\lambda}{\rho} + \frac{m^2\lambda}{\rho^2}) K_m'(\lambda\rho) + \frac{2m^2}{\lambda^3} K(\lambda\rho)]$$
$$\beta_m^I(\lambda\rho) = \lambda^3 I_m'''(\lambda\rho) + \frac{\lambda^2}{\rho} I_m''(\lambda\rho) - (\frac{\lambda}{\rho} + \frac{m^2\lambda}{\rho^2}) I_m'(\lambda\rho) + \frac{2m^2}{\lambda^3} I(\lambda\rho)]$$
$$\gamma_m^Y(\lambda\rho) = m^2[\frac{1}{\rho^2} Y_m(\lambda\rho) - \frac{1}{\rho}\lambda Y_m'(\lambda\rho)]$$
$$\gamma_m^J(\lambda\rho) = m^2[\frac{1}{\rho^2} J_m(\lambda\rho) - \frac{1}{\rho}\lambda J_m'(\lambda\rho)]$$
$$\gamma_m^K(\lambda\rho) = m^2[\frac{1}{\rho^2} K_m(\lambda\rho) - \frac{1}{\rho}\lambda K_m'(\lambda\rho)]$$
$$\gamma_m^I(\lambda\rho) = m^2[\frac{1}{\rho^2} I_m(\lambda\rho) - \frac{1}{\rho}\lambda I_m'(\lambda\rho)]$$

Appendix 2 Recurrence relations of the Bessel function

Bessel function of the first kind (J)

$$
\begin{aligned}
&J_0'(z) = -J_1(z),\\
&J_{n-1}(z) + J_{n+1}(z) = \frac{2n}{z}J_n(z),\\
&J_{n-1}(z) - J_{n+1}(z) = 2J_n'(z),\\
&J_n'(z) = J_{n-1}(z) - \frac{n}{z}J_n(z),\\
&J_n'(z) = -J_{n+1}(z) + \frac{n}{z}J_n(z),\\
&J_n''(z) = \frac{-z^2+n^2-n}{z^2}J_n(z) + \frac{1}{z}J_{n+1}(z),\\
&J_n'''(z) = \frac{-z^2+n^2-n}{z^2}J_n'(z) + \frac{-2n^2+2n}{z^3}J_n(z) + \frac{1}{z}J_{n+1}'(z) - \frac{1}{z^2}J_{n+1}(z),
\end{aligned}
$$

Bessel function of the second kind (Y)

$$
\begin{aligned}
&Y_0'(z) = -Y_1(z),\\
&Y_{n-1}(z) + Y_{n+1}(z) = \frac{2n}{z}Y_n(z),\\
&Y_{n-1}(z) - Y_{n+1}(z) = 2Y_n'(z),\\
&Y_n'(z) = Y_{n-1}(z) - \frac{n}{z}Y_n(z),\\
&Y_n'(z) = -Y_{n+1}(z) + \frac{n}{z}Y_n(z),\\
&Y_n''(z) = \frac{-z^2+n^2-n}{z^2}Y_n(z) + \frac{1}{z}Y_{n+1}(z),\\
&Y_n'''(z) = \frac{-z^2+n^2-n}{z^2}Y_n'(z) + \frac{-2n^2+2n}{z^3}Y_n(z) + \frac{1}{z}Y_{n+1}'(z) - \frac{1}{z^2}Y_{n+1}(z),
\end{aligned}
$$

Modified Bessel function of the first kind (I)

$$
\begin{aligned}
&I_0'(z) = I_1(z),\\
&I_{n-1}(z) - I_{n+1}(z) = \frac{2n}{z} I_n(z),\\
&I_{n-1}(z) + I_{n+1}(z) = 2I_n'(z),\\
&I_n'(z) = I_{n-1}(z) - \frac{n}{z} I_n(z),\\
&I_n'(z) = I_{n+1}(z) + \frac{n}{z} I_n(z),\\
&I_n''(z) = \frac{z^2 + n^2 - n}{z^2} I_n(z) - \frac{1}{z} I_{n+1}(z),\\
&I_n'''(z) = \frac{z^2 + n^2 - n}{z^2} I_n'(z) + \frac{-2n^2 + 2n}{z^3} I_n(z) - \frac{1}{z} I_{n+1}'(z) + \frac{1}{z^2} I_{n+1}(z),
\end{aligned}
$$

Modified Bessel function of the second kind (K)

$$
\begin{aligned}
&K_0'(z) = -K_1(z),\\
&-K_{n-1}(z) + K_{n+1}(z) = \frac{2n}{z} K_n(z),\\
&-K_{n-1}(z) - K_{n+1}(z) = 2K_n'(z),\\
&K_n'(z) = -K_{n-1}(z) - \frac{n}{z} K_n(z),\\
&K_n'(z) = -K_{n+1}(z) + \frac{n}{z} K_n(z),\\
&K_n''(z) = \frac{z^2 + n^2 - n}{z^2} K_n(z) + \frac{1}{z} K_{n+1}(z),\\
&K_n'''(z) = \frac{z^2 + n^2 - n}{z^2} K_n'(z) + \frac{-2n^2 + 2n}{z^3} K_n(z) + \frac{1}{z} K_{n+1}'(z) - \frac{1}{z^2} K_{n+1}(z),
\end{aligned}
$$

Appendix 3 Exact eigenequations for the simply-connected plate

Eigenequation of a simply-connected plate [58]

Clamped circular plate

$$J_n(\lambda)I_{n+1}(\lambda)+I_n(\lambda)J_{n+1}(\lambda)=0,\ n=0,1,2,\ldots.$$

Simply supported circular plate

$$\frac{J_{n+1}(\lambda)}{J_n(\lambda)}+\frac{I_{n+1}(\lambda)}{I_n(\lambda)}=\frac{2\lambda}{1-\nu},\ n=0,1,2,\ldots.$$

Free circular plate

$$\frac{\lambda^2J_n(\lambda)+(1-\nu)[\lambda J_n'(\lambda)-n^2J_n(\lambda)]}{\lambda^2I_n(\lambda)-(1-\nu)[\lambda I_n'(\lambda)-n^2I_n(\lambda)]}=\frac{\lambda^3I_n'(\lambda)+(1-\nu)n^2[\lambda J_n'(\lambda)-J_n(\lambda)]}{\lambda^3I_n'(\lambda)-(1-\nu)n^2[\lambda I_n'(\lambda)-I_n(\lambda)]},\ n=0,1,2,\ldots.$$

It is noted that the eigenequation in the Leissa's book [58] for free case was wrongly typed where the I index in the numerator of the right hand side of the equation should be J [42] as show below:

$$\frac{\lambda^2J_n(\lambda)+(1-\nu)[\lambda J_n'(\lambda)-n^2J_n(\lambda)]}{\lambda^2I_n(\lambda)-(1-\nu)[\lambda I_n'(\lambda)-n^2I_n(\lambda)]}=\frac{\lambda^3J_n'(\lambda)+(1-\nu)n^2[\lambda J_n'(\lambda)-J_n(\lambda)]}{\lambda^3I_n'(\lambda)-(1-\nu)n^2[\lambda I_n'(\lambda)-I_n(\lambda)]},\ n=0,1,2,\ldots.$$

Appendix 4 Mathematical induction for the determinant of the matrix $[M]$

The matrix $[M]_{2n\times 2n}$ is assembled by the four diagonal matrics $[A]_{n\times n}$, $[B]_{n\times n}$, $[C]_{n\times n}$ and $[D]_{n\times n}$, such that

$$[M]_{2n\times 2n} = \left[\begin{array}{cccc|cccc} a_1 & & & & b_1 & & & \\ & a_2 & & & & b_2 & & \\ & & \ddots & & & & \ddots & \\ & & & a_n & & & & b_n \\ \hline c_1 & & & & d_1 & & & \\ & c_2 & & & & d_2 & & \\ & & \ddots & & & & \ddots & \\ & & & c_n & & & & d_n \end{array}\right]_{2n\times 2n}$$

Proof that $det[M] = \prod_{i=1}^{n}(a_i d_i - b_i c_i)$ for any $n \in$ natural number.

Proof:

(1) By mathematical induction, it is easy to proof the case of $n = 1$ that

$$det[M] = det\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} = a_1 d_1 - b_1 c_1 = \prod_{i=1}^{1}(a_i d_i - b_i c_i)$$

Clearly, the result hold if $n = 1$.

(2) Assume that the equation satisfies for $n = N$, it follows from the induction hypothesis that

$$det[M] = det\left[\begin{array}{cccc|cccc} a_1 & & & & b_1 & & & \\ & a_2 & & & & b_2 & & \\ & & \ddots & & & & \ddots & \\ & & & a_N & & & & b_N \\ \hline c_1 & & & & d_1 & & & \\ & c_2 & & & & d_2 & & \\ & & \ddots & & & & \ddots & \\ & & & c_N & & & & d_N \end{array}\right] = \prod_{i=1}^{N}(a_i d_i - b_i c_i)$$

(3) If $n = N + 1$, we have

$$[M] = \left[\begin{array}{ccccc|ccccc} a_1 & & & & & b_1 & & & & \\ & a_2 & & & & & b_2 & & & \\ & & \ddots & & & & & \ddots & & \\ & & & a_N & & & & & b_N & \\ & & & & a_{N+1} & & & & & b_{N+1} \\ \hline c_1 & & & & & d_1 & & & & \\ & c_2 & & & & & d_2 & & & \\ & & \ddots & & & & & \ddots & & \\ & & & c_N & & & & & d_N & \\ & & & & c_{N+1} & & & & & d_{N+1} \end{array}\right]$$

By using the cofactor of the row which contains only two nonzero elements, a_{N+1} and b_{N+1}, we have

$$det[M] = (-1)^{(N+1)+(N+1)} a_{N+1} \; det \left[\begin{array}{cccc|ccccc} a_1 & & & & b_1 & & & & \\ & a_2 & & & & b_2 & & & \\ & & \ddots & & & & \ddots & & \\ & & & a_N & & & & b_N & \\ \hline c_1 & & & & d_1 & & & & \\ & c_2 & & & & d_2 & & & \\ & & \ddots & & & & \ddots & & \\ & & & c_N & & & & d_N & \\ & & & & & & & & d_{N+1} \end{array}\right]$$

$$+ (-1)^{(N+1)+(2N+1)} b_{N+1} \; det \left[\begin{array}{cccc|cccc} a_1 & & & & b_1 & & & \\ & a_2 & & & & b_2 & & \\ & & \ddots & & & & \ddots & \\ & & & a_N & & & & b_N \\ \hline c_1 & & & & d_1 & & & \\ & c_2 & & & & d_2 & & \\ & & \ddots & & & & \ddots & \\ & & & c_N & & & & d_N \\ & & & & c_{N+1} & & & \end{array}\right]$$

By using the cofactor of the row which contains only two nonzero elements, c_{N+1} and d_{N+1} again, we have

$$det[M] = (-1)^{2N+2} a_{N+1} (-1)^{2N+2N} d_{N+1} \; det \left[\begin{array}{cccc|cccc} a_1 & & & & b_1 & & & \\ & a_2 & & & & b_2 & & \\ & & \ddots & & & & \ddots & \\ & & & a_N & & & & b_N \\ \hline c_1 & & & & d_1 & & & \\ & c_2 & & & & d_2 & & \\ & & \ddots & & & & \ddots & \\ & & & c_N & & & & d_N \end{array}\right]$$

$$+ (-1)^{3N+2} b_{N+1} (-1)^{(2N)+(N+1)} c_{N+1} \; det \left[\begin{array}{cccc|cccc} a_1 & & & & b_1 & & & \\ & a_2 & & & & b_2 & & \\ & & \ddots & & & & \ddots & \\ & & & a_N & & & & b_N \\ \hline c_1 & & & & d_1 & & & \\ & c_2 & & & & d_2 & & \\ & & \ddots & & & & \ddots & \\ & & & c_N & & & & d_N \end{array}\right]$$

$$
\begin{aligned}
&= (a_{N+1}d_{N+1} - b_{N+1}c_{N+1}) det \left[\begin{array}{cccc|cccc}
a_1 & & & & b_1 & & & \\
& a_2 & & & & b_2 & & \\
& & \ddots & & & & \ddots & \\
& & & a_N & & & & b_N \\
\hline
c_1 & & & & d_1 & & & \\
& c_2 & & & & d_2 & & \\
& & \ddots & & & & \ddots & \\
& & & c_N & & & & d_N
\end{array}\right] \\
&= (a_{N+1}d_{N+1} - b_{N+1}c_{N+1}) \quad \prod_{i=1}^{N}(a_i d_i - b_i c_i) \\
&= \prod_{i=1}^{N+1}(a_i d_i - b_i c_i)
\end{aligned}
$$

Note:

Let A be an n by n matrix, the A_{ij} denotes the cofactor of a_{ik} for $k = 1, 2, \cdots, n$. M_{ij} denote the $(n-1)$ by $(n-1)$ submatrix obtained from A by deleting the *ith* row and *jth* column element of a_{ij}. The determinant of M_{ij} is called the minor of a_{ij}. We define the cofactor A_{ij} of a_{ij} by

$$A_{ij} = (-1)^{i+j} det(M_{ij})$$

Appendix 5 Properties of the element row operations

Let A be an $n \times n$ matrix. The three row row operations that can be performed on A are

1. Type I operation: Interchange two rows of A.

2. Type II operation: Multiply a row of A by a nonzero real number.

3. Type III operation: Replace a row by its sum with a multiple of another row.

Appendix 6 Exact eigenequations for the multiply-connected plate

Eigenequation of a multiply-connected plate [58]

Annular plates clamped outside and inside

For $n = 0$,

$$\begin{vmatrix} J_0(\lambda) & Y_0(\lambda) & I_0(\lambda) & K_0(\lambda) \\ J_1(\lambda) & Y_1(\lambda) & -I_1(\lambda) & K_1(\lambda) \\ J_0(\alpha\lambda) & Y_0(\alpha\lambda) & I_0(\alpha\lambda) & K_0(\alpha\lambda) \\ J_1(\alpha\lambda) & Y_1(\alpha\lambda) & -I_1(\alpha\lambda) & K_1(\alpha\lambda) \end{vmatrix} = 0$$

where $\alpha = b/a$, a and b are the radius of the outside and inside boundaries, respectively.

For $n = 1$,

$$\begin{vmatrix} J_1(\lambda) & Y_1(\lambda) & I_1(\lambda) & K_1(\lambda) \\ J_0(\lambda) & Y_0(\lambda) & I_0(\lambda) & -K_0(\lambda) \\ J_1(\alpha\lambda) & Y_1(\alpha\lambda) & I_1(\alpha\lambda) & K_1(\alpha\lambda) \\ J_0(\alpha\lambda) & Y_0(\alpha\lambda) & I_0(\alpha\lambda) & -K_0(\alpha\lambda) \end{vmatrix} = 0$$

For $n = 2$,

$$\begin{vmatrix} J_0(\lambda) & Y_0(\lambda) & -I_0(\lambda) + \frac{4}{\lambda}I_1(\lambda) & K_0(\lambda) - \frac{4}{\lambda}K_1(\lambda) \\ J_1(\lambda) & Y_1(\lambda) & I_1(\lambda) & -K_1(\lambda) \\ J_0(\alpha\lambda) & Y_0(\alpha\lambda) & -I_0(\alpha\lambda) + \frac{4}{\alpha\lambda}I_1(\alpha\lambda) & K_0(\alpha\lambda) - \frac{4}{\alpha\lambda}K_1(\alpha\lambda) \\ J_1(\alpha\lambda) & Y_1(\alpha\lambda) & I_1(\alpha\lambda) & -K_1(\alpha\lambda) \end{vmatrix} = 0$$

Annular plates clamped outside and free inside

For $n = 0$

$$\begin{vmatrix} J_0(\lambda) & Y_0(\lambda) & I_0(\lambda) & K_0(\lambda) \\ J_1(\lambda) & -Y_1(\lambda) & I_1(\lambda) & -K_1(\lambda) \\ J_1(\alpha\lambda) & Y_1(\alpha\lambda) & I_1(\alpha\lambda) & -K_1(\alpha\lambda) \\ J_0(\alpha\lambda) & -Y_0(\alpha\lambda) & I_0(\alpha\lambda) + AI_1(\alpha\lambda) & K_0(\alpha\lambda) + BK_1(\alpha\lambda) \end{vmatrix} = 0$$

where $A = -\frac{2(1-\nu)}{\alpha\lambda}$ and $B = \frac{2(1-\nu)}{\alpha\lambda}$.

For $n = 1$

$$\begin{vmatrix} J_0(\lambda) & Y_0(\lambda) & I_0(\lambda) & K_0(\lambda) \\ J_1(\lambda) & Y_1(\lambda) & I_1(\lambda) & -K_1(\lambda) \\ -J_1(\alpha\lambda) & -Y_1(\alpha\lambda) & CJ_0(\alpha\lambda) + DI_1(\alpha\lambda) & -K_1(\alpha\lambda) \\ J_0(\alpha\lambda) & Y_0(\alpha\lambda) & BJ_0(\alpha\lambda) + AI_1(\alpha\lambda) & K_0(\alpha\lambda) + BK_1(\alpha\lambda) \end{vmatrix} = 0$$

where $A = \frac{8(1-\nu)}{(\alpha\lambda)^3}$, $B = -1 + \frac{4(1-\nu)}{(\alpha\lambda)^3}$, $C = -\frac{2(1-\nu)}{\alpha\lambda}$ and $D = 1 + \frac{4(1-\nu)}{(\alpha\lambda)^2}$.

For $n = 2$

$$\begin{vmatrix} J_1(\lambda) & Y_1(\lambda) & I_1(\lambda) & -K_1(\lambda) \\ J_0(\lambda) & Y_0(\lambda) & \frac{4}{\lambda}I_1(\lambda) - I_0(\lambda) & -\frac{4}{\lambda}K_1(\lambda) - K_0(\lambda) \\ J_0(\alpha\lambda) & Y_0(\alpha\lambda) & A^*I_0(\alpha\lambda) - B^*I_1(\alpha\lambda) & A^*K_0(\alpha\lambda) + B^*K_1(\alpha\lambda) \\ J_1(\alpha\lambda) & Y_1(\alpha\lambda) & CI_0(\alpha\lambda) - DI_1(\alpha\lambda) & CK_0(\alpha\lambda) - DK_1(\alpha\lambda) \end{vmatrix} = 0$$

where $A^* = 1 - AC$, $A = \frac{\alpha\lambda}{4} - \frac{3+\nu}{2\alpha\lambda}$, $B^* = B - AD$, $B = \frac{\alpha\lambda}{4} + \frac{3+\nu}{2\alpha\lambda}$, $C = \frac{48(1-\nu)\alpha\lambda}{12(1-\nu)^2-(\alpha\lambda)^4}$ and $D = \frac{12(1-\nu)[(7+\nu)+(\alpha\lambda)^2]-(\alpha\lambda)^4}{12(1-\nu)^2-(\alpha\lambda)^4}$.

Annular plates simply-supported outside and clamped inside

For $n = 0$,

$$\begin{vmatrix} J_0(\lambda) & Y_0(\lambda) & I_0(\lambda) & K_0(\lambda) \\ J_1(\lambda) & Y_1(\lambda) & \frac{2\lambda}{1-\nu}I_0(\lambda) - I_1(\lambda) & -\frac{2\lambda}{1-\nu}K_0(\lambda) - K_1(\lambda) \\ J_0(\alpha\lambda) & Y_0(\alpha\lambda) & I_0(\alpha\lambda) & K_0(\alpha\lambda) \\ J_1(\alpha\lambda) & Y_1(\alpha\lambda) & -I_1(\alpha\lambda) & K_1(\alpha\lambda) \end{vmatrix} = 0$$

For $n = 1$,

$$\begin{vmatrix} J_1(\lambda) & Y_1(\lambda) & I_1(\lambda) & K_1(\lambda) \\ J_0(\lambda) & Y_0(\lambda) & I_0(\lambda) - \frac{2\lambda}{1-\nu}I_1(\lambda) & -K_0(\lambda) - \frac{2\lambda}{1-\nu}K_1(\lambda) \\ J_0(\alpha\lambda) & Y_0(\alpha\lambda) & I_0(\alpha\lambda) & -K_0(\alpha\lambda) \\ J_1(\alpha\lambda) & Y_1(\alpha\lambda) & I_1(\alpha\lambda) & K_1(\alpha\lambda) \end{vmatrix} = 0$$

For $n = 2$,

$$\begin{vmatrix} J_1(\lambda) & Y_1(\lambda) & AI_1(\lambda) - \frac{2\lambda}{1-\nu}I_0(\lambda) & -AK_1(\lambda) - \frac{2\lambda}{1-\nu}K_0(\lambda) \\ J_0(\lambda) & Y_0(\lambda) & \frac{4}{\lambda}BI_1(\lambda) - AI_0(\lambda) & -\frac{4}{\lambda}BK_1(\lambda) - AK_0(\lambda) \\ J_1(\alpha\lambda) & Y_1(\alpha\lambda) & I_1(\alpha\lambda) & -K_1(\alpha\lambda) \\ J_0(\alpha\lambda) & Y_0(\alpha\lambda) & -I_0(\alpha\lambda) + \frac{4}{\alpha\lambda}I_1(\alpha\lambda) & -K_0(\alpha\lambda) - \frac{4}{\alpha\lambda}K_1(\alpha\lambda) \end{vmatrix} = 0$$

where $A = \frac{5-\nu}{1-\nu}$ and $B = \frac{3-\nu}{1-\nu}$.

Annular plates simply-supported outside and free inside

For $n = 0$,

$$\begin{vmatrix} J_0(\lambda) & Y_0(\lambda) & I_0(\lambda) & K_0(\lambda) \\ -J_1(\lambda) & -Y_1(\lambda) & I_1(\lambda) + AI_0(\lambda) & -K_1(\lambda) + AK_0(\lambda) \\ J_1(\alpha\lambda) & Y_1(\alpha\lambda) & I_1(\alpha\lambda) & -K_1(\alpha\lambda) \\ -J_0(\alpha\lambda) & -Y_0(\alpha\lambda) & I_0(\alpha\lambda) - BI_1(\alpha\lambda) & K_0(\alpha\lambda) + BK_1(\alpha\lambda) \end{vmatrix} = 0$$

where $A = -\frac{2\lambda}{1-\nu}$ and $B = \frac{2(1-\nu)}{\alpha\lambda}$.

For $n = 1$,

$$\begin{vmatrix} J_0(\lambda) & Y_0(\lambda) & I_0(\lambda) - EI_1(\lambda) & -K_0(\lambda) - EK_1(\lambda) \\ J_1(\lambda) & Y_1(\lambda) & I_1(\lambda) & K_1(\lambda) \\ -J_1(\alpha\lambda) & -Y_1(\alpha\lambda) & CI_0(\alpha\lambda) + DI_1(\alpha\lambda) & CK_0(\alpha\lambda) + DK_1(\alpha\lambda) \\ J_0(\alpha\lambda) & Y_0(\alpha\lambda) & BI_0(\alpha\lambda) + AI_1(\alpha\lambda) & -BK_0(\alpha\lambda) + AK_1(\alpha\lambda) \end{vmatrix} = 0$$

where $A = -\frac{8(1-\nu)}{(\alpha\lambda)^2}$, $B = -1 + \frac{4(1-\nu)}{(\alpha\lambda)^3}$, $C = -\frac{2(1-\nu)}{\alpha\lambda}$, $D = 1 + \frac{4(1-\nu)}{(\alpha\lambda)^2}$ and $E = \frac{2\lambda}{1-\nu}$.

For $n = 2$,

$$\begin{vmatrix} J_1(\lambda) & Y_1(\lambda) & EI_1(\lambda) - FI_0(\lambda) & -EK_1(\lambda) - FK_0(\lambda) \\ J_0(\lambda) & Y_0(\lambda) & GI_1(\lambda) - EI_0(\lambda) & -GK_1(\lambda) - EK_0(\lambda) \\ J_0(\alpha\lambda) & Y_0(\alpha\lambda) & A^*I_0(\alpha\lambda) - B^*I_1(\alpha\lambda) & A^*K_0(\alpha\lambda) + B^*K_1(\alpha\lambda) \\ J_1(\alpha\lambda) & Y_1(\alpha\lambda) & CI_0(\lambda) - DI_1(\alpha\lambda) & CK_0(\lambda) + DK_1(\alpha\lambda) \end{vmatrix} = 0$$

where $A^* = 1 - AC$, $A = -\frac{3+\nu}{2\alpha} + \frac{\alpha\lambda}{4}$, $B^* = B - AD$, $B = \frac{3+\nu}{2\alpha} + \frac{\alpha\lambda}{4}$, $C = \frac{48(1-\nu)\alpha\lambda}{12(1-\nu)^2-(\alpha\lambda)^4}$, $D = \frac{12(1-\nu)[(7+\nu)+(\alpha\lambda)^2]-(\alpha\lambda)^4}{12(1-\nu)^2-(\alpha\lambda)^4}$, $E = \frac{5-\nu}{1-\nu}$, $F = \frac{2\lambda}{1-\nu}$ and $G = \frac{4(3-\nu)}{\lambda(1-\nu)}$.

Annular plates free outside and clamped inside

For $n = 0$,

$$\begin{vmatrix} J_0(\lambda) & Y_0(\lambda) & -I_0(\lambda) + \frac{2(1-\nu)}{\lambda}I_1(\lambda) & -K_0(\lambda) - \frac{2(1-\nu)}{\lambda}K_1(\lambda) \\ J_1(\lambda) & Y_1(\lambda) & I_1(\lambda) & -K_1(\lambda) \\ J_0(\alpha\lambda) & Y_0(\alpha\lambda) & I_0(\alpha\lambda) & K_0(\alpha\lambda) \\ J_1(\alpha\lambda) & Y_1(\alpha\lambda) & -I_1(\alpha\lambda) & K_1(\alpha\lambda) \end{vmatrix} = 0$$

For $n = 1$,

$$\begin{vmatrix} J_0(\lambda) & Y_0(\lambda) & AI_0(\lambda) - BI_1(\lambda) & -K_0(\lambda) - \frac{2(1-\nu)}{\lambda}K_1(\lambda) \\ J_1(\lambda) & Y_1(\lambda) & I_1(\lambda) & -K_1(\lambda) \\ J_0(\alpha\lambda) & Y_0(\alpha\lambda) & I_0(\alpha\lambda) & K_0(\alpha\lambda) \\ J_1(\alpha\lambda) & Y_1(\alpha\lambda) & I_1(\alpha\lambda) & K_1(\alpha\lambda) \end{vmatrix} = 0$$

where $A = -1 + \frac{4(1-\nu)}{\lambda^2}$ and $B = \frac{8(1-\nu)}{\lambda^3}$.

For $n = 2$,

$$\begin{vmatrix} J_0(\lambda) & Y_0(\lambda) & (1-4AB\lambda)I_0(\lambda) - DI_1(\lambda) & (1-4AB\lambda)K_0(\lambda) + DK_1(\lambda) \\ J_1(\lambda) & Y_1(\lambda) & 4B\lambda I_0(\lambda) - CI_1(\lambda) & 4B\lambda K_0(\lambda) + CK_1(\lambda) \\ J_0(\alpha\lambda) & Y_0(\alpha\lambda) & -I_0(\alpha\lambda) + \frac{4}{\alpha\lambda}I_1(\alpha\lambda) & -K_0(\alpha\lambda) - \frac{4}{\alpha\lambda}K_1(\alpha\lambda) \\ J_1(\alpha\lambda) & Y_1(\alpha\lambda) & I_1(\alpha\lambda) & -K_1(\alpha\lambda) \end{vmatrix} = 0$$

where $A = \frac{\lambda}{4} - \frac{3+\nu}{2\lambda}$, $B = \frac{12(1-\nu)}{12(1-\nu^2)-\lambda^4}$, $C = \frac{12(1-\nu)(7+\nu+\lambda^2)-\lambda^4}{12(1-\nu^2)-\lambda^4}$ and $E = \frac{\lambda}{4} + \frac{3+\nu}{2\lambda} - AC$.

Annular plates free outside and inside

For $n = 0$,

$$\begin{vmatrix} J_0(\lambda) & Y_0(\lambda) & -I_0(\lambda) + AI_1(\lambda) & -K_0(\lambda) - AK_1(\lambda) \\ J_1(\lambda) & Y_1(\lambda) & I_1(\lambda) & -K_1(\lambda) \\ J_0(\alpha\lambda) & Y_0(\alpha\lambda) & -I_0(\alpha\lambda) + BI_1(\alpha\lambda) & -K_0(\alpha\lambda) - BK_1(\alpha\lambda) \\ J_1(\alpha\lambda) & Y_1(\alpha\lambda) & I_1(\alpha\lambda) & -K_1(\alpha\lambda) \end{vmatrix} = 0$$

where $A = \frac{2(1-\nu)}{\lambda}$ and $B = \frac{2(1-\nu)}{\alpha\lambda}$.

For $n = 1$,

$$\begin{vmatrix} J_0(\lambda) & Y_0(\lambda) & (-1+\frac{\lambda}{2}A)I_0(\lambda) - AI_1(\lambda) & -(-1+\frac{\lambda}{2}A)K_0(\lambda) - AK_1(\lambda) \\ J_1(\lambda) & Y_1(\lambda) & \frac{\lambda^2}{4}AI_0(\lambda) - (1+\frac{\lambda}{2}A)I_1(\lambda) & -\frac{\lambda^2}{4}AK_0(\lambda) - (1+\frac{\lambda}{2}A)K_1(\lambda) \\ J_0(\alpha\lambda) & Y_0(\alpha\lambda) & (-1+\frac{\alpha\lambda}{2}B)I_0(\alpha\lambda) - BI_1(\alpha\lambda) & -(-1+\frac{\alpha\lambda}{2}B)K_0(\alpha\lambda) - BK_1(\alpha\lambda) \\ J_1(\alpha\lambda) & Y_1(\alpha\lambda) & \frac{(\alpha\lambda)^2}{4}BI_0(\alpha\lambda) - (1+\frac{\alpha\lambda}{2}B)I_1(\alpha\lambda) & -\frac{(\alpha\lambda)^2}{4}BK_0(\alpha\lambda) - (1+\frac{\alpha\lambda}{2}B)K_1(\alpha\lambda) \end{vmatrix} = 0$$

where $A = \frac{8(1-\nu)}{\lambda^3}$ and $B = \frac{8(1-\nu)}{(\alpha\lambda)^3}$.

For $n = 2$,

$$\begin{vmatrix} J_0(\lambda) & Y_0(\lambda) & AI_0(\lambda) - BI_1(\lambda) & AK_0(\lambda) + BK_1(\lambda) \\ J_1(\lambda) & Y_1(\lambda) & CI_0(\lambda) - DI_1(\lambda) & CK_0(\lambda) + DK_1(\lambda) \\ J_0(\alpha\lambda) & Y_0(\alpha\lambda) & A^*I_0(\alpha\lambda) - B^*I_1(\alpha\lambda) & A^*K_0(\alpha\lambda) + B^*K_1(\alpha\lambda) \\ J_1(\alpha\lambda) & Y_1(\alpha\lambda) & C^*I_0(\alpha\lambda) - D^*I_1(\alpha\lambda) & C^*K_0(\lambda) + D^*K_1(\alpha\lambda) \end{vmatrix} = 0$$

where $A = 1 - (\frac{\lambda}{4} - \frac{3+\nu}{2\lambda})C,\ B = \frac{\lambda}{4} + \frac{3+\nu}{2\lambda} - (\frac{\lambda}{4} - \frac{3+\nu}{2\alpha\lambda})D,\ C = \frac{48(1-\nu)\lambda}{12(1-\nu^2)-\lambda},\ D = \frac{12(1-\nu)(7+\nu+\lambda^2)-\lambda^4}{12(1-\nu^2)-\lambda^4}$, $A^* = 1-(\frac{\alpha\lambda}{4}-\frac{3+\nu}{2\lambda})$, $B^* = \frac{\alpha\lambda}{4}+\frac{3+\nu}{2\alpha\lambda}-(\frac{\alpha\lambda}{4}-\frac{3+\nu}{2\alpha\lambda})D^*$, $C^* = \frac{48(1-\nu)(\alpha\lambda)}{12(1-\nu^2)-(\alpha\lambda)^4}$ and $D^* = \frac{12(1-\nu)[7+\nu+(\alpha\lambda)^2)]-(\alpha\lambda)^4}{12(1-\nu^2)-\lambda^4}$.

Appendix 7 The determinant of the matrix [M]

The Matrix $\left[M\right]_{8\times 4}$ is

$$[M]=\begin{bmatrix} a1 & a2 & a3 & a4 \\ b1 & b2 & b3 & b4 \\ c1 & c2 & c3 & c4 \\ d1 & d2 & d3 & d4 \\ p1 & p2 & p3 & p4 \\ q1 & q2 & q3 & q4 \\ r1 & r2 & r3 & r4 \\ s1 & s2 & s3 & s4 \end{bmatrix}$$

$$Det[[M]^T[M]]$$

$$\left(Det\begin{bmatrix} a1 & a2 & a3 & a4 \\ b1 & b2 & b3 & b4 \\ c1 & c2 & c3 & c4 \\ d1 & d2 & d3 & d4 \end{bmatrix}\right)^2 + \left(Det\begin{bmatrix} a1 & a2 & a3 & a4 \\ b1 & b2 & b3 & b4 \\ c1 & c2 & c3 & c4 \\ p1 & p2 & p3 & p4 \end{bmatrix}\right)^2 + \left(Det\begin{bmatrix} a1 & a2 & a3 & a4 \\ b1 & b2 & b3 & b4 \\ c1 & c2 & c3 & c4 \\ q1 & q2 & q3 & q4 \end{bmatrix}\right)^2 + \left(Det\begin{bmatrix} a1 & a2 & a3 & a4 \\ b1 & b2 & b3 & b4 \\ c1 & c2 & c3 & c4 \\ r1 & r2 & r3 & r4 \end{bmatrix}\right)^2 + \left(Det\begin{bmatrix} a1 & a2 & a3 & a4 \\ b1 & b2 & b3 & b4 \\ c1 & c2 & c3 & c4 \\ s1 & s2 & s3 & s4 \end{bmatrix}\right)^2$$

$$+\left(Det\begin{bmatrix} a1 & a2 & a3 & a4 \\ b1 & b2 & b3 & b4 \\ d1 & d2 & d3 & d4 \\ p1 & p2 & p3 & p4 \end{bmatrix}\right)^2 + \left(Det\begin{bmatrix} a1 & a2 & a3 & a4 \\ b1 & b2 & b3 & b4 \\ d1 & d2 & d3 & d4 \\ q1 & q2 & q3 & q4 \end{bmatrix}\right)^2 + \left(Det\begin{bmatrix} a1 & a2 & a3 & a4 \\ b1 & b2 & b3 & b4 \\ d1 & d2 & d3 & d4 \\ r1 & r2 & r3 & r4 \end{bmatrix}\right)^2 + \left(Det\begin{bmatrix} a1 & a2 & a3 & a4 \\ b1 & b2 & b3 & b4 \\ d1 & d2 & d3 & d4 \\ s1 & s2 & s3 & s4 \end{bmatrix}\right)^2 + \left(Det\begin{bmatrix} a1 & a2 & a3 & a4 \\ b1 & b2 & b3 & b4 \\ p1 & p2 & p3 & p4 \\ q1 & q2 & q3 & q4 \end{bmatrix}\right)^2$$

$$+\left(Det\begin{bmatrix} a1 & a2 & a3 & a4 \\ b1 & b2 & b3 & b4 \\ p1 & p2 & p3 & p4 \\ r1 & r2 & r3 & r4 \end{bmatrix}\right)^2 + \left(Det\begin{bmatrix} a1 & a2 & a3 & a4 \\ b1 & b2 & b3 & b4 \\ p1 & p2 & p3 & p4 \\ s1 & s2 & s3 & s4 \end{bmatrix}\right)^2 + \left(Det\begin{bmatrix} a1 & a2 & a3 & a4 \\ b1 & b2 & b3 & b4 \\ q1 & q2 & q3 & q4 \\ r1 & r2 & r3 & r4 \end{bmatrix}\right)^2 + \left(Det\begin{bmatrix} a1 & a2 & a3 & a4 \\ b1 & b2 & b3 & b4 \\ q1 & q2 & q3 & q4 \\ s1 & s2 & s3 & s4 \end{bmatrix}\right)^2 + \left(Det\begin{bmatrix} a1 & a2 & a3 & a4 \\ b1 & b2 & b3 & b4 \\ r1 & r2 & r3 & r4 \\ s1 & s2 & s3 & s4 \end{bmatrix}\right)^2$$

$$+\left(Det\begin{bmatrix} a1 & a2 & a3 & a4 \\ c1 & c2 & c3 & c4 \\ d1 & d2 & d3 & d4 \\ p1 & p2 & p3 & p4 \end{bmatrix}\right)^2 + \left(Det\begin{bmatrix} a1 & a2 & a3 & a4 \\ c1 & c2 & c3 & c4 \\ d1 & d2 & d3 & d4 \\ q1 & q2 & q3 & q4 \end{bmatrix}\right)^2 + \left(Det\begin{bmatrix} a1 & a2 & a3 & a4 \\ c1 & c2 & c3 & c4 \\ d1 & d2 & d3 & d4 \\ r1 & r2 & r3 & r4 \end{bmatrix}\right)^2 + \left(Det\begin{bmatrix} a1 & a2 & a3 & a4 \\ c1 & c2 & c3 & c4 \\ d1 & d2 & d3 & d4 \\ s1 & s2 & s3 & s4 \end{bmatrix}\right)^2 + \left(Det\begin{bmatrix} a1 & a2 & a3 & a4 \\ c1 & c2 & c3 & c4 \\ p1 & p2 & p3 & p4 \\ q1 & q2 & q3 & q4 \end{bmatrix}\right)^2$$

$$+\left(Det\begin{bmatrix} a1 & a2 & a3 & a4 \\ c1 & c2 & c3 & c4 \\ p1 & p2 & p3 & p4 \\ r1 & r2 & r3 & r4 \end{bmatrix}\right)^2 + \left(Det\begin{bmatrix} a1 & a2 & a3 & a4 \\ c1 & c2 & c3 & c4 \\ p1 & p2 & p3 & p4 \\ s1 & s2 & s3 & s4 \end{bmatrix}\right)^2 + \left(Det\begin{bmatrix} a1 & a2 & a3 & a4 \\ c1 & c2 & c3 & c4 \\ q1 & q2 & q3 & q4 \\ r1 & r2 & r3 & r4 \end{bmatrix}\right)^2 + \left(Det\begin{bmatrix} a1 & a2 & a3 & a4 \\ c1 & c2 & c3 & c4 \\ q1 & q2 & q3 & q4 \\ s1 & s2 & s3 & s4 \end{bmatrix}\right)^2 + \left(Det\begin{bmatrix} a1 & a2 & a3 & a4 \\ c1 & c2 & c3 & c4 \\ r1 & r2 & r3 & r4 \\ s1 & s2 & s3 & s4 \end{bmatrix}\right)^2$$

$$+\left(Det\begin{bmatrix} a1 & a2 & a3 & a4 \\ d1 & d2 & d3 & d4 \\ p1 & p2 & p3 & p4 \\ q1 & q2 & q3 & q4 \end{bmatrix}\right)^2 + \left(Det\begin{bmatrix} a1 & a2 & a3 & a4 \\ d1 & d2 & d3 & d4 \\ p1 & p2 & p3 & p4 \\ r1 & r2 & r3 & r4 \end{bmatrix}\right)^2 + \left(Det\begin{bmatrix} a1 & a2 & a3 & a4 \\ d1 & d2 & d3 & d4 \\ p1 & p2 & p3 & p4 \\ s1 & s2 & s3 & s4 \end{bmatrix}\right)^2 + \left(Det\begin{bmatrix} a1 & a2 & a3 & a4 \\ d1 & d2 & d3 & d4 \\ q1 & q2 & q3 & q4 \\ r1 & r2 & r3 & r4 \end{bmatrix}\right)^2 + \left(Det\begin{bmatrix} a1 & a2 & a3 & a4 \\ d1 & d2 & d3 & d4 \\ q1 & q2 & q3 & q4 \\ s1 & s2 & s3 & s4 \end{bmatrix}\right)^2$$

$$+\left(Det\begin{bmatrix} a1 & a2 & a3 & a4 \\ d1 & d2 & d3 & d4 \\ r1 & r2 & r3 & r4 \\ s1 & s2 & s3 & s4 \end{bmatrix}\right)^2 + \left(Det\begin{bmatrix} a1 & a2 & a3 & a4 \\ p1 & p2 & p3 & p4 \\ q1 & q2 & q3 & q4 \\ r1 & r2 & r3 & r4 \end{bmatrix}\right)^2 + \left(Det\begin{bmatrix} a1 & a2 & a3 & a4 \\ p1 & p2 & p3 & p4 \\ q1 & q2 & q3 & q4 \\ s1 & s2 & s3 & s4 \end{bmatrix}\right)^2 + \left(Det\begin{bmatrix} a1 & a2 & a3 & a4 \\ p1 & p2 & p3 & p4 \\ r1 & r2 & r3 & r4 \\ s1 & s2 & s3 & s4 \end{bmatrix}\right)^2 + \left(Det\begin{bmatrix} a1 & a2 & a3 & a4 \\ q1 & q2 & q3 & q4 \\ r1 & r2 & r3 & r4 \\ s1 & s2 & s3 & s4 \end{bmatrix}\right)^2$$

$$+\left(Det\begin{bmatrix} b1 & b2 & b3 & b4 \\ c1 & c2 & c3 & c4 \\ d1 & d2 & d3 & d4 \\ p1 & p2 & p3 & p4 \end{bmatrix}\right)^2 + \left(Det\begin{bmatrix} b1 & b2 & b3 & b4 \\ c1 & c2 & c3 & c4 \\ d1 & d2 & d3 & d4 \\ q1 & q2 & q3 & q4 \end{bmatrix}\right)^2 + \left(Det\begin{bmatrix} b1 & b2 & b3 & b4 \\ c1 & c2 & c3 & c4 \\ d1 & d2 & d3 & d4 \\ r1 & r2 & r3 & r4 \end{bmatrix}\right)^2 + \left(Det\begin{bmatrix} b1 & b2 & b3 & b4 \\ c1 & c2 & c3 & c4 \\ d1 & d2 & d3 & d4 \\ s1 & s2 & s3 & s4 \end{bmatrix}\right)^2 + \left(Det\begin{bmatrix} b1 & b2 & b3 & b4 \\ c1 & c2 & c3 & c4 \\ p1 & p2 & p3 & p4 \\ q1 & q2 & q3 & q4 \end{bmatrix}\right)^2$$

$$+\left(Det\begin{bmatrix} b1 & b2 & b3 & b4 \\ c1 & c2 & c3 & c4 \\ p1 & p2 & p3 & p4 \\ r1 & r2 & r3 & r4 \end{bmatrix}\right)^2 + \left(Det\begin{bmatrix} b1 & b2 & b3 & b4 \\ c1 & c2 & c3 & c4 \\ p1 & p2 & p3 & p4 \\ s1 & s2 & s3 & s4 \end{bmatrix}\right)^2 + \left(Det\begin{bmatrix} b1 & b2 & b3 & b4 \\ c1 & c2 & c3 & c4 \\ q1 & q2 & q3 & q4 \\ r1 & r2 & r3 & r4 \end{bmatrix}\right)^2 + \left(Det\begin{bmatrix} b1 & b2 & b3 & b4 \\ c1 & c2 & c3 & c4 \\ q1 & q2 & q3 & q4 \\ s1 & s2 & s3 & s4 \end{bmatrix}\right)^2 + \left(Det\begin{bmatrix} b1 & b2 & b3 & b4 \\ c1 & c2 & c3 & c4 \\ r1 & r2 & r3 & r4 \\ s1 & s2 & s3 & s4 \end{bmatrix}\right)^2$$

$$+\left(Det\begin{bmatrix} b1 & b2 & b3 & b4 \\ d1 & d2 & d3 & d4 \\ p1 & p2 & p3 & p4 \\ q1 & q2 & q3 & q4 \end{bmatrix}\right)^2 + \left(Det\begin{bmatrix} b1 & b2 & b3 & b4 \\ d1 & d2 & d3 & d4 \\ p1 & p2 & p3 & p4 \\ r1 & r2 & r3 & r4 \end{bmatrix}\right)^2 + \left(Det\begin{bmatrix} b1 & b2 & b3 & b4 \\ d1 & d2 & d3 & d4 \\ p1 & p2 & p3 & p4 \\ s1 & s2 & s3 & s4 \end{bmatrix}\right)^2 + \left(Det\begin{bmatrix} b1 & b2 & b3 & b4 \\ d1 & d2 & d3 & d4 \\ q1 & q2 & q3 & q4 \\ r1 & r2 & r3 & r4 \end{bmatrix}\right)^2 + \left(Det\begin{bmatrix} b1 & b2 & b3 & b4 \\ d1 & d2 & d3 & d4 \\ q1 & q2 & q3 & q4 \\ s1 & s2 & s3 & s4 \end{bmatrix}\right)^2$$

$$+\left(Det\begin{bmatrix} b1 & b2 & b3 & b4 \\ d1 & d2 & d3 & d4 \\ r1 & r2 & r3 & r4 \\ s1 & s2 & s3 & s4 \end{bmatrix}\right)^2 + \left(Det\begin{bmatrix} b1 & b2 & b3 & b4 \\ p1 & p2 & p3 & p4 \\ q1 & q2 & q3 & q4 \\ r1 & r2 & r3 & r4 \end{bmatrix}\right)^2 + \left(Det\begin{bmatrix} b1 & b2 & b3 & b4 \\ p1 & p2 & p3 & p4 \\ q1 & q2 & q3 & q4 \\ s1 & s2 & s3 & s4 \end{bmatrix}\right)^2 + \left(Det\begin{bmatrix} b1 & b2 & b3 & b4 \\ p1 & p2 & p3 & p4 \\ r1 & r2 & r3 & r4 \\ s1 & s2 & s3 & s4 \end{bmatrix}\right)^2 + \left(Det\begin{bmatrix} b1 & b2 & b3 & b4 \\ q1 & q2 & q3 & q4 \\ r1 & r2 & r3 & r4 \\ s1 & s2 & s3 & s4 \end{bmatrix}\right)^2$$

$$+\left(Det\begin{bmatrix} c1 & c2 & c3 & c4 \\ d1 & d2 & d3 & d4 \\ p1 & p2 & p3 & p4 \\ q1 & q2 & q3 & q4 \end{bmatrix}\right)^2 + \left(Det\begin{bmatrix} c1 & c2 & c3 & c4 \\ d1 & d2 & d3 & d4 \\ p1 & p2 & p3 & p4 \\ r1 & r2 & r3 & r4 \end{bmatrix}\right)^2 + \left(Det\begin{bmatrix} c1 & c2 & c3 & c4 \\ d1 & d2 & d3 & d4 \\ p1 & p2 & p3 & p4 \\ s1 & s2 & s3 & s4 \end{bmatrix}\right)^2 + \left(Det\begin{bmatrix} c1 & c2 & c3 & c4 \\ d1 & d2 & d3 & d4 \\ q1 & q2 & q3 & q4 \\ r1 & r2 & r3 & r4 \end{bmatrix}\right)^2 + \left(Det\begin{bmatrix} c1 & c2 & c3 & c4 \\ d1 & d2 & d3 & d4 \\ q1 & q2 & q3 & q4 \\ s1 & s2 & s3 & s4 \end{bmatrix}\right)^2$$

$$+\left(Det\begin{bmatrix} c1 & c2 & c3 & c4 \\ d1 & d2 & d3 & d4 \\ r1 & r2 & r3 & r4 \\ s1 & s2 & s3 & s4 \end{bmatrix}\right)^2 + \left(Det\begin{bmatrix} c1 & c2 & c3 & c4 \\ p1 & p2 & p3 & p4 \\ q1 & q2 & q3 & q4 \\ r1 & r2 & r3 & r4 \end{bmatrix}\right)^2 + \left(Det\begin{bmatrix} c1 & c2 & c3 & c4 \\ p1 & p2 & p3 & p4 \\ q1 & q2 & q3 & q4 \\ s1 & s2 & s3 & s4 \end{bmatrix}\right)^2 + \left(Det\begin{bmatrix} c1 & c2 & c3 & c4 \\ p1 & p2 & p3 & p4 \\ r1 & r2 & r3 & r4 \\ s1 & s2 & s3 & s4 \end{bmatrix}\right)^2 + \left(Det\begin{bmatrix} c1 & c2 & c3 & c4 \\ q1 & q2 & q3 & q4 \\ r1 & r2 & r3 & r4 \\ s1 & s2 & s3 & s4 \end{bmatrix}\right)^2$$

$$+\left(Det\begin{bmatrix} d1 & d2 & d3 & d4 \\ p1 & p2 & p3 & p4 \\ q1 & q2 & q3 & q4 \\ r1 & r2 & r3 & r4 \end{bmatrix}\right)^2 + \left(Det\begin{bmatrix} d1 & d2 & d3 & d4 \\ p1 & p2 & p3 & p4 \\ q1 & q2 & q3 & q4 \\ s1 & s2 & s3 & s4 \end{bmatrix}\right)^2 + \left(Det\begin{bmatrix} d1 & d2 & d3 & d4 \\ p1 & p2 & p3 & p4 \\ r1 & r2 & r3 & r4 \\ s1 & s2 & s3 & s4 \end{bmatrix}\right)^2 + \left(Det\begin{bmatrix} d1 & d2 & d3 & d4 \\ q1 & q2 & q3 & q4 \\ r1 & r2 & r3 & r4 \\ s1 & s2 & s3 & s4 \end{bmatrix}\right)^2 + \left(Det\begin{bmatrix} p1 & p2 & p3 & p4 \\ q1 & q2 & q3 & q4 \\ r1 & r2 & r3 & r4 \\ s1 & s2 & s3 & s4 \end{bmatrix}\right)^2$$

The determinant of the matrix $[M]^T[M]$ can be decomposed into the summation of the square determinant in the 70 (C_4^8) matrices. Each matrix was assembled by any four rows of the matrix $[M]$(no repeat).

Table 2-1 True eigenequations for a circular plate ($a=1$).

		True eigenequations for circular plate
clamped	Leissa	$I_{\ell+1}J_{\ell}+I_{\ell}J_{\ell+1}=0$
	Real-part and imaginary-part BEMs	$I_{\ell+1}J_{\ell}+I_{\ell}J_{\ell+1}=0$
Simply-supported	Leissa	$\frac{J_{\ell+1}}{J_{\ell}}+\frac{I_{\ell+1}}{I_{\ell}}=\frac{2\lambda}{(1-\nu)}$
	Real-part and imaginary-part BEMs	$(1-\nu)(I_{\ell}J_{\ell+1}+I_{\ell+1}J_{\ell})-2\lambda I_{\ell}J_{\ell}=0$
Free	Leissa	$\frac{\lambda^2 J_{\ell}+(1-\nu)[\lambda J'_{\ell}-\ell^2 J_{\ell}]}{\lambda^2 I_{\ell}-(1-\nu)[\lambda I'_{\ell}-\ell^2 I_{\ell}]}=\frac{\lambda^3 J'_{\ell}+(1-\nu)\ell^2[\lambda J'_{\ell}-J_{\ell}]}{\lambda^3 I'_{\ell}-(1-\nu)\ell^2[\lambda I'_{\ell}-I_{\ell}]}$
	Real-part and imaginary-part BEMs	$\lambda(1-\nu)[-4\ell^2(\ell-1)I_{\ell}J_{\ell}-2\lambda^2 I_{\ell+1}J_{\ell+1}]$ $+2\ell\lambda^2(1-\nu)(1-\ell)(I_{\ell+1}J_{\ell}-I_{\ell}J_{\ell+1})$ $+[\ell^2(1-\nu)^2(\ell^2-1)+\lambda^4](I_{\ell+1}J_{\ell}+I_{\ell}J_{\ell+1})=0$

Table 2-2 Spurious eigenequations by using the six formulations in the real-part BEM

	Spurious eigenequations in the real-part BEM
u,θ **formulation**	$K_{\ell+1}Y_{\ell}-K_{\ell}Y_{\ell+1}=0$
u,m **formulation**	$(1-\nu)(K_{\ell}Y_{\ell+1}-K_{\ell+1}Y_{\ell})-2\lambda a K_{\ell}Y_{\ell}=0$
u,v **formulation**	$\ell^2(1-\nu)(K_{\ell}Y_{\ell+1}-K_{\ell+1}Y_{\ell})-2\lambda a\ell K_{\ell}Y_{\ell}+\lambda^2 a^2(K_{\ell+1}Y_{\ell}+K_{\ell}Y_{\ell+1})=0$
θ,m **formulation**	$\ell^2(1-\nu)(K_{\ell}Y_{\ell+1}-K_{\ell+1}Y_{\ell})-2\lambda a\,\ell\,K_{\ell}Y_{\ell}+\lambda^2 a^2(K_{\ell+1}Y_{\ell}+K_{\ell}Y_{\ell+1})=0$
θ,v **formulation**	$2\lambda a(\ell^2 K_{\ell}Y_{\ell}+\lambda^2\rho^2 K_{\ell+1}Y_{\ell+1})-2\lambda^2 a^2\,\ell\,(K_{\ell+1}Y_{\ell}+K_{\ell}Y_{\ell+1})$ $+\ell^2(1-\nu)(K_{\ell+1}Y_{\ell}-K_{\ell}Y_{\ell+1})=0$
m,v **formulation**	$\lambda a(1-\nu)[4\ell^2(\ell-1)K_{\ell}Y_{\ell}-2\lambda^2 a^2 K_{\ell+1}Y_{\ell+1}]$ $+2\ell\lambda^2 a^2(1-\nu)(1-\ell)(K_{\ell+1}Y_{\ell}+K_{\ell}Y_{\ell+1})$ $+[\ell^2(1-\nu)^2(\ell^2-1)+\lambda^4 a^4](K_{\ell+1}Y_{\ell}-K_{\ell}Y_{\ell+1})=0$

where $\ell=0,\pm1,\pm2,\pm3,......$

Table 2-3 True and Spurious eigenequations for the membrane by using the real-part and imaginary-part BEMs

		u = 0 (Dirichlet)	t = 0 (Neumann)
***UT* formulation**	**Re**	$\{J\}[Y]=0$	$\{J'\}[Y]=0$
	Im	$\{J\}[J]=0$	$\{J'\}[J]=0$
***LM* formulation**	**Re**	$\{J\}[Y']=0$	$\{J'\}[Y']=0$
	Im	$\{J\}[J']=0$	$\{J'\}[J']=0$

[ ] **and** { } **mean true and spurious eigenequation.**

Table 2-4 Spurious eigenequations by using the six formulations in the imaginary-part BEM

	Spurious eigenequations in the imaginary-part BEM
u,θ **formulation**	$I_{\ell+1}J_{\ell}+I_{\ell}J_{\ell+1}=0$
u,m **formulation**	$(1-\nu)(I_{\ell}J_{\ell+1}+I_{\ell+1}J_{\ell})-2\lambda\rho I_{\ell}J_{\ell}=0$
u,v **formulation**	$\ell^2(1-\nu)(I_{\ell}J_{\ell+1}+I_{\ell+1}J_{\ell})-2\lambda\rho\,\ell\,I_{\ell}J_{\ell}+\lambda^2\rho^2(I_{\ell}J_{\ell+1}-I_{\ell+1}J_{\ell})=0$
θ,m **formulation**	$\ell^2(1-\nu)(I_{\ell}J_{\ell+1}+I_{\ell+1}J_{\ell})-2\lambda\rho\,\ell\,I_{\ell}J_{\ell}+\lambda^2\rho^2(I_{\ell}J_{\ell+1}-I_{\ell+1}J_{\ell})=0$
θ,v **formulation**	$2\lambda\rho(\ell^2 I_{\ell}J_{\ell}-\lambda^2\rho^2 I_{\ell+1}J_{\ell+1})+2\lambda^2\rho^2\,\ell(I_{\ell+1}J_{\ell}-I_{\ell}J_{\ell+1})$ $-\ell^2(1-\nu)](I_{\ell+1}J_{\ell}+I_{\ell}J_{\ell+1})=0$
m,v **formulation**	$\lambda\rho(1-\nu)[-4\ell^2(\ell-1)I_{\ell}J_{\ell}-2\lambda^2\rho^2 I_{\ell+1}J_{\ell+1}]$ $+2\ell\lambda^2\rho^2(1-\nu)(1-\ell)(I_{\ell+1}J_{\ell}-I_{\ell}J_{\ell+1})$ $+[\ell^2(1-\nu)^2(\ell^2-1)+\lambda^4\rho^4](I_{\ell+1}J_{\ell}+I_{\ell}J_{\ell+1})=0$

where $\ell=0,\pm1,\pm2,\pm3,......$

Table 2-5 True eigenvalues (λ) for the clamped circular plate ($a=1$).

	λ for values of n of					
m	0	1	2	3	4	5
0	3.19	4.61	5.90	7.14	8.34	9.52
1	6.30	7.80	9.19	10.53	11.83	13.10
2	9.44	10.95	12.40	13.79	15.15	16.47
3	12.57	14.10	15.58	17.00	18.39	19.75
4	15.71	17.25	18.74	20.19	21.60	22.99

where n refers to the number of nodal diameters and m is the number of nodal circles, not including the boundary circle.

Table 2-6 True eigenvalues (λ) for the simply-supported circular plate ($a=1, \nu=0.33$).

	λ for values of n of					
m	0	1	2	3	4	5
0	2.23	3.73	5.06	6.32	7.54	8.73
1	5.45	6.96	8.37	9.72	11.03	12.31
2	8.61	10.14	11.59	12.98	14.34	15.67
3	11.76	13.29	14.77	16.20	17.59	18.96
4	14.90	16.45	17.94	19.39	20.81	22.20

where n refers to the number of nodal diameters and m is the number of nodal circles, not including the boundary circle.

Table 2-7 True eigenvalues (λ) for the free circular plate ($a=1, \nu=0.33$).

	λ for values of n of					
m	0	1	2	3	4	5
0			2.29	3.50	4.64	5.75
1	3.012	4.53	5.93	7.27	8.56	9.82
2	6.20	7.73	9.18	10.57	11.93	13.25
3	9.37	10.91	12.38	13.80	15.19	16.55
4	12.52	14.06	15.55	17.00	18.42	19.81

where n refers to the number of nodal diameters and m is the number of nodal circles, not including the boundary circle.

Table 3-1(a) The term $A(\lambda)$ of the $A(\lambda)+iB(\lambda)$ by using the real-part BEM in conjunction with the Burton & Miller method for the simply-connected plate.

	$A(\lambda)$
u,θ **formulation**	$\frac{-K_{n+1}\{\{n^4(-1+\nu)^2-2n\lambda^2(-1+\nu)\rho^2+(1+\lambda^4)\rho^4-n^2[1+\nu^2+2\lambda^2\rho^2-2\nu(1+\lambda^2\rho^2)]\}Y_n+2\lambda^3(-1+\nu)\rho^3Y_{n+1}\}}{32\pi\lambda^2\rho^4}+\frac{K_n\{4(-1+n)n^2\lambda(-1+\nu)\rho Y_n+\{n^4(-1+\nu)^2+2n\lambda^2(-1+\nu)\rho^2+(1+\lambda^4)\rho^4-n^2[1+\nu^2-2\lambda^2\rho^2+2\nu(-1+\lambda^2\rho^2)]\}Y_{n+1}\}}{32\pi\lambda^2\rho^4}$
u,m **formulation**	$\frac{K_{n+1}\{[n^2(-1+\nu)+2n\lambda^2\rho^2-(-1+\nu)\rho^2]Y_n-2\lambda^3\rho^3Y_{n+1}\}+K_n\{2\lambda\rho(-n^2+\rho^2)Y_n+[-n^2(-1+\nu)+2n\lambda^2\rho^2+(-1+\nu)\rho^2]Y_{n+1}\}}{32\pi\lambda^2\rho^4}$
u,v **formulation**	0
θ,m **formulation**	$\frac{K_{n+1}[(n^2-n^2\nu+\rho^2-2n\lambda^2\rho^2-\nu\rho^2)Y_n+2\lambda^3\rho^3Y_{n+1}]+K_n\{2\lambda\rho(n^2+\rho^2)Y_n+[n^2(-1+\nu)-2n\lambda^2\rho^2+(-1+\nu)\rho^2]Y_{n+1}\}}{32\pi\lambda^2\rho^3}$
θ,v **formulation**	$\frac{[-n^2(-1+\nu)-\lambda^2\rho^2]K_{n+1}Y_n+K_n\{2n\lambda\rho Y_n+[n^2(-1+\nu)-\lambda^2\rho^2]Y_{n+1}\}}{16\pi\lambda^2\rho^3}$
m,v **formulation**	0

Table 3-1(b) The term $B(\lambda)$ of the $A(\lambda)+iB(\lambda)$ by using the real-part BEM in conjunction with the Burton & Miller method for the simply-connected plate.

	$B(\lambda)$
u,θ **formulation**	0
u, m **formulation**	$\frac{[-n^2(-1+\nu)-\lambda^2\rho^2]K_{n+1}Y_n+K_n\{2n\lambda\rho Y_n+[n^2(-1+\nu)-\lambda^2\rho^2]Y_{n+1}\}}{16\pi\lambda^2\rho^3}$
u, v **formulation**	$\frac{K_{n+1}[(n^2-n^2\nu+\rho^2-2n\lambda^2\rho^2-\nu\rho^2)Y_n+2\lambda^3\rho^3Y_{n+1}]+K_n\{2\lambda\rho(n^2+\rho^2)Y_n+[n^2(-1+\nu)-2n\lambda^2\rho^2+(-1+\nu)\rho^2]Y_{n+1}\}}{32\pi\lambda^2\rho^3}$
θ, m **formulation**	0
θ, v **formulation**	$\frac{K_{n+1}\{[n^2(-1+\nu)+2n\lambda^2\rho^2-(-1+\nu)\rho^2]Y_n-2\lambda^3\rho^3Y_{n+1}\}+K_n\{2\lambda\rho(-n^2+\rho^2)Y_n+[-n^2(-1+\nu)+2n\lambda^2\rho^2+(-1+\nu)\rho^2]Y_{n+1}\}}{32\pi\lambda^2\rho^4}$
m, v **formulation**	$\frac{-K_{n+1}\{\{n^4(-1+\nu)^2-2n\lambda^2(-1+\nu)\rho^2+(1+\lambda^4)\rho^4-n^2[1+\nu^2+2\lambda^2\rho^2-2\nu(1+\lambda^2\rho^2)]\}Y_n+2\lambda^3(-1+\nu)\rho^3Y_{n+1}\}}{32\pi\lambda^2\rho^4}$ $+\frac{K_n\{4(-1+n)n^2\lambda(-1+\nu)\rho Y_n+\{n^4(-1+\nu)^2+2n\lambda^2(-1+\nu)\rho^2+(1+\lambda^4)\rho^4-n^2[1+\nu^2-2\lambda^2\rho^2+2\nu(-1+\lambda^2\rho^2)]\}Y_{n+1}\}}{32\pi\lambda^2\rho^4}$

Table 3-2(a) The term $A(\lambda)$ of the $A(\lambda)+iB(\lambda)$ by using the imaginary-part BEM in conjunction with the Burton & Miller method for the simply-connected plate.

	$A(\lambda)$
u,θ **formulation**	$\frac{(-1)^n\{J_{n+1}\{[n^4(-1+\nu)^2+2n\lambda^2(-1+\nu)\rho^2+(1+\lambda^4)\rho^4-n^2(1-2\nu+\nu^2-2\lambda^2\rho^2+2\lambda^2\nu\rho^2)]I_n+2\lambda^3(-1+\nu)\rho^3 I_{n+1}\}}{32\pi\lambda^2\rho^4}$ $+\frac{(-1)^n\{J_n\{4(-1+n)n^2\lambda(-1+\nu)\rho I_n+\{n^4(-1+\nu)^2-2n\lambda^2(-1+\nu)\rho^2+(1+\lambda^4)\rho^4-n^2[1+\nu^2+2\lambda^2\rho^2-2\nu(1+\lambda^2\rho^2)]\}I_{n+1}\}\}}{32\pi\lambda^2\rho^4}$
u,m **formulation**	$\frac{(-1)^n\{J_{n+1}\{[-n^2(-1+\nu)+2n\lambda^2\rho^2+(-1+\nu)\rho^2]I_n+2\lambda^3\rho^3 I_{n+1}\}\}+(-1)^n\{J_n\{2\lambda\rho(-n^2+\rho^2)I_n+[-n^2(-1+\nu)-2n\lambda^2\rho^2+(-1+\nu)\rho^2]I_{n+1}\}\}}{32\pi\lambda^2\rho^3}$
u,v **formulation**	0
θ,m **formulation**	$\frac{(-1)^n\{J_{n+1}\{[n^2(-1+\nu)-2n\lambda^2\rho^2+(-1+\nu)\rho^2]I_n-2\lambda^3\rho^3 I_{n+1}\}+(-1)^n\{J_n\{2\lambda\rho(n^2+\rho^2)I_n+[n^2(-1+\nu)+2n\lambda^2\rho^2+(-1+\nu)\rho^2]I_{n+1}\}\}}{32\pi\lambda^2\rho^4}$
θ,v **formulation**	$\frac{(-1)^n\{[n^2(-1+\nu)-\lambda^2\rho^2]J_{n+1}I_n\}+(-1)^n\{J_n\{2n\lambda\rho I_n+[n^2(-1+\nu)+\lambda^2\rho^2]I_{n+1}\}\}}{16\pi\lambda^2\rho^2}$
m,v **formulation**	0

Table 3-2(b) The term $B(\lambda)$ of the $A(\lambda)+iB(\lambda)$ by using the imaginary-part BEM in conjunction with the Burton & Miller method for the simply-connected plate.

	$B(\lambda)$
u, θ **formulation**	0
u, m **formulation**	$\frac{(-1)^n\{[n^2(-1+\nu)-\lambda^2\rho^2]J_{n+1}I_n\}+(-1)^n\{J_n\{2n\lambda\rho I_n+[n^2(-1+\nu)+\lambda^2\rho^2]I_{n+1}\}\}}{16\pi\lambda^2\rho^2}$
u, v **formulation**	$\frac{(-1)^n\{J_{n+1}\{[n^2(-1+\nu)-2n\lambda^2\rho^2+(-1+\nu)\rho^2]I_n-2\lambda^3\rho^3I_{n+1}\}+(-1)^n\{J_n\{2\lambda\rho(n^2+\rho^2)I_n+[n^2(-1+\nu)+2n\lambda^2\rho^2+(-1+\nu)\rho^2]I_{n+1}\}\}}{32\pi\lambda^2\rho^4}$
θ, m **formulation**	0
θ, v **formulation**	$\frac{(-1)^n\{J_{n+1}\{[-n^2(-1+\nu)+2n\lambda^2\rho^2+(-1+\nu)\rho^2]I_n+2\lambda^3\rho^3I_{n+1}\}\}+(-1)^n\{J_n\{2\lambda\rho(-n^2+\rho^2)I_n+[-n^2(-1+\nu)-2n\lambda^2\rho^2+(-1+\nu)\rho^2]I_{n+1}\}\}}{32\pi\lambda^2\rho^3}$
m, v **formulation**	$\frac{(-1)^n\{J_{n+1}\{[n^4(-1+\nu)^2+2n\lambda^2(-1+\nu)\rho^2+(1+\lambda^4)\rho^4-n^2(1-2\nu+\nu^2-2\lambda^2\rho^2+2\lambda^2\nu\rho^2)]I_n+2\lambda^3(-1+\nu)\rho^3I_{n+1}\}}{32\pi\lambda^2\rho^4}$ $+\frac{(-1)^n\{J_n\{4(-1+n)n^2\lambda(-1+\nu)\rho I_n+\{n^4(-1+\nu)^2-2n\lambda^2(-1+\nu)\rho^2+(1+\lambda^4)\rho^4-n^2[1+\nu^2+2\lambda^2\rho^2-2\nu(1+\lambda^2\rho^2)]\}I_{n+1}\}\}}{32\pi\lambda^2\rho^4}\}$

Table 3-3(a) The term $A(\lambda)$ of the $A(\lambda)+iB(\lambda)$ by using the complex-valued BEM for the simply-connected plate.

	$A(\lambda)$
u,θ **formulation**	$-\frac{(-1)^n J_{n+1}I_n+(-1)^n J_n I_{n+1}+K_{n+1}Y_n-K_nY_{n+1}}{32\pi\lambda^2}$
u,m **formulation**	$-\frac{(-1)^n(-1+\nu)J_{n+1}I_n+(-1)^n J_n[2\lambda\rho I_n+(-1+\nu)I_{n+1}]+2\lambda\rho K_nY_n+K_{n+1}Y_n-\nu K_{n+1}Y_n-K_nY_{n+1}+\nu K_nY_{n+1}}{32\pi\lambda^2\rho}$
u,v **formulation**	$-\frac{(-1)^n[n^2(-1+\nu)-\lambda^2\rho^2]J_{n+1}I_n+(-1)^n J_n\{2n\lambda\rho I_n+[n^2(-1+\nu)+\lambda^2\rho^2]I_{n+1}\}}{32\pi\lambda^2\rho^2}$ $+\frac{-2n\lambda\rho K_nY_n-n^2K_{n+1}Y_n+n^2\nu K_{n+1}Y_n+\lambda^2\rho^2K_{n+1}Y_n+n^2K_nY_{n+1}-n^2\nu K_nY_{n+1}+\lambda^2\rho^2K_nY_{n+1}}{32\pi\lambda^2\rho^2}$
θ,m **formulation**	$-\frac{(-1)^n[n^2(-1+\nu)-\lambda^2\rho^2]J_{n+1}I_n+(-1)^n J_n\{2n\lambda\rho I_n+[n^2(-1+\nu)+\lambda^2\rho^2]I_{n+1}\}}{32\pi\lambda^2\rho^2}$ $-\frac{-2n\lambda\rho K_nY_n-n^2K_{n+1}Y_n+n^2\nu K_{n+1}Y_n+\lambda^2\rho^2K_{n+1}Y_n+n^2K_nY_{n+1}-n^2\nu K_nY_{n+1}+\lambda^2\rho^2K_nY_{n+1}}{32\pi\lambda^2\rho^2}$
θ,v **formulation**	$\frac{(-1)^n J_{n+1}\{n[n(-1+\nu)-2\lambda^2\rho^2]I_n-2\lambda^3\rho^3I_{n+1}\}+(-1)^n nJ_n\{2n\lambda\rho I_n+[n(-1+\nu)+2\lambda^2\rho^2]I_{n+1}\}}{32\pi\lambda^2\rho^3}$ $+\frac{2n^2\lambda\rho K_nY_n+n^2K_{n+1}Y_n-n^2\nu K_{n+1}Y_n-2n\lambda^2\rho^2K_{n+1}Y_n-n^2K_nY_{n+1}+n^2\nu K_nY_{n+1}-2n\lambda^2\rho^2K_nY_{n+1}+2\lambda^3\rho^3K_{n+1}Y_{n+1}}{32\pi\lambda^2\rho^3}$
m,v **formulation**	$\frac{1}{32\pi\lambda^2\rho^4}\{(-1)^n J_{n+1}\{[n^4(-1+\nu)^2+2n\lambda^2(-1+\nu)\rho^2+\lambda^4\rho^4-n^2(1-2\nu+\nu^2-2\lambda^2\rho^2+2\lambda^2\nu\rho^2]I_n+2\lambda^3(-1+\nu)\rho^3I_{n+1}\}$ $+(-1)^n J_n\{4(-1+n)n^2\lambda(-1+\nu)\rho I_n+\{n^4(-1+\nu)^2-2n\lambda^2(-1+\nu)\rho^2+\lambda^4\rho^4-n^2(1+\nu^2+2\lambda^2\rho^2-2\nu(1+\lambda^2\rho^2))]\}I_{n+1}\}$ $-4n^2\lambda\rho K_nY_n+4n^3\lambda\rho K_nY_n+4n^2\lambda\nu\rho K_nY_n-4n^3\lambda\nu\rho K_nY_n-n^2K_{n+1}Y_n+n^4K_{n+1}Y_n+2n^2\nu K_{n+1}Y_n-2n^4\nu K_{n+1}Y_n-n^2\nu^2K_{n+1}Y_n+n^4\nu^2K_{n+1}Y_n$ $+2n\lambda^2\rho^2K_{n+1}Y_n-2n^2\lambda^2\rho^2K_{n+1}Y_n-2n\lambda^2\nu\rho^2K_{n+1}Y_n+2n^2\lambda^2\nu\rho^2K_{n+1}Y_n+\lambda^4\rho^4K_{n+1}Y_n+n^2K_nY_{n+1}-n^4K_nY_{n+1}-2n^2\nu K_nY_{n+1}+2n^4\nu K_nY_{n+1}+n^2\nu^2K_nY_{n+1}$ $-n^4\nu^2K_nY_{n+1}+2n\lambda^2\rho^2K_nY_{n+1}-2n^2\lambda^2\rho^2K_nY_{n+1}-2n\lambda^2\nu\rho^2K_nY_{n+1}+2n^2\lambda^2\nu\rho^2K_nY_{n+1}-\lambda^4\rho^4K_nY_{n+1}-2\lambda^3\rho^3K_{n+1}Y_{n+1}+2\lambda^3\nu\rho^3K_{n+1}Y_{n+1}\}$

Table 3-3(b) The term $B(\lambda)$ of the $A(\lambda)+iB(\lambda)$ by using the complex-valued BEM for the simply-connected plate.

	$B(\lambda)$
u, θ **formulation**	$\frac{J_{n+1}K_n - J_nK_{n+1} + (-1)^n(I_{n+1}Y_n + I_nY_{n+1})}{32\pi\lambda^2}$
u, m **formulation**	$\frac{(-1+\nu)J_{n+1}K_n + J_n[2\lambda\rho K_n - (-1+\nu)K_{n+1}] + (-1)^n\{(-1+\nu)I_{n+1}Y_n + I_n[2\lambda\rho Y_n + (-1+\nu)Y_n]\}}{32\pi\lambda^2\rho}$
u, v **formulation**	$\frac{[n^2(-1+\nu) - \lambda^2\rho^2]J_{n+1}K_n + J_n[2n\lambda\rho K_n + (n^2 - n^2\nu - \lambda^2\rho^2)K_{n+1}]}{32\pi\lambda^2\rho^2} + \frac{(-1)^n\{[n^2(-1+\nu) + \lambda^2\rho^2]I_{n+1}Y_n + I_n\{2n\lambda\rho Y_n + [n^2(-1+\nu) - \lambda^2\rho^2]Y_{n+1}\}\}}{32\pi\lambda^2\rho^2}$
θ, m **formulation**	$\frac{[n^2(-1+\nu) - \lambda^2\rho^2]J_{n+1}K_n + J_n[2n\lambda\rho K_n + (n^2 - n^2\nu - \lambda^2\rho^2)K_{n+1}]}{32\pi\lambda^2\rho^2} + \frac{(-1)^n\{[n^2(-1+\nu) + \lambda^2\rho^2]I_{n+1}Y_n + I_n\{2n\lambda\rho Y_n + [n^2(-1+\nu) - \lambda^2\rho^2]Y_{n+1}\}\}}{32\pi\lambda^2\rho^2}$
θ, v **formulation**	$\frac{J_{n+1}\{n[n(-1+\nu) - 2\lambda^2\rho^2]K_n + 2\lambda^3\rho^3K_{n+1}\} + nJ_n[2n\lambda\rho K_n + (n - n\nu - 2\lambda^2\rho^2)K_{n+1}]}{32\pi\lambda^2\rho^3} + \frac{(-1)^n\{I_{n+1}\{n[n(-1+\nu) + 2\lambda^2\rho^2]Y_n - 2\lambda^3\rho^3Y_{n+1}\} + nI_n\{2n\lambda\rho Y_n + [n(-1+\nu) - 2\lambda^2\rho^2]Y_{n+1}\}\}}{32\pi\lambda^2\rho^3}$
m, v **formulation**	$-\frac{1}{32\pi\lambda^2\rho^4}\{J_{n+1}\{[n^4(-1+\nu)^2 + 2n\lambda^2(-1+\nu)\rho^2 + \lambda^4\rho^4 - n^2(1-2\nu+\nu^2 - 2\lambda^2\rho^2 + 2\lambda^2\nu\rho^2]K_n - 2\lambda^3(-1+\nu)\rho^3K_{n+1}\}$ $- J_n\{-4(-1+n)n^2\lambda(-1+\nu)\rho K_n + \{n^4(-1+\nu)^2 - 2n\lambda^2(-1+\nu)\rho^2 + \lambda^4\rho^4 - n^2[1+\nu^2 + 2\lambda^2\rho^2 - 2\nu(1+\lambda^2\rho^2)]\}K_{n+1}\}$ $+ (-1)^n\{I_{n+1}\{\{n^4(-1+\nu)^2 - 2n\lambda^2(-1+\nu)\rho^2 + \lambda^4\rho^4 - n^2[1+\nu^2 + 2\lambda^2\rho^2 - 2\nu(1+\lambda^2\rho^2)]\}Y_n + 2\lambda^3(-1+\nu)\rho^3Y_{n+1}\}$ $+ I_n\{4(-1+n)n^2\lambda(-1+\nu)\rho Y_n + \{n^4(-1+\nu)^2 + 2n\lambda^2(-1+\nu)\rho^2 + \lambda^4\rho^4 - n^2[1+\nu^2 - 2\lambda^2\rho^2 + 2\nu(-1+\lambda^2\rho^2)]\}Y_{n+1}\}\}\}$

Table 4-1 True eigenequations for the annular plate.

Cases	**True eigenequation** $[T_n]$
C-C	$\begin{bmatrix} J_n(\lambda a) & J_n(\lambda b) & J_n'(\lambda a) & J_n'(\lambda b) \\ Y_n(\lambda a) & Y_n(\lambda b) & Y_n'(\lambda a) & Y_n'(\lambda b) \\ I_n(\lambda a) & I_n(\lambda b) & I_n'(\lambda a) & I_n'(\lambda b) \\ K_n(\lambda a) & K_n(\lambda b) & K_n'(\lambda a) & K_n'(\lambda b) \end{bmatrix}$
C-S	$\begin{bmatrix} J_n(\lambda a) & J_n(\lambda b) & J_n'(\lambda a) & \alpha_n^J(\lambda b) \\ Y_n(\lambda a) & Y_n(\lambda b) & Y_n'(\lambda a) & \alpha_n^Y(\lambda b) \\ I_n(\lambda a) & I_n(\lambda b) & I_n'(\lambda a) & \alpha_n^I(\lambda b) \\ K_n(\lambda a) & K_n(\lambda b) & K_n'(\lambda a) & \alpha_n^K(\lambda b) \end{bmatrix}$
C-F	$\begin{bmatrix} J_n(\lambda a) & \alpha_n^J(\lambda b) & J_n'(\lambda a) & \beta_n^J(\lambda b)+\frac{1-\nu}{b}\gamma_n^J(\lambda b) \\ Y_n(\lambda a) & \alpha_n^Y(\lambda b) & Y_n'(\lambda a) & \beta_n^Y(\lambda b)+\frac{1-\nu}{b}\gamma_n^Y(\lambda b) \\ I_n(\lambda a) & \alpha_n^I(\lambda b) & I_n'(\lambda a) & \beta_n^I(\lambda b)+\frac{1-\nu}{b}\gamma_n^I(\lambda b) \\ K_n(\lambda a) & \alpha_n^K(\lambda b) & K_n'(\lambda a) & \beta_n^K(\lambda b)+\frac{1-\nu}{b}\gamma_n^K(\lambda b) \end{bmatrix}$
S-C	$\begin{bmatrix} J_n(\lambda a) & J_n(\lambda b) & \alpha_n^J(\lambda a) & J_n'(\lambda b) \\ Y_n(\lambda a) & Y_n(\lambda b) & \alpha_n^Y(\lambda a) & Y_n'(\lambda b) \\ I_n(\lambda a) & I_n(\lambda b) & \alpha_n^I(\lambda a) & I_n'(\lambda b) \\ K_n(\lambda a) & K_n(\lambda b) & \alpha_n^K(\lambda a) & K_n'(\lambda b) \end{bmatrix}$
S-S	$\begin{bmatrix} J_n(\lambda a) & J_n(\lambda b) & \alpha_n^J(\lambda a) & \alpha_n^J(\lambda b) \\ Y_n(\lambda a) & Y_n(\lambda b) & \alpha_n^Y(\lambda a) & \alpha_n^Y(\lambda b) \\ I_n(\lambda a) & I_n(\lambda b) & \alpha_n^I(\lambda a) & \alpha_n^I(\lambda b) \\ K_n(\lambda a) & K_n(\lambda b) & \alpha_n^K(\lambda a) & \alpha_n^K(\lambda b) \end{bmatrix}$
S-F	$\begin{bmatrix} J_n(\lambda a) & \alpha_n^J(\lambda b) & \alpha_n^J(\lambda a) & \beta_n^J(\lambda b)+\frac{1-\nu}{b}\gamma_n^J(\lambda b) \\ Y_n(\lambda a) & \alpha_n^Y(\lambda b) & \alpha_n^Y(\lambda a) & \beta_n^Y(\lambda b)+\frac{1-\nu}{b}\gamma_n^Y(\lambda b) \\ I_n(\lambda a) & \alpha_n^I(\lambda b) & \alpha_n^I(\lambda a) & \beta_n^I(\lambda b)+\frac{1-\nu}{b}\gamma_n^I(\lambda b) \\ K_n(\lambda a) & \alpha_n^K(\lambda b) & \alpha_n^K(\lambda a) & \beta_n^K(\lambda b)+\frac{1-\nu}{b}\gamma_n^K(\lambda b) \end{bmatrix}$

F-C	$\begin{bmatrix} \alpha_n^J(\lambda a) & J_n(\lambda b) & \beta_n^J(\lambda a)+\frac{1-\nu}{b}\gamma_n^J(\lambda a) & J_n'(\lambda b) \\ \alpha_n^Y(\lambda a) & Y_n(\lambda b) & \beta_n^Y(\lambda a)+\frac{1-\nu}{b}\gamma_n^Y(\lambda a) & Y_n'(\lambda b) \\ \alpha_n^I(\lambda a) & I_n(\lambda b) & \beta_n^I(\lambda a)+\frac{1-\nu}{b}\gamma_n^I(\lambda a) & I_n'(\lambda b) \\ \alpha_n^K(\lambda a) & K_n(\lambda b) & \beta_n^K(\lambda a)+\frac{1-\nu}{b}\gamma_n^K(\lambda a) & K_n'(\lambda b) \end{bmatrix}$
F-S	$\begin{bmatrix} \alpha_n^J(\lambda a) & J_n(\lambda b) & \beta_n^J(\lambda a)+\frac{1-\nu}{b}\gamma_n^J(\lambda a) & \alpha_n^J(\lambda b) \\ \alpha_n^Y(\lambda a) & Y_n(\lambda b) & \beta_n^Y(\lambda a)+\frac{1-\nu}{b}\gamma_n^Y(\lambda a) & \alpha_n^Y(\lambda b) \\ \alpha_n^I(\lambda a) & I_n(\lambda b) & \beta_n^I(\lambda a)+\frac{1-\nu}{b}\gamma_n^I(\lambda a) & \alpha_n^I(\lambda b) \\ \alpha_n^K(\lambda a) & K_n(\lambda b) & \beta_n^K(\lambda a)+\frac{1-\nu}{b}\gamma_n^K(\lambda a) & \alpha_n^K(\lambda b) \end{bmatrix}$
F-F	$\begin{bmatrix} \alpha_n^J(\lambda a) & \alpha_n^J(\lambda b) & \beta_n^J(\lambda a)+\frac{1-\nu}{b}\gamma_n^J(\lambda a) & \beta_n^J(\lambda b)+\frac{1-\nu}{b}\gamma_n^J(\lambda b) \\ \alpha_n^Y(\lambda a) & \alpha_n^Y(\lambda b) & \beta_n^Y(\lambda a)+\frac{1-\nu}{b}\gamma_n^Y(\lambda a) & \beta_n^Y(\lambda b)+\frac{1-\nu}{b}\gamma_n^Y(\lambda b) \\ \alpha_n^I(\lambda a) & \alpha_n^I(\lambda b) & \beta_n^I(\lambda a)+\frac{1-\nu}{b}\gamma_n^I(\lambda a) & \beta_n^I(\lambda b)+\frac{1-\nu}{b}\gamma_n^I(\lambda b) \\ \alpha_n^K(\lambda a) & \alpha_n^K(\lambda b) & \beta_n^K(\lambda a)+\frac{1-\nu}{b}\gamma_n^K(\lambda a) & \beta_n^K(\lambda b)+\frac{1-\nu}{b}\gamma_n^K(\lambda b) \end{bmatrix}$

Table 4-2(a) Spurious eigenequations for the annular plate.

formulation	$\left[S_n\right]$
u, θ **formulation**	$\begin{bmatrix} Y_n(\lambda a)+iJ_n(\lambda a) & 0 & K_n(\lambda a)+i(-1)^n I_n(\lambda a) & 0 \\ iJ_n(\lambda b) & J_n(\lambda b) & i(-1)^n I_n(\lambda a) & I_n(\lambda b) \\ \lambda(Y_n'(\lambda a)+iJ_n'(\lambda a)) & 0 & \lambda(K_n'(\lambda a)+i(-1)^n I_n'(\lambda a)) & 0 \\ i\lambda J_n'(\lambda b) & \lambda J_n'(\lambda b) & i\lambda I_n'(\lambda b) & \lambda(I_n'(\lambda b)) \end{bmatrix}$
u, m **formulation**	$\begin{bmatrix} Y_n(\lambda a)+iJ_n(\lambda a) & 0 & K_n(\lambda a)+i(-1)^n I_n(\lambda a) & 0 \\ iJ_n(\lambda b) & J_n(\lambda b) & i(-1)^n I_n(\lambda a) & I_n(\lambda b) \\ \alpha_n^Y(\lambda a)+i\alpha_n^J(\lambda a)) & 0 & \alpha_n^K(\lambda a)+i(-1)^n\alpha_n^I(\lambda a) & 0 \\ i\alpha_n^J(\lambda b) & \alpha_n^J(\lambda b) & i\alpha_n^I(\lambda b) & \alpha_n^I(\lambda b) \end{bmatrix}$
u, v **formulation**	$\begin{bmatrix} Y_n(\lambda a)+iJ_n(\lambda a) & 0 & K_n(\lambda a)+i(-1)^n I_n(\lambda a) & 0 \\ iJ_n(\lambda b) & iJ_n(\lambda b) & i(-1)^n I_n(\lambda a) & I_n(\lambda b) \\ [\beta_n^Y(\lambda a)+\frac{(1-\nu)}{a}\gamma_n^Y(\lambda a)]+i[\beta_n^J(\lambda a)+\frac{(1-\nu)}{a}\gamma_n^J(\lambda a)] & 0 & \lambda[\beta_n^K(\lambda a)+\frac{(1-\nu)}{a}\gamma_n^K(\lambda a)]+i(-1)^n[\beta_n^I(\lambda a)+\frac{(1-\nu)}{a}\gamma_n^I(\lambda a)] & 0 \\ i[\beta_n^J(\lambda b)+\frac{(1-\nu)}{b}\gamma_n^J(\lambda b)] & [\beta_n^J(\lambda b)+\frac{(1-\nu)}{b}\gamma_n^J(\lambda b)] & i[\beta_n^I(\lambda b)+\frac{(1-\nu)}{b}\gamma_n^I(\lambda b)] & \beta_n^I(\lambda b)+\frac{(1-\nu)}{b}\gamma_n^I(\lambda b)] \end{bmatrix}$
θ, m **formulation**	$\begin{bmatrix} \lambda(Y_n'(\lambda a)+iJ_n'(\lambda a)) & 0 & \lambda(K_n'(\lambda a)+i(-1)^n I_n'(\lambda a)) & 0 \\ i\lambda J_n'(\lambda b) & \lambda J_n'(\lambda b) & i(-1)^n\lambda I_n'(\lambda a) & \lambda I_n'(\lambda b) \\ \alpha_n^Y(\lambda a)+i\alpha_n^J(\lambda a)) & 0 & \alpha_n^K(\lambda a)+i(-1)^n\alpha_n^I(\lambda a) & 0 \\ i\alpha_n^J(\lambda b) & \alpha_n^J(\lambda b) & i\alpha_n^I(\lambda b) & \alpha_n^I(\lambda b) \end{bmatrix}$

θ, ν **formulation**	$\begin{bmatrix} \lambda(Y_n'(\lambda a)+iJ_n'(\lambda a)) & 0 & \lambda(K_n'(\lambda a)+i(-1)^n I_n'(\lambda a)) & 0 \\ i\lambda J_n'(\lambda b) & \lambda J_n'(\lambda b) & i(-1)^n \lambda I_n'(\lambda a) & \lambda I_n'(\lambda b) \\ [\beta_n^Y(\lambda a)+\frac{(1-\nu)}{a}\gamma_n^Y(\lambda a)]+i[\beta_n^J(\lambda a)+\frac{(1-\nu)}{a}\gamma_n^J(\lambda a)] & 0 & \lambda[\beta_n^K(\lambda a)+\frac{(1-\nu)}{a}\gamma_n^K(\lambda a)]+i(-1)^n[\beta_n^I(\lambda a)+\frac{(1-\nu)}{a}\gamma_n^I(\lambda a)] & 0 \\ i[\beta_n^J(\lambda b)+\frac{(1-\nu)}{b}\gamma_n^J(\lambda b)] & [\beta_n^J(\lambda b)+\frac{(1-\nu)}{b}\gamma_n^J(\lambda b)] & i[\beta_n^I(\lambda b)+\frac{(1-\nu)}{b}\gamma_n^I(\lambda b)] & \beta_n^I(\lambda b)+\frac{(1-\nu)}{b}\gamma_n^I(\lambda b)] \end{bmatrix}$
m, ν **formulation**	$\begin{bmatrix} \alpha_n^Y(\lambda a)+i\alpha_n^J(\lambda a) & 0 & \alpha_n^K(\lambda a)+i(-1)^n \alpha_n^I(\lambda a) & 0 \\ i\alpha_n^J(\lambda b) & \alpha_n^J(\lambda b) & i(-1)^n \alpha_n^I(\lambda a) & \alpha_n^I(\lambda b) \\ [\beta_n^Y(\lambda a)+\frac{(1-\nu)}{a}\gamma_n^Y(\lambda a)]+i[\beta_n^J(\lambda a)+\frac{(1-\nu)}{a}\gamma_n^J(\lambda a)] & 0 & \lambda[\beta_n^K(\lambda a)+\frac{(1-\nu)}{a}\gamma_n^K(\lambda a)]+i(-1)^n[\beta_n^I(\lambda a)+\frac{(1-\nu)}{a}\gamma_n^I(\lambda a)] & 0 \\ i[\beta_n^J(\lambda b)+\frac{(1-\nu)}{b}\gamma_n^J(\lambda b)] & [\beta_n^J(\lambda b)+\frac{(1-\nu)}{b}\gamma_n^J(\lambda b)] & i[\beta_n^I(\lambda b)+\frac{(1-\nu)}{b}\gamma_n^I(\lambda b)] & \beta_n^I(\lambda b)+\frac{(1-\nu)}{b}\gamma_n^I(\lambda b)] \end{bmatrix}$

Table 4-2(b) Spurious eigenequations for the annular plate.

	$[Sa_n]$
u,θ **formulation**	$\begin{bmatrix} Y_n(\lambda a)+iJ_n(\lambda a) & K_n(\lambda a)+i(-1)^n I_n(\lambda a) \\ \lambda(Y_n'(\lambda a)+iJ_n'(\lambda a)) & \lambda(K_n'(\lambda a)+i(-1)^n I_n'(\lambda a)) \end{bmatrix}$
u,m **formulation**	$\begin{bmatrix} Y_n(\lambda a)+iJ_n(\lambda a) & K_n(\lambda a)+i(-1)^n I_n(\lambda a) \\ \alpha_n^Y(\lambda a)+i\alpha_n^J(\lambda a)) & \alpha_n^K(\lambda a)+i(-1)^n \alpha_n^I(\lambda a) \end{bmatrix}$
u,v **formulation**	$\begin{bmatrix} Y_n(\lambda a)+iJ_n(\lambda a) & K_n(\lambda a)+i(-1)^n I_n(\lambda a) \\ [\beta_n^Y(\lambda a)+\frac{(1-\nu)}{a}\gamma_n^Y(\lambda a)]+i[\beta_n^J(\lambda a)+\frac{(1-\nu)}{a}\gamma_n^J(\lambda a)] & [\beta_n^K(\lambda a)+\frac{(1-\nu)}{a}\gamma_n^K(\lambda a)]+i(-1)^n[\beta_n^I(\lambda a)+\frac{(1-\nu)}{a}\gamma_n^I(\lambda a)] \end{bmatrix}$
θ,m **formulation**	$\begin{bmatrix} \lambda(Y_n'(\lambda a)+iJ_n'(\lambda a)) & \lambda(K_n'(\lambda a)+i(-1)^n I_n'(\lambda a)) \\ \alpha_n^Y(\lambda a)+i\alpha_n^J(\lambda a)) & \alpha_n^K(\lambda a)+i(-1)^n \alpha_n^I(\lambda a) \end{bmatrix}$
θ,v **formulation**	$\begin{bmatrix} \lambda(Y_n'(\lambda a)+iJ_n'(\lambda a)) & \lambda(K_n'(\lambda a)+i(-1)^n I_n'(\lambda a)) \\ \beta_n^Y(\lambda a)+\frac{(1-\nu)}{a}\gamma_n^Y(\lambda a)]+i[\beta_n^J(\lambda a)+\frac{(1-\nu)}{a}\gamma_n^J(\lambda a)] & \lambda[\beta_n^K(\lambda a)+\frac{(1-\nu)}{a}\gamma_n^K(\lambda a)]+i(-1)^n[\beta_n^I(\lambda a)+\frac{(1-\nu)}{a}\gamma_n^I(\lambda a)] \end{bmatrix}$
m,v **formulation**	$\begin{bmatrix} \alpha_n^Y(\lambda a)+i\alpha_n^J(\lambda a) & \alpha_n^K(\lambda a)+i(-1)^n \alpha_n^I(\lambda a) \\ [\beta_n^Y(\lambda a)+\frac{(1-\nu)}{a}\gamma_n^Y(\lambda a)]+i[\beta_n^J(\lambda a)+\frac{(1-\nu)}{a}\gamma_n^J(\lambda a)] & [\beta_n^K(\lambda a)+\frac{(1-\nu)}{a}\gamma_n^K(\lambda a)]+i(-1)^n[\beta_n^I(\lambda a)+\frac{(1-\nu)}{a}\gamma_n^I(\lambda a) \end{bmatrix}$

Table 4-2(c) Spurious eigenequations for the annular plate.

formulation	$[Sb_n]$	**B.C. of the simply-connected plate**
u,θ **formulation**	$\begin{bmatrix} J_n(\lambda b) & I_n(\lambda b) \\ \lambda(J_n'(\lambda b)) & \lambda(I_n'(\lambda b)) \end{bmatrix}$	$u=0,\quad \theta=0$
u,m **formulation**	$\begin{bmatrix} J_n(\lambda b) & I_n(\lambda b) \\ \alpha_n^J(\lambda b) & \alpha_n^I(\lambda b) \end{bmatrix}$	$u=0,\quad m=0$
u,v **formulation**	$\begin{bmatrix} J_n(\lambda b) & I_n(\lambda b) \\ [\beta_n^J(\lambda b)+\frac{(1-\nu)}{b}\gamma_n^J(\lambda b)] & [\beta_n^I(\lambda b)+\frac{(1-\nu)}{b}\gamma_n^I(\lambda b)] \end{bmatrix}$	$u=0,\quad v=0$
θ,m **formulation**	$\begin{bmatrix} \lambda J_n'(\lambda b) & \lambda I_n'(\lambda b) \\ \alpha_n^J(\lambda b) & \alpha_n^I(\lambda b) \end{bmatrix}$	$\theta=0,\quad m=0$
θ,v **formulation**	$\begin{bmatrix} \lambda J_n'(\lambda b) & \lambda I_n'(\lambda b) \\ [\beta_n^J(\lambda b)+\frac{(1-\nu)}{b}\gamma_n^J(\lambda b)] & [\beta_n^I(\lambda b)+\frac{(1-\nu)}{b}\gamma_n^I(\lambda b)] \end{bmatrix}$	$\theta=0,\quad v=0$
m,v **formulation**	$\begin{bmatrix} \alpha_n^J(\lambda b) & \alpha_n^I(\lambda b) \\ [\beta_n^J(\lambda b)+\frac{(1-\nu)}{b}\gamma_n^J(\lambda b)] & [\beta_n^I(\lambda b)+\frac{(1-\nu)}{b}\gamma_n^I(\lambda b) \end{bmatrix}$	$m=0,\quad v=0$

Table 4-3(a) True eigenvalues (λ) for the C-C case ($a = 1, \quad b = 0.5$).

	λ for values of n of					
m	0	1	2	3	4	5
0	9.447	9.499	9.660	9.945	10.370	10.940
1	15.694	15.739	15.873	16.098	16.415	16.827

Table 4-3(b) True eigenvalues (λ) for the S-S case ($a = 1, \quad b = 0.5, \quad \nu = 1/3$).

	λ for values of n of					
m	0	1	2	3	4	5
0	6.325	6.463	6.861	7.480	8.269	9.180
1	12.592	12.669	12.895	13.265	13.767	14.391

Table 4-3(c) True eigenvalues (λ) for the F-F case ($a = 1, \quad b = 0.5, \quad \nu = 1/3$).

	λ for values of n of					
m	0	1	2	3	4	5
0	3.037	4.115	2.050	3.355	4.557	5.704
1	9.603	9.800	5.541	6.854	8.139	9.429

Table 4-3(d) True eigenvalues (λ) for the C-S case ($a = 1, \quad b = 0.5, \quad \nu = 1/3$).

	λ for values of n of					
m	0	1	2	3	4	5
0	7.99	8.085	8.369	8.837	9.476	10.256
1	14.210	14.272	14.459	14.768	15.195	15.733

Table 4-3(e) True eigenvalues (λ) for the C-F case ($a = 1, \quad b = 0.5, \quad \nu = 1/3$).

	λ for values of n of					
m	0	1	2	3	4	5
0	4.194	4.666	5.637	6.744	7.923	9.147
1	9.675	9.850	10.339	11.054	11.915	12.87

Table 4-3(f) True eigenvalues (λ) for the S-C case ($a = 1, \quad b = 0.5, \quad \nu = 1/3$).

	λ for values of n of					
m	0	1	2	3	4	5
0	7.739	7.814	8.044	8.436	8.992	9.698
1	14.075	14.127	14.286	14.552	14.924	15.401

Table 4-3(g) The true eigenvalues (λ) for the S-F case ($a = 1, \quad b = 0.5, \quad \nu = 1/3$).

	λ for values of n of					
m	0	1	2	3	4	5
0	2.245	3.385	4.709	5.959	7.209	8.464
1	8.104	8.344	8.977	9.843	10.837	11.914

Table 4-3(h) True eigenvalues (λ) for the F-C case ($a = 1, \quad b = 0.5, \quad \nu = 1/3$).

	λ for values of n of					
m	0	1	2	3	4	5
0	3.617	3.648	3.821	4.278	5.020	5.932
1	9.229	9.318	9.579	10.003	10.573	11.272

Table 4-3(i) True eigenvalues (λ) for the F-S case ($a = 1, \quad b = 0.5, \quad \nu = 1/3$).

	λ for values of n of					
m	0	1	2	3	4	5
0	2.016	2.183	2.793	3.708	4.732	5.791
1	7.813	7.952	8.348	8.955	9.722	10.605

where n refers to the number of nodal diameters and m is the number of nodal circles, not including the boundary circle.

Table 4-4(a) True eigenvalues (λ) for the C-C case ($a = 1$).

	Radius b of the inner boundary				
n	**0.1**	**0.2**	**0.3**	**0.4**	**0.5**
0	5.223	5.883	6.734	7.866	9.447
1	5.377	6.009	6.830	7.937	9.499
2	6.051	6.467	7.151	8.165	9.660
3	7.157	7.307	7.748	8.581	9.945
4	8.347	8.382	8.016	9.198	10.370
5	9.526	9.532	9.617	9.994	10.940

Table 4-4(b) True eigenvalues (λ) for the S-S case ($a = 1, \quad \nu = 1/3$).

	Radius b of the inner boundary				
n	**0.1**	**0.2**	**0.3**	**0.4**	**0.5**
0	3.799	4.089	4.585	5.300	6.325
1	4.093	4.380	4.825	5.484	6.463
2	5.096	5.220	5.502	6.012	6.861
3	6.326	6.351	6.475	6.806	7.480
4	7.542	7.546	7.588	7.773	8.269
5	8.732	8.732	8.744	8.837	9.180

Table 4-4(c) True eigenvalues (λ) for the F-F case ($a=1, \quad \nu=1/3$).

	Radius b **of the inner boundary**				
n	**0.1**	**0.2**	**0.3**	**0.4**	**0.5**
0	2.969	2.90	2.883	2.923	3.037
1	4.520	4.435	4.262	4.127	4.115
2	2.279	2.246	2.195	2.128	2.050
3	3.495	3.490	3.472	3.429	3.355
4	4.635	4.634	4.630	4.610	4.557
5	5.746	5.746	5.745	5.737	5.704

Table 4-4(d) True eigenvalues (λ) for the C-S case ($a=1, \quad \nu=1/3$).

	Radius b **of the inner boundary**				
n	**0.1**	**0.2**	**0.3**	**0.4**	**0.5**
0	4.751	5.159	5.801	6.703	7.991
1	5.021	5.400	5.984	6.836	8.085
2	5.949	6.130	6.532	7.236	8.369
3	7.146	7.189	7.388	7.884	8.837
4	8.347	8.354	8.429	8.731	9.476
5	9.526	9.527	9.550	9.713	10.256

Table 4-4(e) True eigenvalues (λ) for the C-F case ($a=1, \quad \nu=1/3$).

	Radius b **of the inner boundary**				
n	**0.1**	**0.2**	**0.3**	**0.4**	**0.5**
0	3.183	3.216	3.366	3.673	4.194
1	4.602	4.526	4.403	4.402	4.666
2	5.878	5.811	5.706	5.598	5.637
3	7.141	7.109	7.008	6.011	6.744
4	8.346	8.339	8.284	8.123	7.923
5	9.526	9.524	9.503	9.389	9.147

Table 4-4(f) True eigenvalues (λ) for the S-C case ($a=1, \quad \nu=1/3$).

	Radius b **of the inner boundary**				
n	**0.1**	**0.2**	**0.3**	**0.4**	**0.5**
0	4.222	4.771	5.481	6.423	7.739
1	4.408	4.931	5.609	6.523	7.814
2	5.173	5.489	6.025	6.834	8.044
3	6.332	6.432	6.746	7.371	8.436
4	7.542	7.562	7.702	8.119	8.992
5	8.732	8.735	8.785	9.029	9.698

Table 4-4(g) True eigenvalues (λ) for the S-F case ($a = 1$, $\nu = 1/3$).

	Radius b of the inner boundary				
n	**0.1**	**0.2**	**0.3**	**0.4**	**0.5**
0	2.210	2.174	2.158	2.177	2.245
1	3.729	3.685	3.573	3.450	3.385
2	5.044	4.994	4.914	4.805	4.709
3	6.323	6.302	6.233	6.105	5.959
4	7.542	7.537	7.504	7.395	7.209
5	8.732	8.731	8.719	8.650	8.464

Table 4-4(h) True eigenvalues (λ) for the F-C case ($a = 1$, $\nu = 1/3$).

	Radius b of the inner boundary				
n	**0.1**	**0.2**	**0.3**	**0.4**	**0.5**
0	2.064	2.282	2.588	3.011	3.617
1	1.864	2.194	2.560	3.020	3.648
2	2.348	2.518	2.404	2.404	3.821
3	3.497	3.520	3.611	3.836	4.278
4	4.635	4.637	4.660	4.754	5.020
5	5.746	5.746	5.751	5.787	5.932

Table 4-4(i) True eigenvalues (λ) for the F-S case ($a = 1$, $\nu = 1/3$).

	Radius b of the inner boundary				
n	**0.1**	**0.2**	**0.3**	**0.4**	**0.5**
0	1.854	1.819	1.839	1.904	2.016
1	1.551	1.685	1.820	1.979	2.183
2	2.305	2.352	2.439	2.579	2.404
3	3.495	3.499	3.518	3.576	3.708
4	4.635	4.635	4.639	4.660	4.732
5	5.746	5.746	5.746	5.753	5.791

where n refers to the number of nodal diameters.

Table 5-1 The terms of $\left[Sb_n^1\right]+i\left[Sb_n^2\right]$ for the annular plate by using the complex-valued BEM in conjunction the Burton & Miller method

	$\left[Sb_n^1\right]+i\left[Sb_n^2\right]$
u,θ **formulation**	$\begin{bmatrix} J_n(\lambda b) & I_n(\lambda b) \\ \lambda(J_n'(\lambda b)) & \lambda(I_n'(\lambda b)) \end{bmatrix} + i \begin{bmatrix} \alpha_n^J(\lambda b) & \alpha_n^I(\lambda b) \\ [\beta_n^J(\lambda b)+\frac{(1-\nu)}{b}\gamma_n^J(\lambda b)] & [\beta_n^I(\lambda b)+\frac{(1-\nu)}{b}\gamma_n^I(\lambda b)] \end{bmatrix}$
u,m **formulation**	$\begin{bmatrix} J_n(\lambda b) & I_n(\lambda b) \\ \alpha_n^J(\lambda b) & \alpha_n^I(\lambda b) \end{bmatrix} + i \begin{bmatrix} \lambda J_n'(\lambda b) & \lambda I_n'(\lambda b) \\ [\beta_n^J(\lambda b)+\frac{(1-\nu)}{b}\gamma_n^J(\lambda b)] & [\beta_n^I(\lambda b)+\frac{(1-\nu)}{b}\gamma_n^I(\lambda b)] \end{bmatrix}$
u,v **formulation**	$\begin{bmatrix} J_n(\lambda b) & I_n(\lambda b) \\ [\beta_n^J(\lambda b)+\frac{(1-\nu)}{b}\gamma_n^J(\lambda b)] & [\beta_n^I(\lambda b)+\frac{(1-\nu)}{b}\gamma_n^I(\lambda b)] \end{bmatrix} + i \begin{bmatrix} \lambda J_n'(\lambda b) & \lambda I_n'(\lambda b) \\ \alpha_n^J(\lambda b) & \alpha_n^I(\lambda b) \end{bmatrix}$
θ,m **formulation**	$\begin{bmatrix} \lambda J_n'(\lambda b) & \lambda I_n'(\lambda b) \\ \alpha_n^J(\lambda b) & \alpha_n^I(\lambda b) \end{bmatrix} + i \begin{bmatrix} J_n(\lambda b) & I_n(\lambda b) \\ [\beta_n^J(\lambda b)+\frac{(1-\nu)}{b}\gamma_n^J(\lambda b)] & [\beta_n^I(\lambda b)+\frac{(1-\nu)}{b}\gamma_n^I(\lambda b)] \end{bmatrix}$
θ,v **formulation**	$\begin{bmatrix} \lambda J_n'(\lambda b) & \lambda I_n'(\lambda b) \\ [\beta_n^J(\lambda b)+\frac{(1-\nu)}{b}\gamma_n^J(\lambda b)] & [\beta_n^I(\lambda b)+\frac{(1-\nu)}{b}\gamma_n^I(\lambda b)] \end{bmatrix} + i \begin{bmatrix} J_n(\lambda b) & I_n(\lambda b) \\ \alpha_n^J(\lambda b) & \alpha_n^I(\lambda b) \end{bmatrix}$
m,v **formulation**	$\begin{bmatrix} \alpha_n^J(\lambda b) & \alpha_n^I(\lambda b) \\ [\beta_n^J(\lambda b)+\frac{(1-\nu)}{b}\gamma_n^J(\lambda b)] & [\beta_n^I(\lambda b)+\frac{(1-\nu)}{b}\gamma_n^I(\lambda b)] \end{bmatrix} + i \begin{bmatrix} J_n(\lambda b) & I_n(\lambda b) \\ \lambda(J_n'(\lambda b)) & \lambda(I_n'(\lambda b)) \end{bmatrix}$

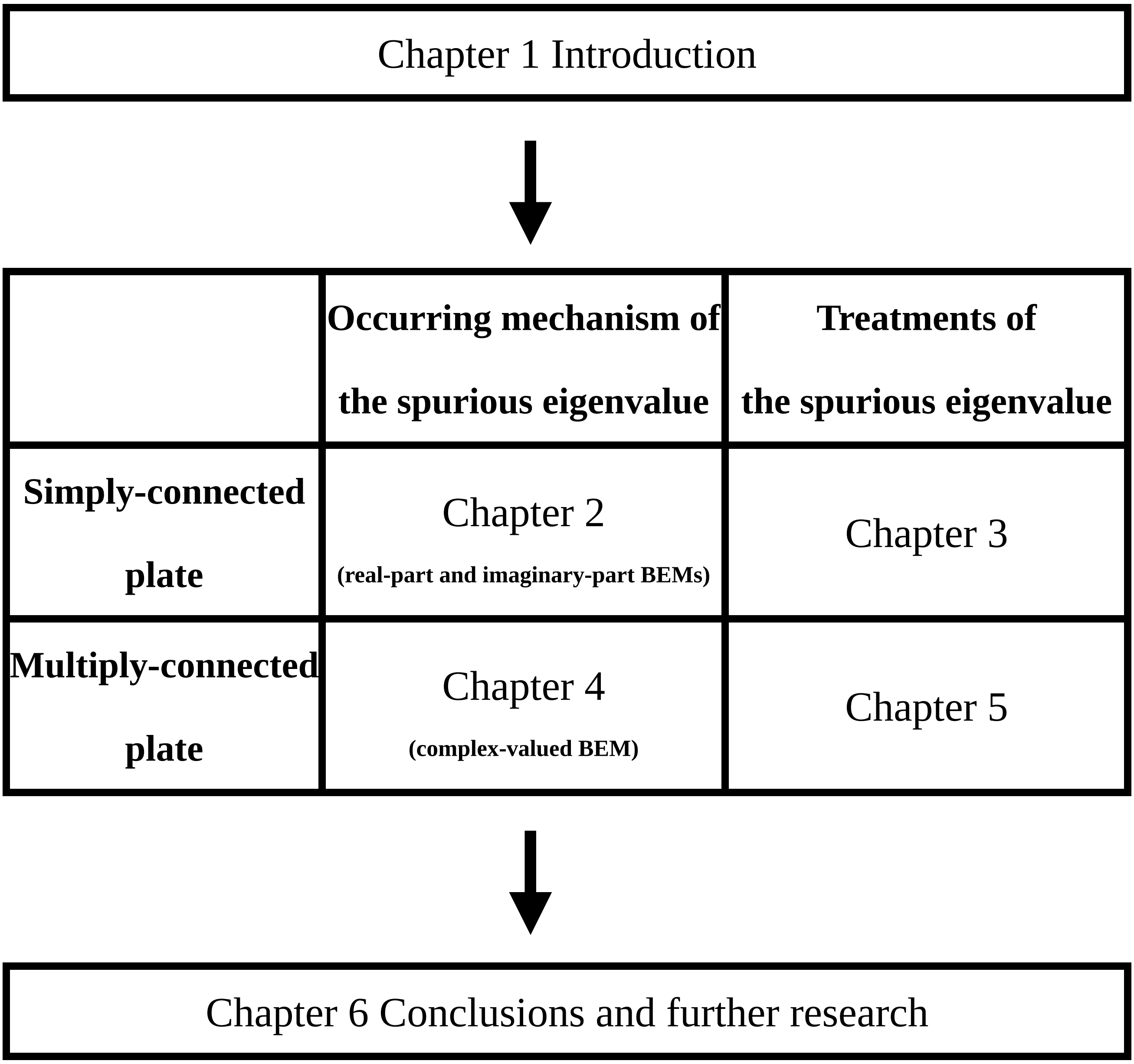


Figure 1-1 The frame of the thesis

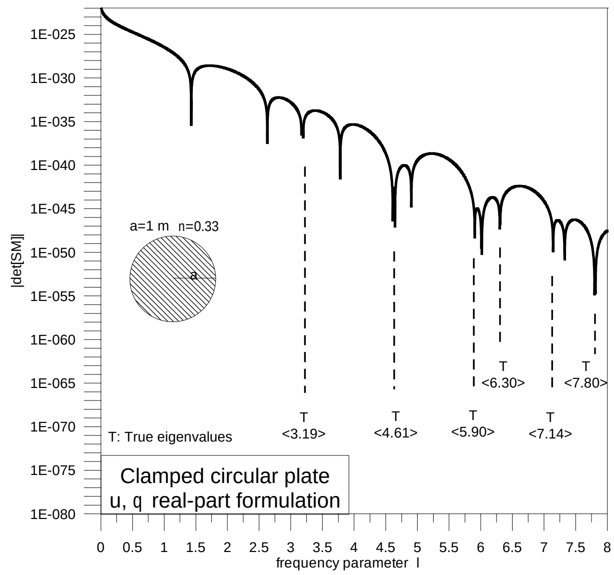


Figure 2-1.(a)

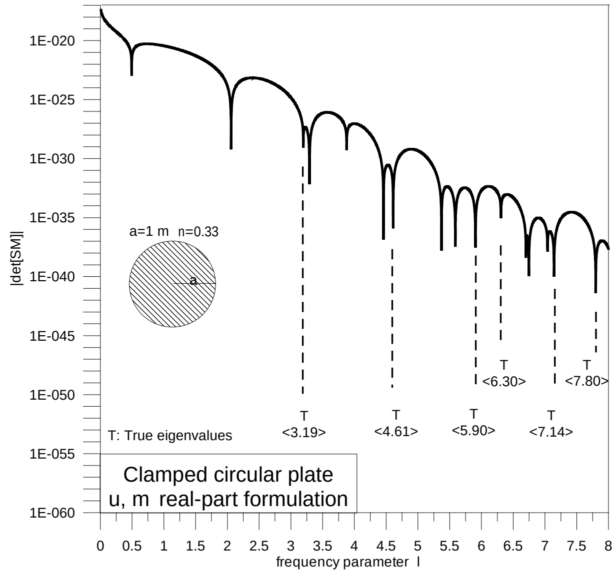


Figure 2-1.(b)

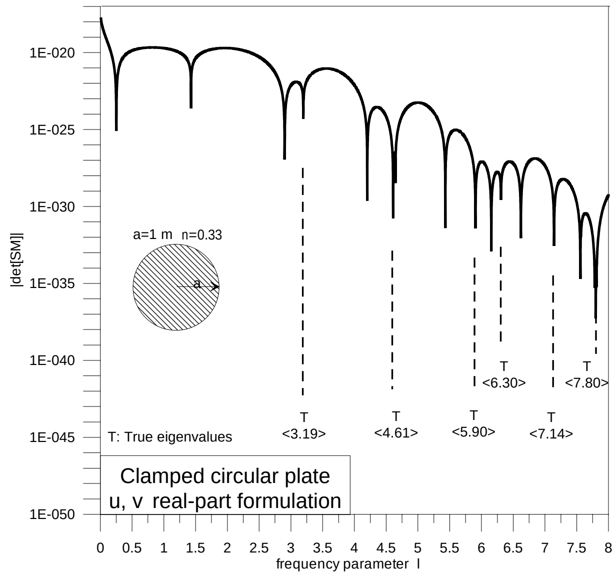


Figure 2-1.(c)

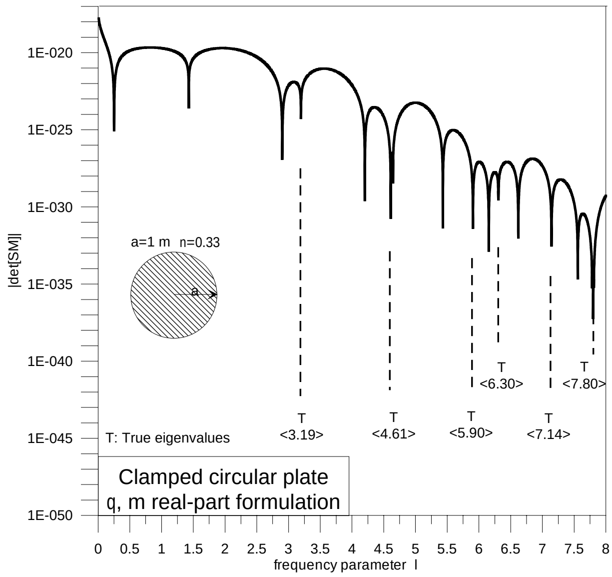


Figure 2-1.(d)

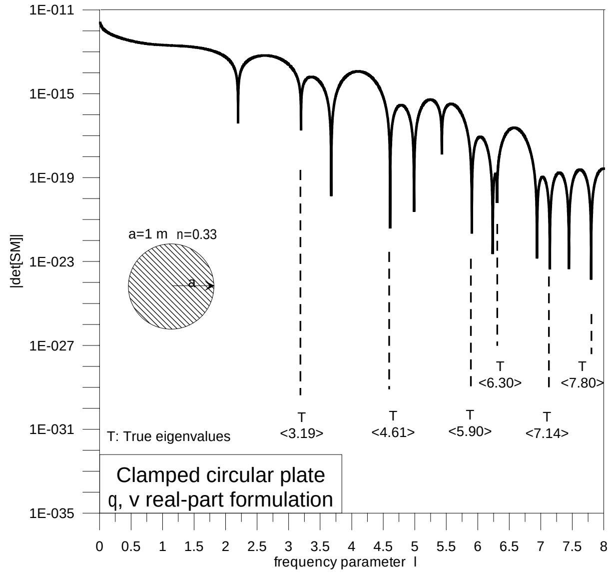


Figure 2-1.(e)

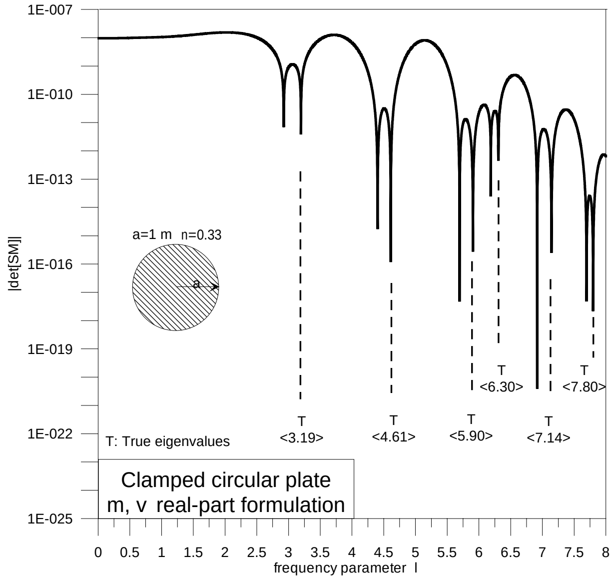


Figure 2-1.(f)

Figure 2-1 The determinant of $[SM^c]$ versus frequency parameter λ for the clamped circular plate using the six real-part formulations.

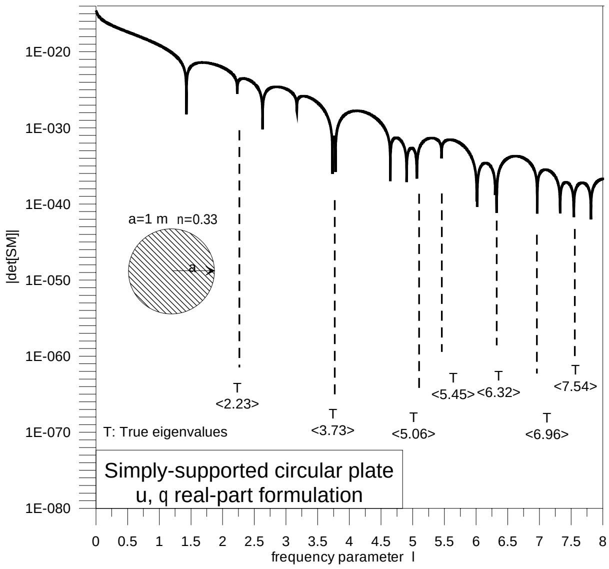


Figure 2-2.(a)

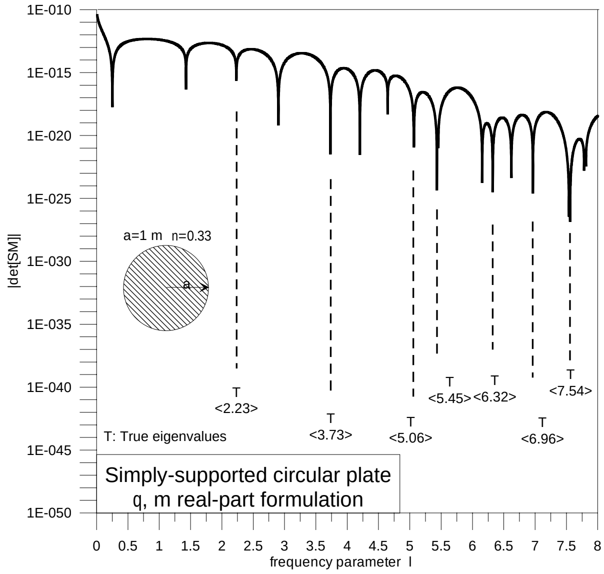


Figure 2-2.(a)

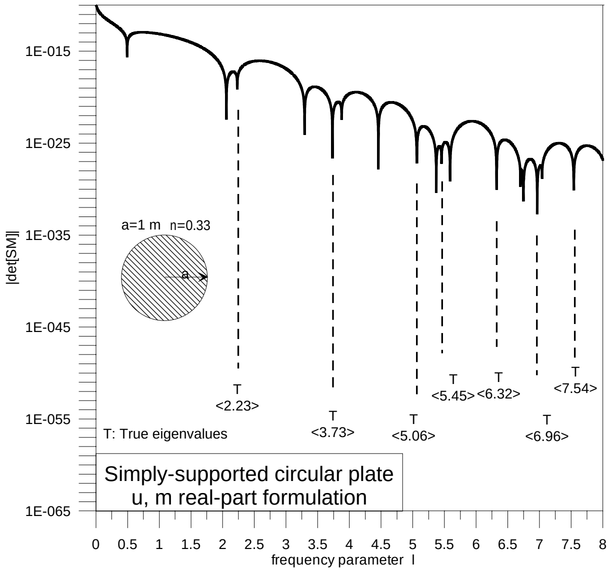


Figure 2-2.(a)

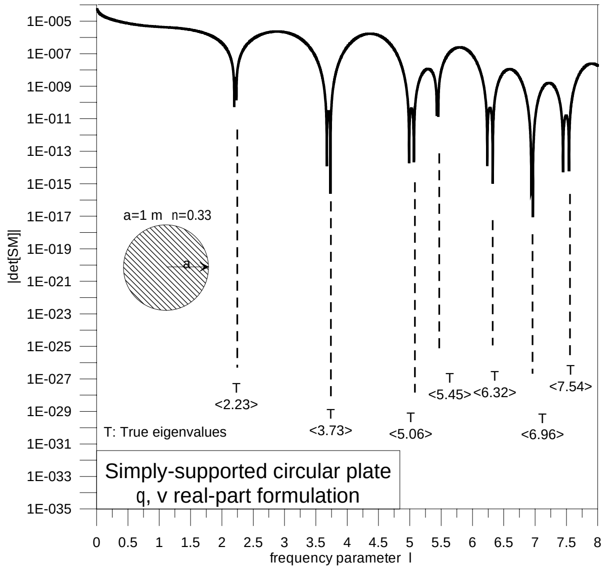


Figure 2-2.(a)

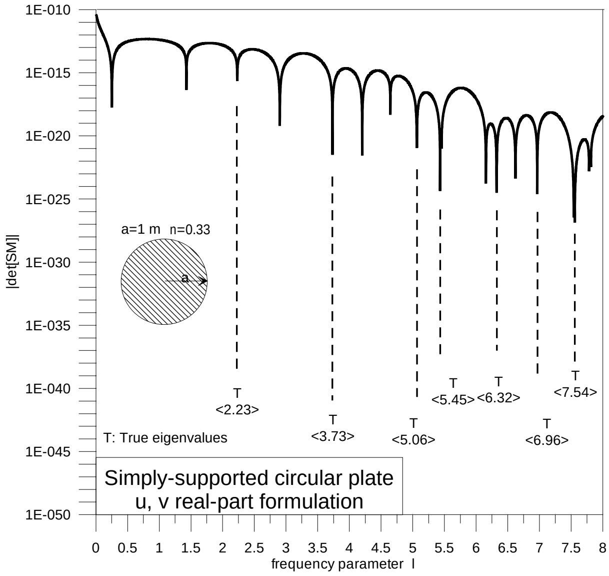


Figure 2-2.(a)

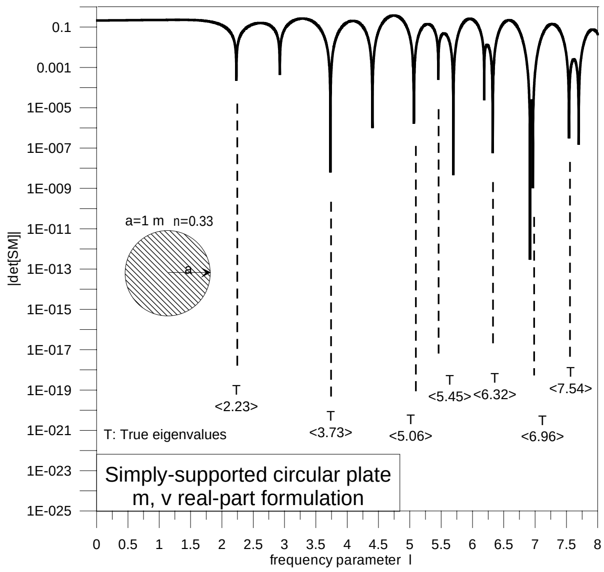


Figure 2-2.(a)

Figure 2-2 The determinant of $[SM^s]$ versus frequency parameter λ for the simply-supported circular plate using the six real-part formulations.

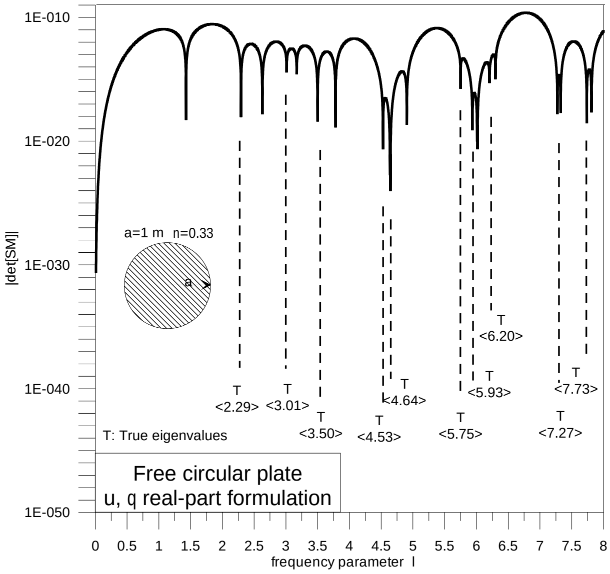


Figure 2-3.(a)

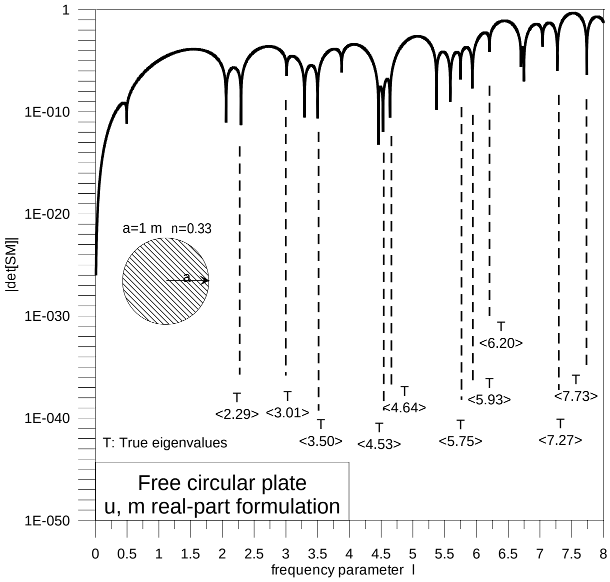


Figure 2-3.(b)

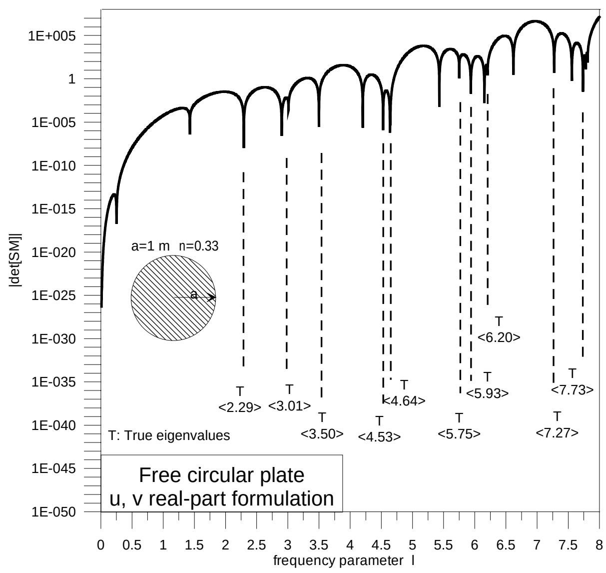
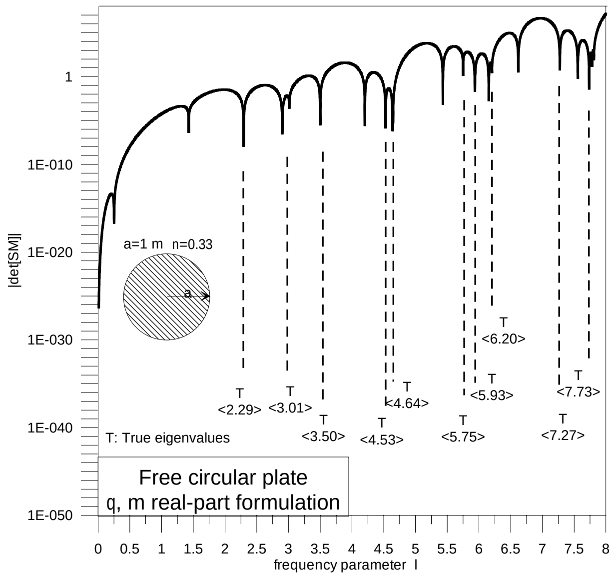


Figure 2-3.(c)



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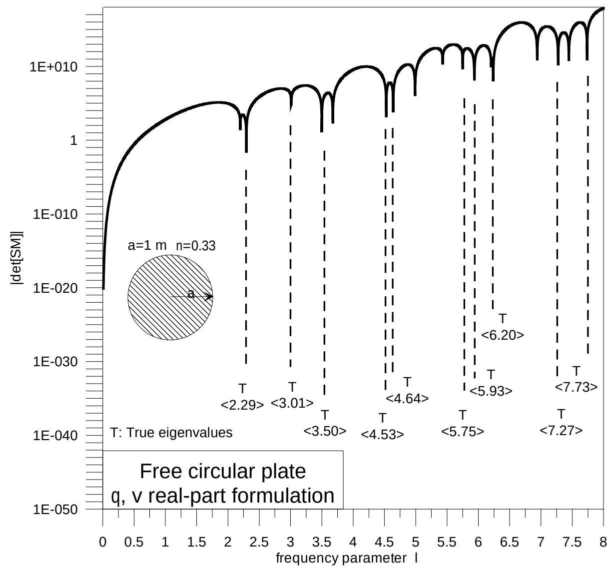

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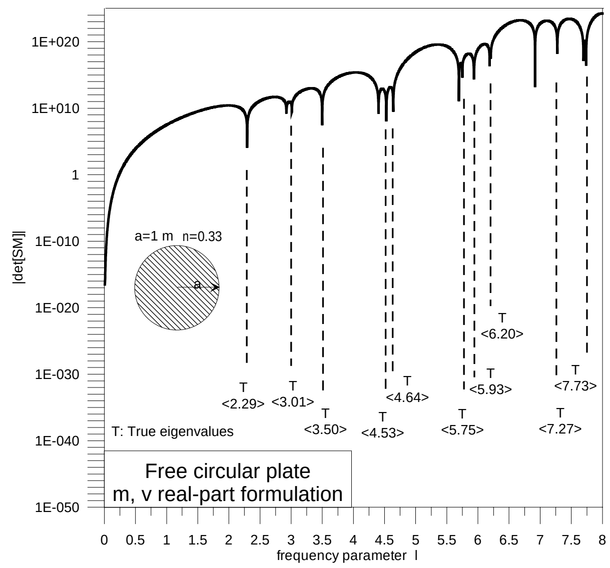


Figure 2-3.(f)

Figure 2-3 The determinant of $[SM^f]$ versus frequency parameter λ for the free circular plate using the six real-part formulations.

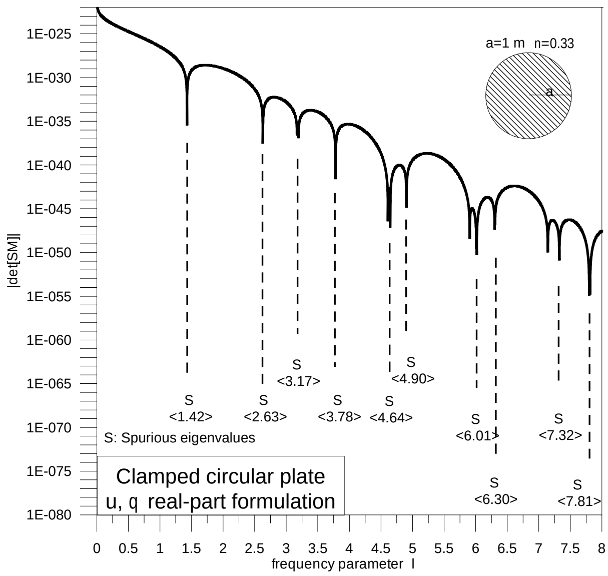
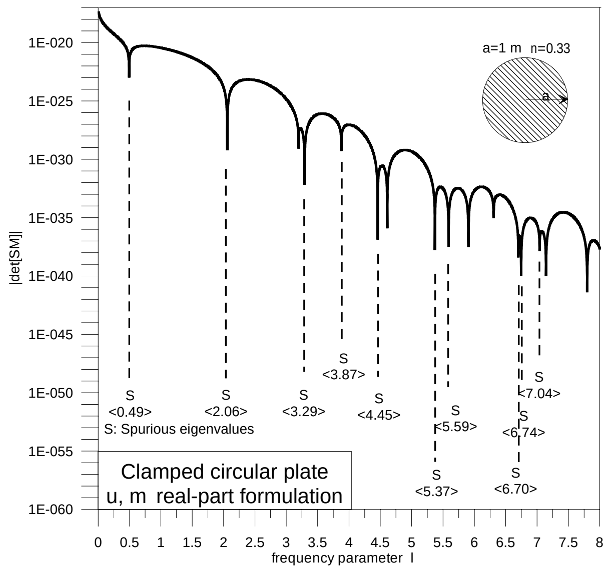


Figure 2-4.(a)



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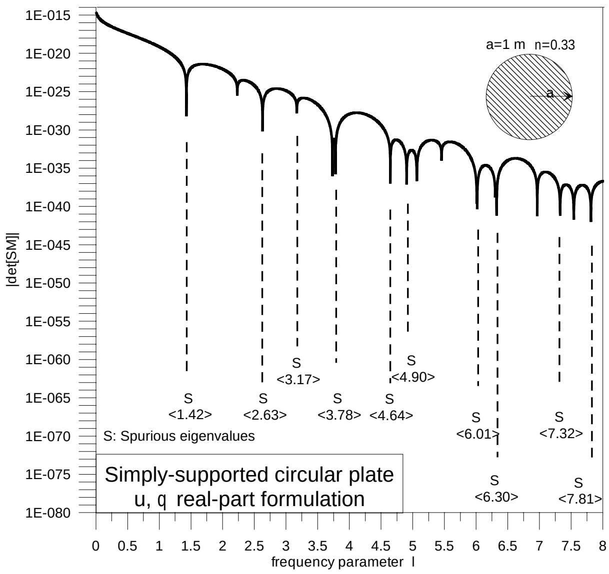

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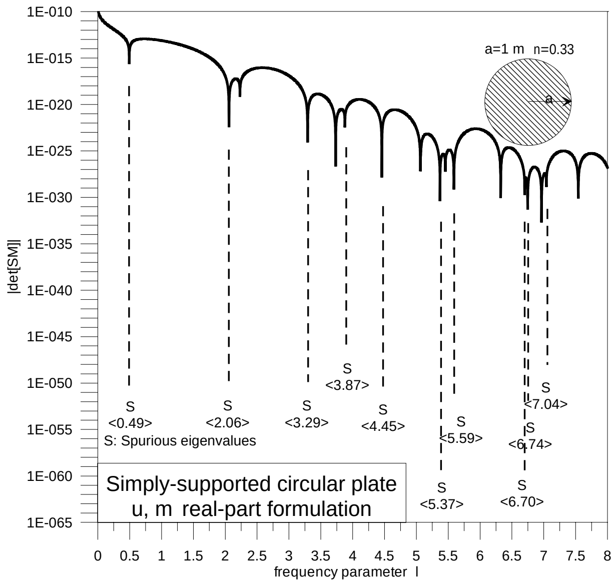

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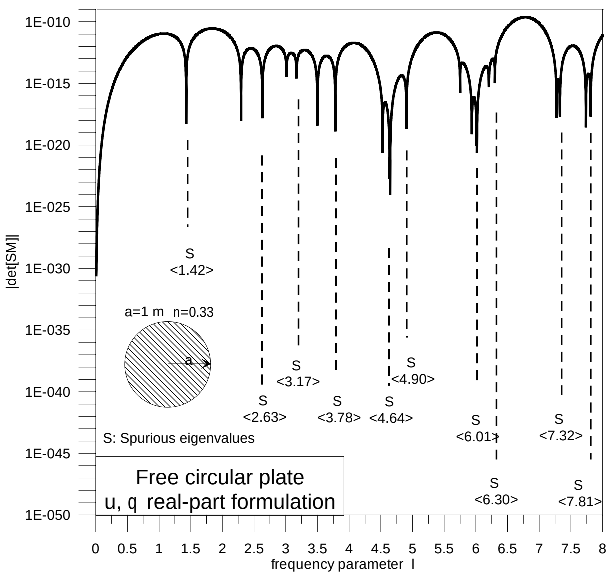

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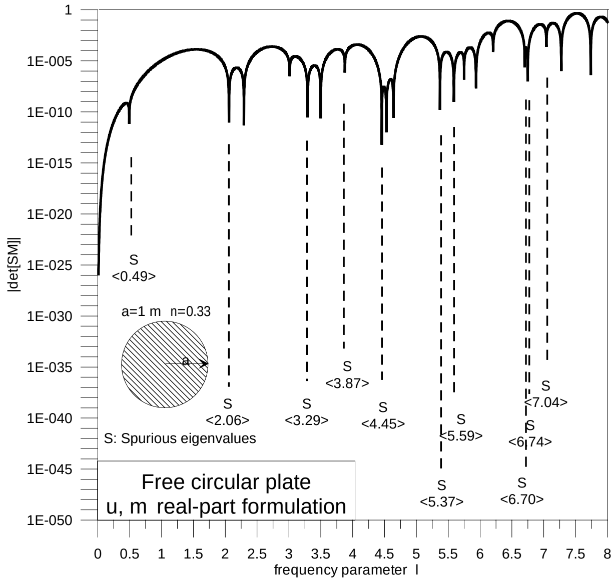


Figure 2-4.(f)

Figure 2-4 The determinant of $[SM]$ versus frequency parameter λ using the real-part formulation (u, θ or u, m formulation) to solve plates subject to different boundary conditions.

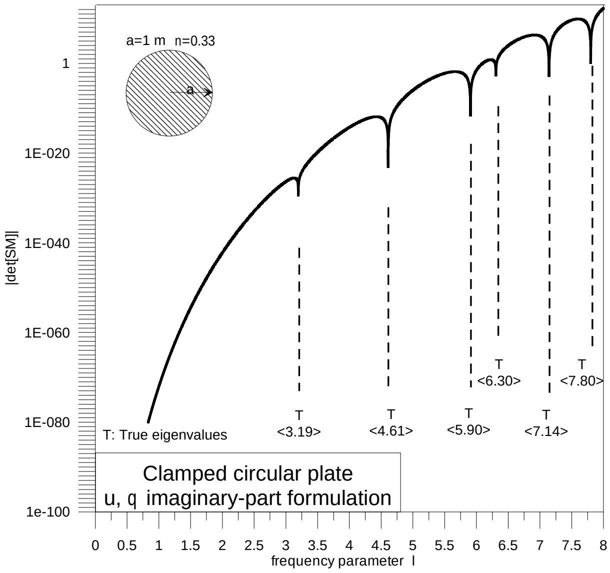
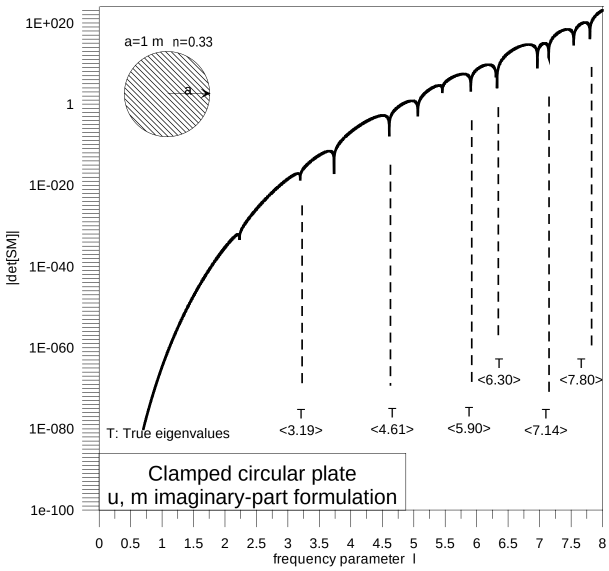


Figure 2-5.(a)



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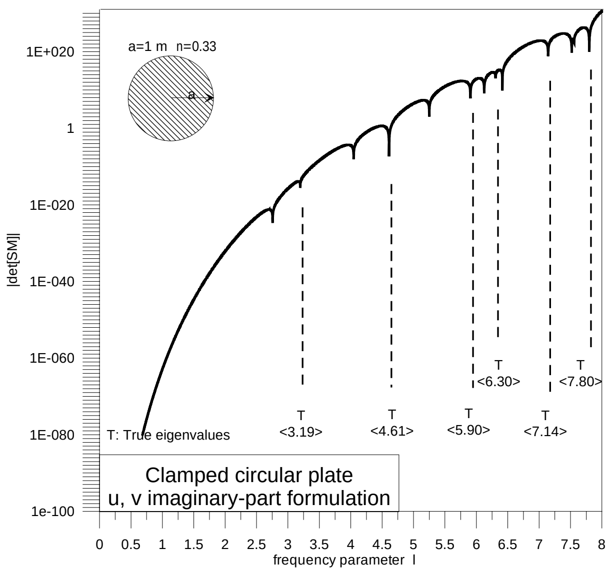

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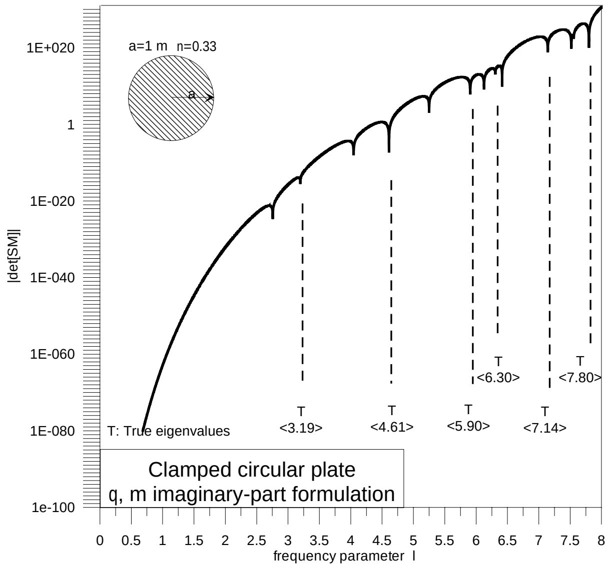

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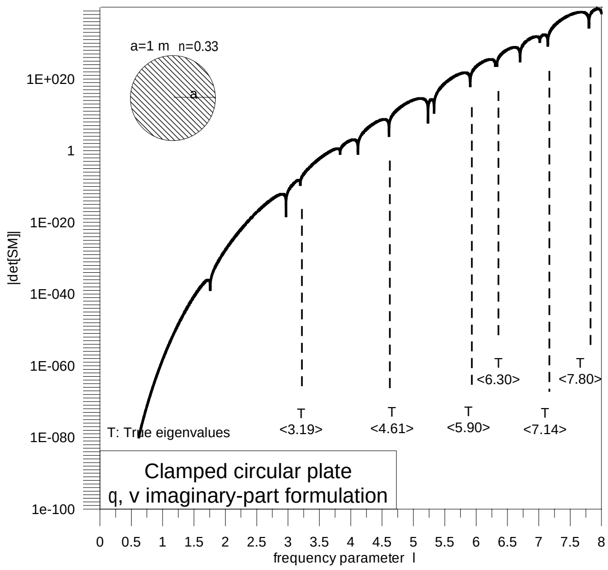

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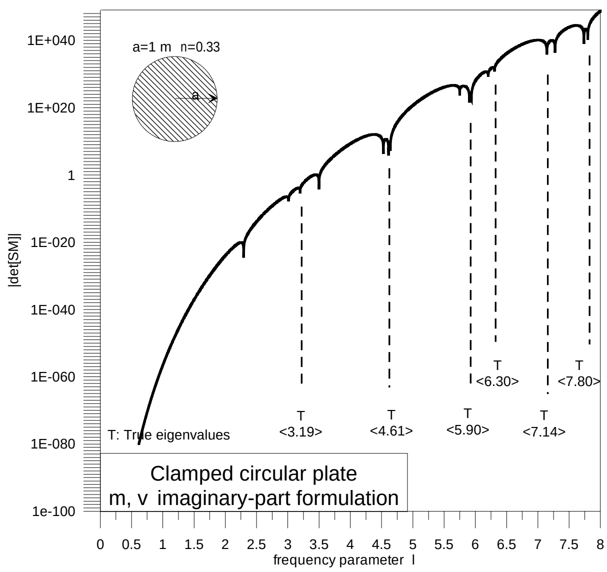


Figure 2-5.(f)

Figure 2-5 The determinant of $[SM^c]$ versus frequency parameter λ for the clamped circular plate using the six imaginary-part formulations.

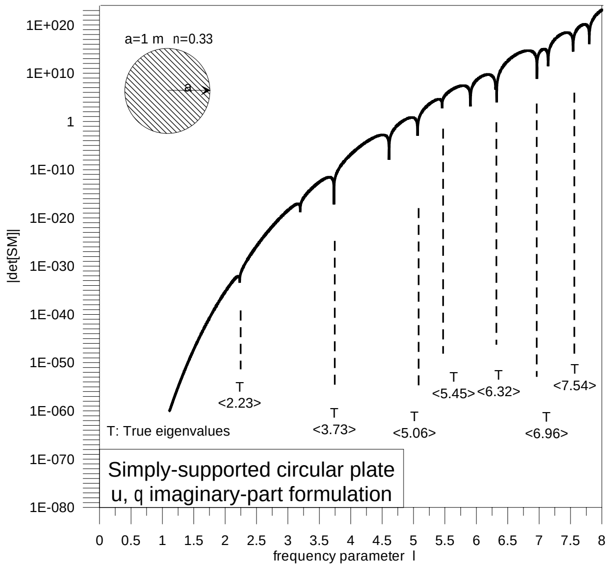
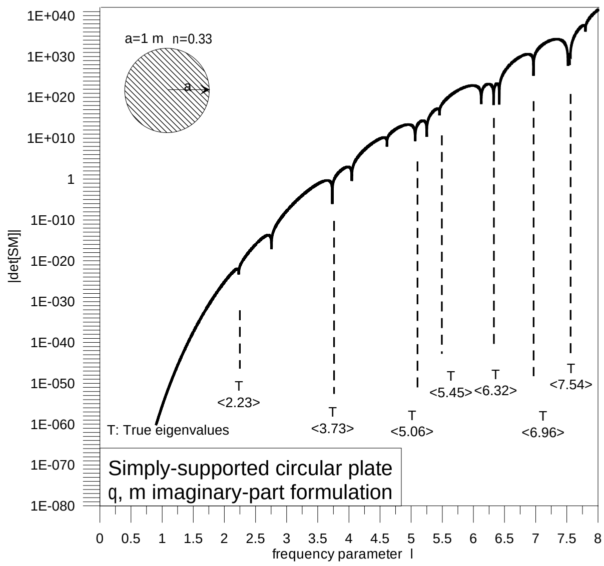


Figure 2-6.(a)



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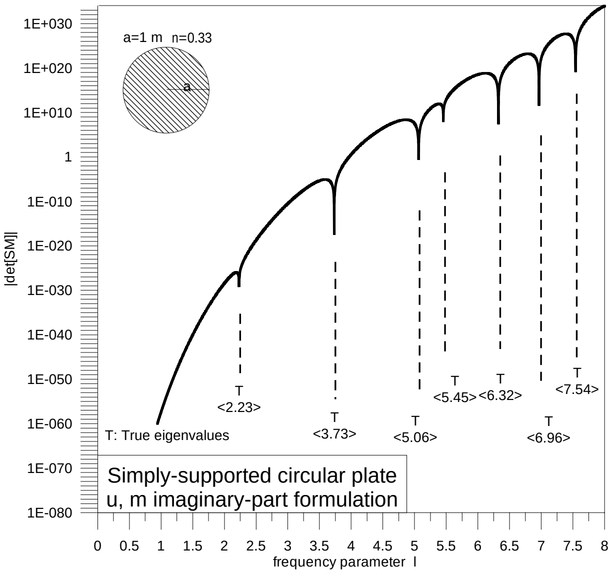

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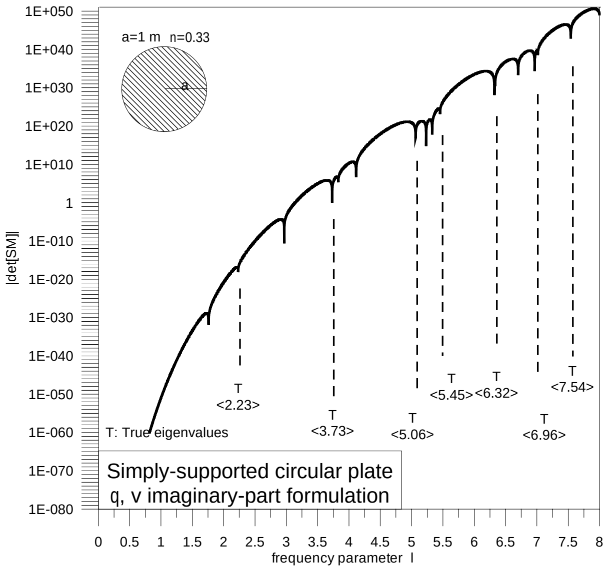

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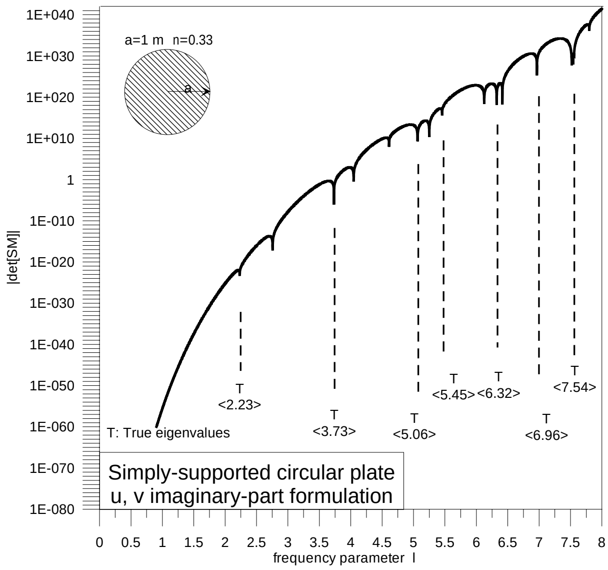

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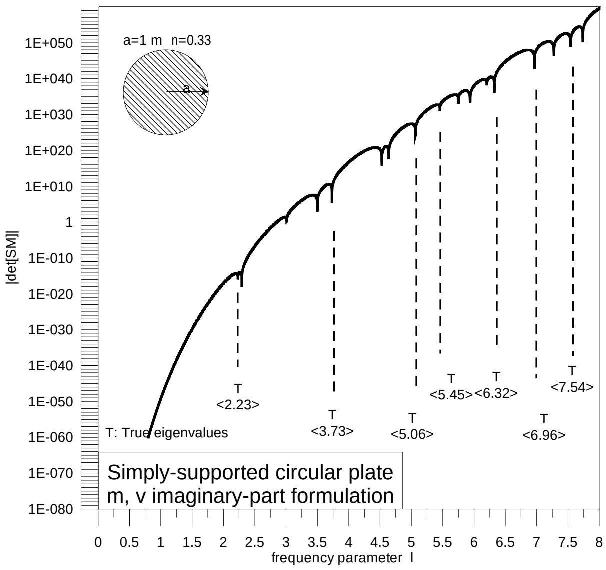


Figure 2-6.(f)

Figure 2-6 The determinant of $[SM^s]$ versus frequency parameter λ for the simply-supported circular plate using the six imaginary-part formulations.

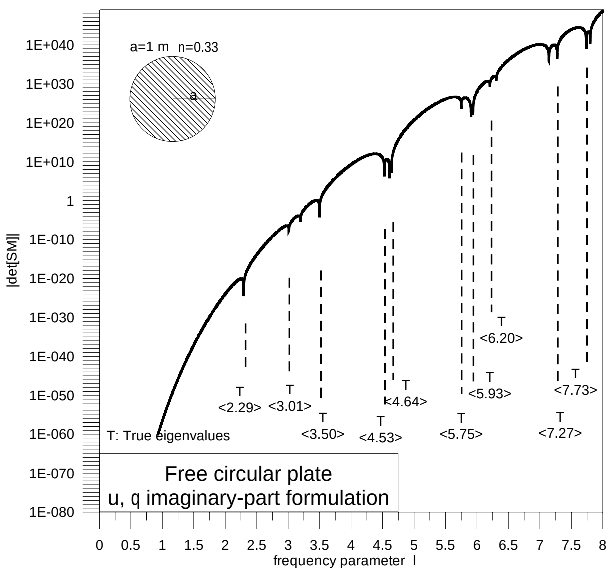


Figure 2-7.(a)

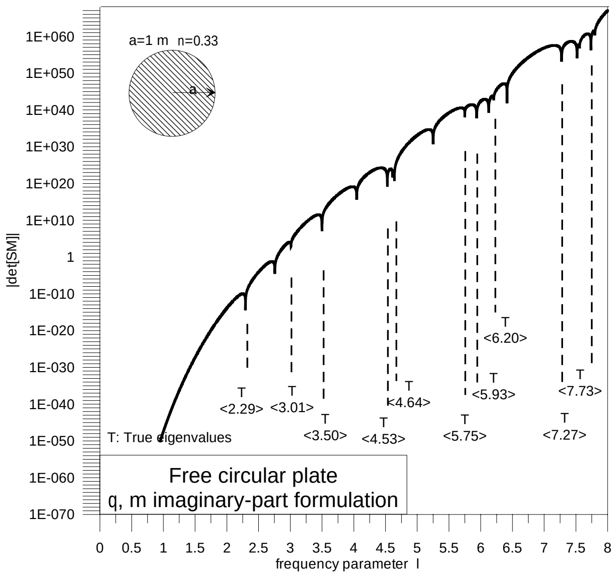
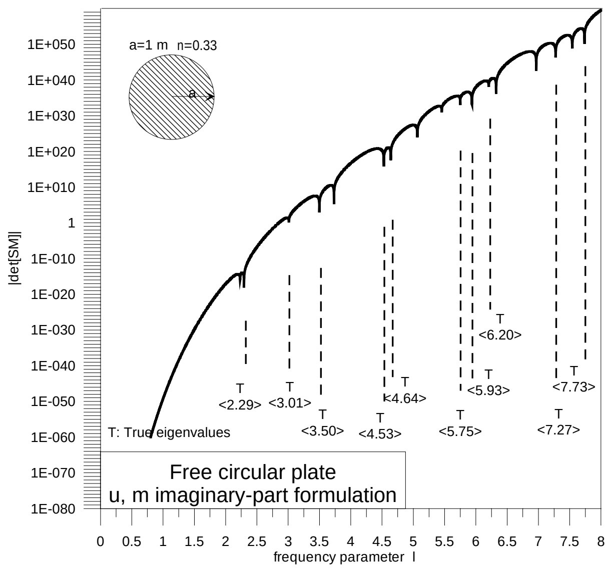


Figure 2-7.(d)



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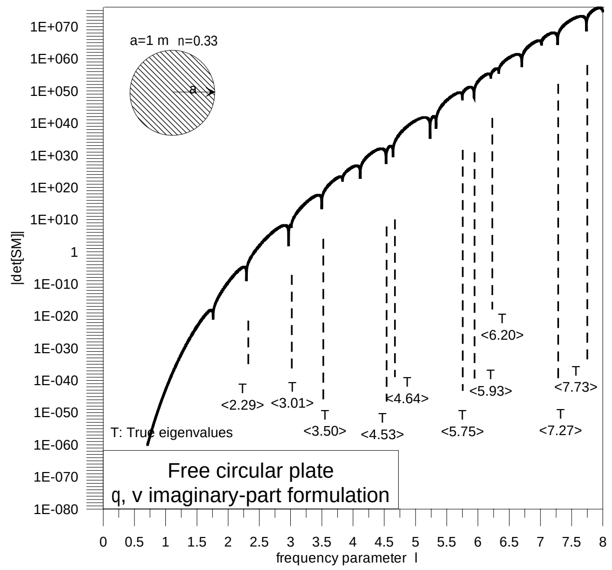

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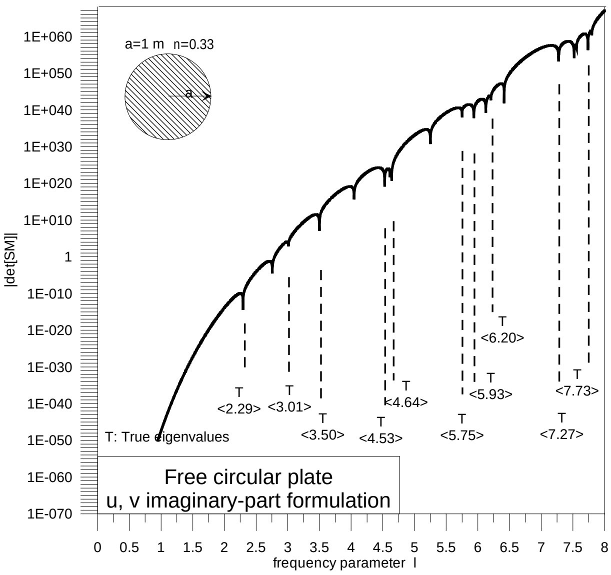

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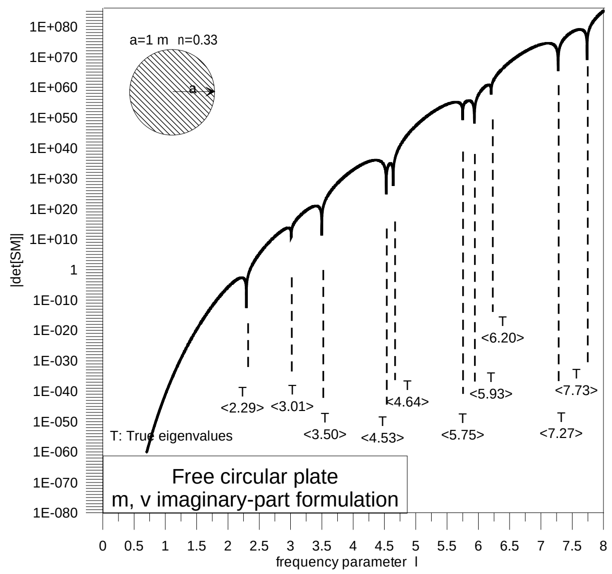


Figure 2-7.(f)

Figure 2-7 The determinant of $[SM^f]$ versus frequency parameter λ for the free circular plate using the six imaginary-part formulations.

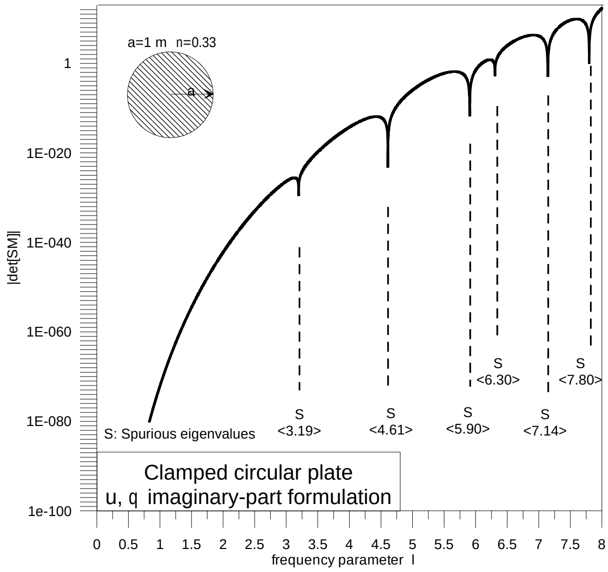


Figure 2-8.(a)

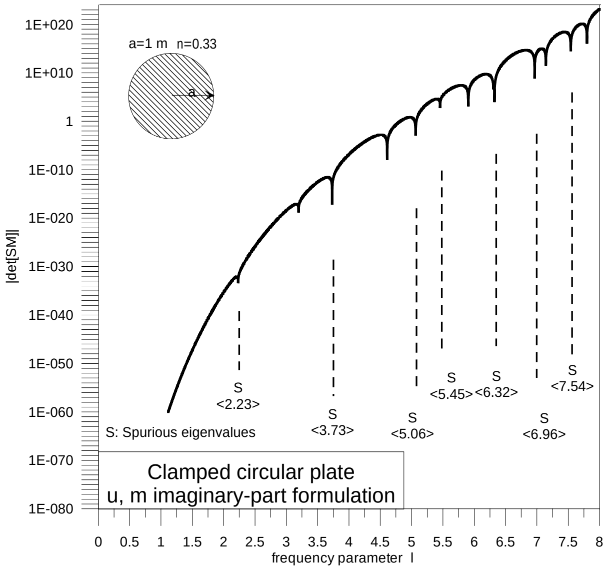
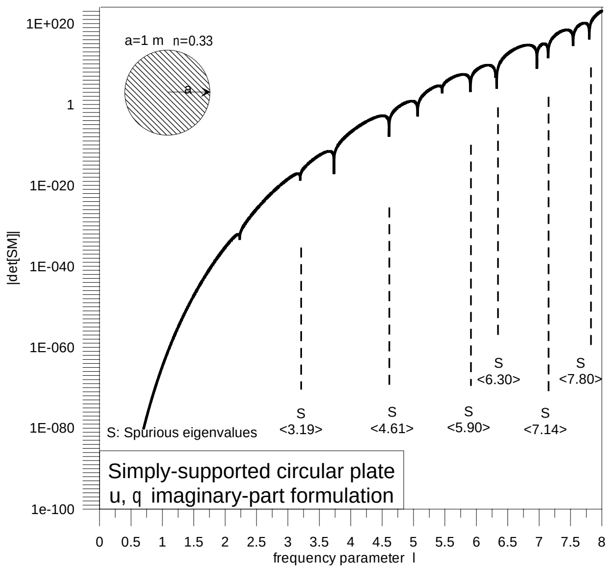


Figure 2-8.(d)



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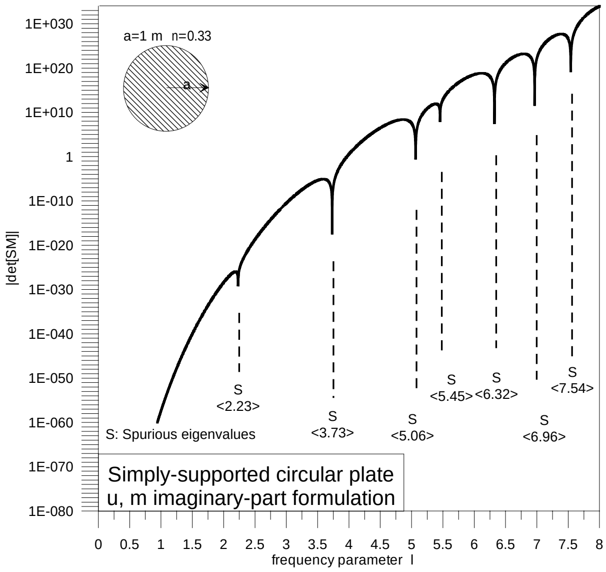

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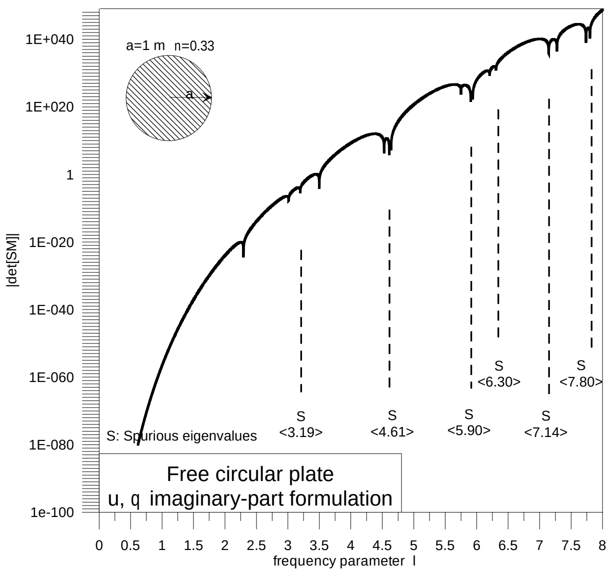

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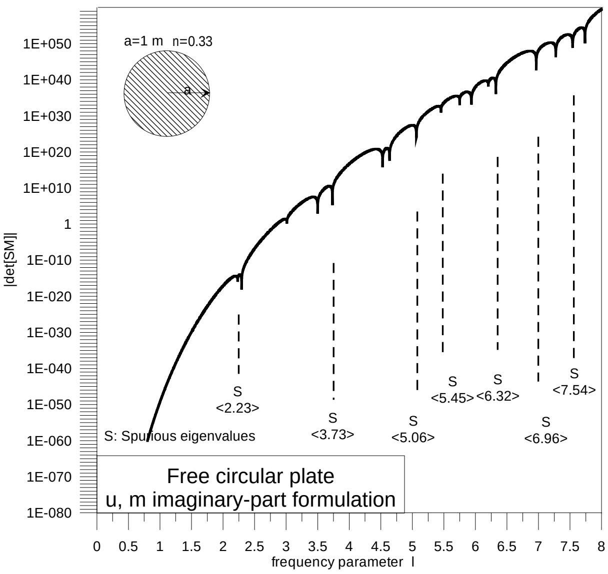


Figure 2-8.(f)

Figure 2-8 The determinant of $[SM]$ versus frequency parameter λ using the imaginary-part formulation (u, θ or u, m formulation) to solve plates subject to different boundary conditions.

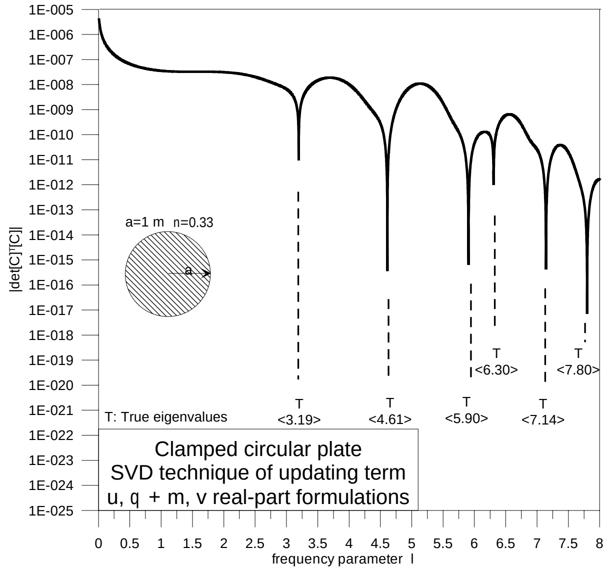


Figure 3-1.(a)

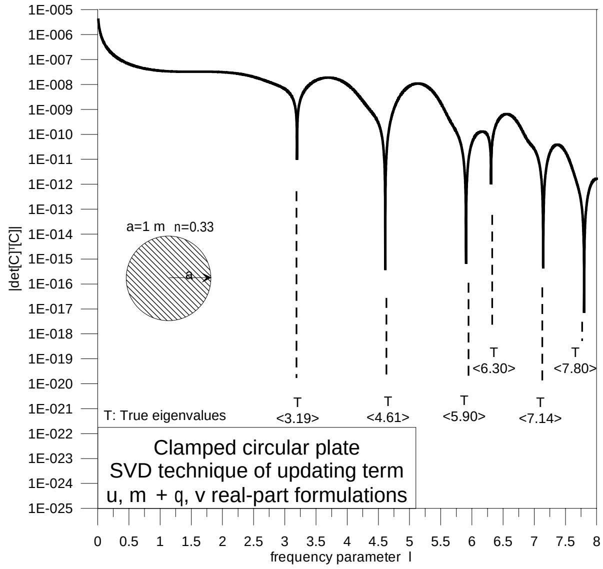
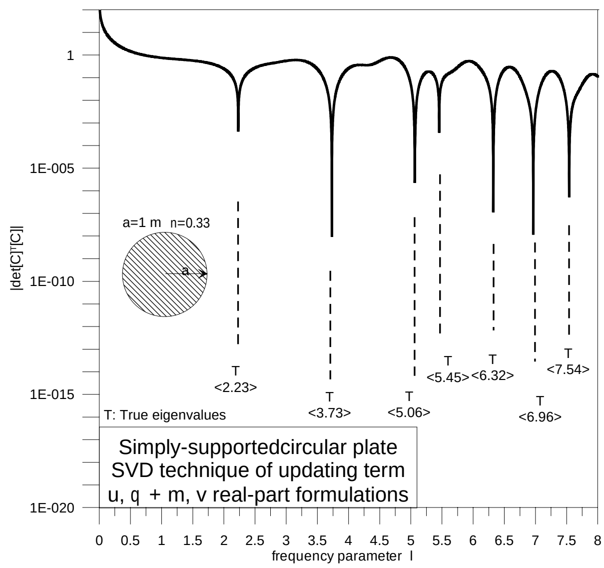


Figure 3-1.(d)



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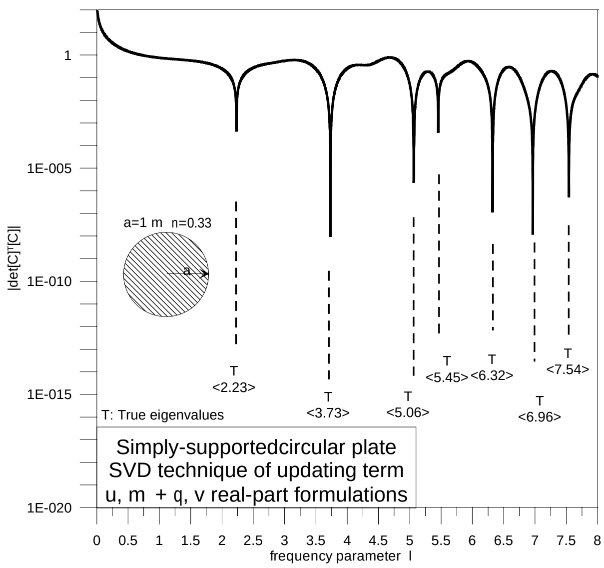

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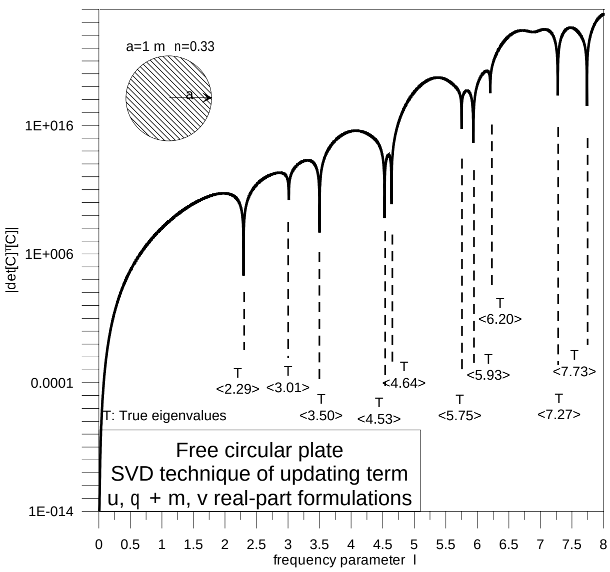

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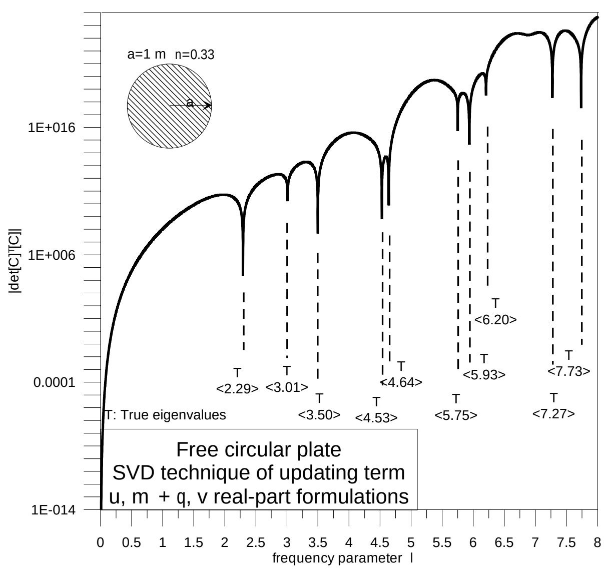


Figure 3-1.(f)

Figure 3-1 The determinant of the of the $[C]^T[C]$ versus frequency parameter λ for the circular plates by using the real-part formulations with the SVD technique of updating term.

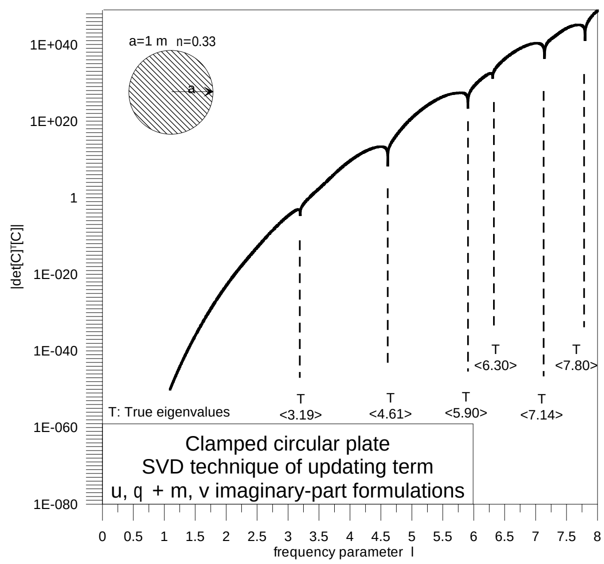


Figure 3-2.(a)

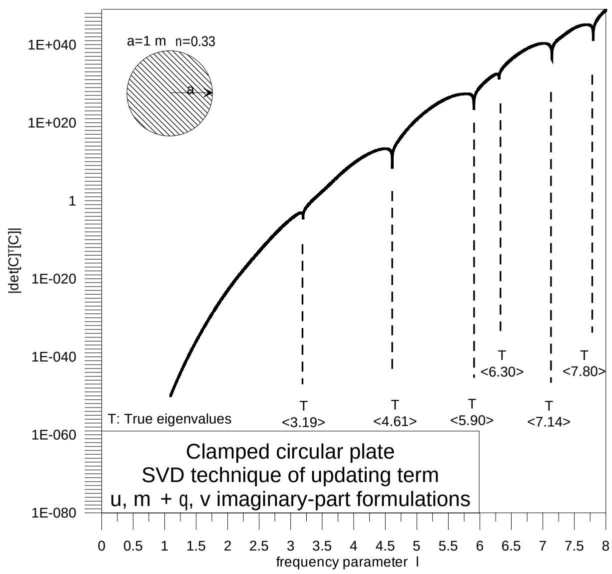


Figure 3-2.(d)

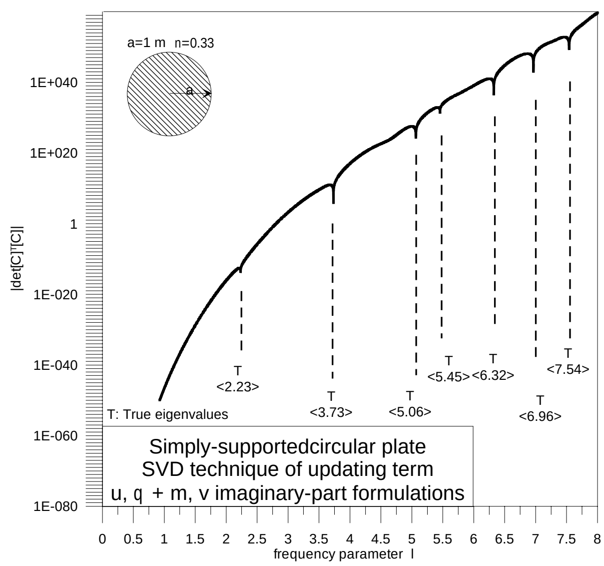


Figure 3-2.(b)

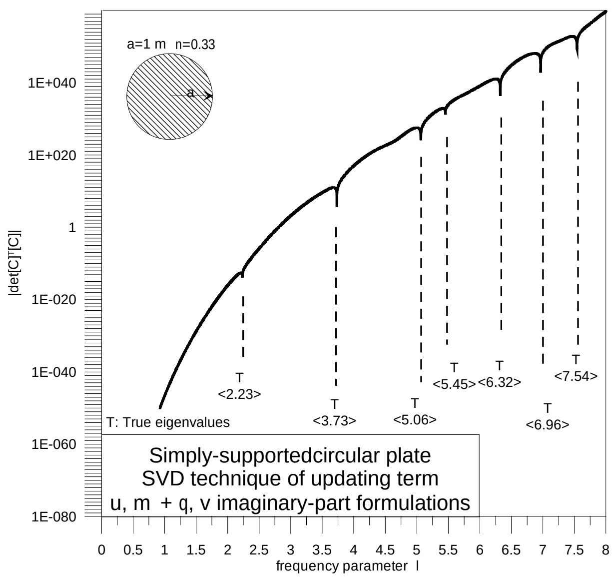
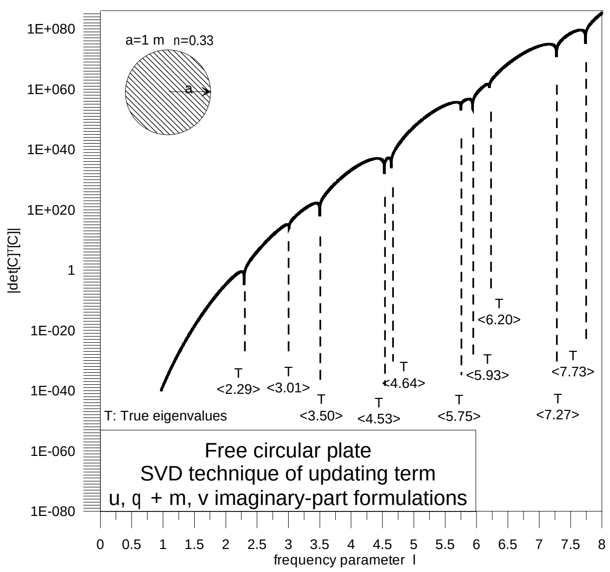


Figure 3-2.(e)



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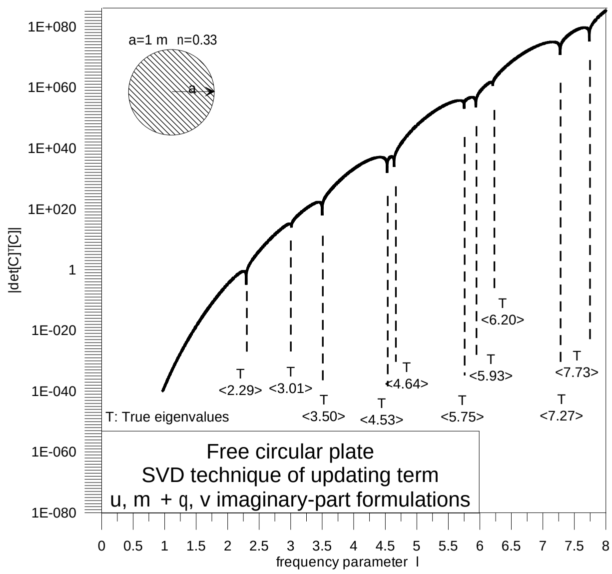


Figure 3-2.(f)

Figure 3-2 The determinant of the of the $[C]^T[C]$ versus frequency parameter λ for the circular plates by using the imaginary-part formulations with the SVD technique of updating term.

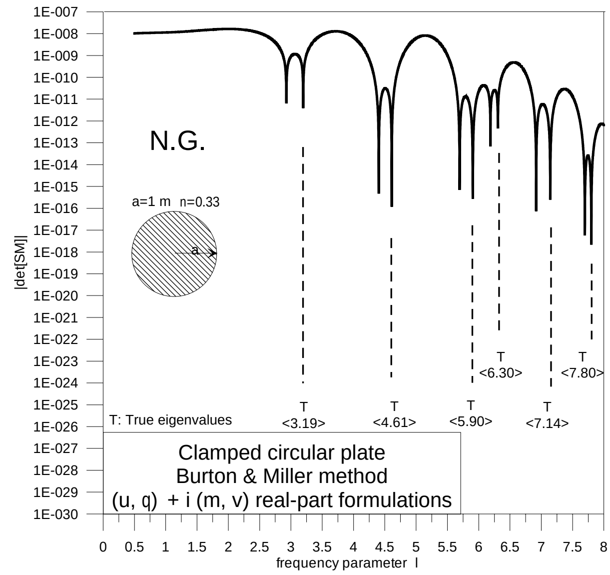


Figure 3-3.(a)

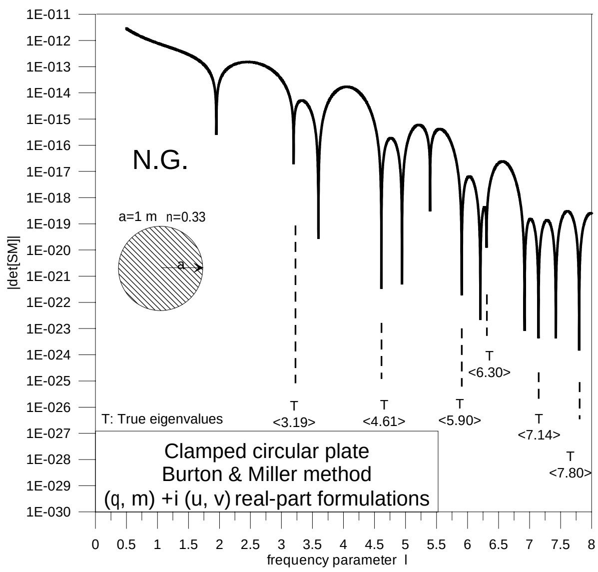


Figure 3-3.(d)

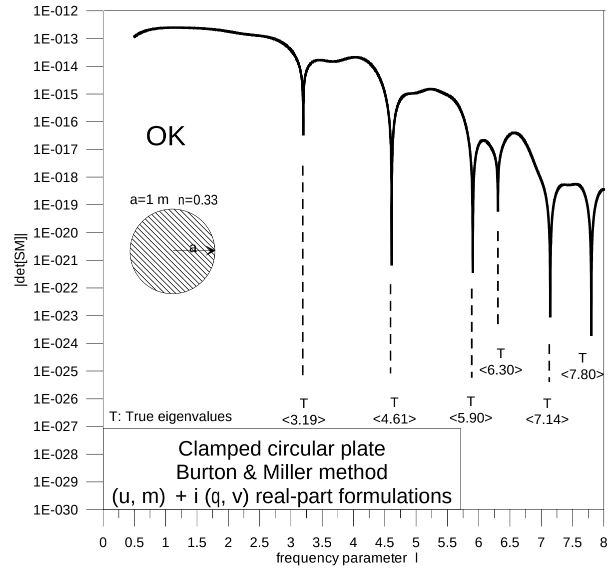
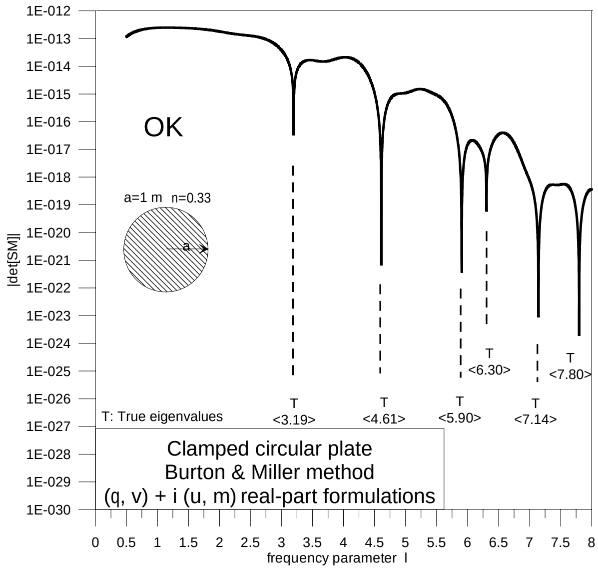


Figure 3-3.(b)



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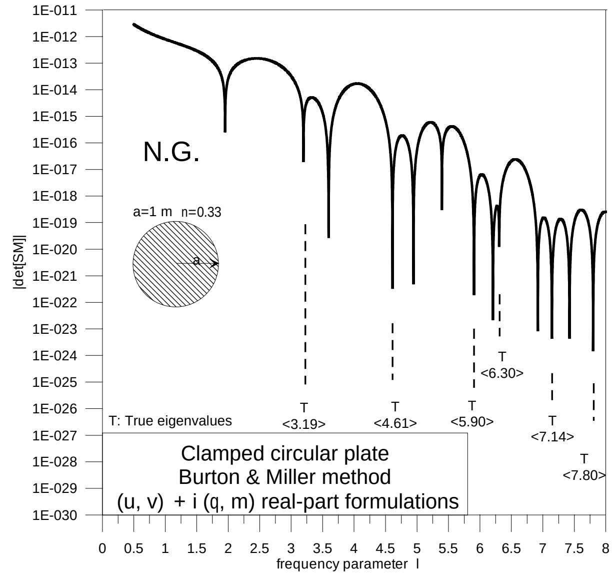

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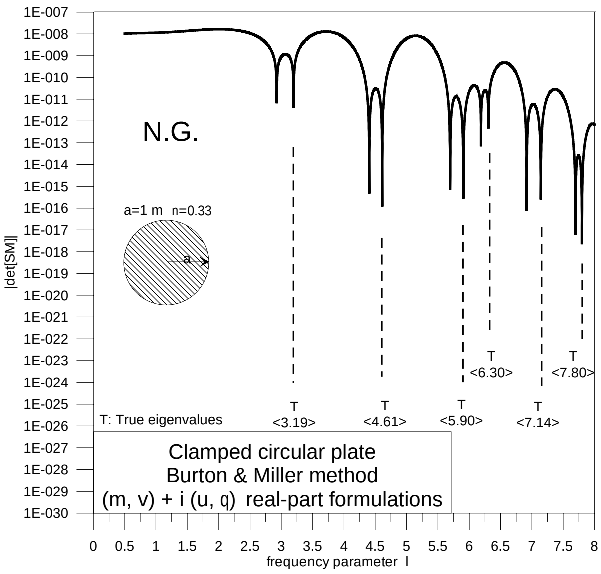


Figure 3-3.(f)

Figure 3-3 The determinant of the $[SM^c]$ versus frequency parameter λ
for the clamped circular plate using the six real-part formulations with the Burton & Miller concept.

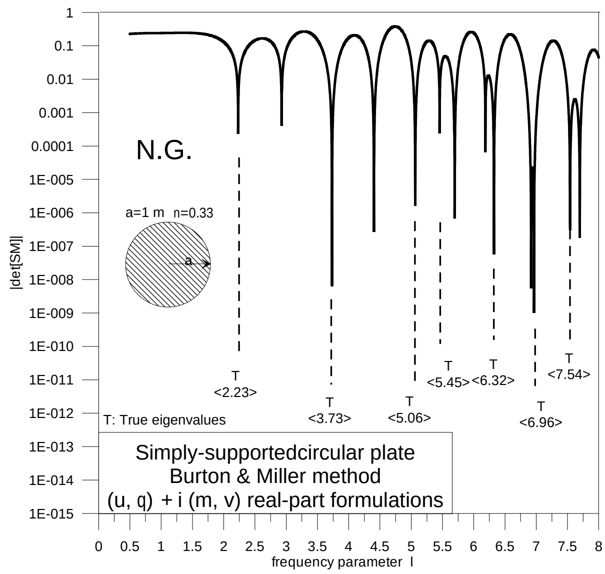
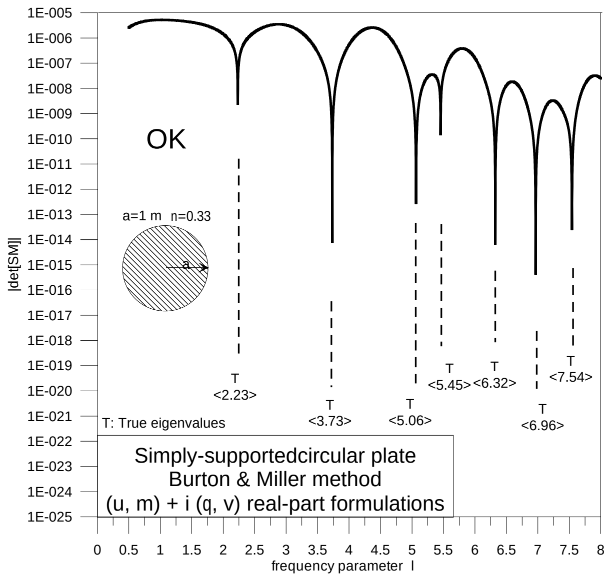


Figure 3-4.(a)



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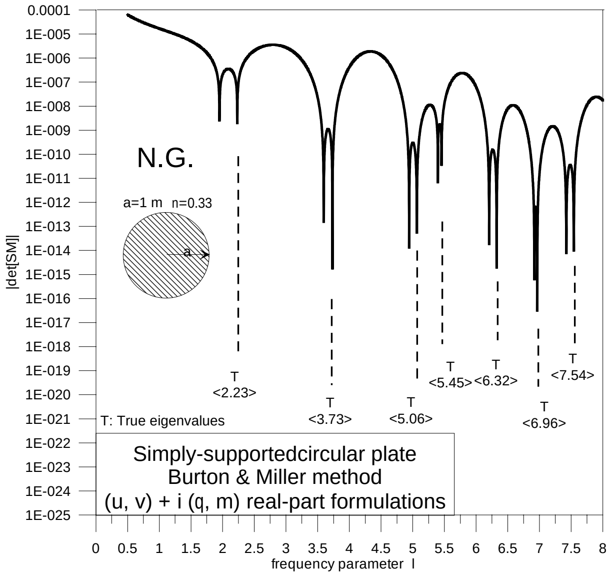

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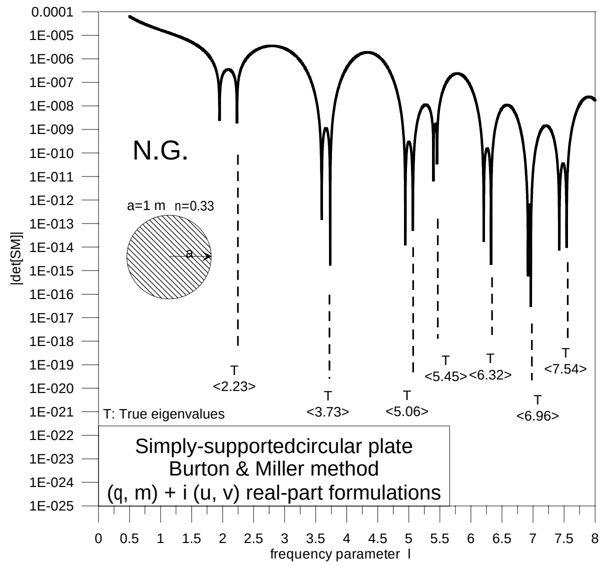

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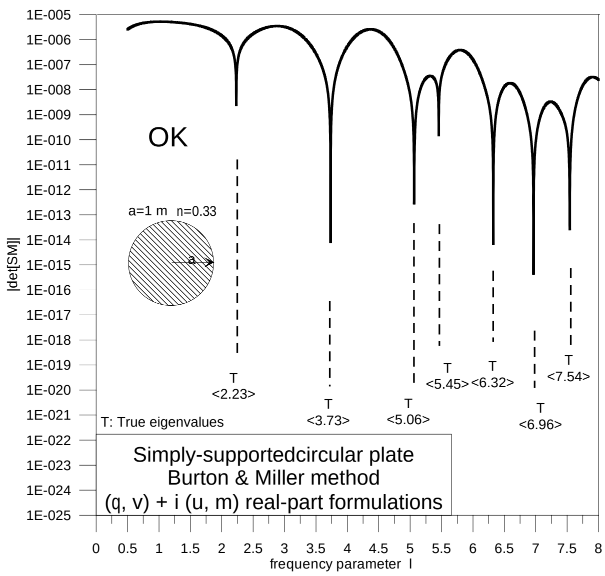


Figure 3-4.(e)

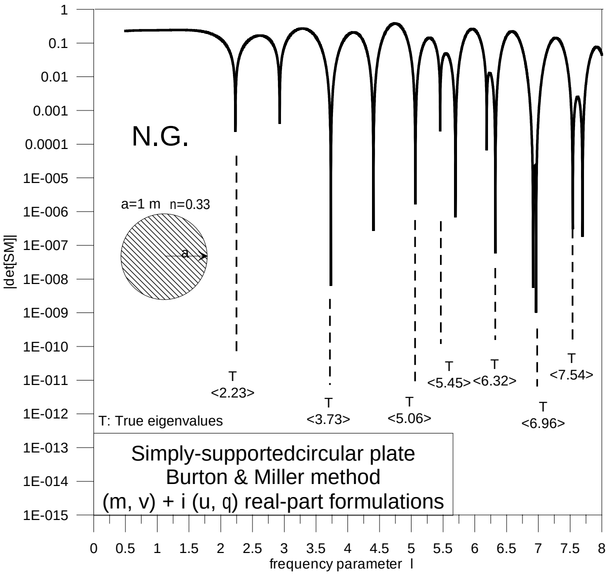


Figure 3-4.(f)

Figure 3-4 The determinant of the $[SM^s]$ versus frequency parameter λ for the simply-supported circular plate using the six real-part formulations with the Burton & Miller concept.

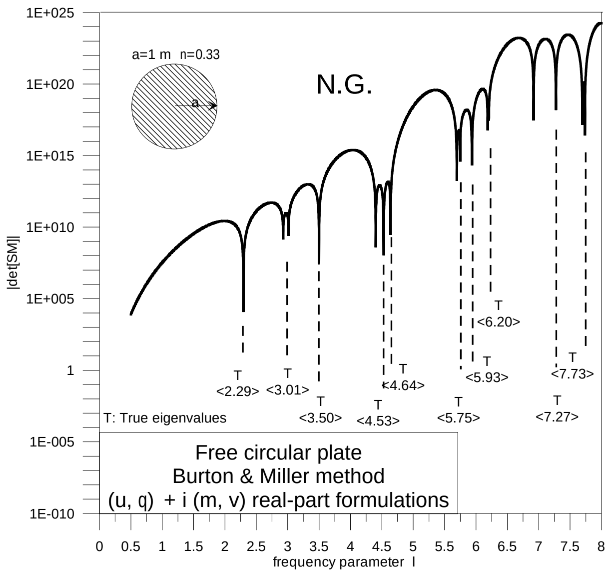


Figure 3-5.(a)

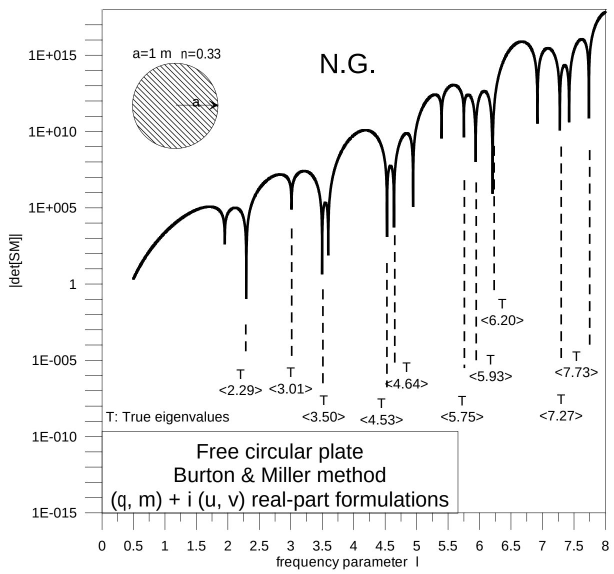
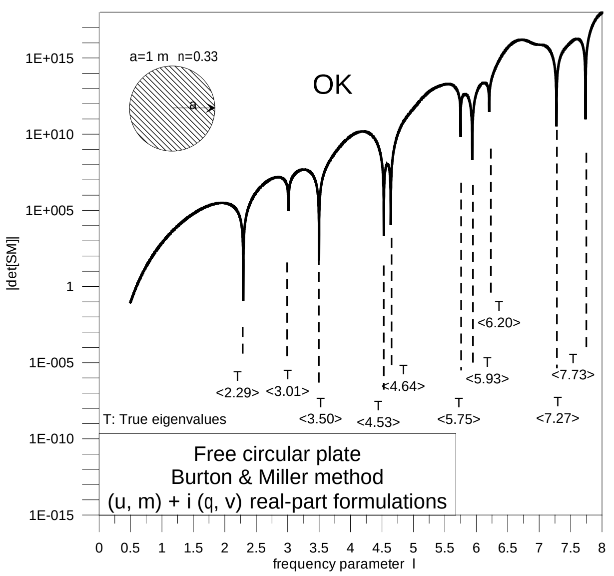


Figure 3-5.(d)



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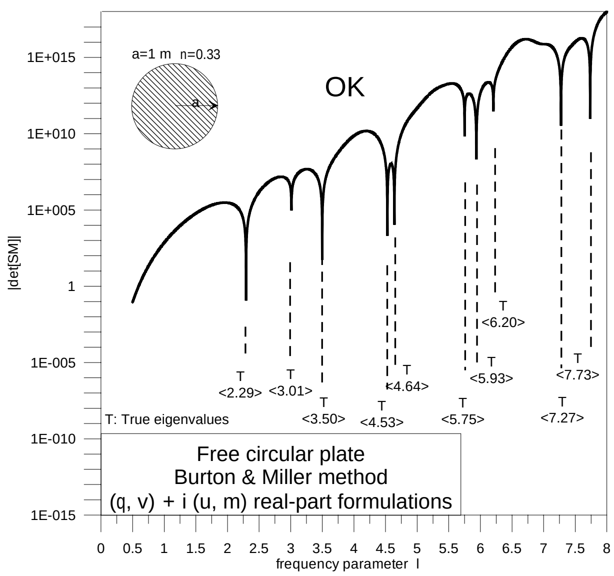

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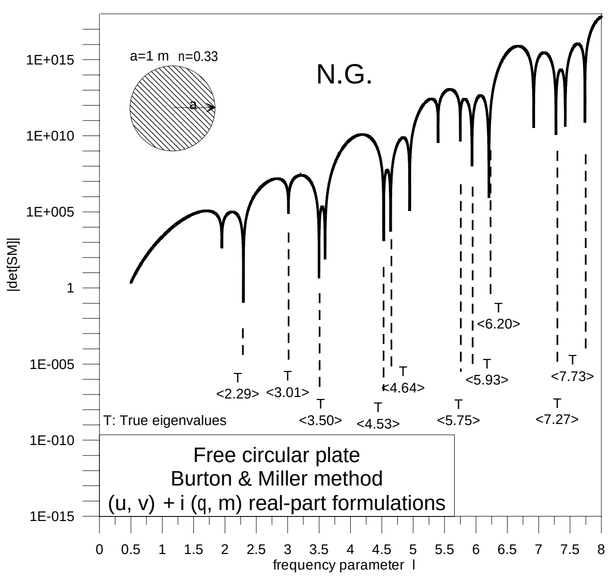

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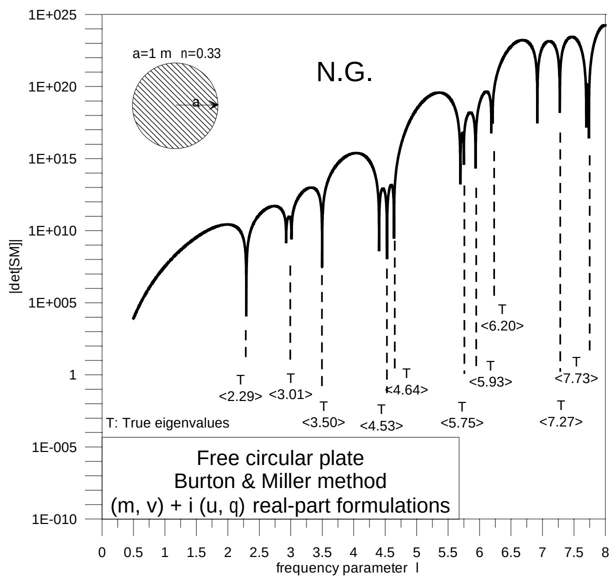


Figure 3-5.(f)

Figure 3-5 The determinant of the $[SM^f]$ versus frequency parameter λ for the free circular plate using the six real-part formulations with the Burton & Miller concept.

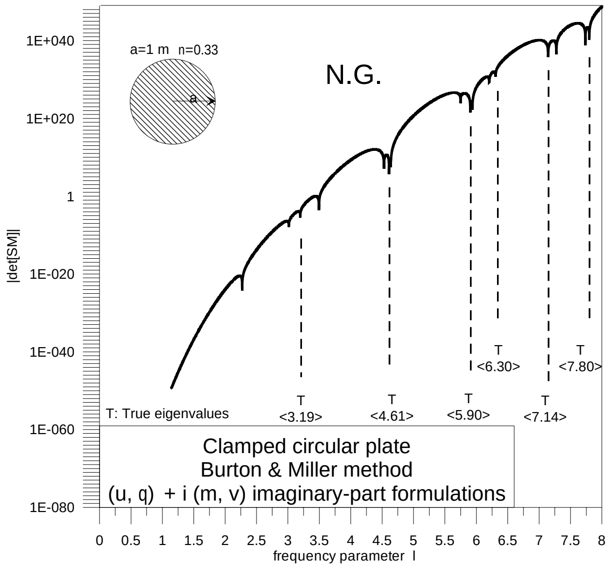


Figure 3-6.(a)

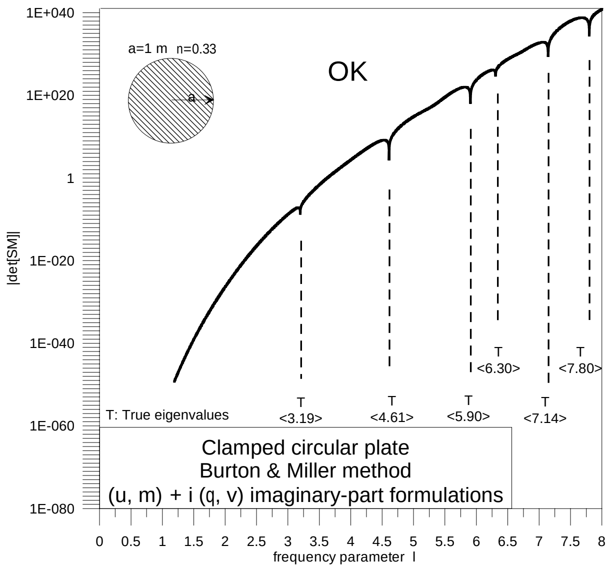


Figure 3-6.(b)

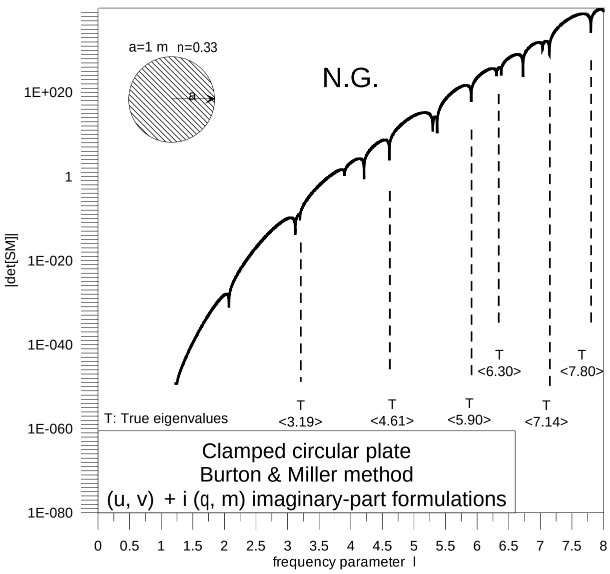


Figure 3-6.(c)

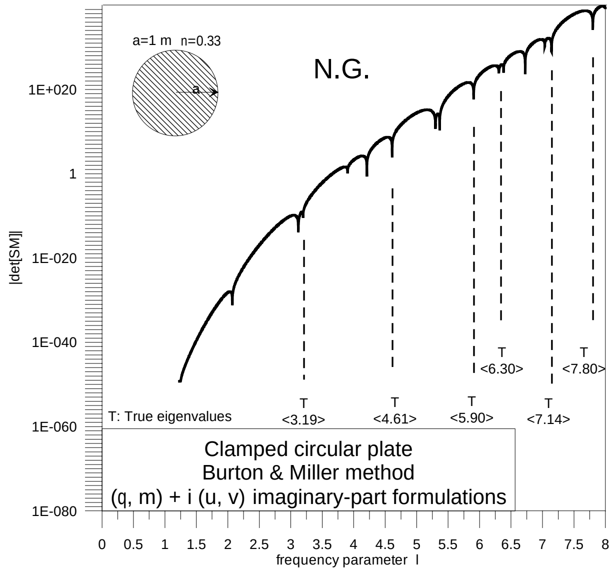


Figure 3-6.(d)

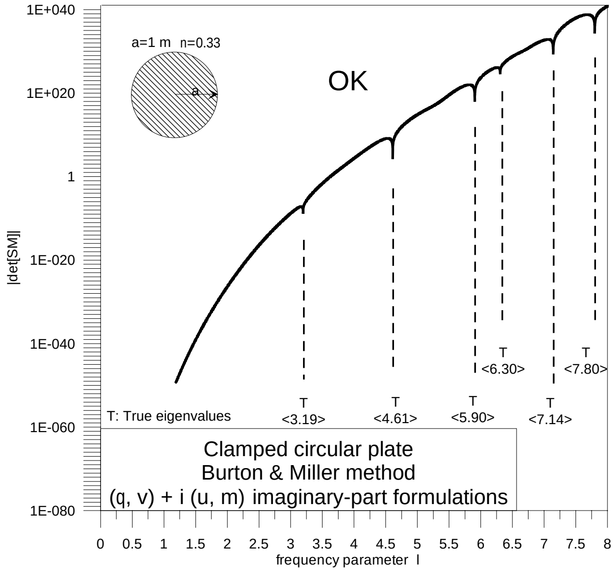


Figure 3-6.(e)

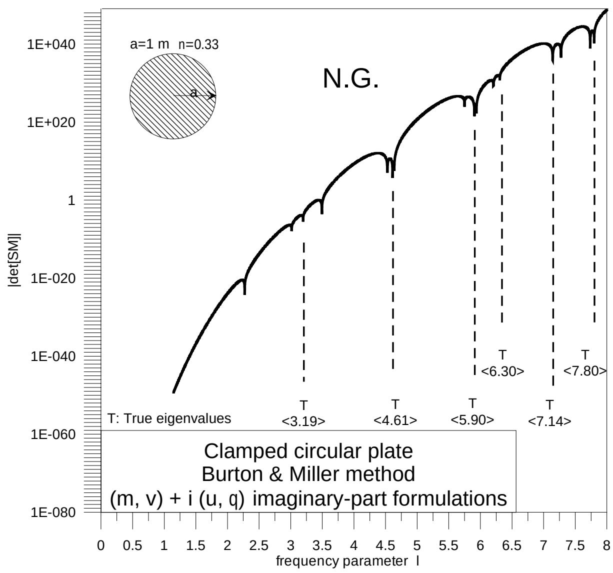


Figure 3-6.(f)

Figure 3-6 The determinant of the $[SM^c]$ versus frequency parameter λ for the clamped circular plate using the six imaginary-part formulations with the Burton & Miller concept.

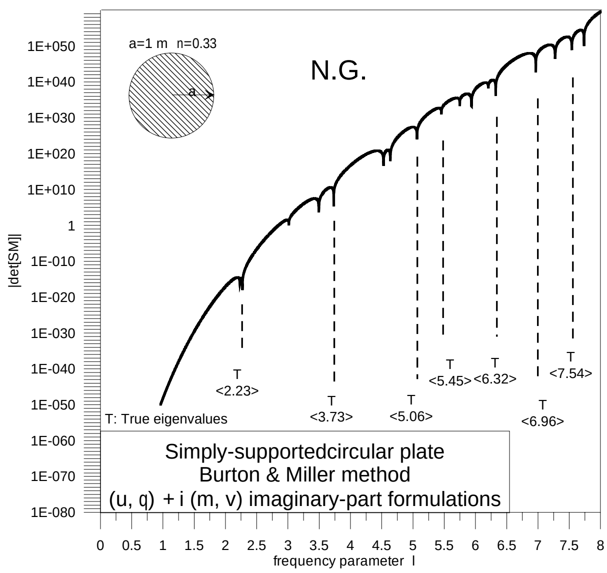


Figure 3-7.(a)

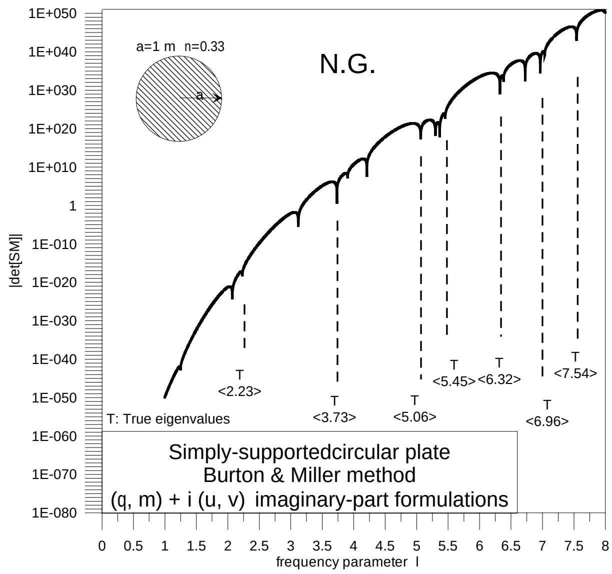


Figure 3-7.(d)

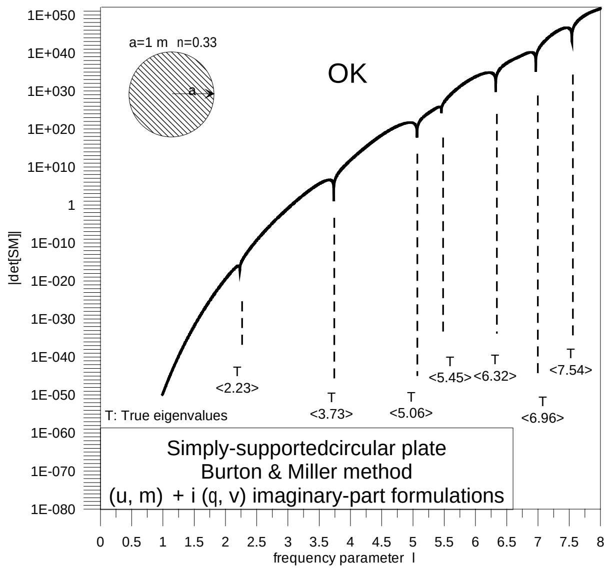


Figure 3-7.(b)

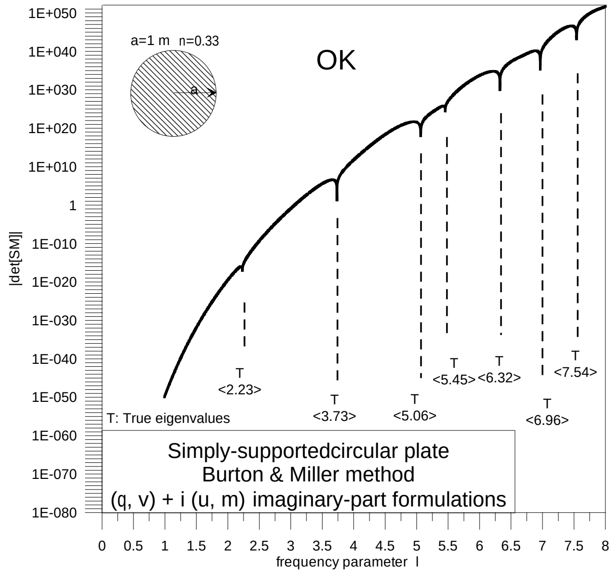


Figure 3-7.(e)

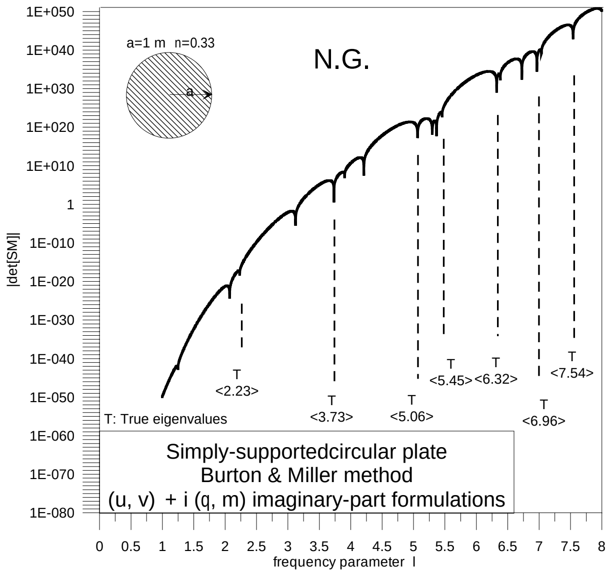


Figure 3-7.(c)

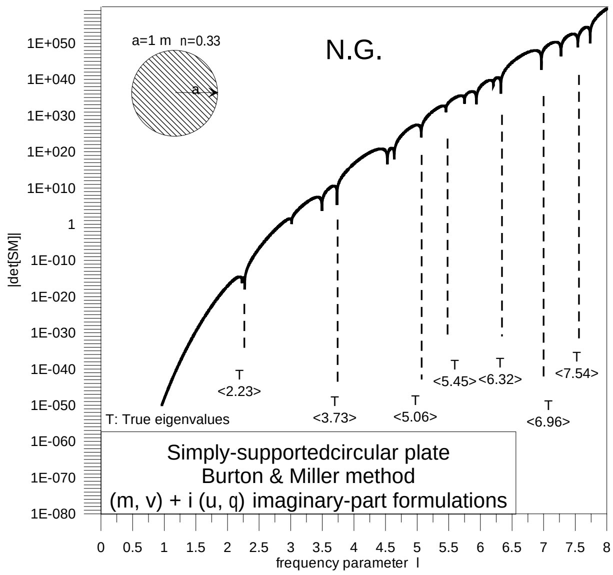


Figure 3-7.(f)

Figure 3-7 The determinant of the $[SM^s]$ versus frequency parameter λ for the simply-supported circular plate using the six imaginary-part formulations with the Burton & Miller concept.

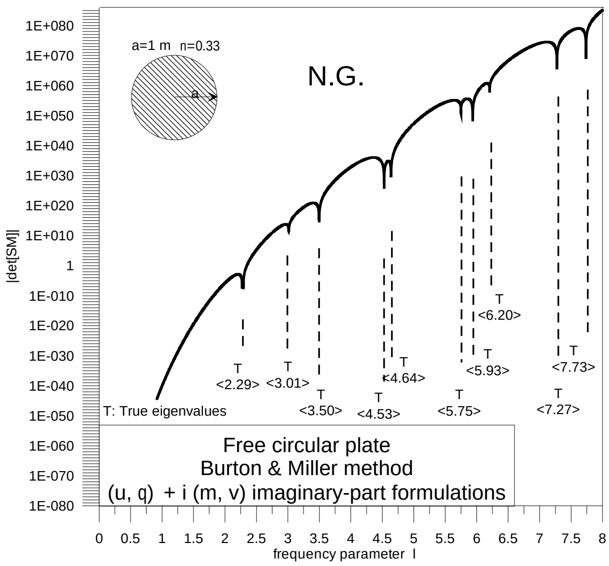


Figure 3-8.(a)

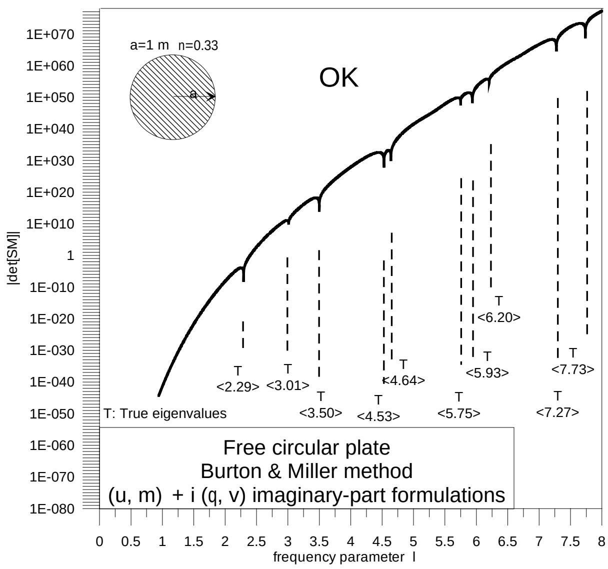


Figure 3-8.(b)

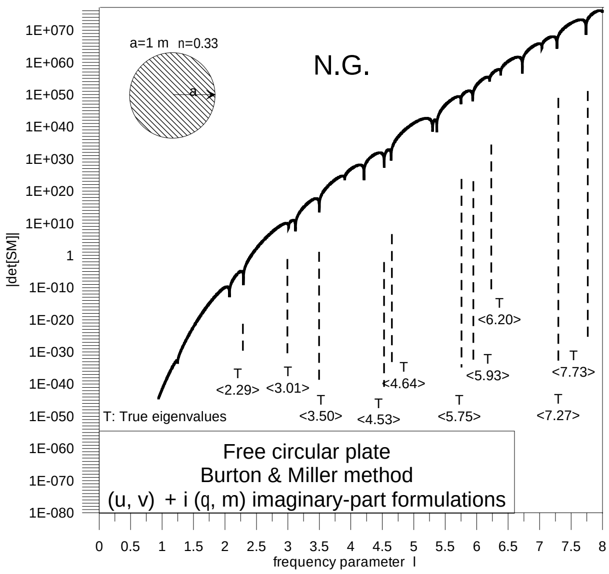


Figure 3-8.(c)

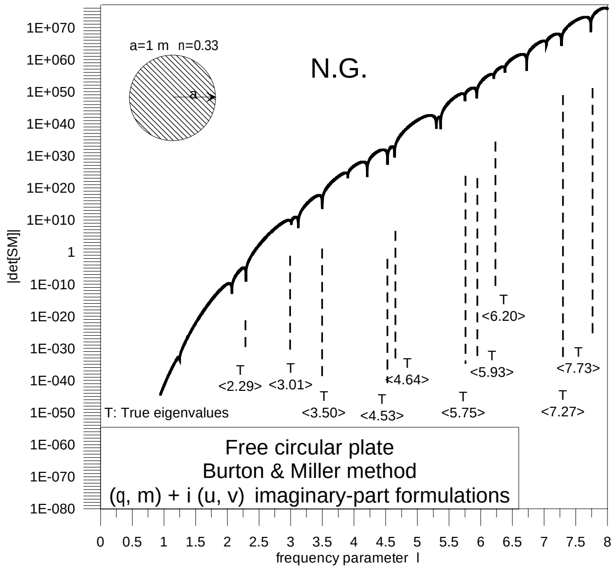


Figure 3-8.(d)

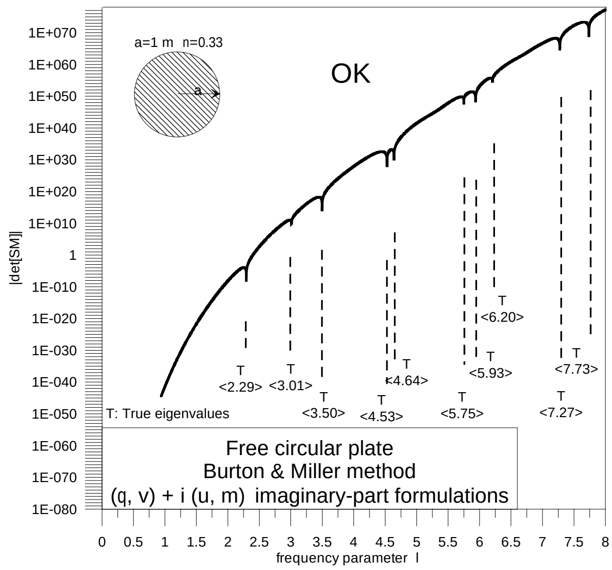


Figure 3-8.(e)

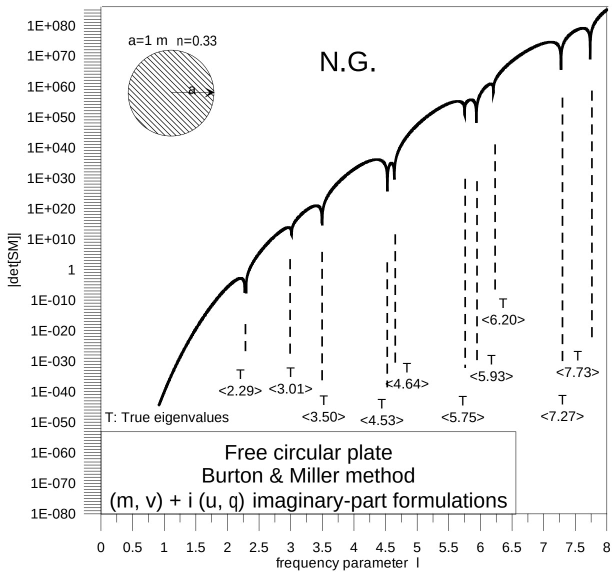


Figure 3-8.(f)

Figure 3-8 The determinant of the $[SM^f]$ versus frequency parameter λ for the free circular plate using the six imaginary-part formulations with the Burton & Miller concept.

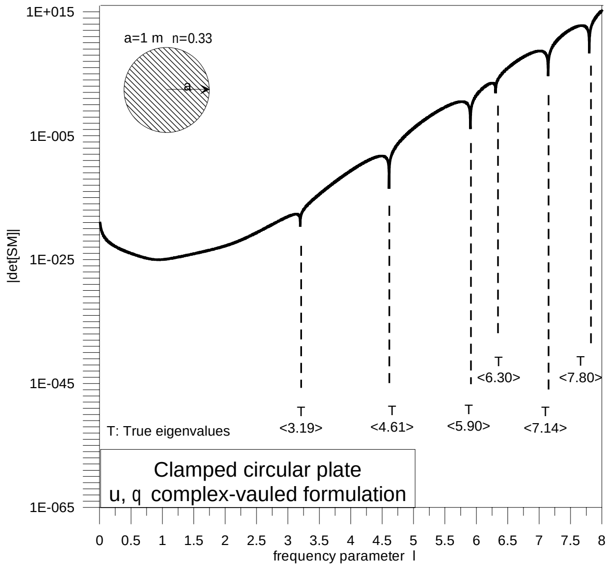


Figure 3-9.(a)

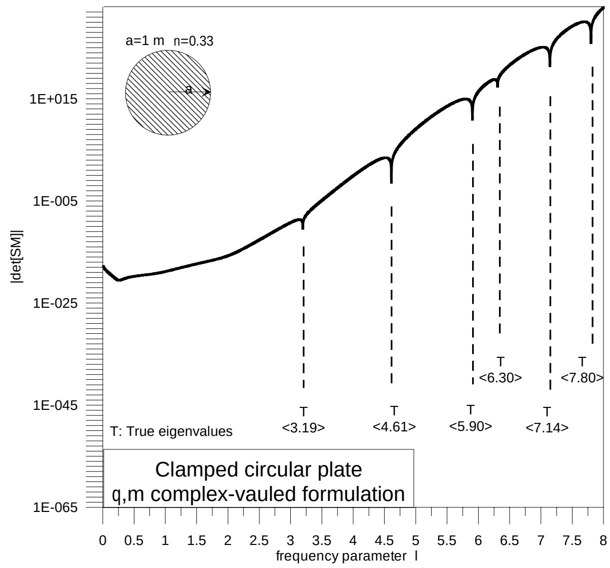


Figure 3-9.(d)

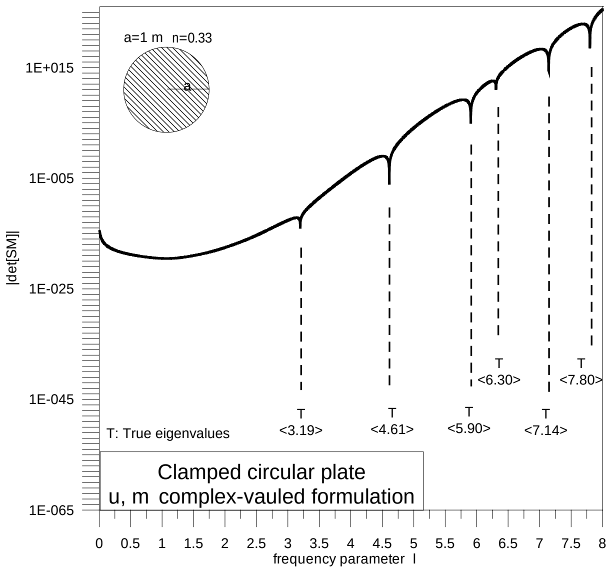


Figure 3-9.(b)

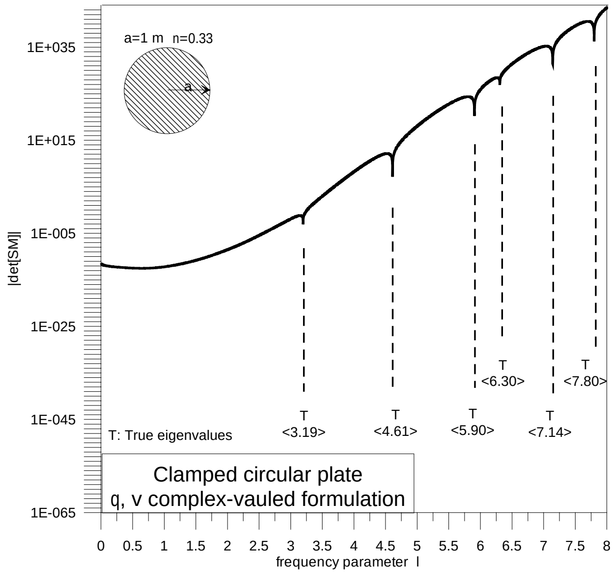


Figure 3-9.(e)

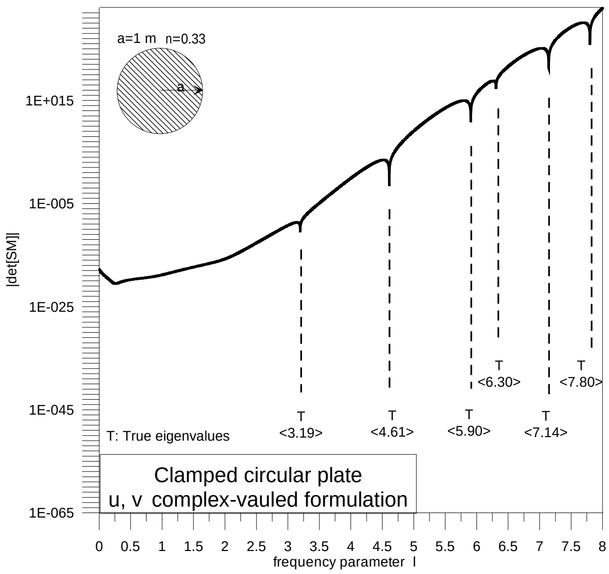


Figure 3-9.(c)

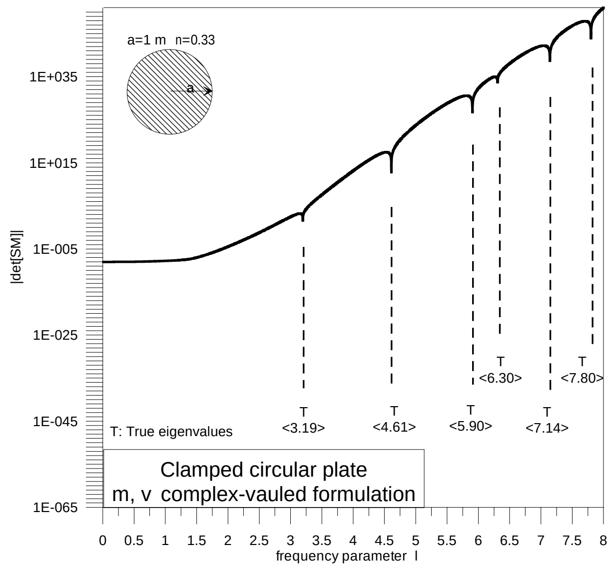


Figure 3-9.(f)

Figure 3-9 The determinant of the $[SM^c]$ versus frequency parameter λ for the clamped circular plate using the six complex-valued BEM.

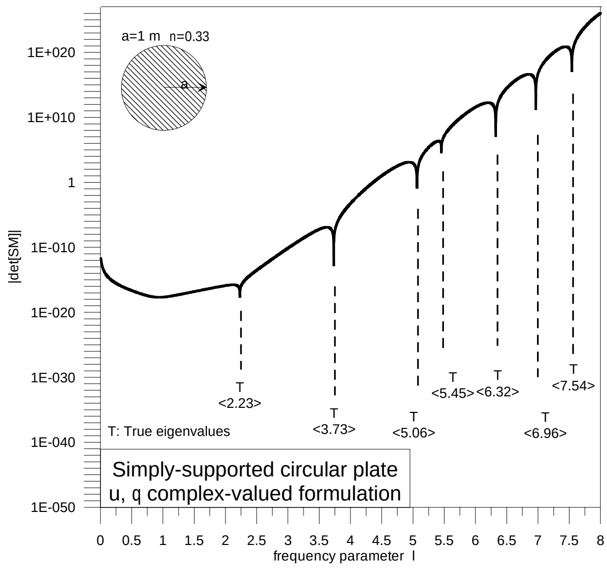


Figure 3-10.(a)

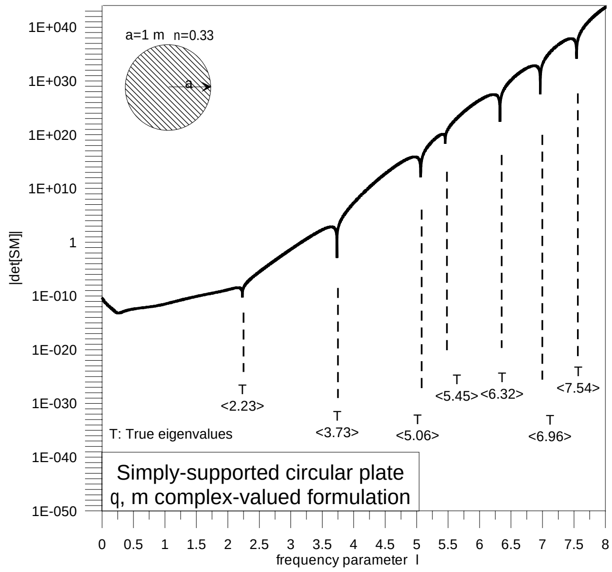


Figure 3-10.(d)

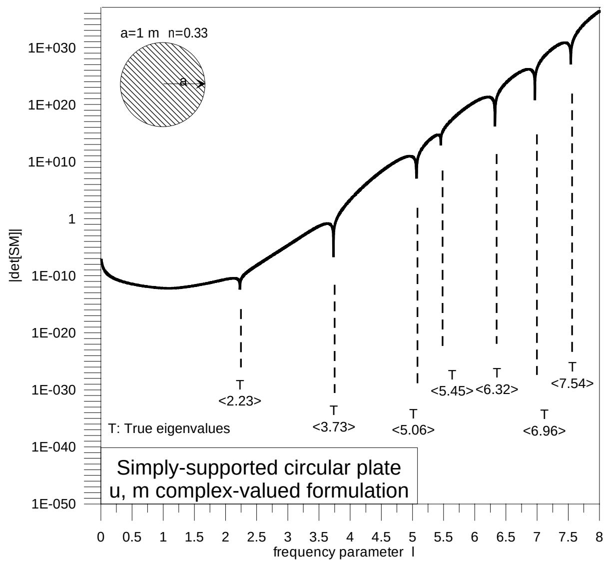


Figure 3-10.(b)

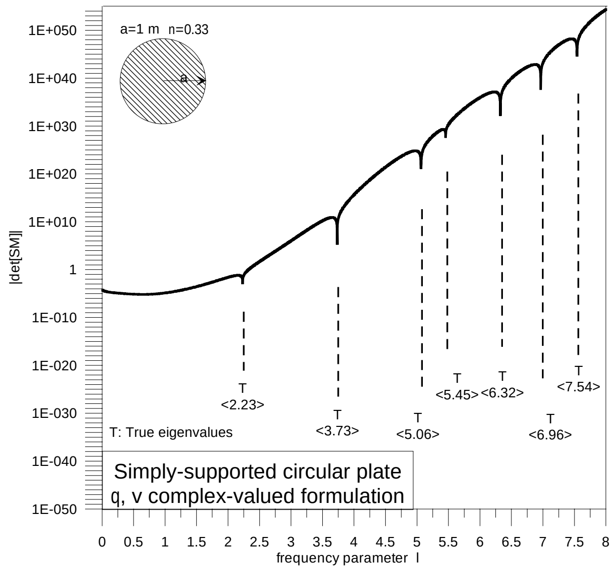


Figure 3-10.(e)

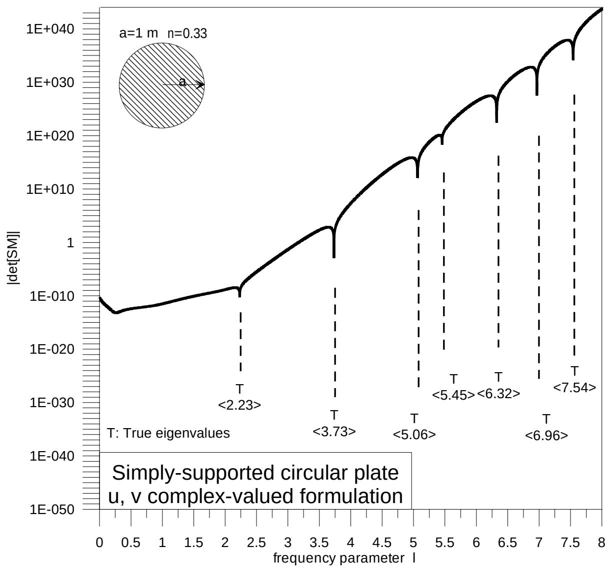


Figure 3-10.(c)

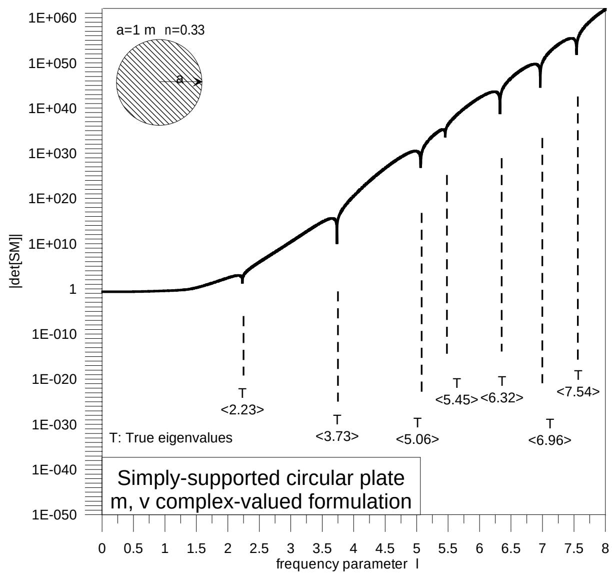


Figure 3-10.(f)

Figure 3-10 The determinant of the $[SM^s]$ versus frequency parameter λ for the simply-supported circular using the six complex-valued BEM.

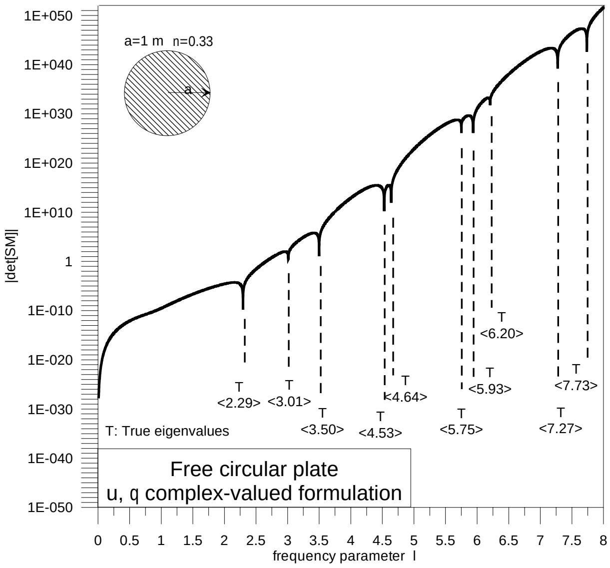


Figure 3-11.(a)

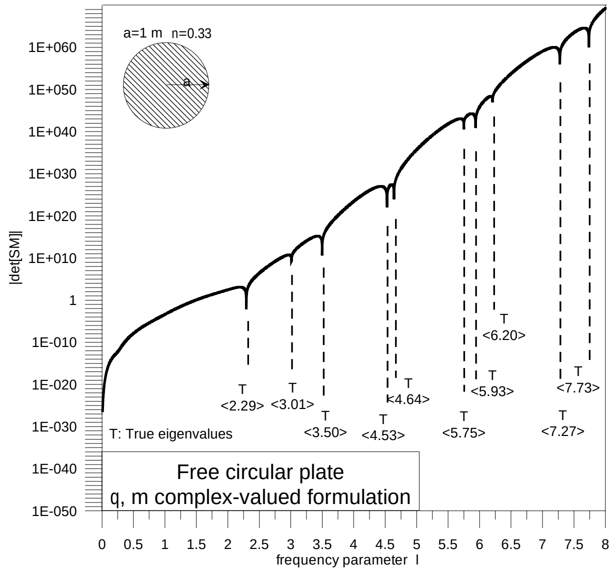


Figure 3-11.(d)

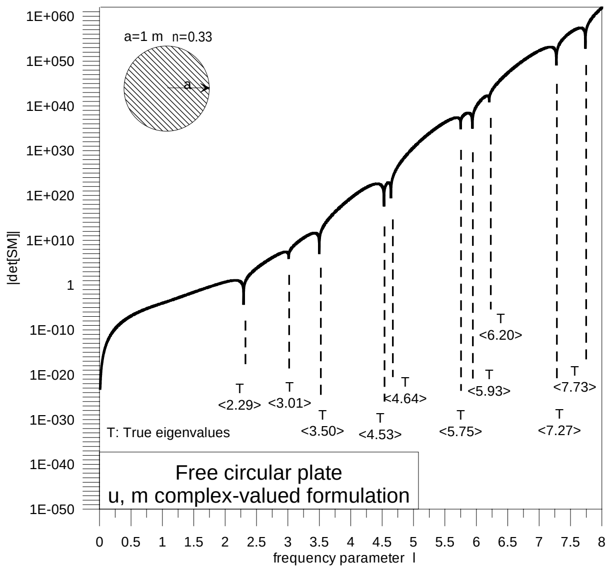


Figure 3-11.(b)

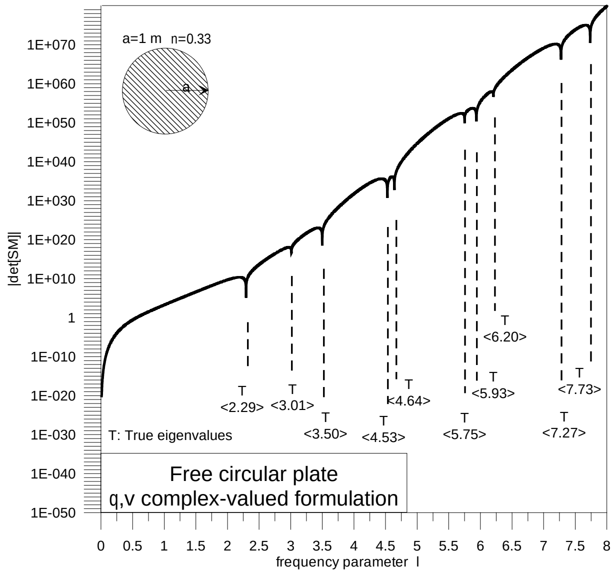


Figure 3-11.(e)

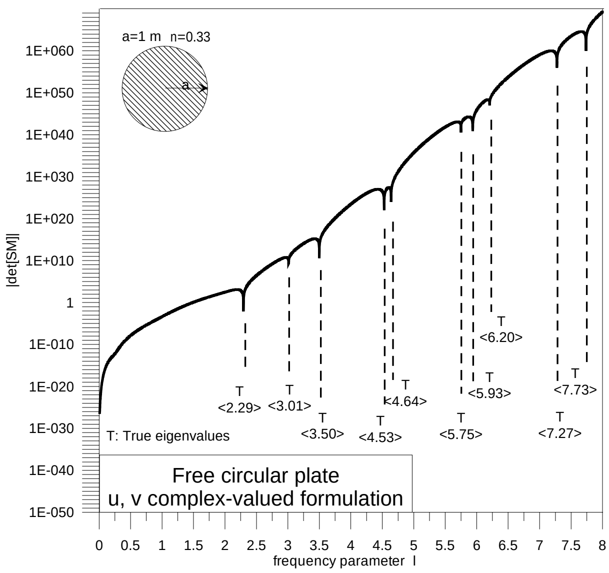


Figure 3-11.(c)

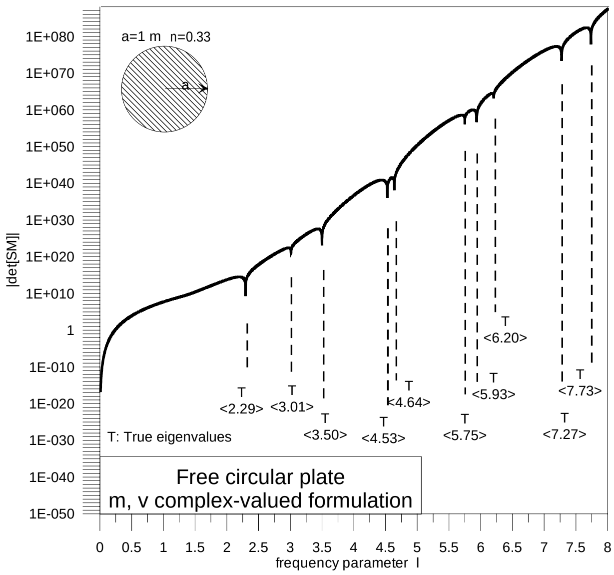


Figure 3-11.(f)

Figure 3-11 The determinant of the $[SM^f]$ versus frequency parameter λ for the free circular plate using the six complex-valued BEM.

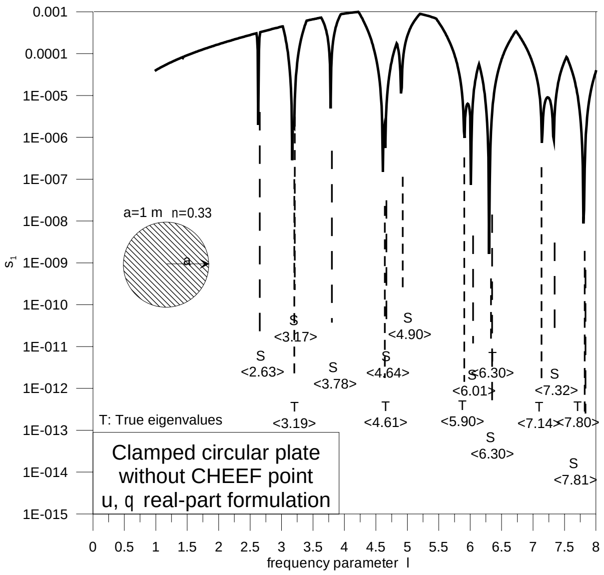


Figure 3-12.(a)

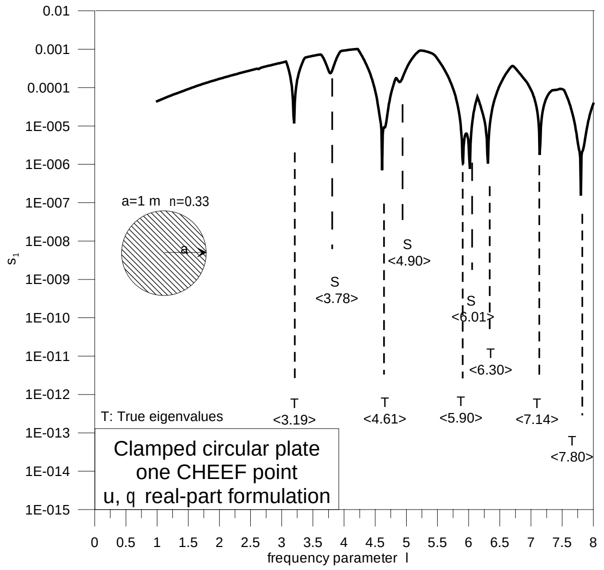


Figure 3-12.(b)

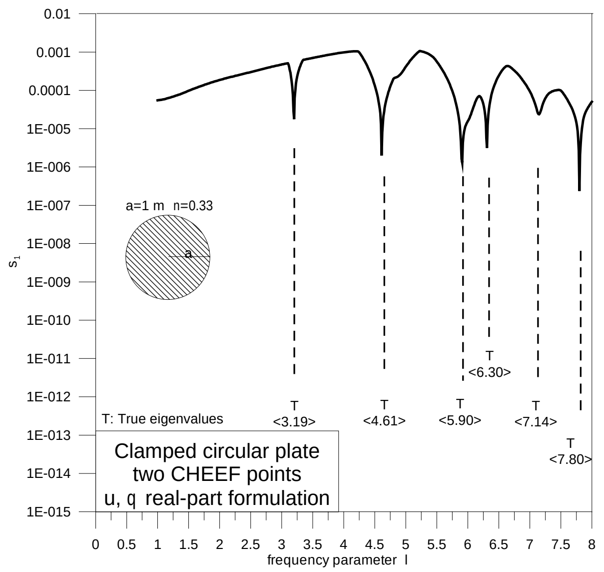


Figure 3-12.(c)

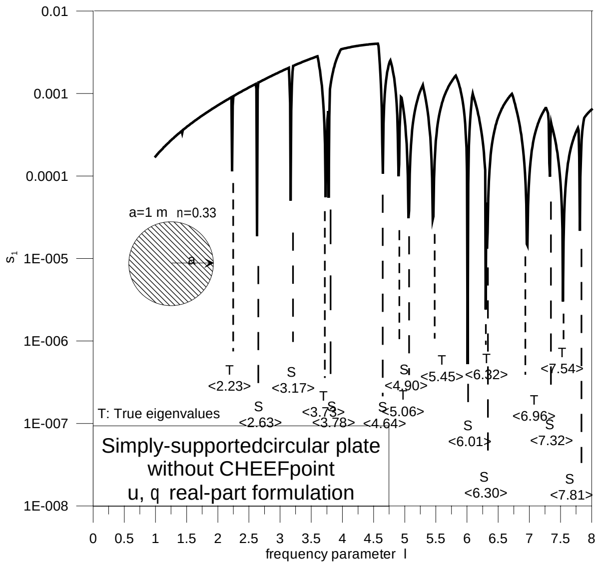


Figure 3-12.(d)

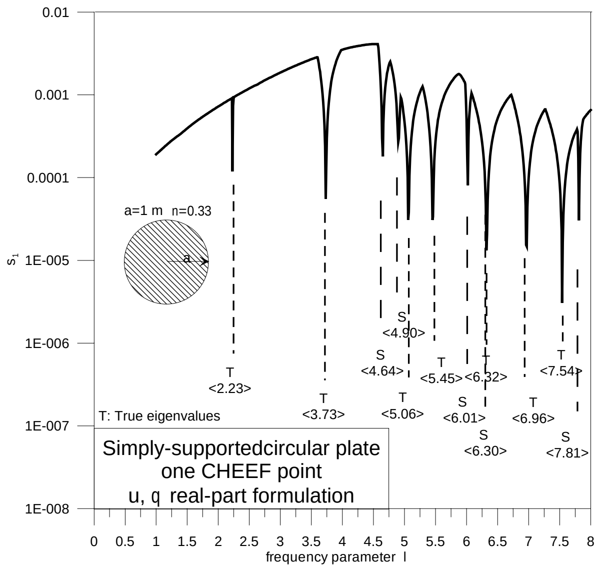


Figure 3-12.(e)

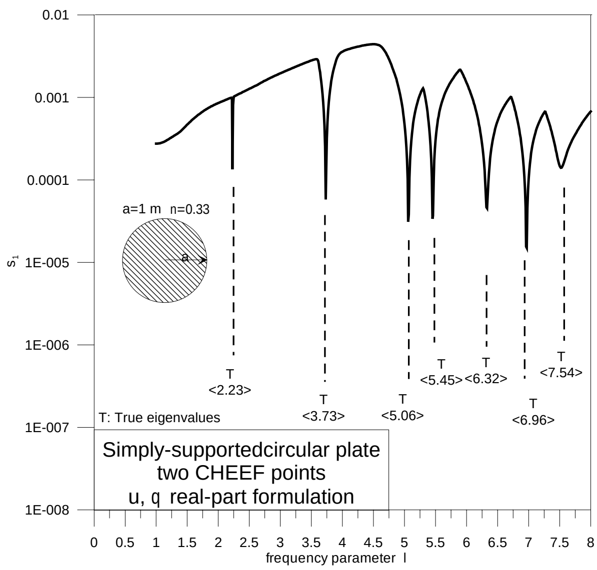


Figure 3-12.(f)

Figure 3-12 The minimum singularvalue σ_1 of the $[C^*]$ versus frequency parameter λ for the clamped and simply-supported circular plates by using the real-part BEM in conjunction with CHEEF method.

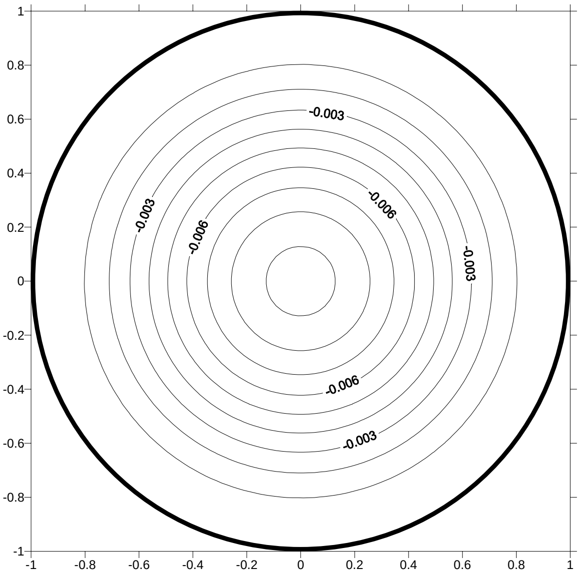


Figure 3-13.(a) $\lambda = 3.197$, n = 0

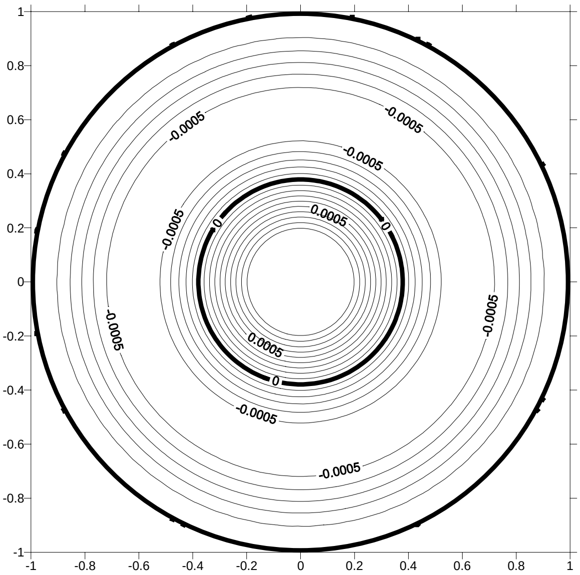


Figure 3-13.(d) $\lambda = 6.307$, n = 0

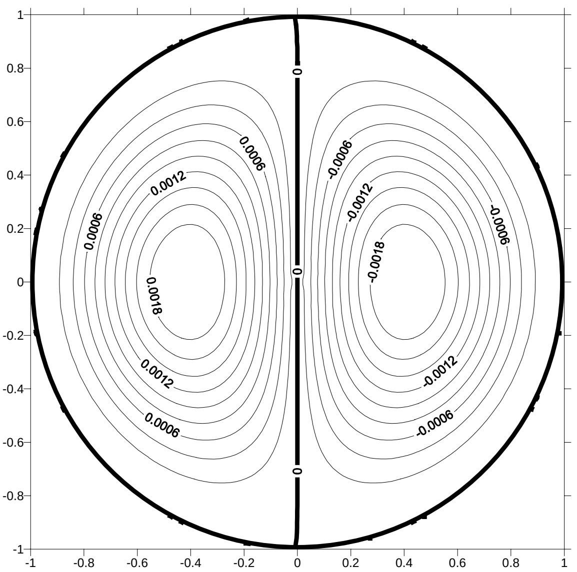


Figure 3-13.(b) $\lambda = 4.611$, n = 1

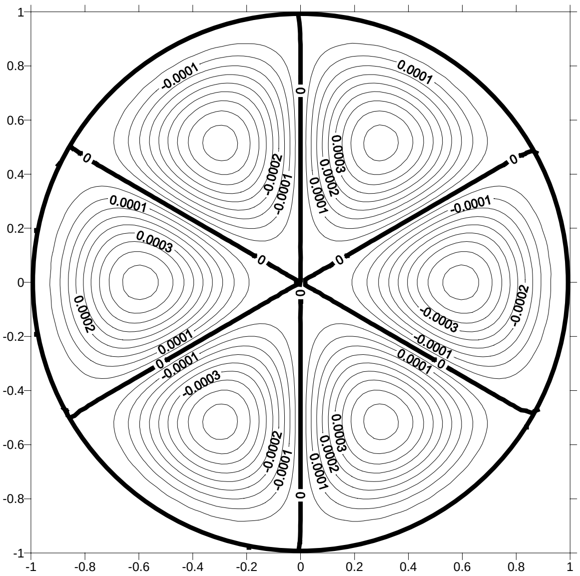


Figure 3-13.(e) $\lambda = 7.144$, n = 3

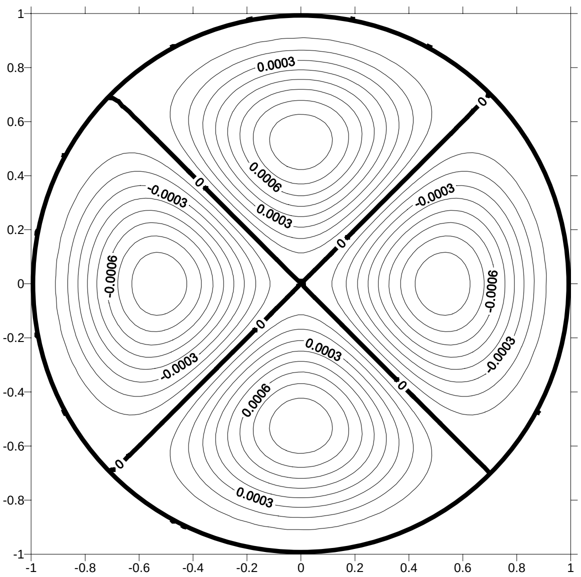


Figure 3-13.(c) $\lambda = 5.9067$, n = 2

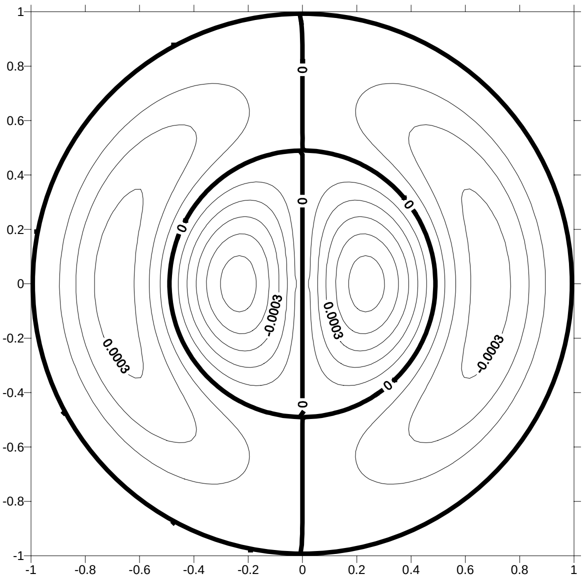


Figure 3-13.(f) $\lambda = 7.800$, n = 1

Figure 3-13 The former six modes of the exact solution for the clamped circular plate.

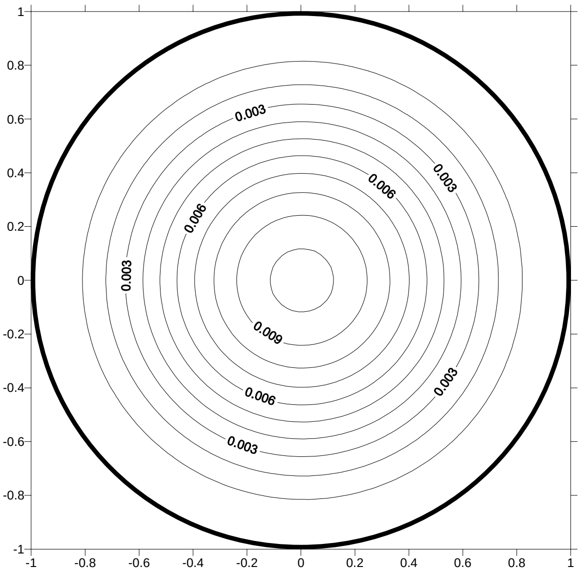


Figure 3-14.(a) $\lambda = 3.197$, n = 0

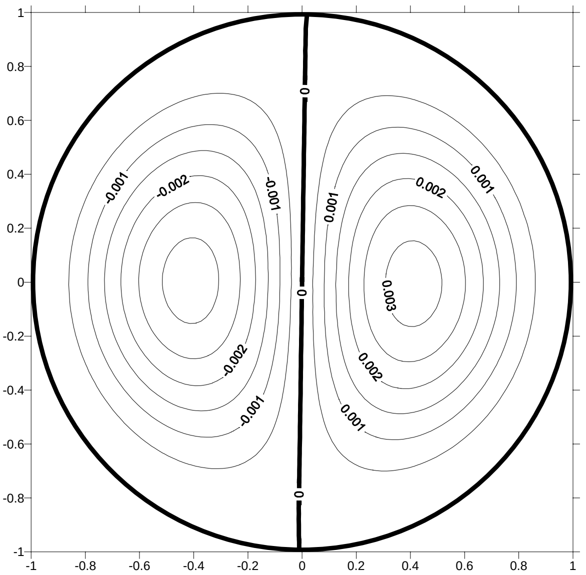


Figure 3-14.(b) $\lambda = 4.611$, n = 1

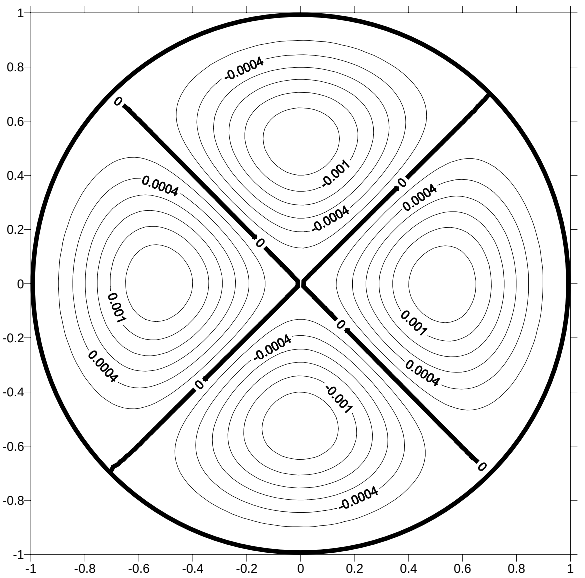


Figure 3-14.(c) $\lambda = 5.9067$, n = 2

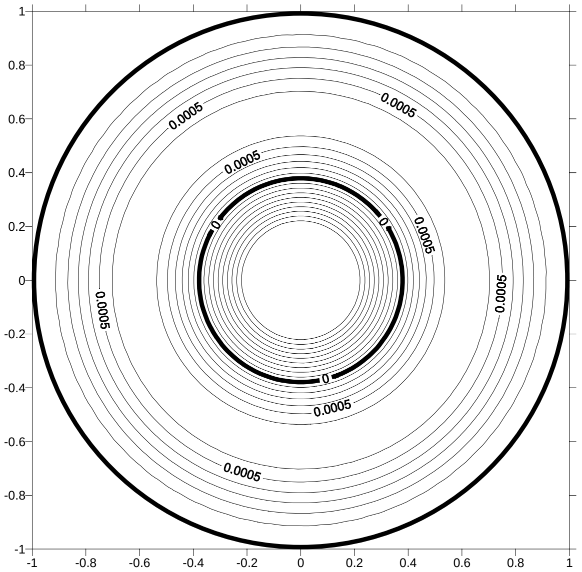


Figure 3-14.(d) $\lambda = 6.307$, n = 0

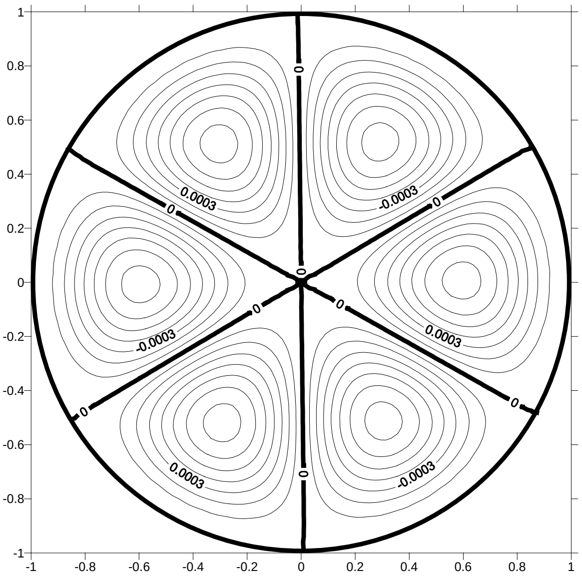


Figure 3-14.(e) $\lambda = 7.144$, n = 3

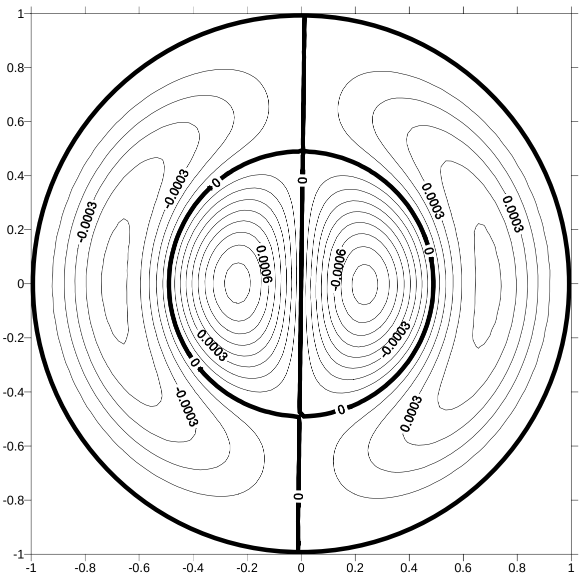


Figure 3-14.(f) $\lambda = 7.800$, n = 1

Figure 3-14 The former six modes for the clamped circular plate by using the real-part BEM.

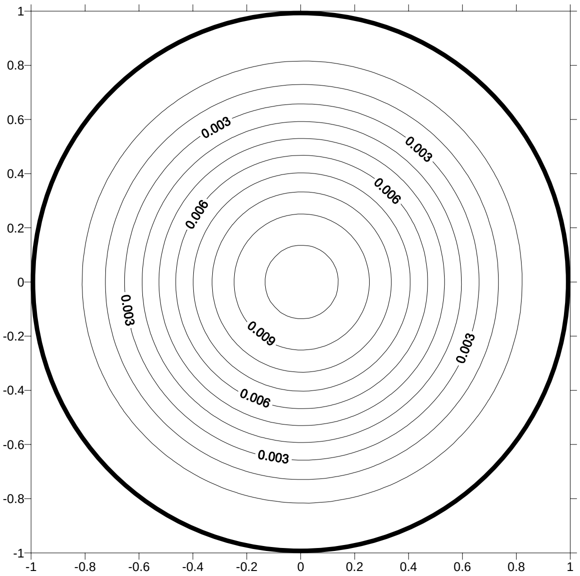


Figure 3-15.(a) $\lambda = 3.197$, n = 0

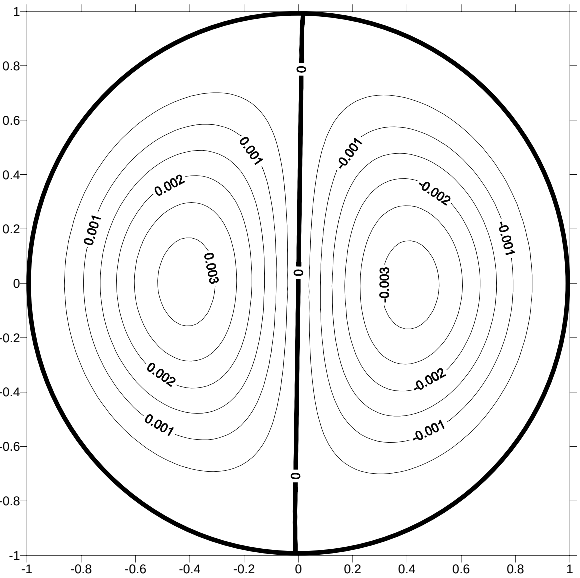


Figure 3-15.(b) $\lambda = 4.611$, n = 1

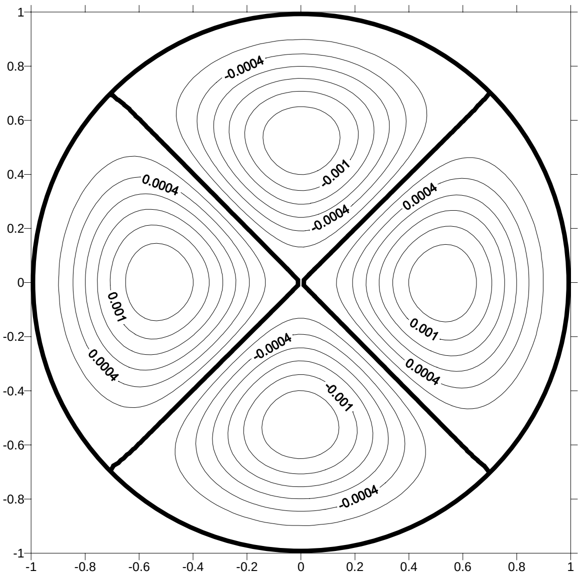


Figure 3-15.(c) $\lambda = 5.9067$, n = 2

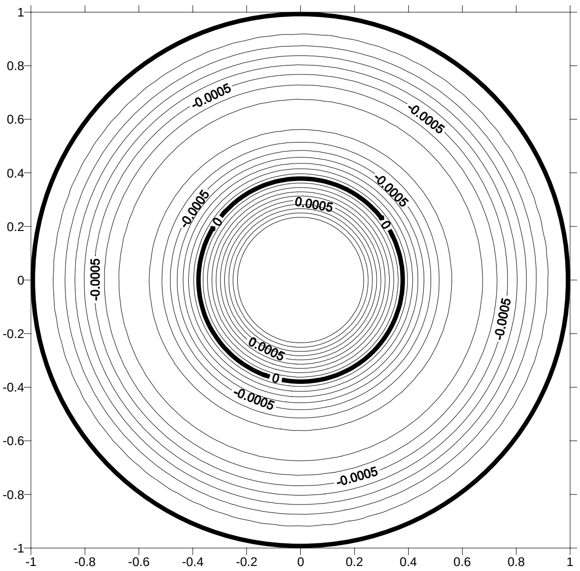


Figure 3-15.(d) $\lambda = 6.307$, n = 0

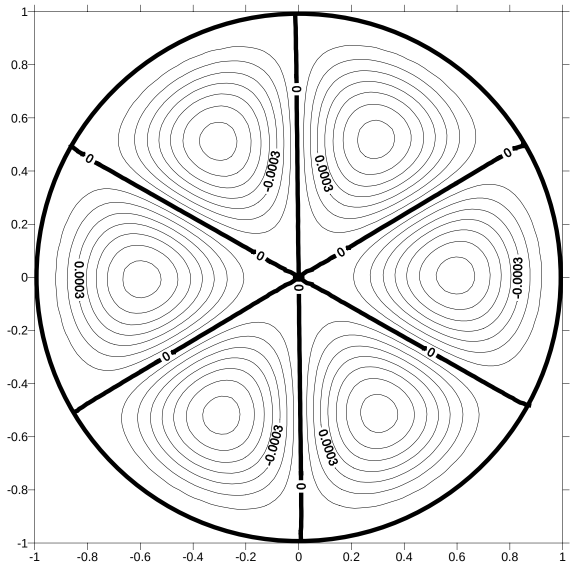


Figure 3-15.(e) $\lambda = 7.144$, n = 3

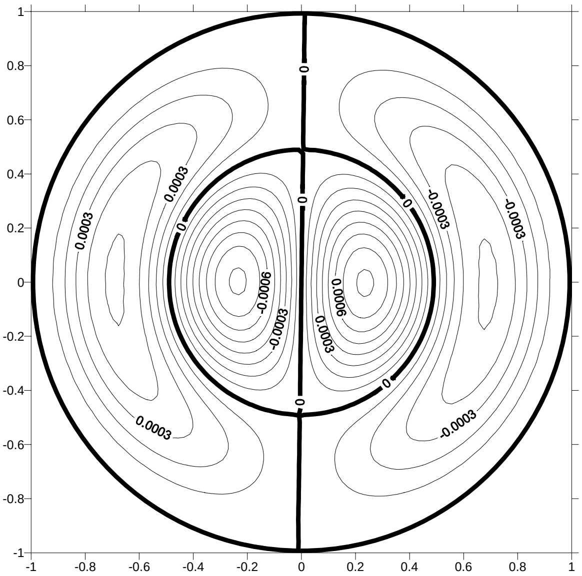


Figure 3-15.(f) $\lambda = 7.800$, n = 1

Figure 3-15 The former six modes for the clamped circular plate by using the complex-valued BEM.

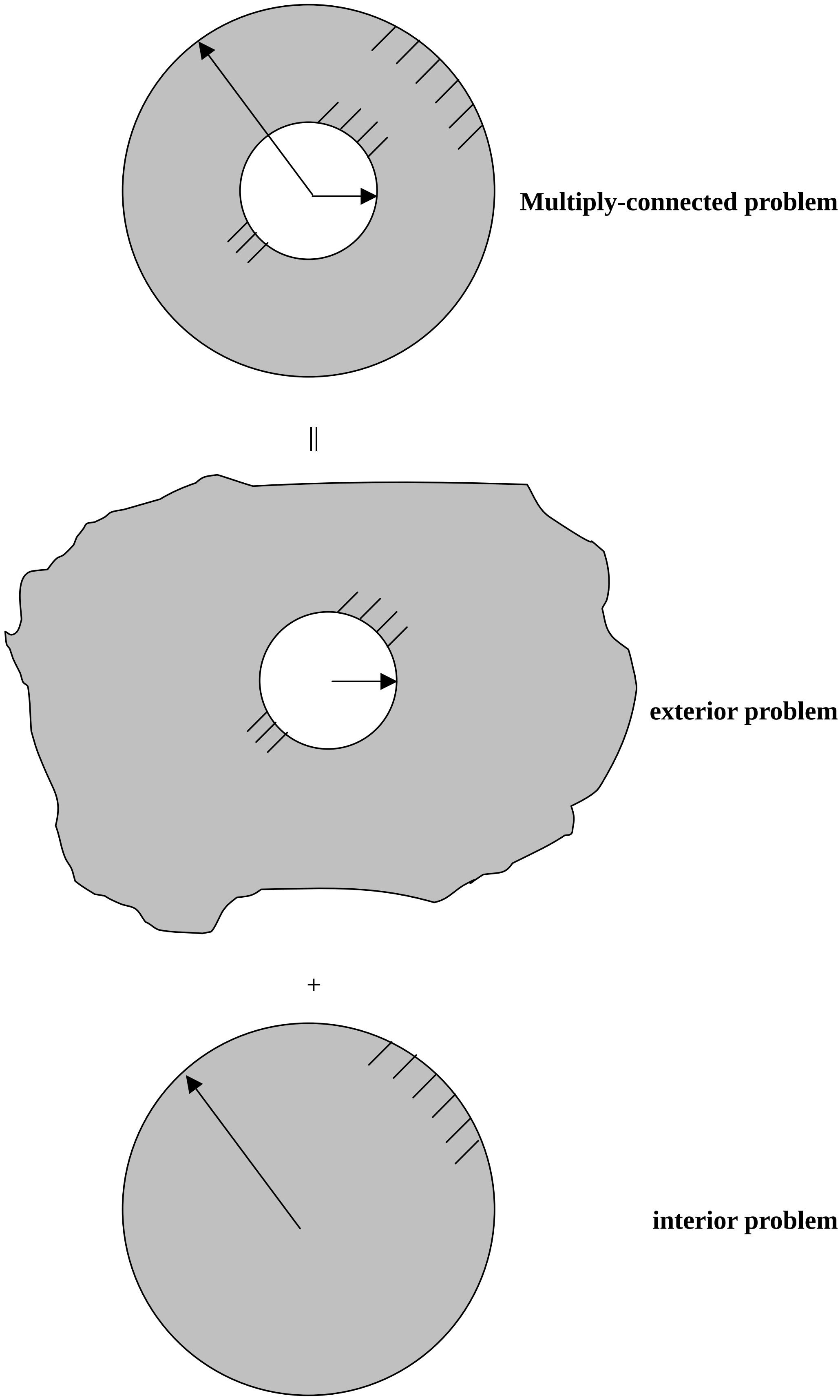


Figure 4-1 The relationship among the multiply-connected problem, exterior problem and interior problem.

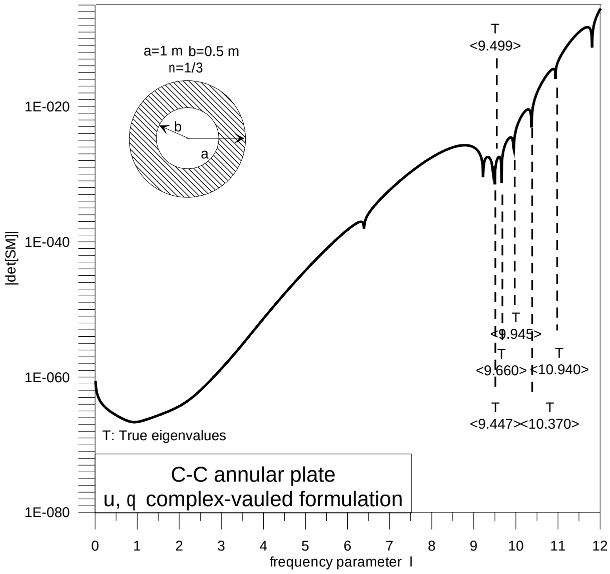


Figure 4-2.(a)

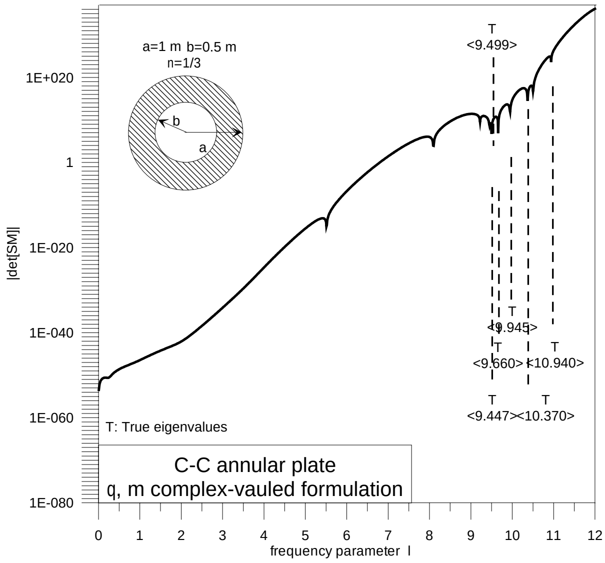


Figure 4-2.(d)

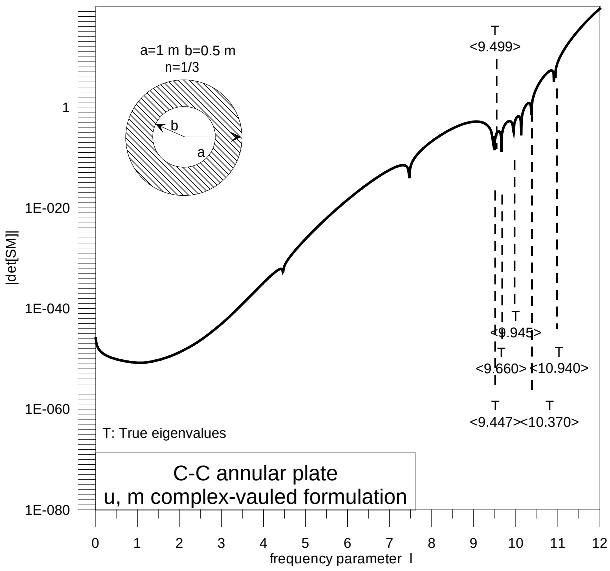


Figure 4-2.(b)

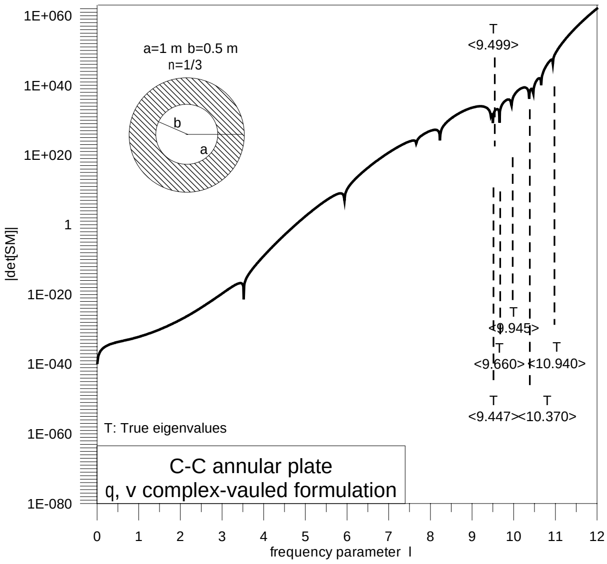


Figure 4-2.(e)

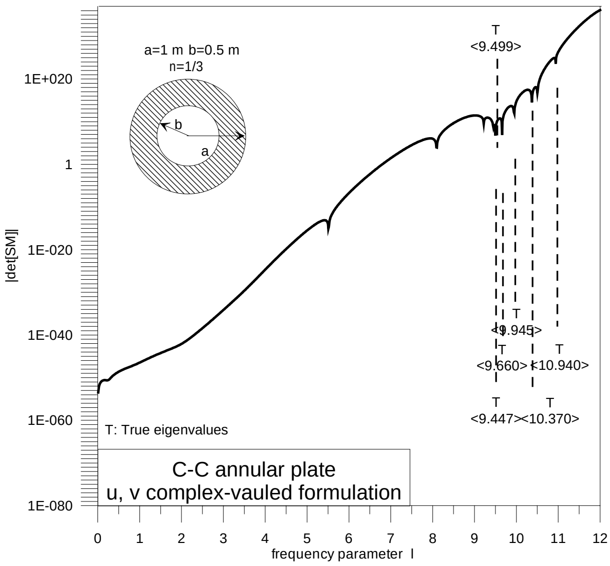


Figure 4-2.(c)

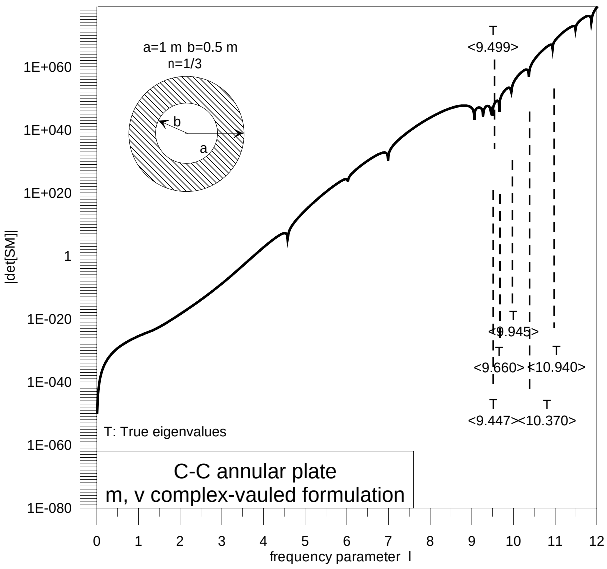


Figure 4-2.(f)

Figure 4-2 The determinant of the of the $[SM^{cc}]$ versus frequency parameter λ for the C-C annular plate by using the complex-valued formulations.

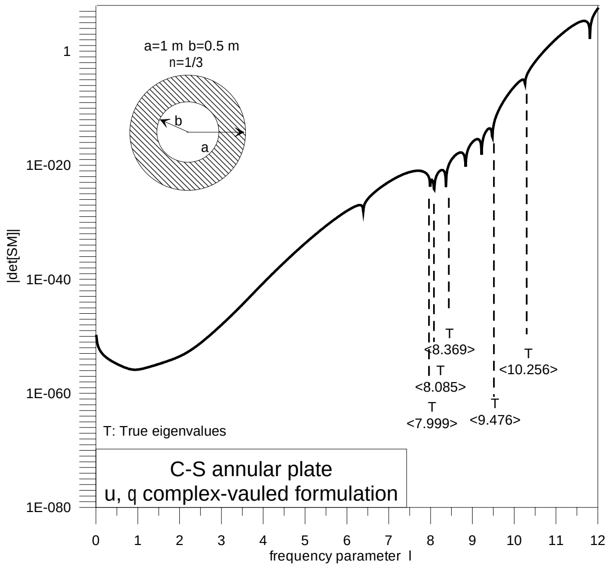


Figure 4-3.(a)

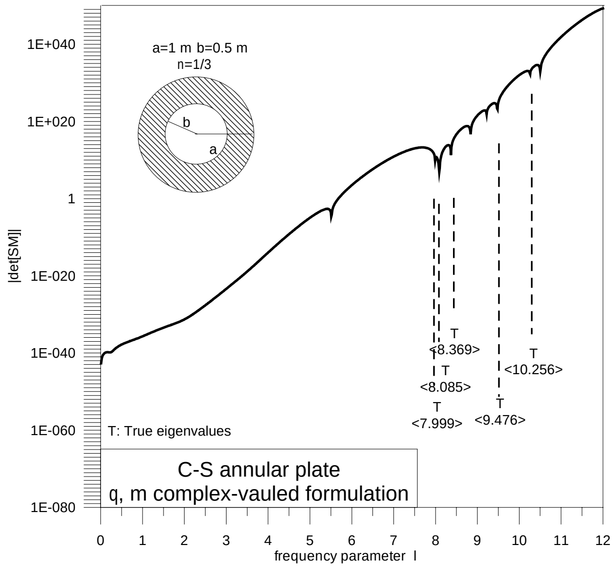


Figure 4-3.(d)

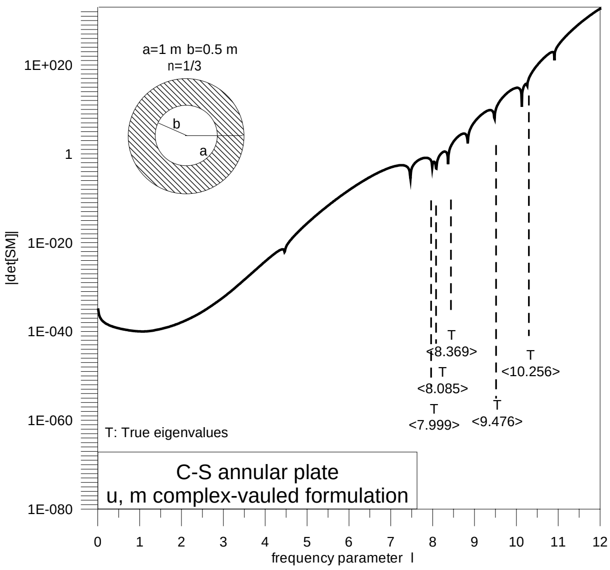


Figure 4-3.(b)

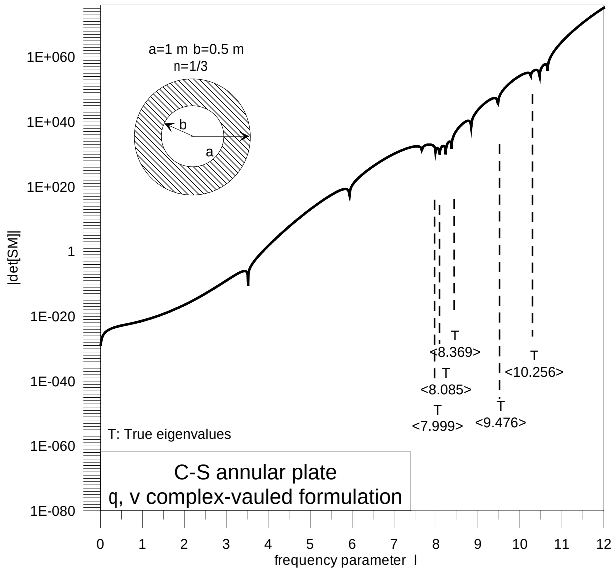


Figure 4-3.(e)

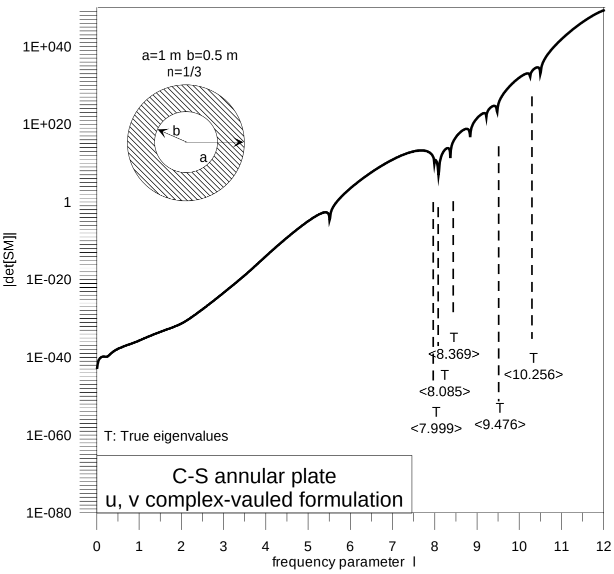


Figure 4-3.(c)

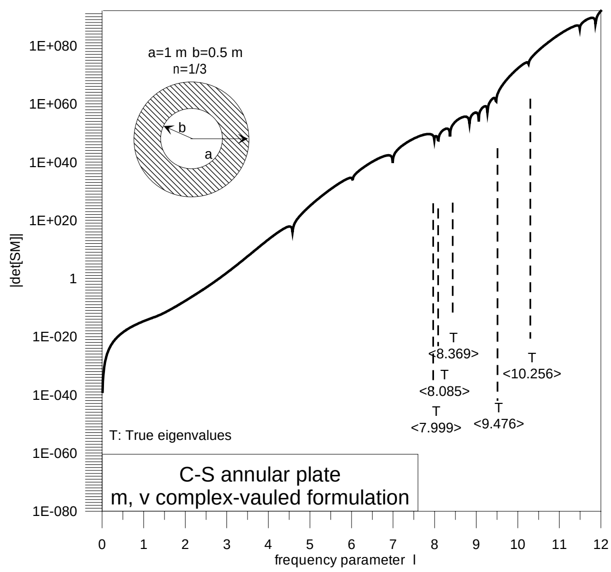


Figure 4-3.(f)

Figure 4-3 The determinant of the of the $[SM^{cs}]$ versus frequency parameter λ for the C-S annular plate by using the complex-valued formulations.

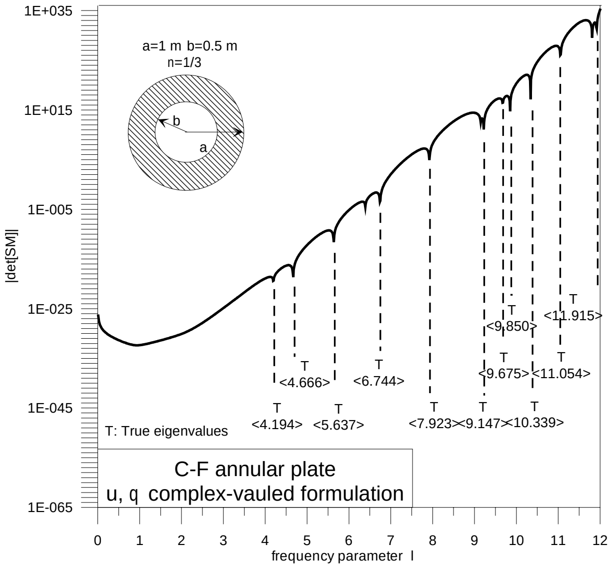


Figure 4-4.(a)

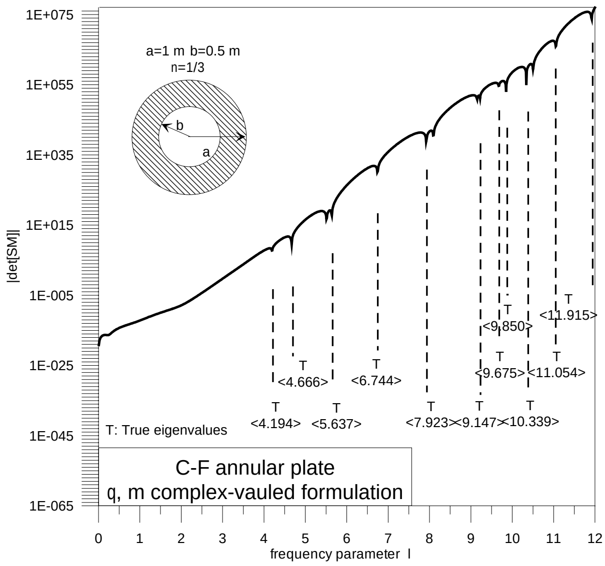


Figure 4-4.(d)

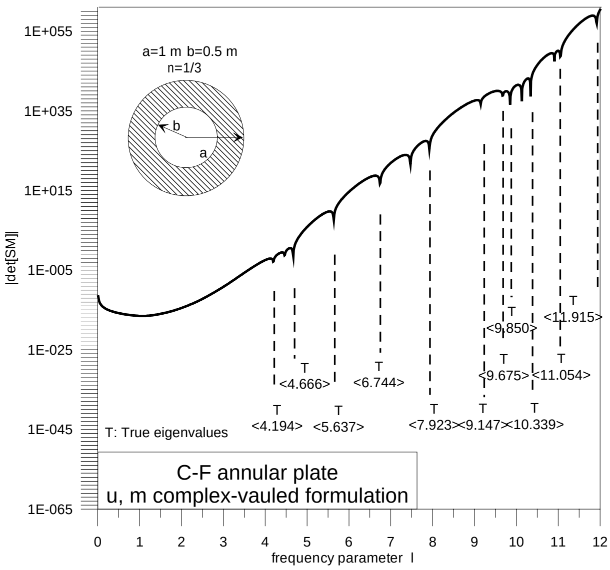


Figure 4-4.(b)

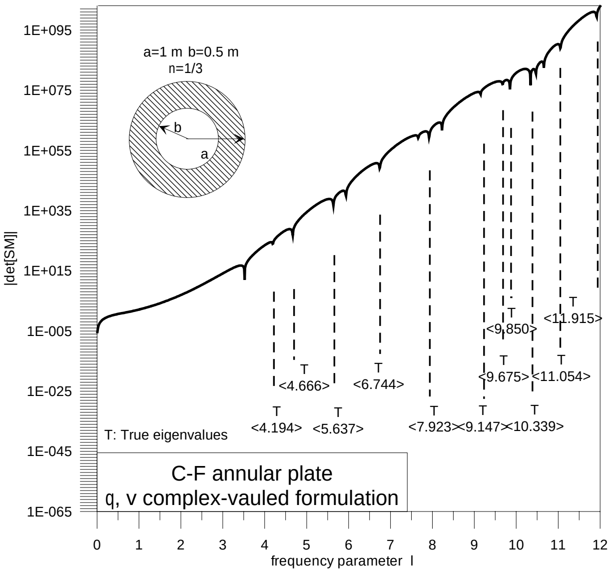


Figure 4-4.(e)

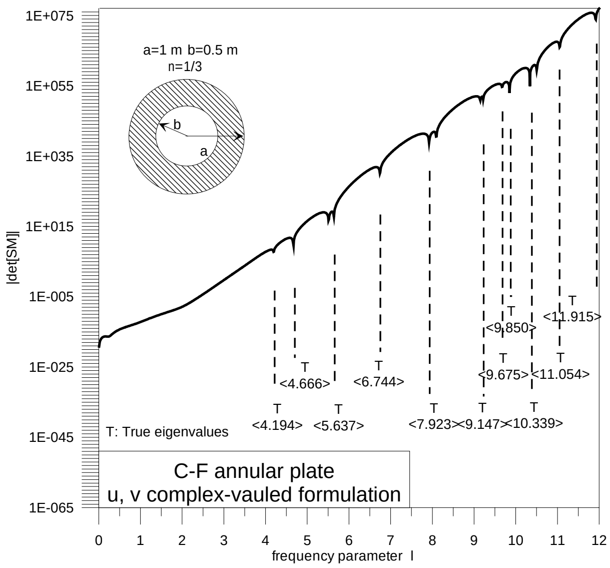


Figure 4-4.(c)

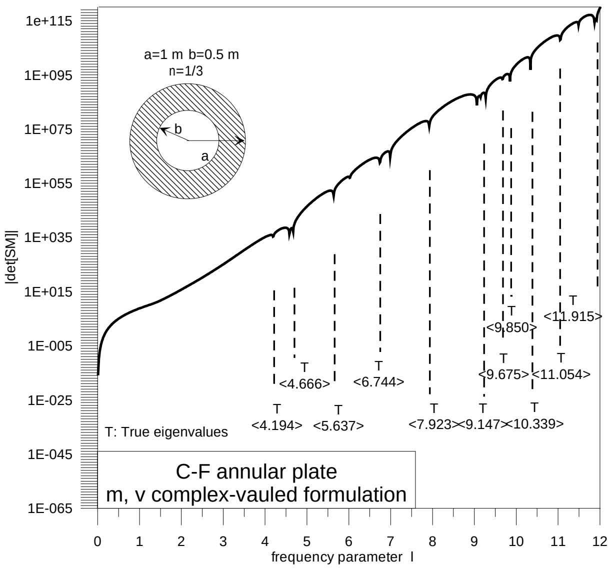


Figure 4-4.(f)

Figure 4-4 The determinant of the $[SM^{cf}]$ versus frequency parameter λ for the C-F annular plate by using the complex-valued formulations.

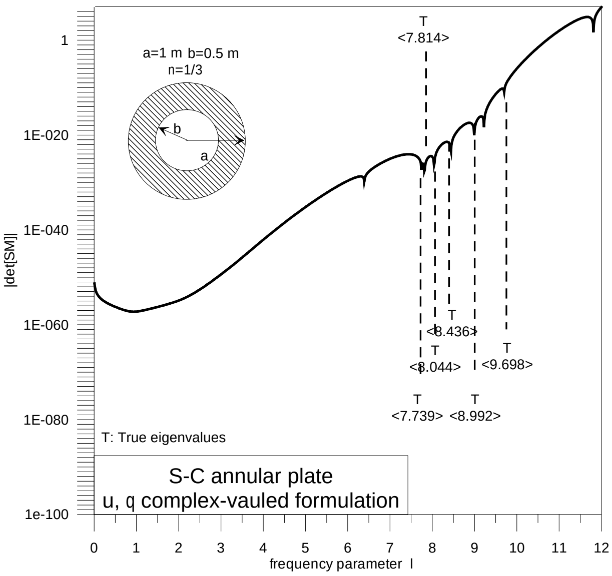


Figure 4-5.(a)

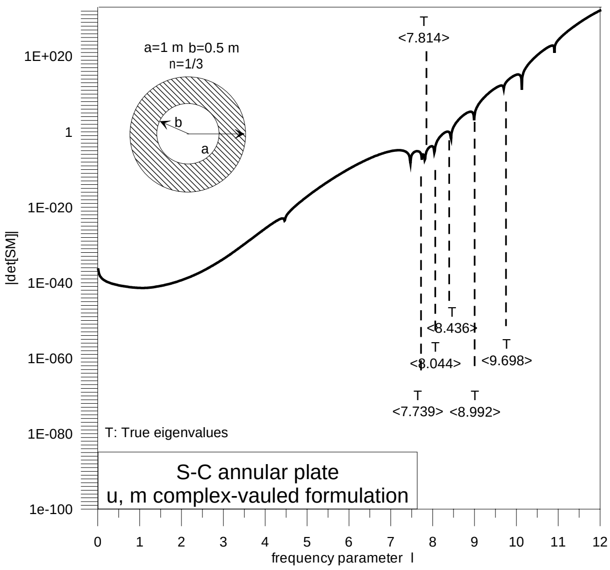


Figure 4-5.(b)

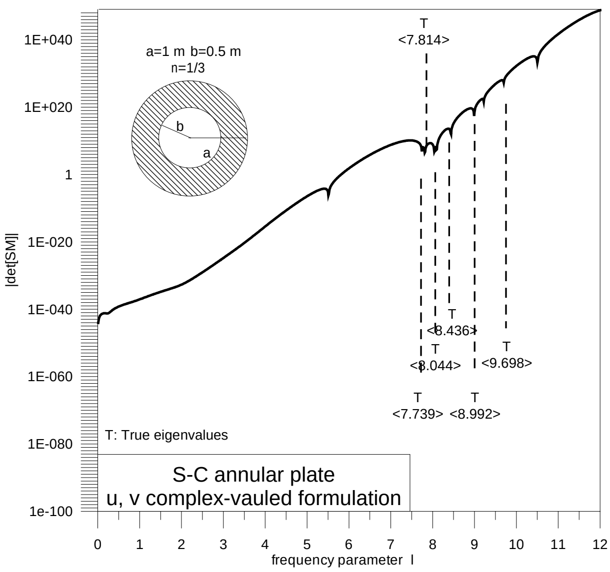


Figure 4-5.(c)

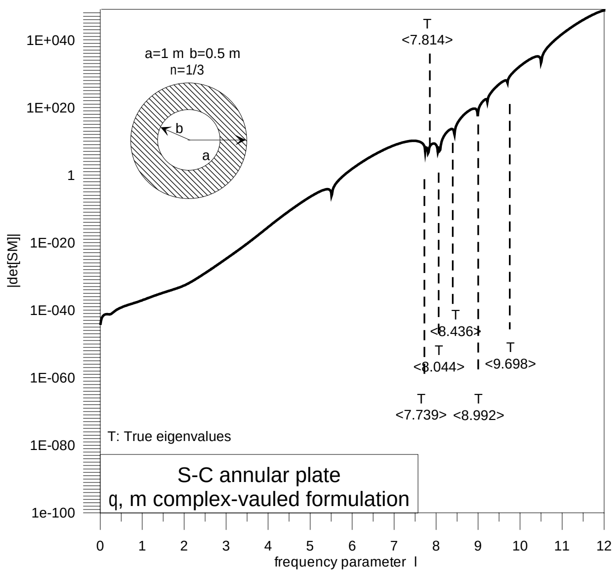


Figure 4-5.(d)

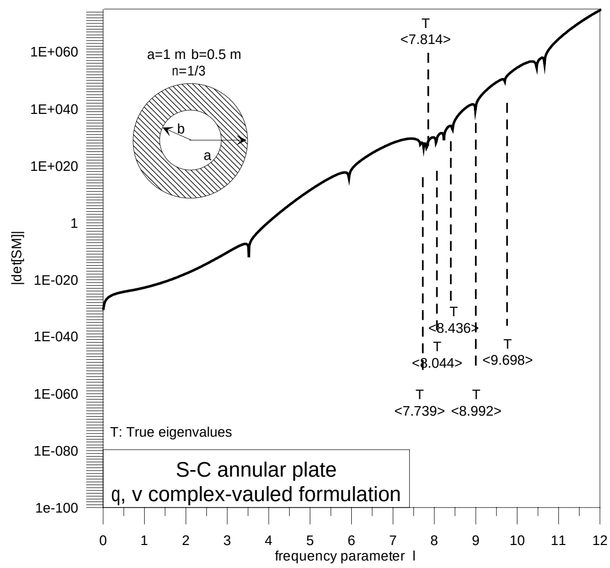


Figure 4-5.(e)

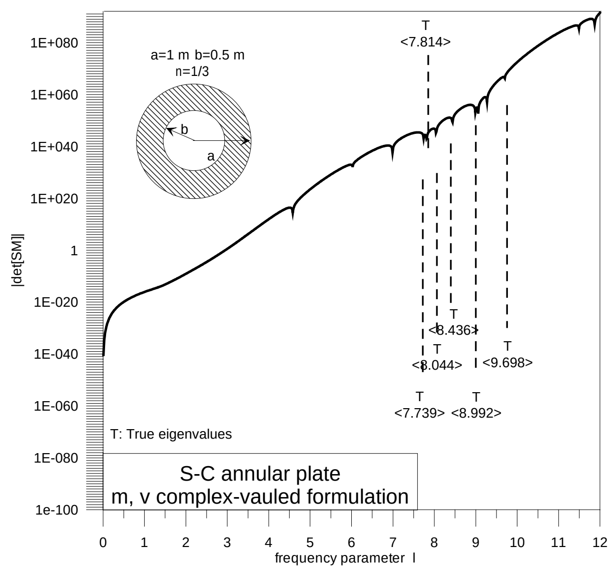


Figure 4-5.(f)

Figure 4-5 The determinant of the $[SM^{sc}]$ versus frequency parameter λ for the S-C annular plate by using the complex-valued formulations.

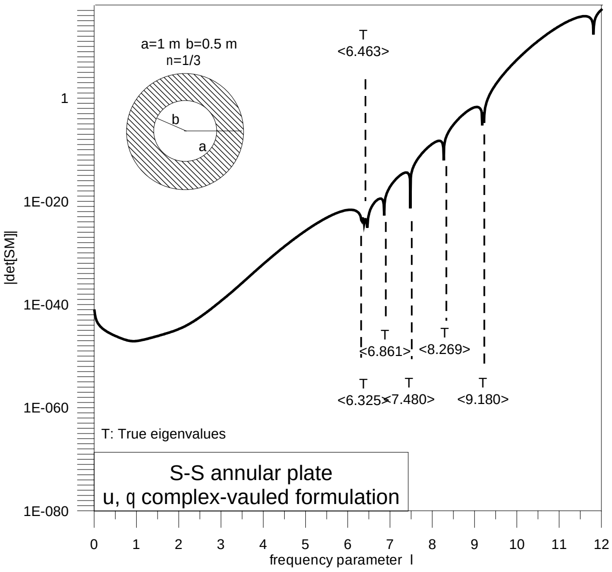


Figure 4-6.(a)

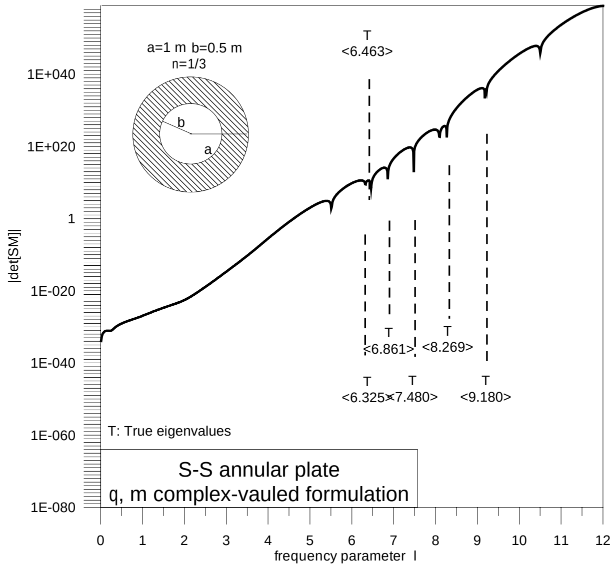


Figure 4-6.(d)

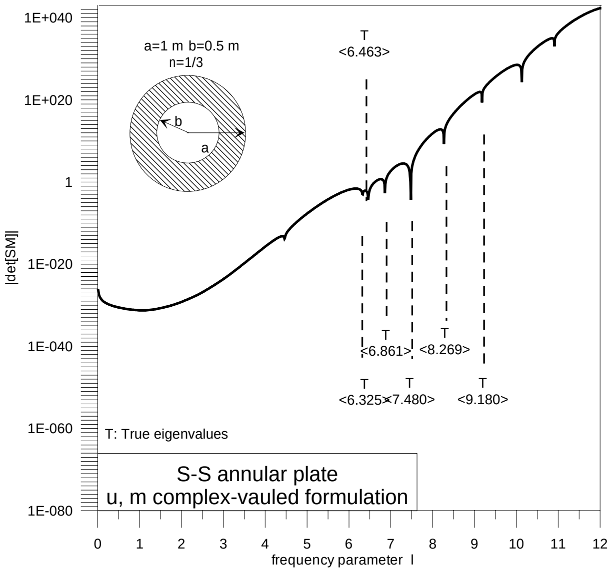


Figure 4-6.(b)

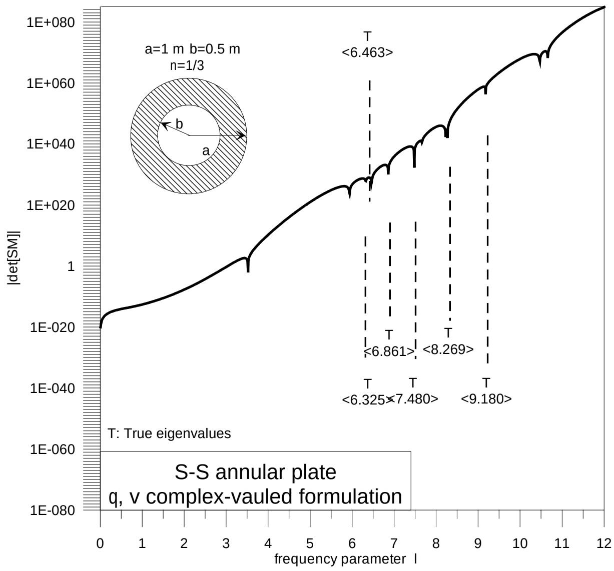


Figure 4-6.(e)

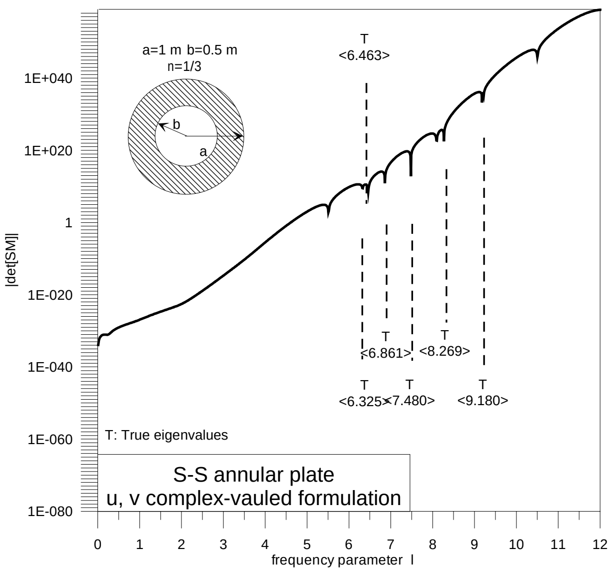


Figure 4-6.(c)

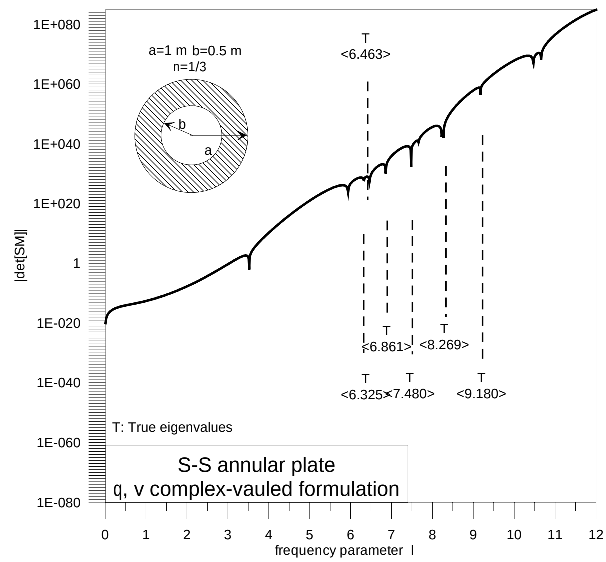


Figure 4-6.(f)

Figure 4-6 The determinant of the $[SM^{ss}]$ versus frequency parameter λ for the S-S annular plate by using the complex-valued formulations.

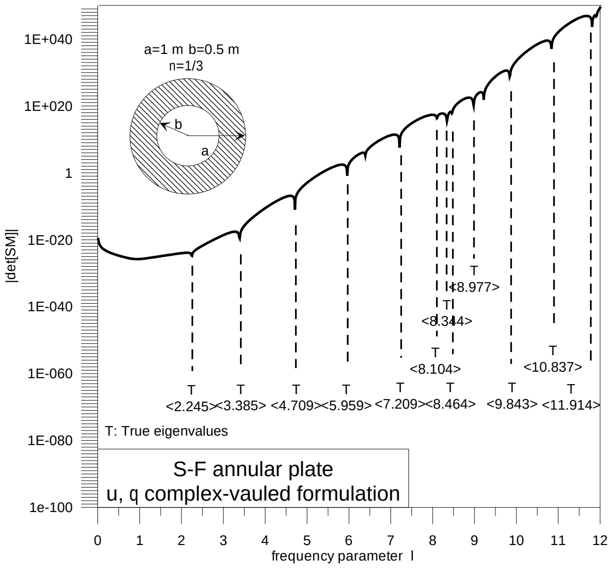


Figure 4-7.(a)

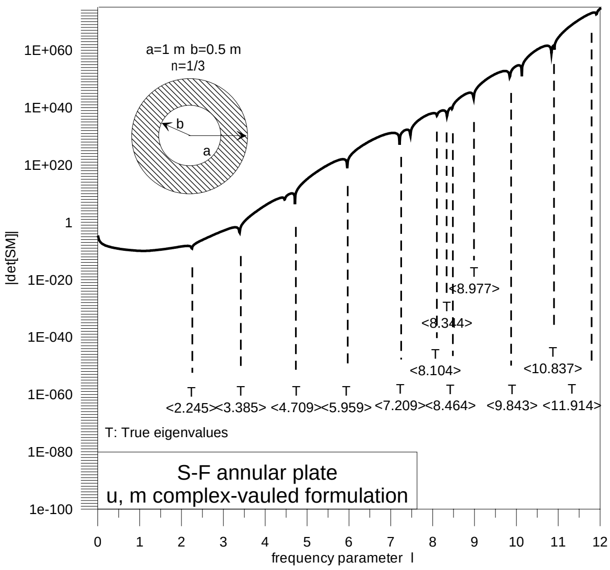


Figure 4-7.(b)

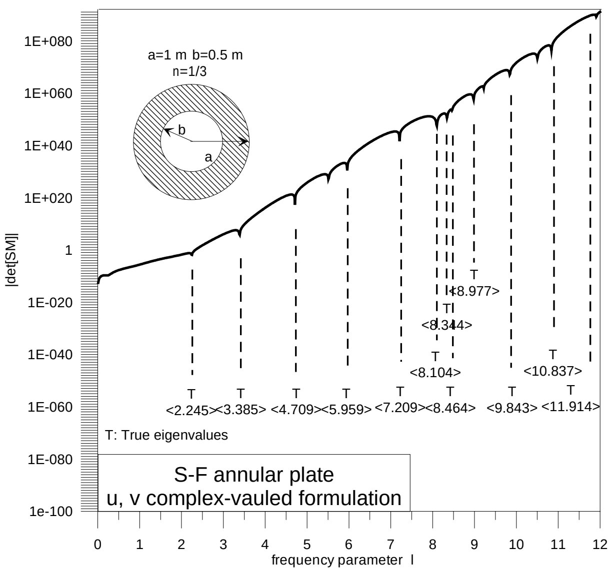


Figure 4-7.(c)

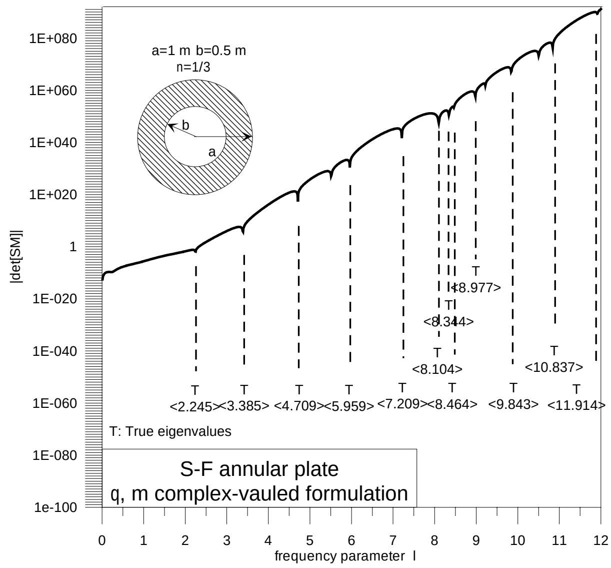


Figure 4-7.(d)

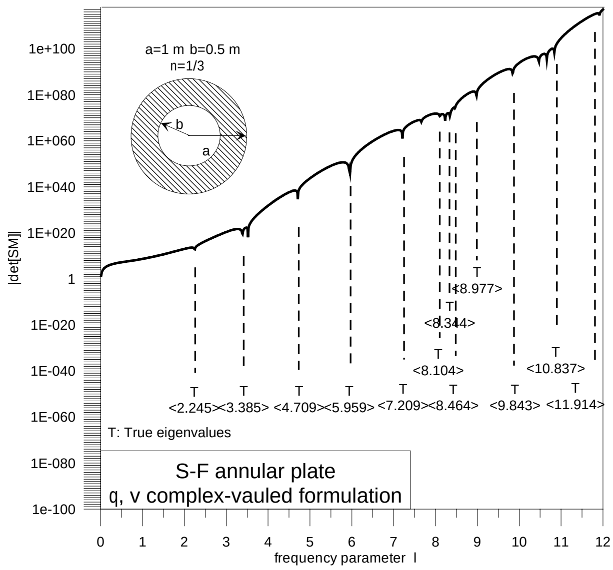


Figure 4-7.(e)

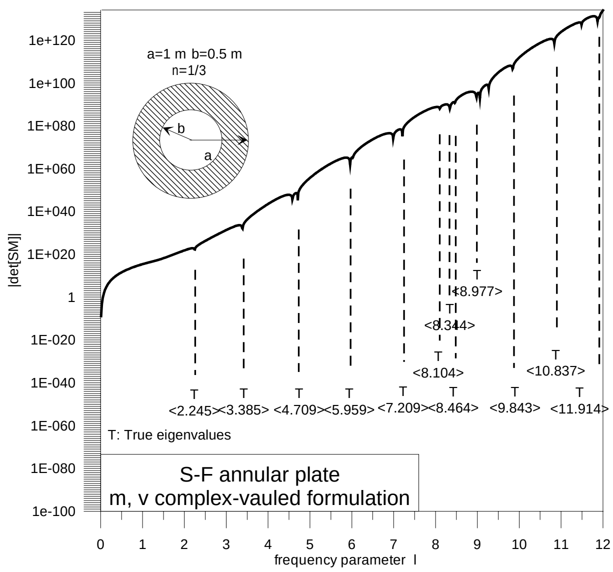


Figure 4-7.(f)

Figure 4-7 The determinant of the $[SM^{sf}]$ versus frequency parameter λ for the S-F annular plate by using the complex-valued formulations.

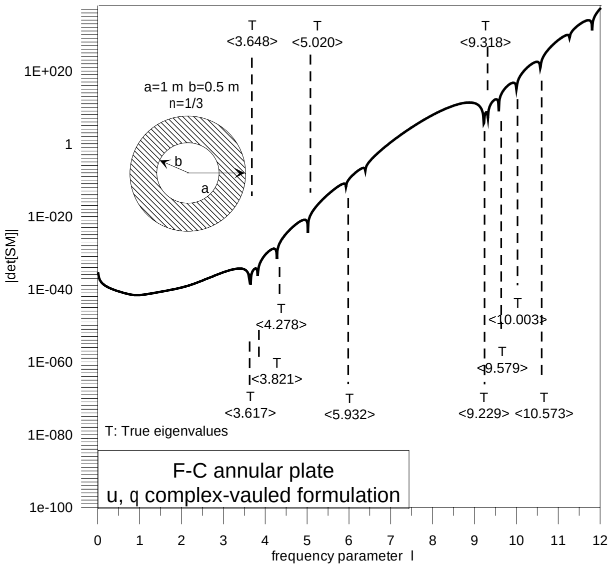


Figure 4-8.(a)

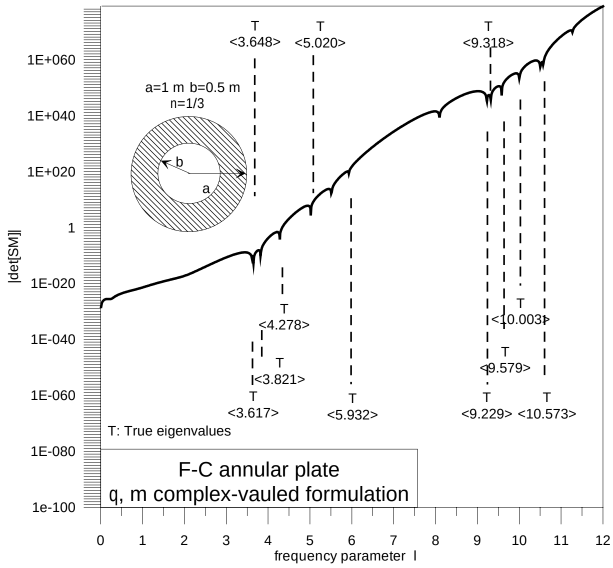


Figure 4-8.(d)

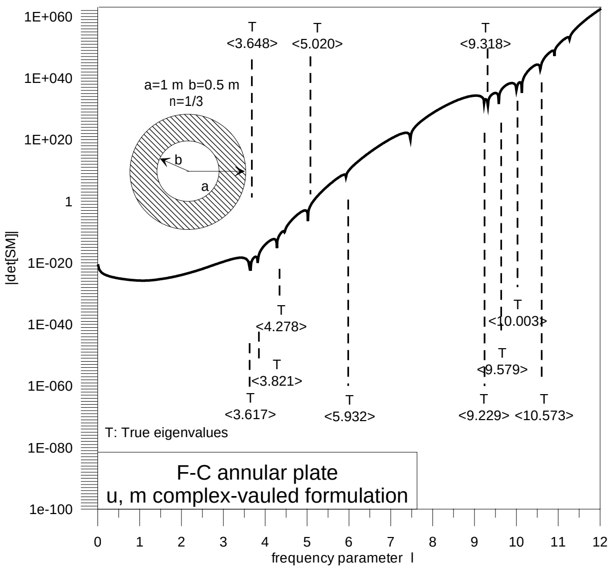


Figure 4-8.(b)

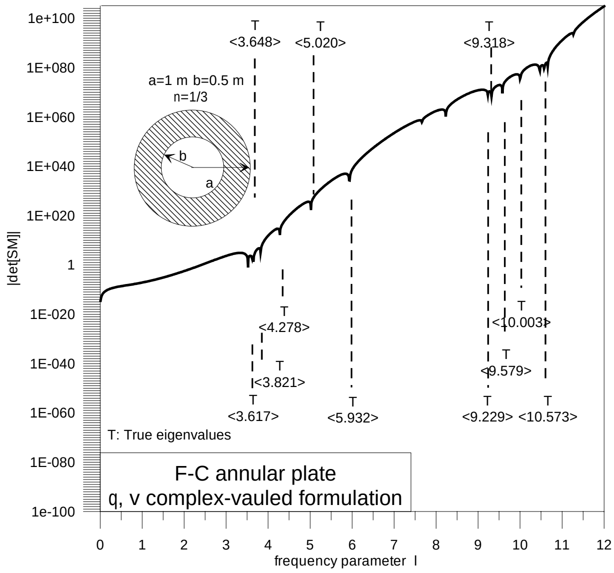


Figure 4-8.(e)

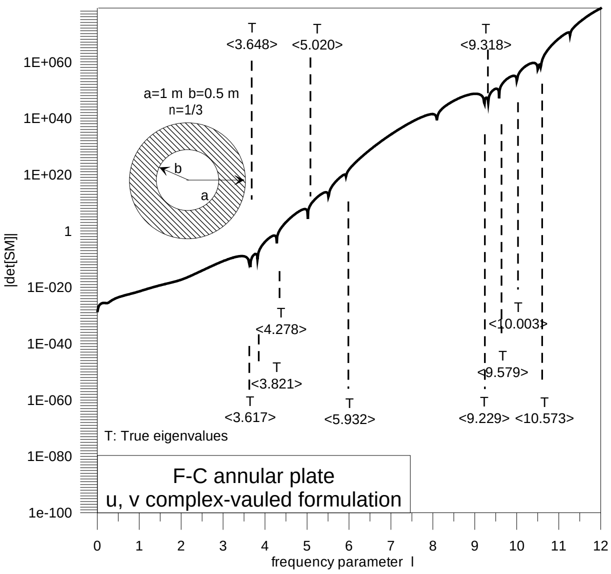


Figure 4-8.(c)

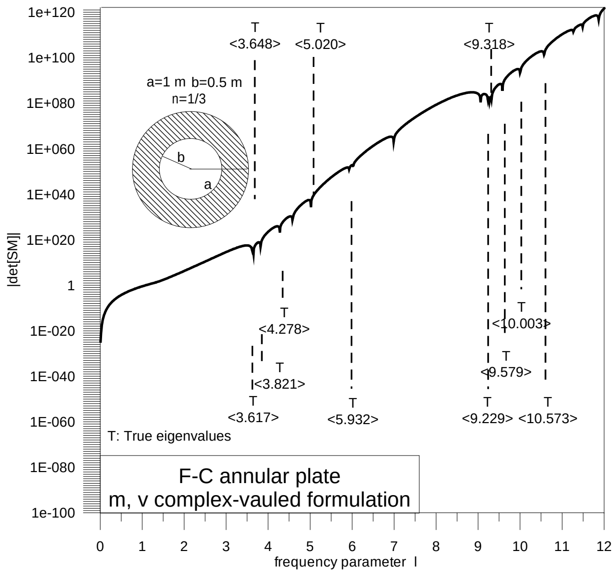


Figure 4-8.(f)

Figure 4-8 The determinant of the $[SM^{fc}]$ versus frequency parameter λ for the F-C annular plate by using the complex-valued formulations.

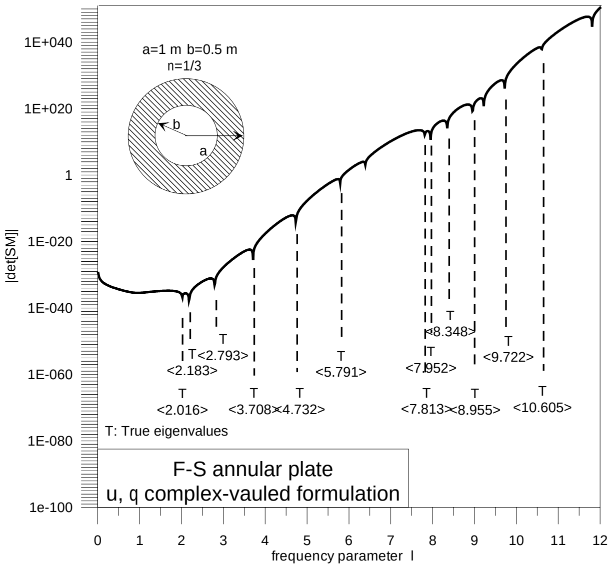


Figure 4-9.(a)

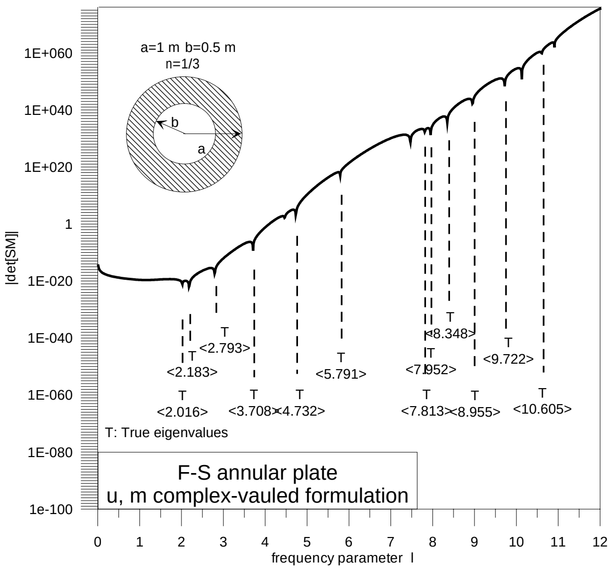


Figure 4-9.(b)

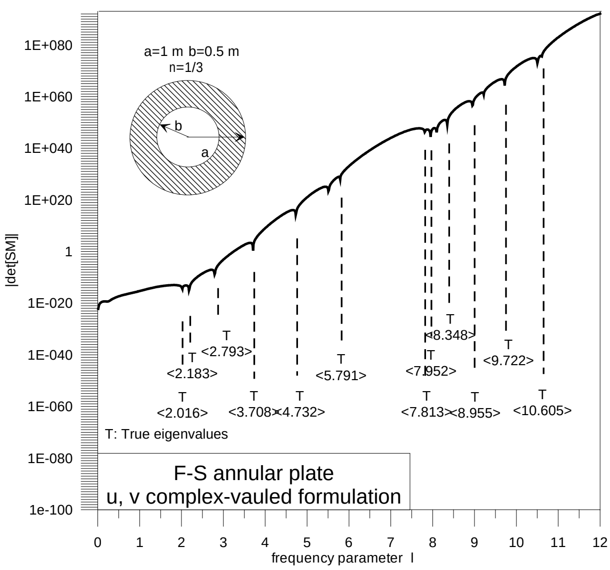


Figure 4-9.(c)

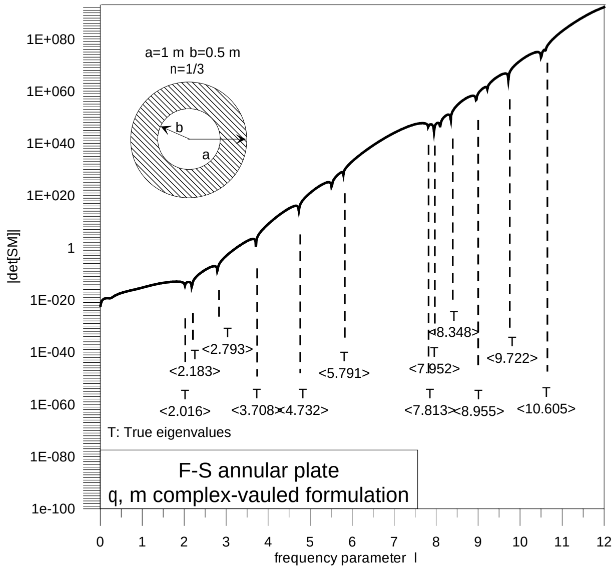


Figure 4-9.(d)

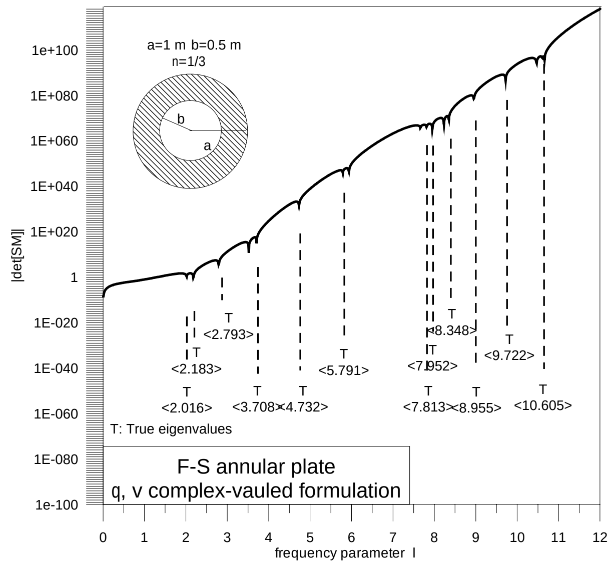


Figure 4-9.(e)

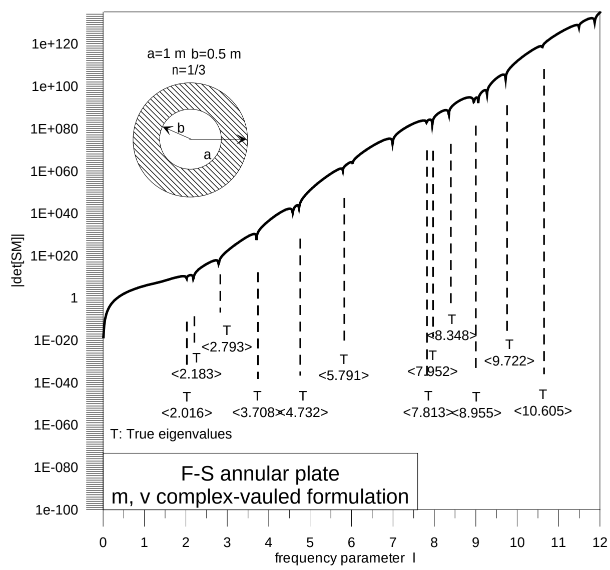


Figure 4-9.(f)

Figure 4-9 The determinant of the $[SM^{fs}]$ versus frequency parameter λ for the F-S annular plate by using the complex-valued formulations.

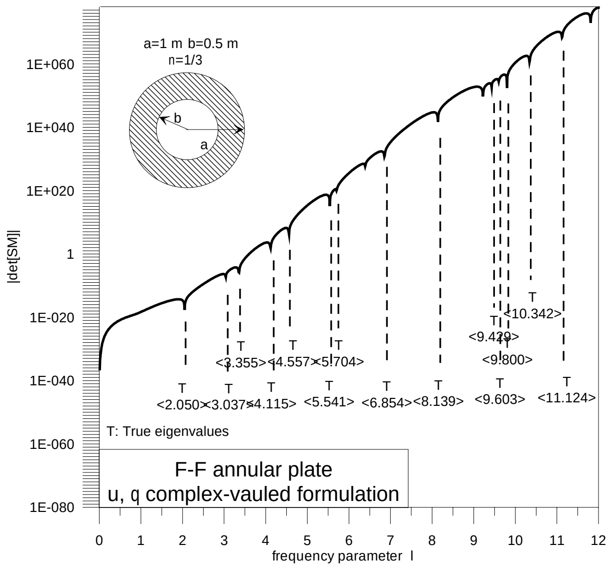


Figure 4-10.(a)

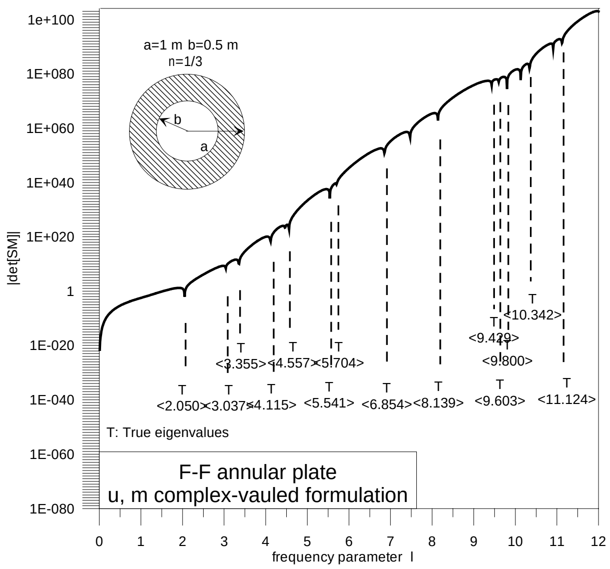


Figure 4-10.(b)

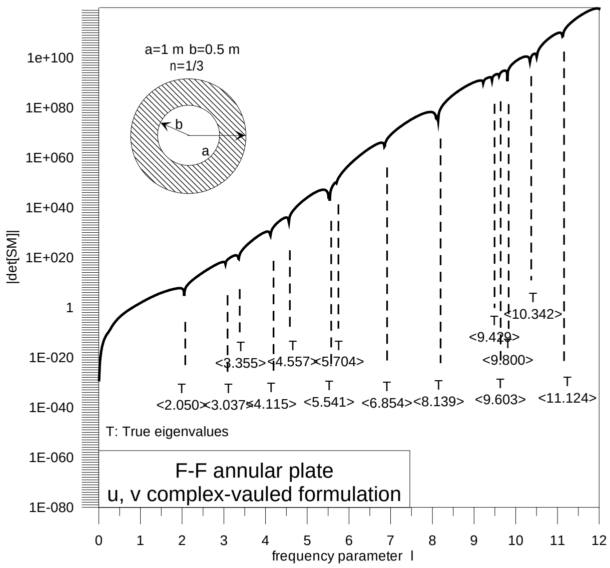


Figure 4-10.(c)

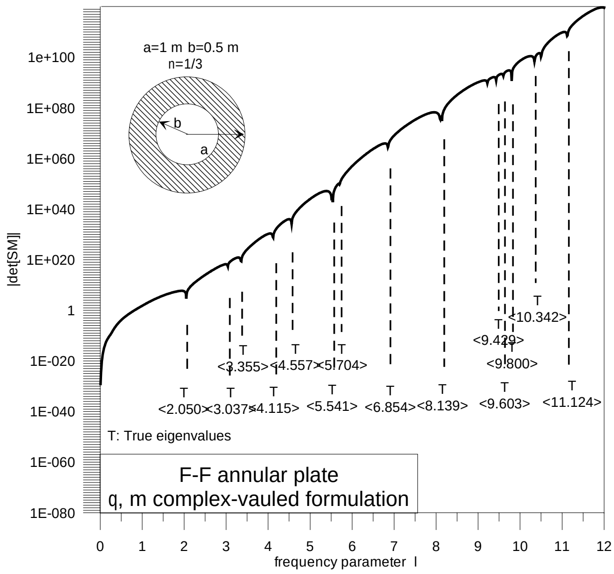


Figure 4-10.(d)

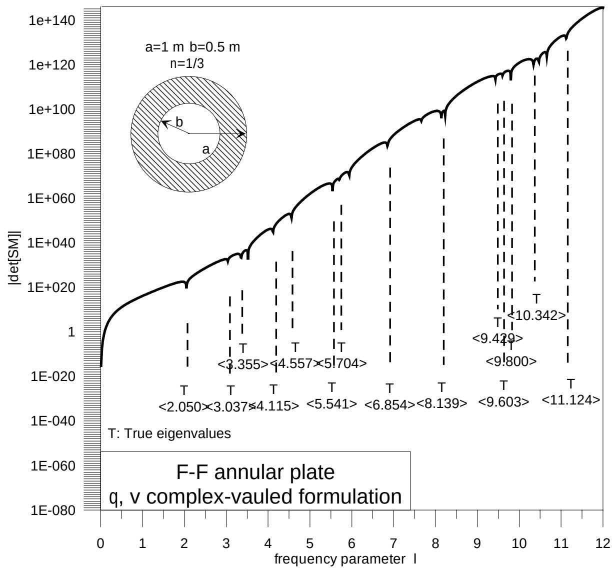


Figure 4-10.(e)

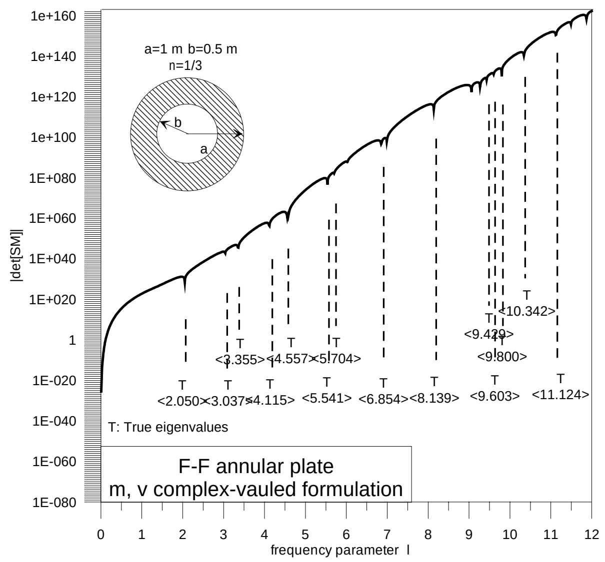


Figure 4-10.(f)

Figure 4-10 The determinant of the $[SM^{ff}]$ versus frequency parameter λ for the F-F annular plate by using the complex-valued formulations.

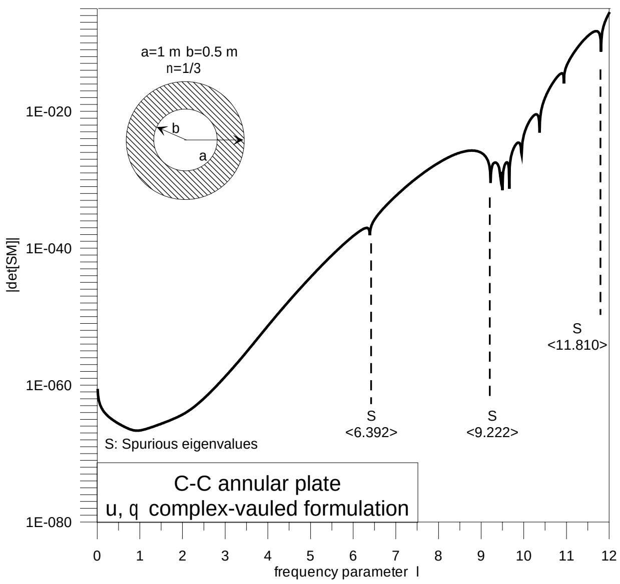


Figure 4-11.(a)

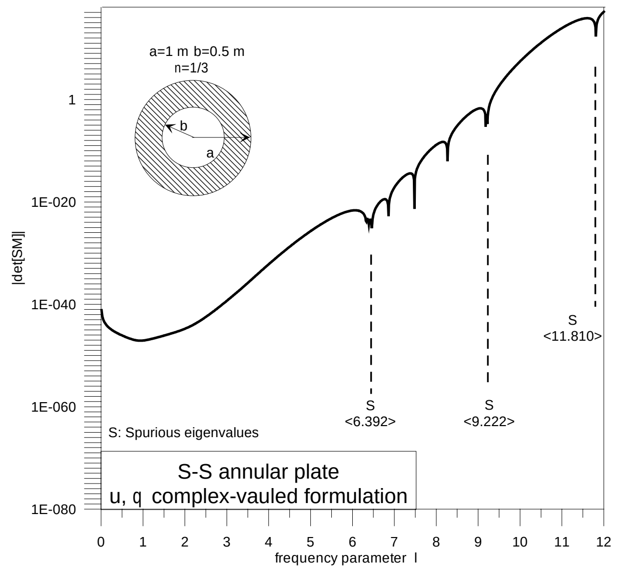


Figure 4-11.(b)

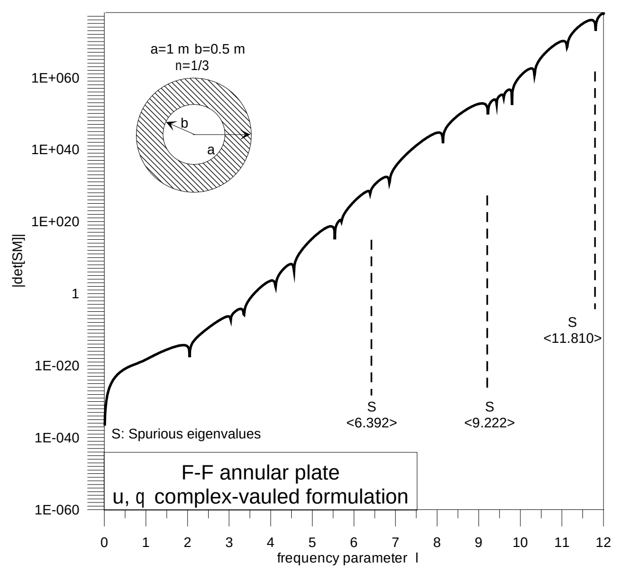


Figure 4-11.(c)

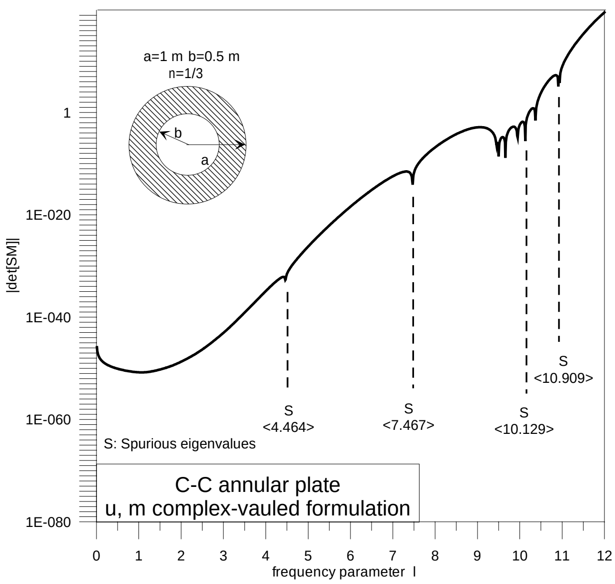


Figure 4-11.(d)

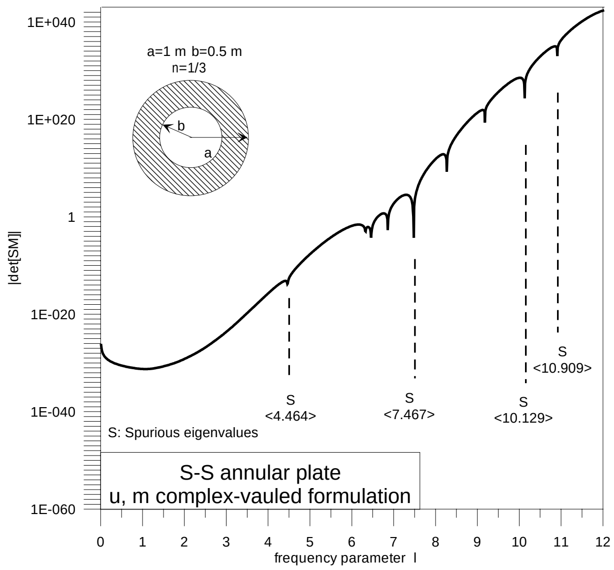


Figure 4-11.(e)

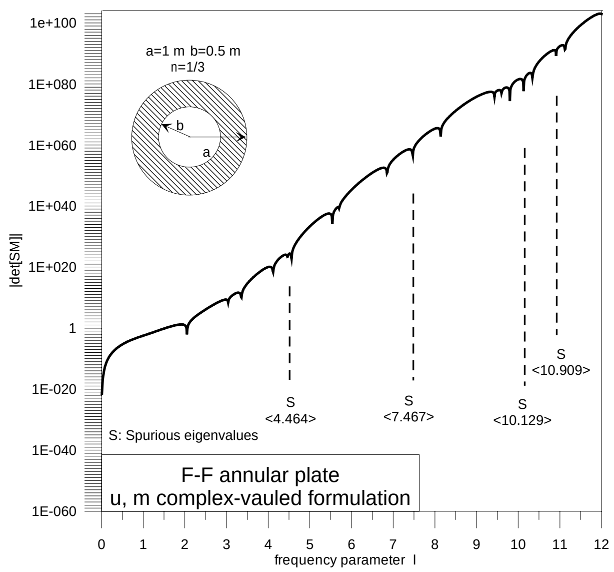


Figure 4-11.(f)

Figure 4-11 The determinant of $[SM]$ versus frequency parameter λ using the same formulation to solve plates subject to different boundary conditions.

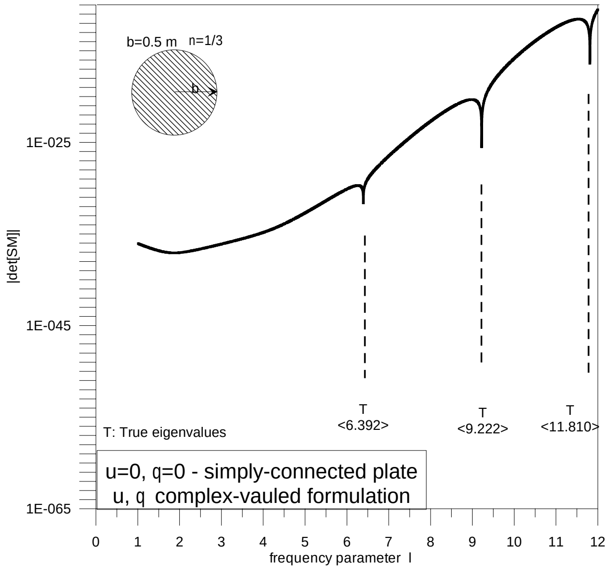


Figure 4-12.(a)

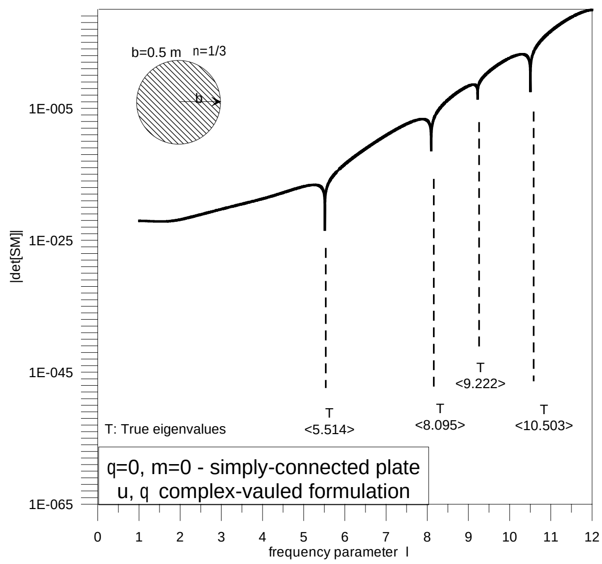


Figure 4-12.(d)

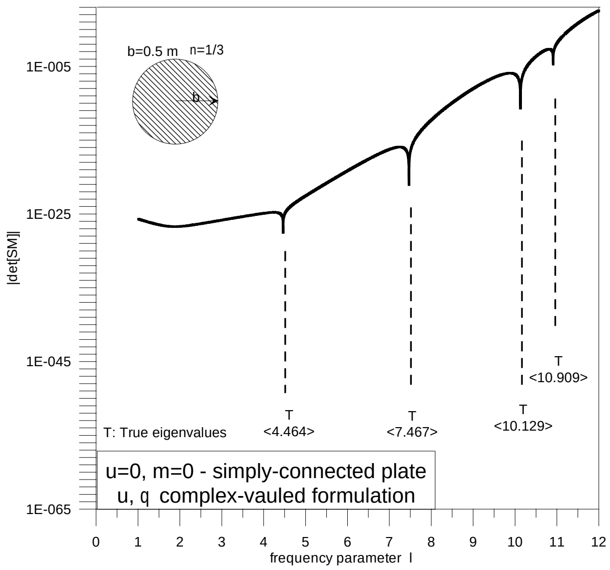


Figure 4-12.(b)

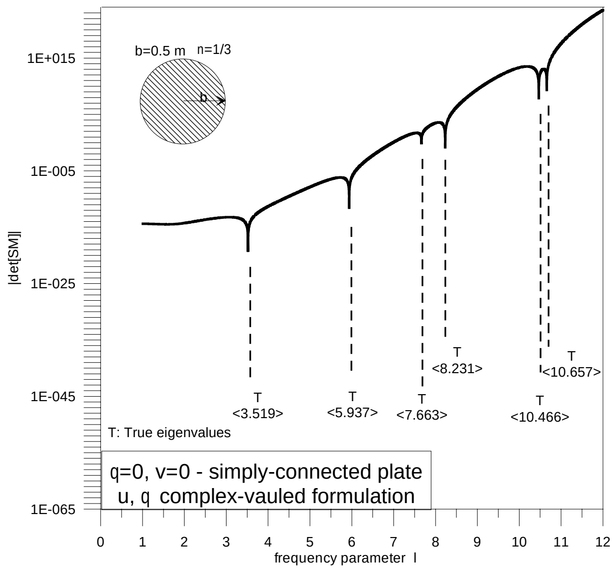


Figure 4-12.(e)

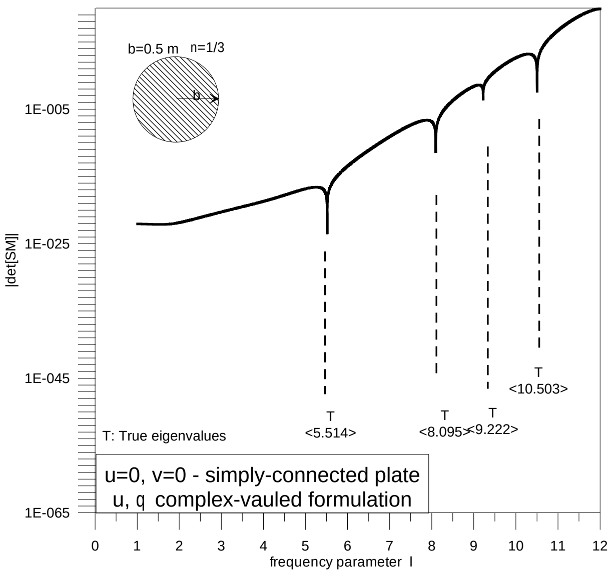


Figure 4-12.(c)

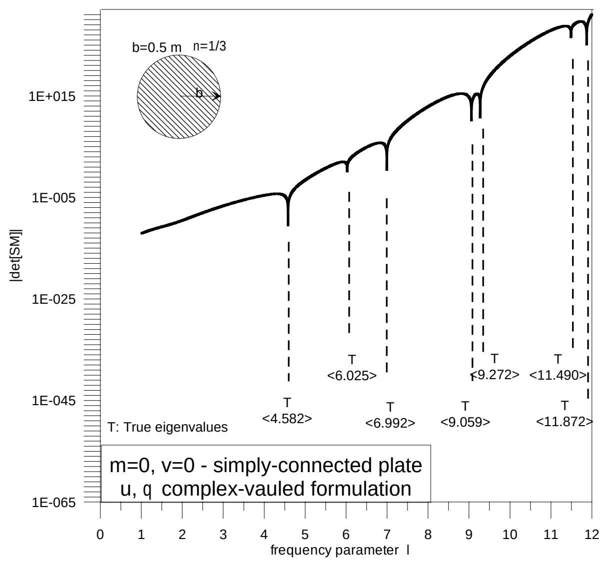


Figure 4-12.(f)

Figure 4-12 The determinant of $[SM]$ versus frequency parameter λ using the complex-valued formulation to solve plates subject to different boundary conditions for the simply-connected plate with a radius b.

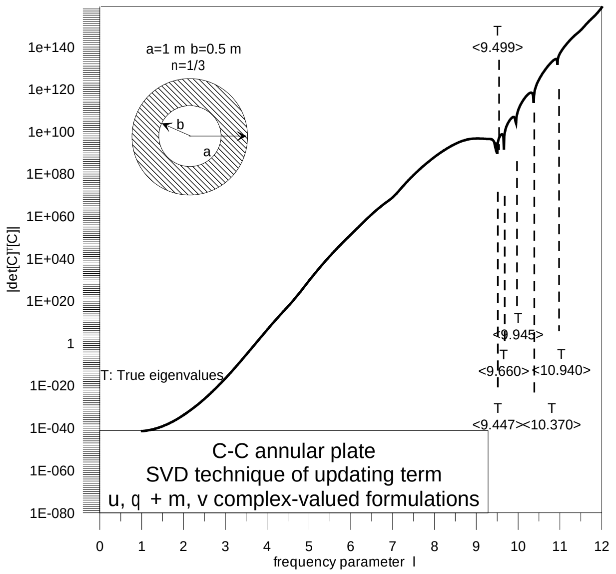


Figure 5-1.(a)

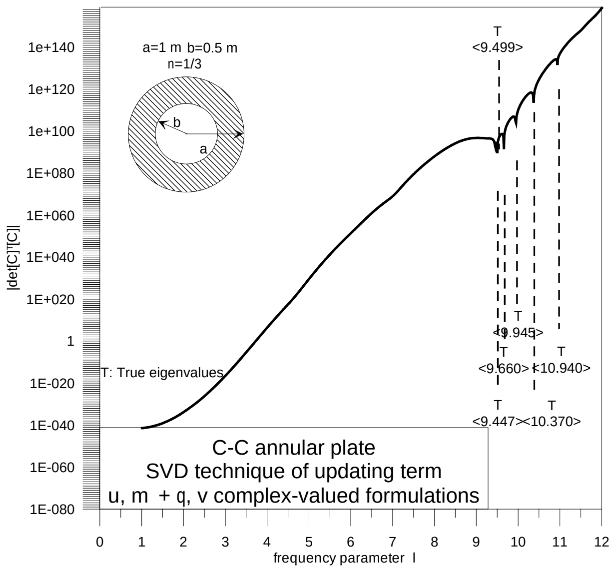


Figure 5-1.(b)

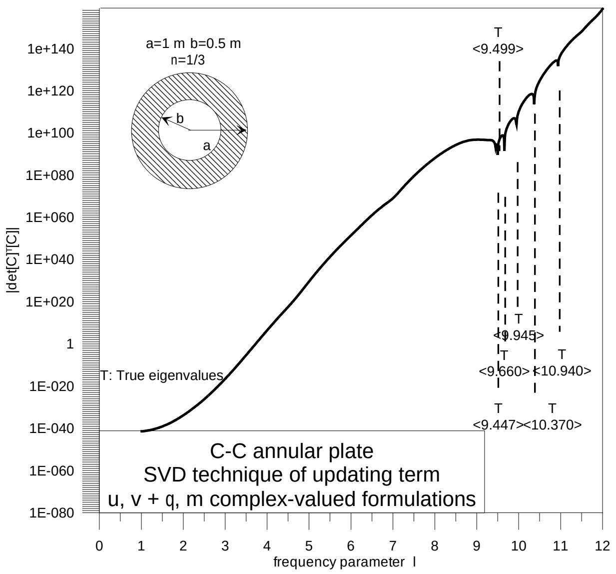


Figure 5-1.(c)

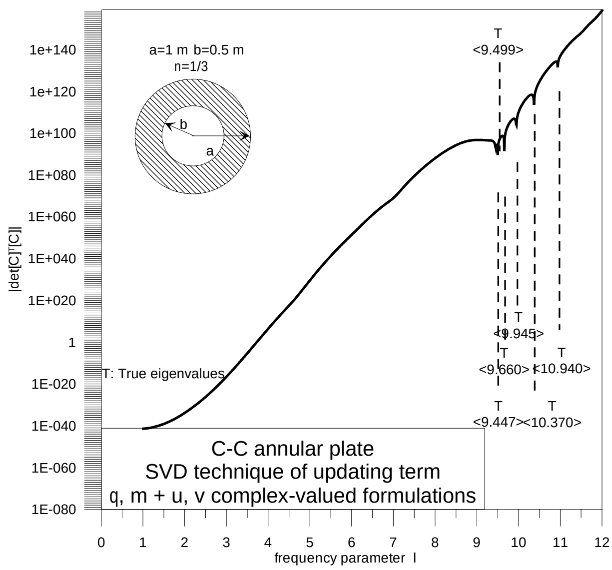


Figure 5-1.(d)

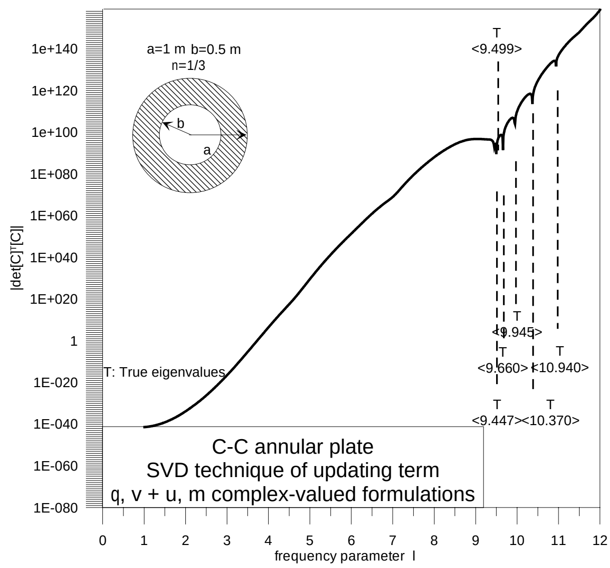


Figure 5-1.(e)

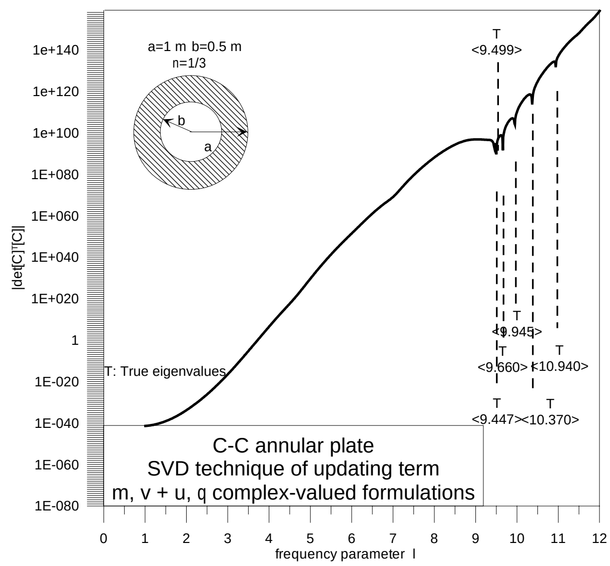


Figure 5-1.(f)

Figure 5-1 The determinant of the of the $[C]^T[C]$ versus frequency paramete for the C-C annular plate by using the complex-valued formulations with the SVD technique of updating term.

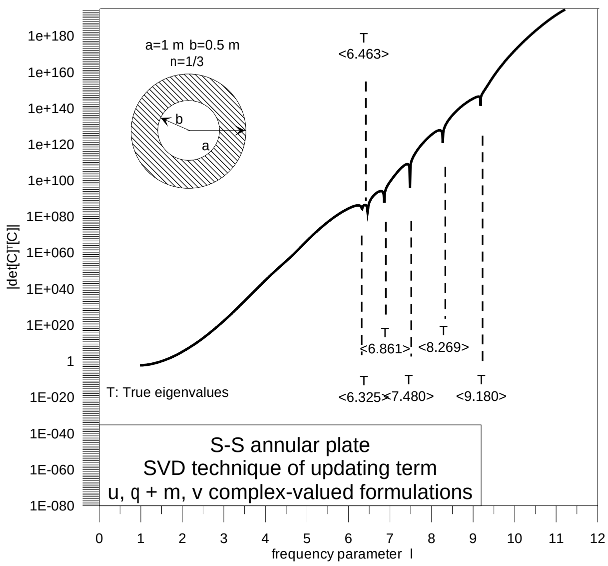


Figure 5-2.(a)

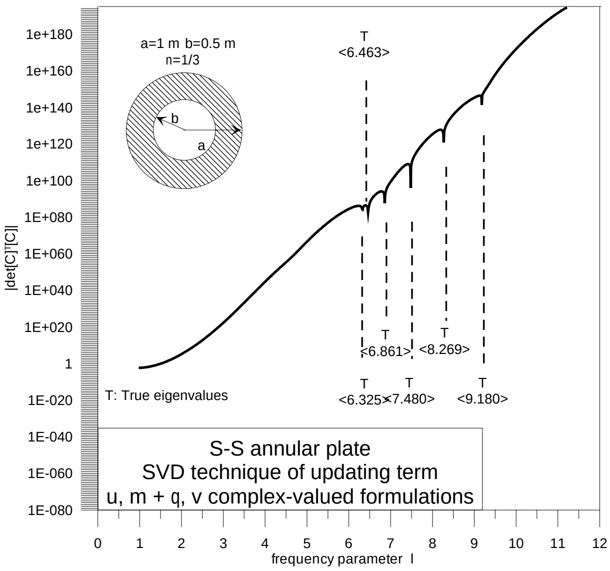


Figure 5-2.(b)

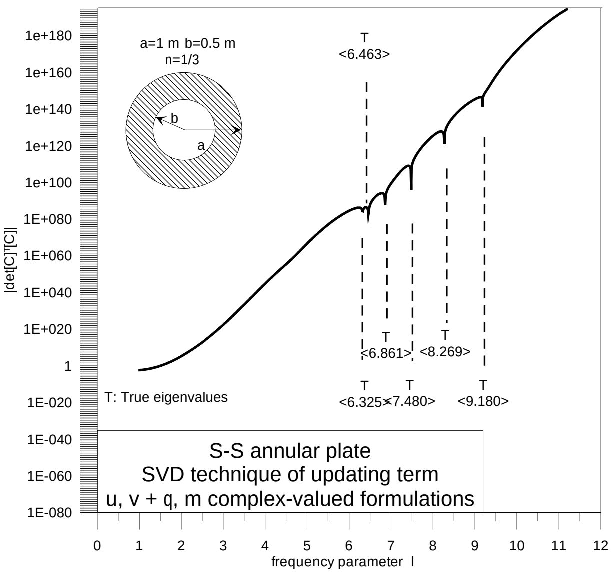


Figure 5-2.(c)

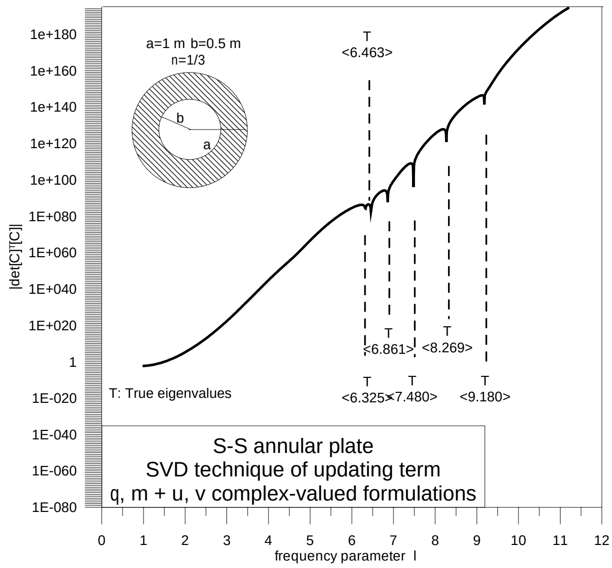


Figure 5-2.(d)

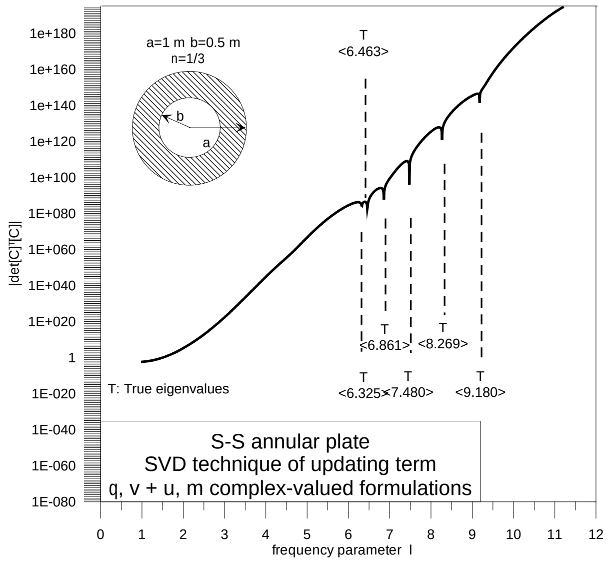


Figure 5-2.(e)

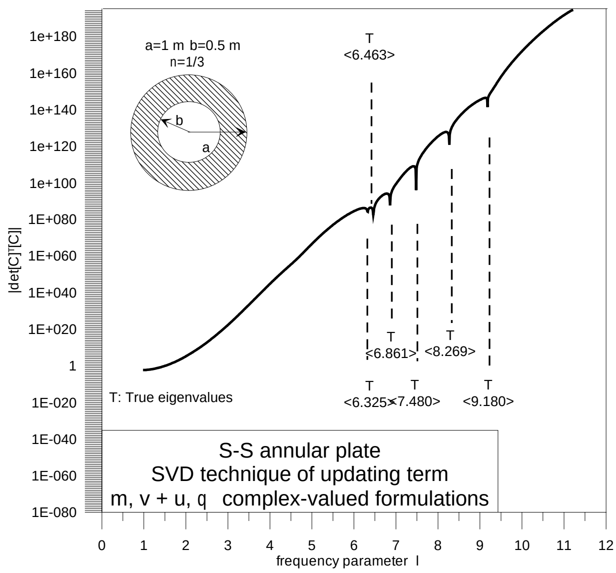
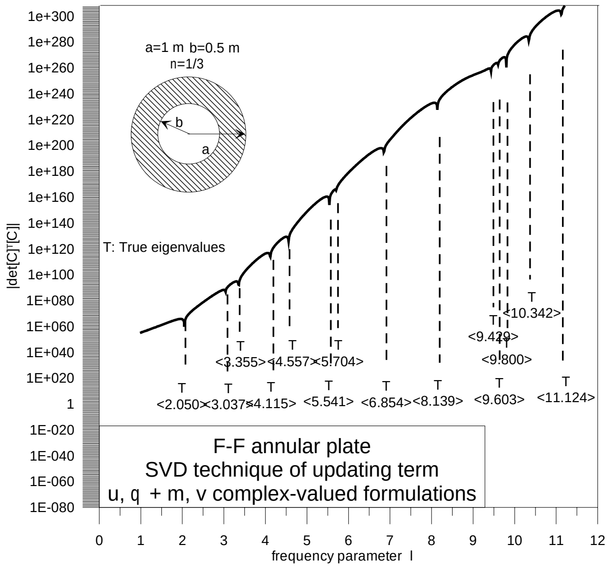


Figure 5-2.(f)

Figure 5-2 The determinant of the of the $[C]^T[C]$ versus frequency parameter λ for the S-S annular plate by using the complex-valued formulations with the SVD technique of updating term.



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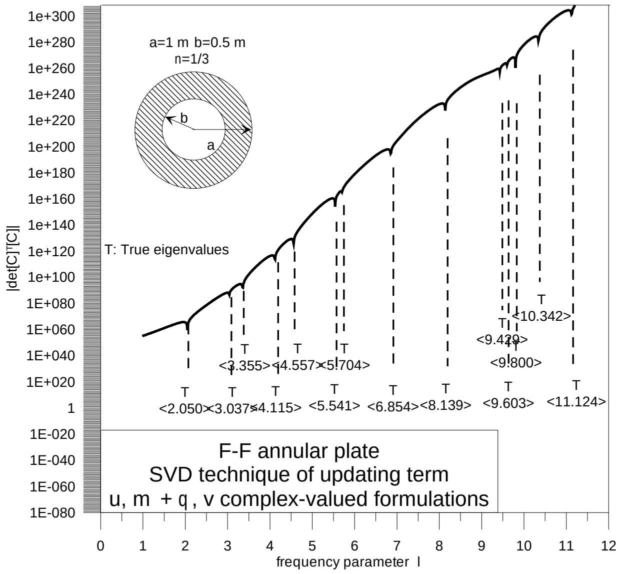


Figure 5-3.(b)

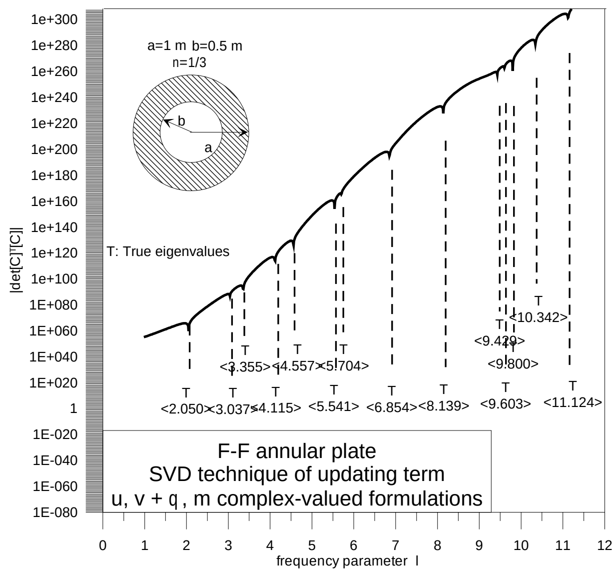


Figure 5-3.(c)

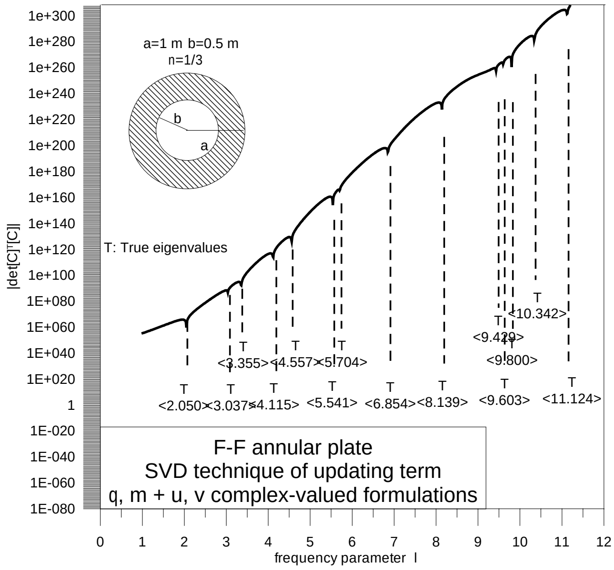


Figure 5-3.(d)

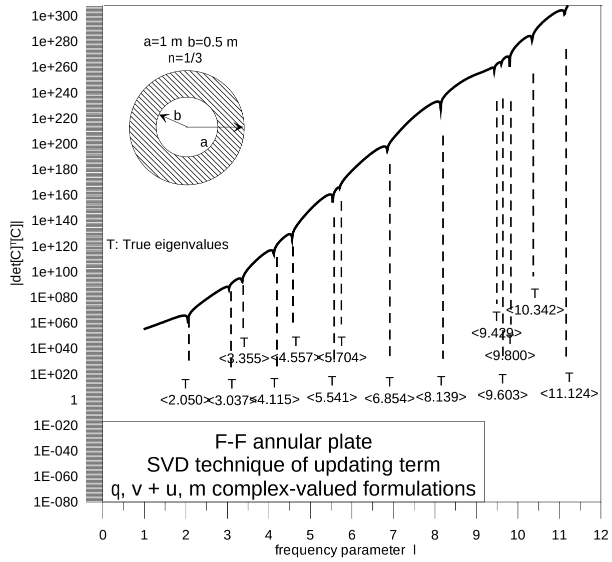


Figure 5-3.(e)

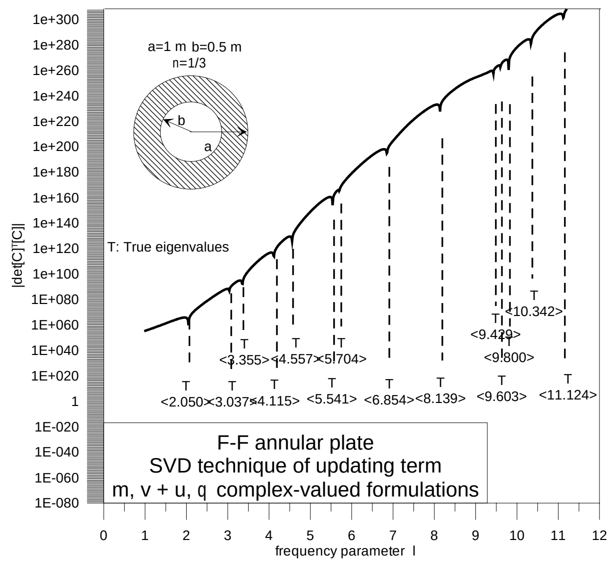


Figure 5-3.(f)

Figure 5-3 The determinant of the of the $[C]^T[C]$ versus frequency parameter λ for the F-F annular plate by using the complex-valued formulations with the SVD technique of updating term.

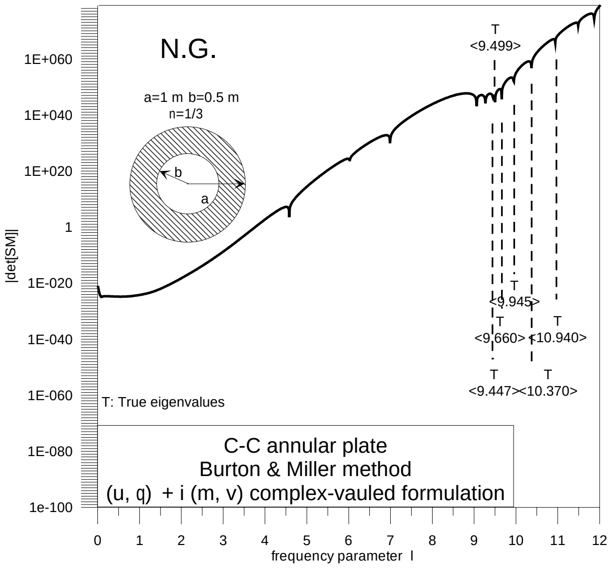
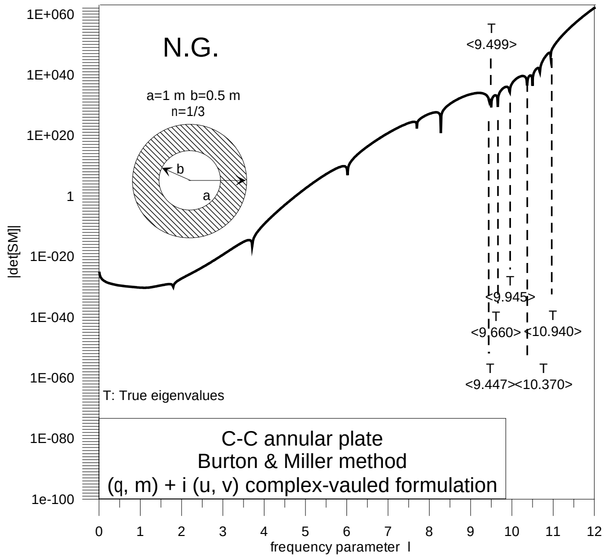


Figure 5-4.(a)



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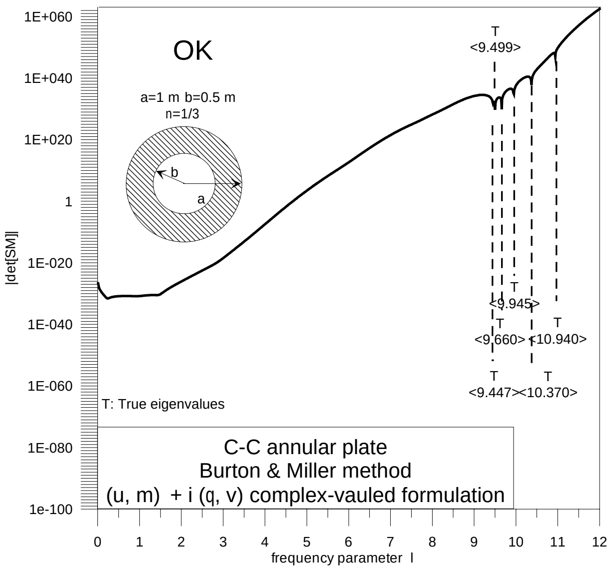

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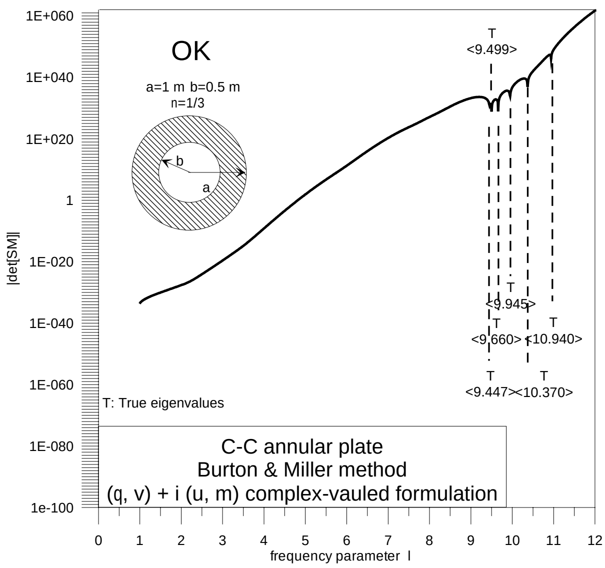


Figure 5-4.(e)

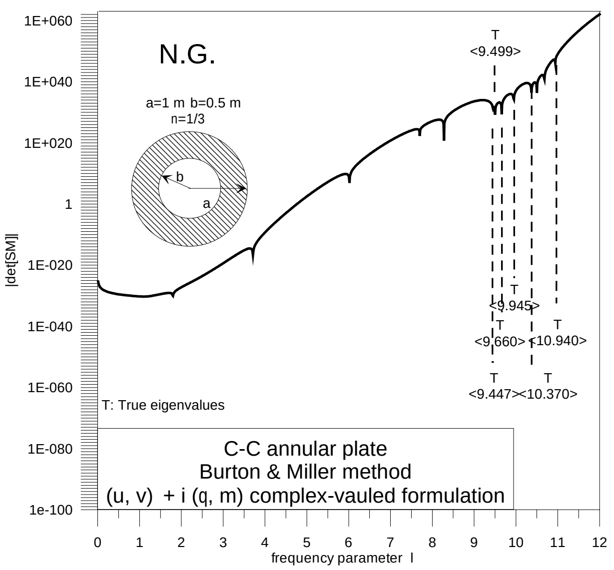


Figure 5-4.(c)

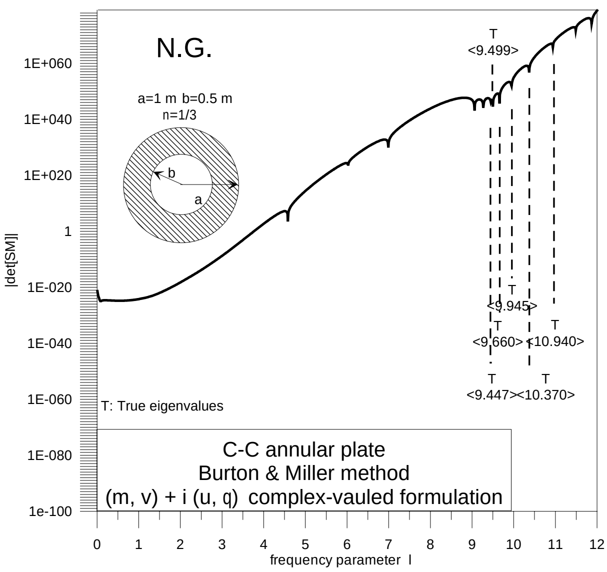


Figure 5-4.(f)

Figure 5-4 The determinant of the $[SM^c]$ versus frequency parameter λ for the C-C annular plate using the six complex-valued formulations with the Burton & Miller concept.

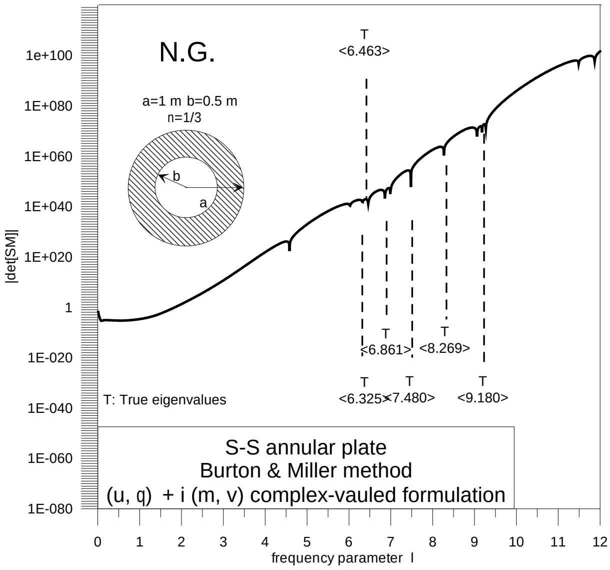


Figure 5-5.(a)

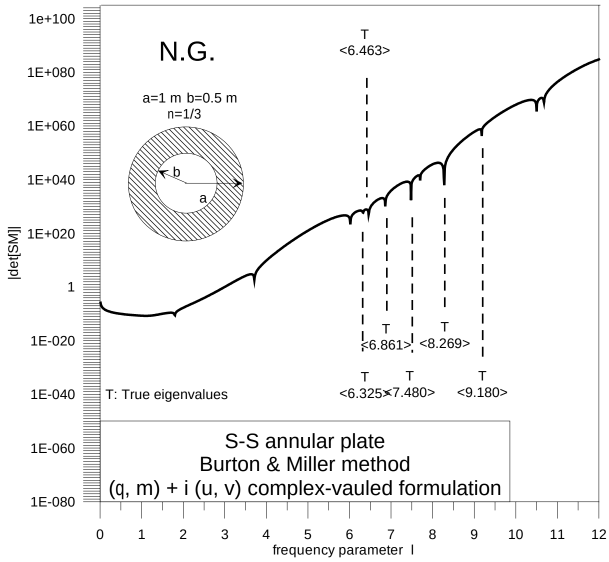
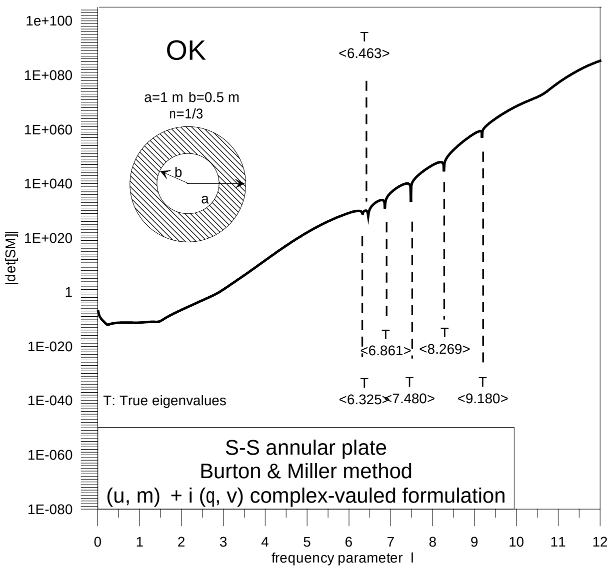


Figure 5-5.(d)



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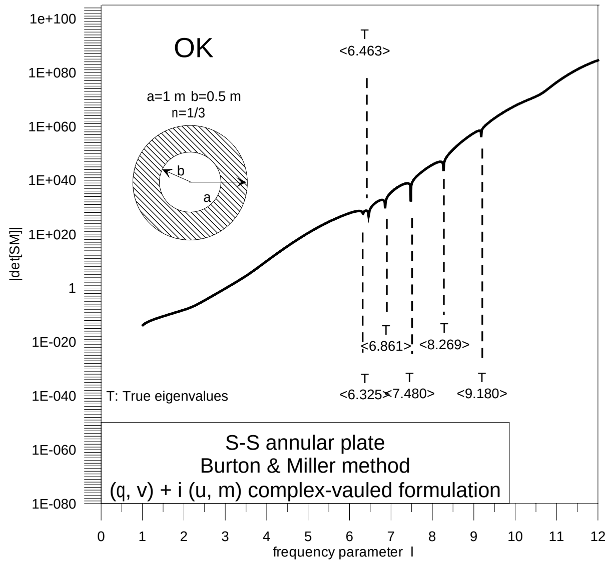

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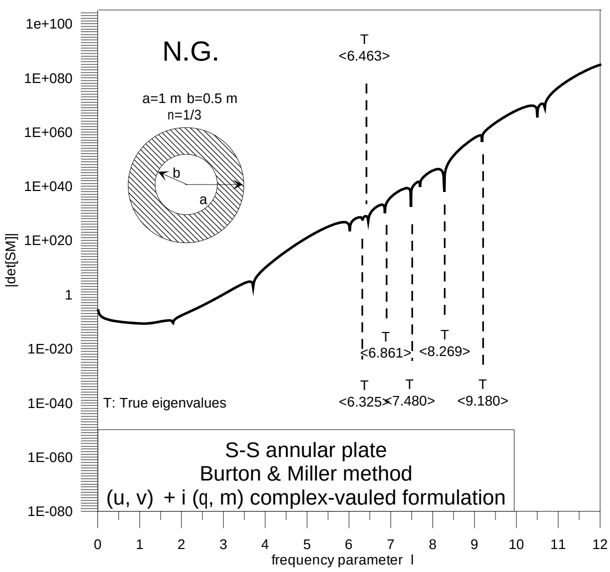


Figure 5-5.(c)

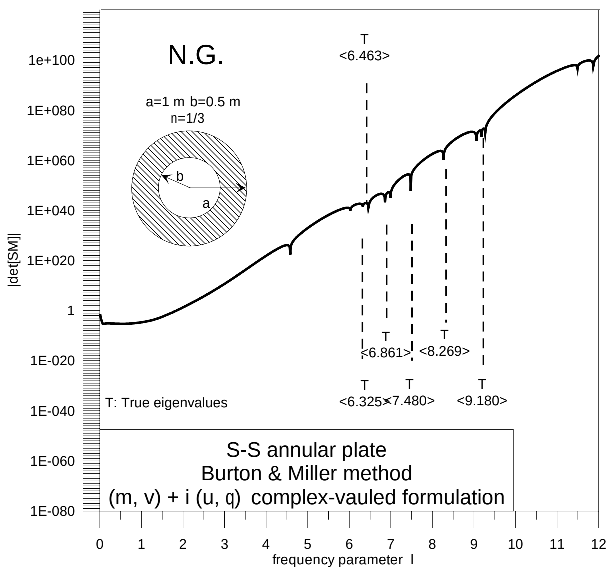


Figure 5-5.(f)

Figure 5-5 The determinant of the $[SM^s]$ versus frequency parameter λ for the S-S annular plate using the six complex-valued formulations with the Burton & Miller concept.

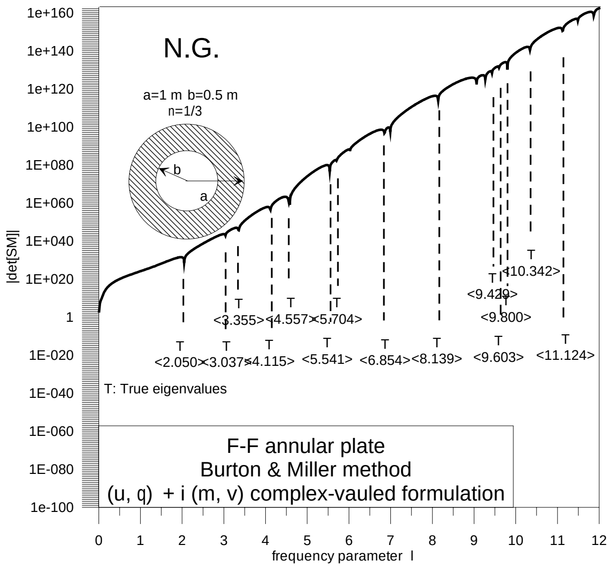
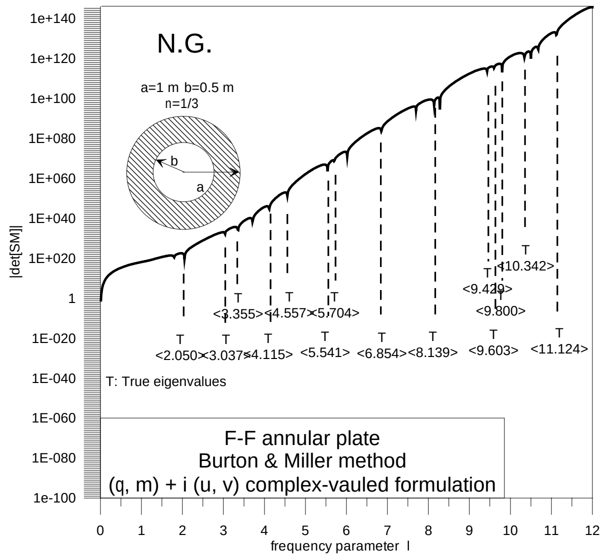


Figure 5-6.(a)



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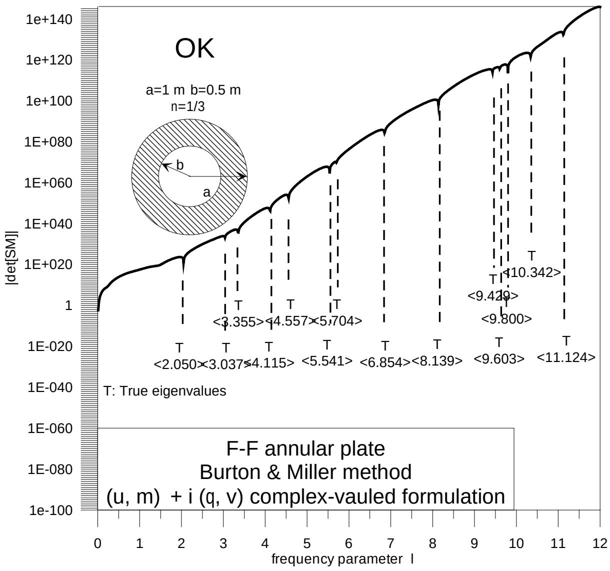

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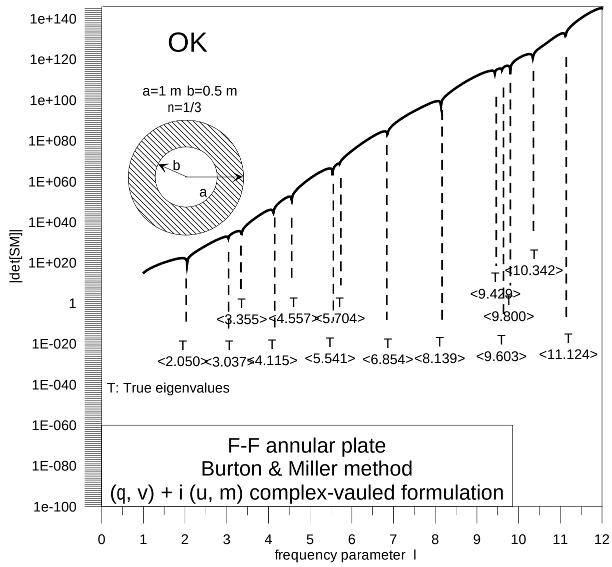
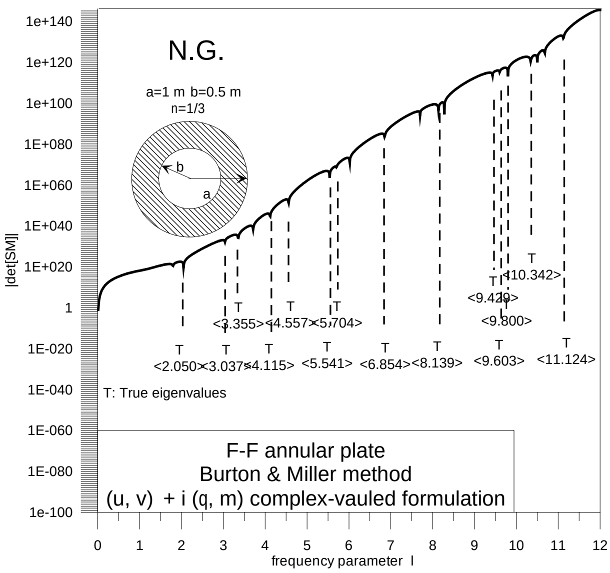


Figure 5-6.(e)



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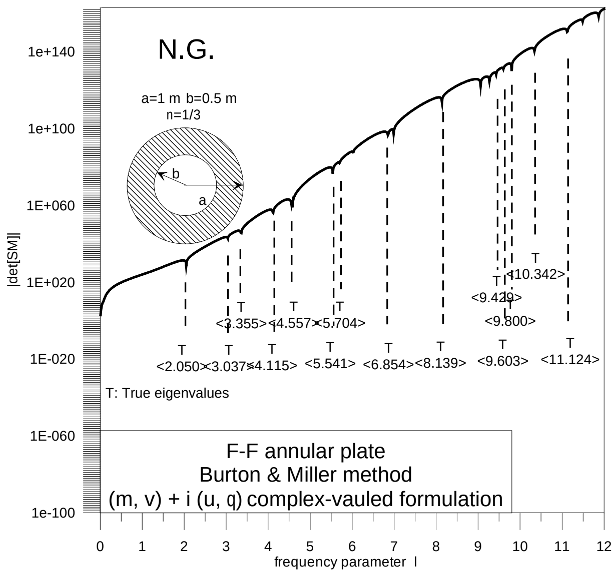


Figure 5-6.(f)

Figure 5-6 The determinant of the $[SM^f]$ versus frequency parameter λ for the F-F annular plate using the six complex-valued formulations with the Burton & Miller concept.

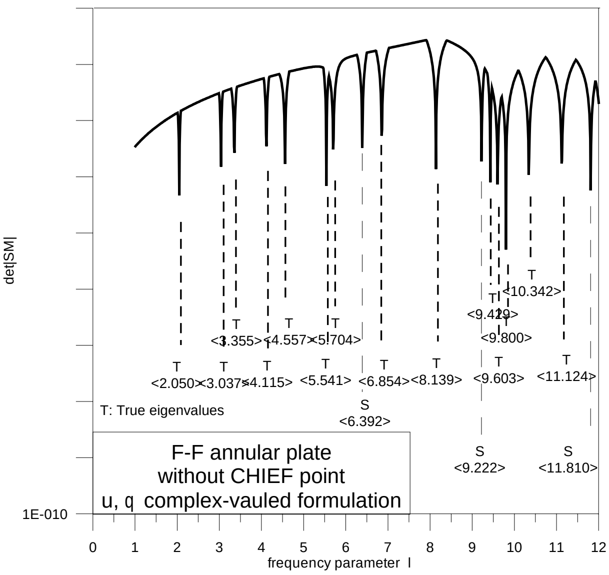


Figure 5-7.(a)

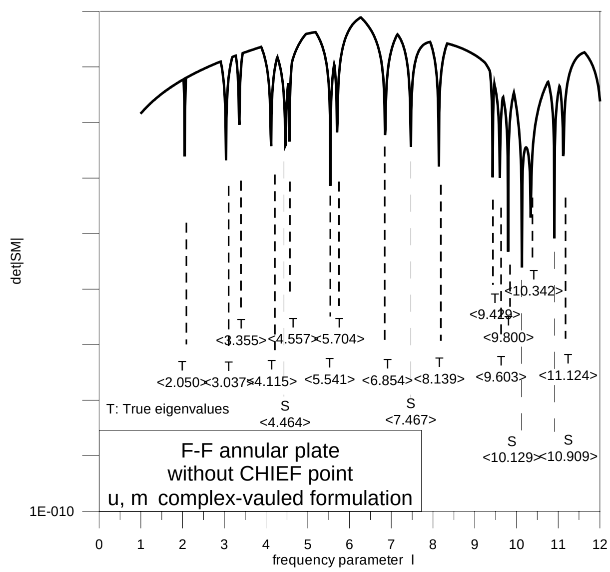


Figure 5-7.(d)

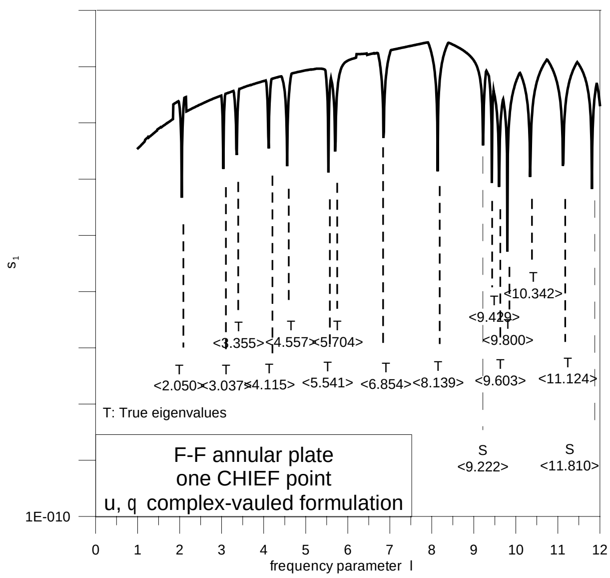


Figure 5-7.(b)

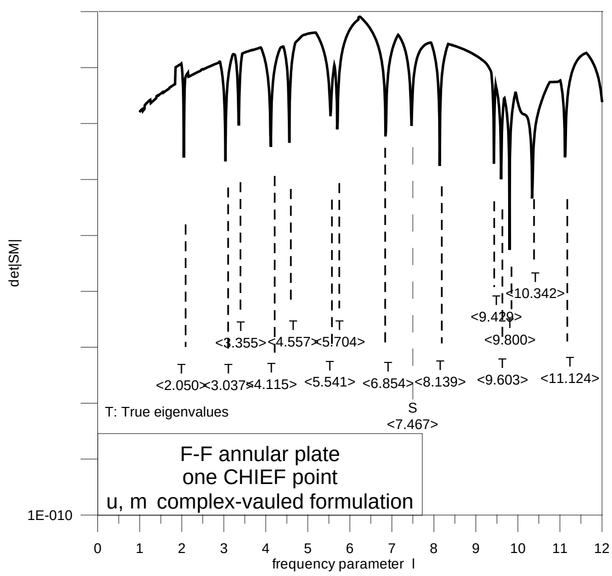
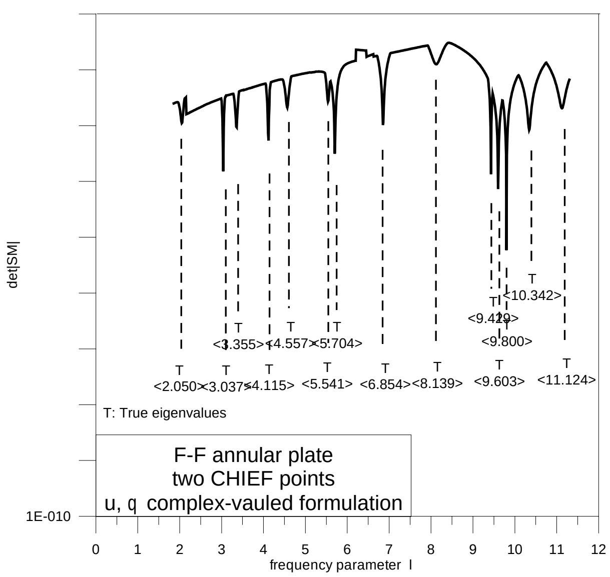


Figure 5-7.(e)



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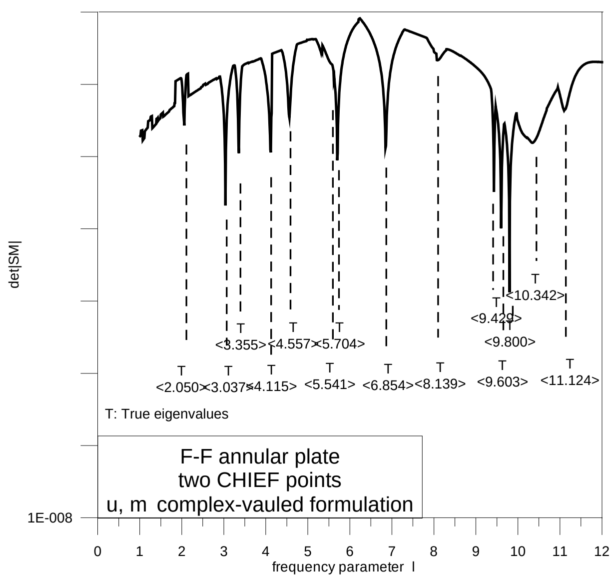

Figure 5-7.(f)

Figure 5-7 The minimum singularvalue σ_1 of the $[C^*]$ versus frequency parameter λ for the F-F annular plate by using the complex-valued BEM in conjunction with the CHIEF method.

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著　　作：

(A) 期刊論文：

1. J. T. Chen, C. F. Lee and S. Y. Lin, A new point of view for the polar decomposition using singular value decomposition, International Journal of Computational and Numerical Analysis and Applications, Vol.2, No. 3, pp. 257-264, 2002.
2. K. H. Chen, J. T. Chen, S. Y. Lin and Y. T. Lee, Dual boundary element analysis of normal incident wave passing a thin submerged breakwater with rigid, absorbing and permeable boundaries, Journal of Waterway, Port, Coastal and Ocean Engineering, ASCE, Revised, 2003.
3. J. T. Chen, S. Y. Lin, K. H. Chen and I. L. Chen, Mathematical analysis and numerical study of true and spurious eigensolutions for free vibration of plate using real-part BEM, Submitted, 2003.
4. J. T. Chen, S. Y. Lin, I. L. Chen and Y. T. Lee, Mathematical analysis and numerical study for free vibration of simply-connected circular plate using the imaginary-part BEM, Submitted, 2003.
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using BEM, Submitted, 2003.

(B) 會議論文：

1. 林盛益, 朱雅雯與陳正宗, 邊界元素法在拉普拉斯方程反算問題之應用, 中華民國第二十六屆全國力學會議, 虎尾, 2002.
2. S. Y. Lin, Y. T. Lee, K. H. Chen and J. T. Chen, Mathematical analysis of the true and spurious eigensolutions for free vibration of plate using real-part BEM, 第十五屆輪機工程暨造船研討會, pp.545-553, 高雄, 2003.
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5. C. S. Wu, S. Y. Lin, S. R. Lin and J. T. Chen, On the equivalence of method of fundamental solutions and Trefftz method for Laplace equation, 第十五屆工程暨造船研討會, pp. 538-544, 高雄, 2003.
6. J. T. Chen, S. Y. Lin, K. H. Chen and I. L. Chen, Spurious and true eigenequations for free vibration of plate using real-part BEM, International Conference on Computational & Experimental Engineering and Sciences, Greece, 2003.

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2. 朱雅雯, 林盛益與陳正宗, 邊界元素法在拉普拉斯方程反算問題之應用, 國科會大專生參與專題研究計畫成果報告, NSC 91-2815-C-019-011-E, 國立海洋大學河海工程學系, 2003.