A new concept of modal participation factor for numerical instability in the dual BEM for exterior acoustics

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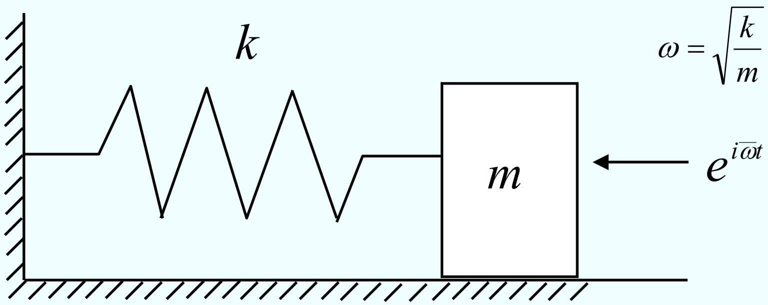
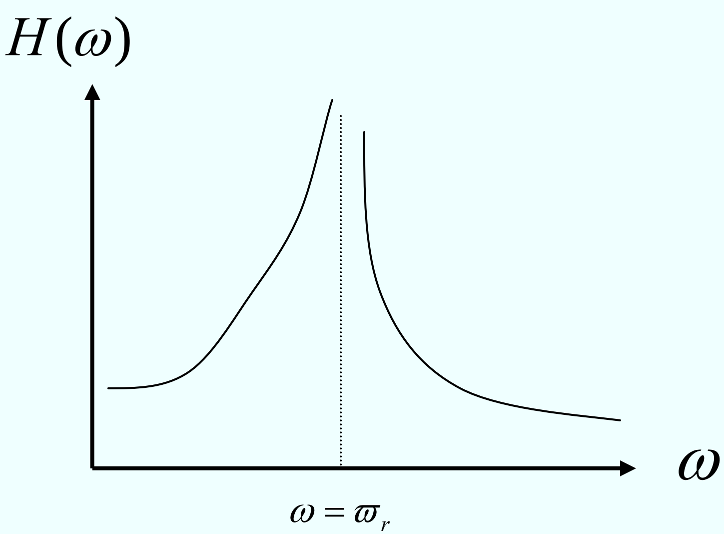
Physical and numerical resonance

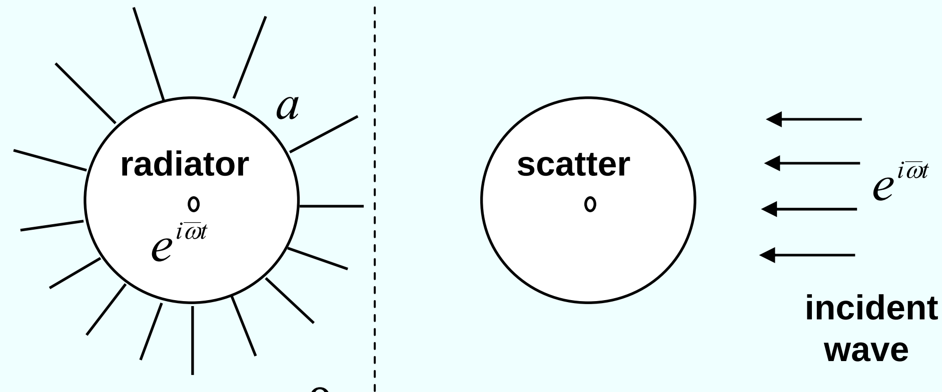
Physical resonance



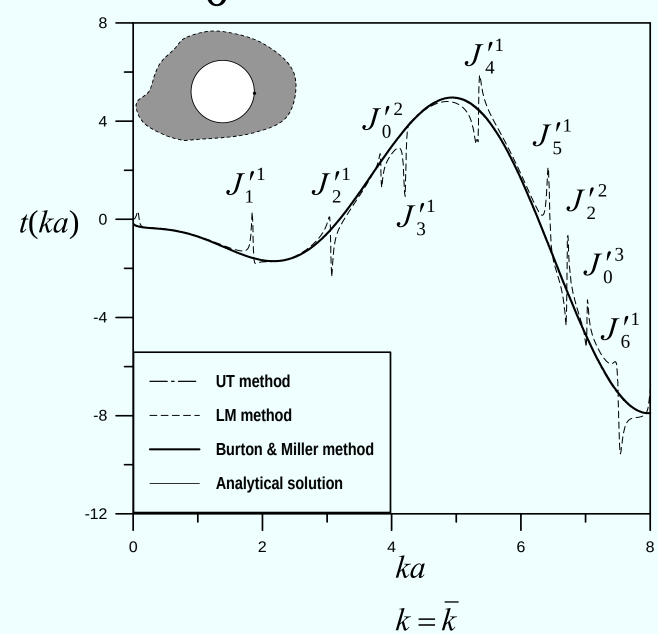
$$u = \frac{finite}{(\omega^2 - \bar{\omega}^2)} \rightarrow \infty, \text{ if } \bar{\omega} \rightarrow \omega$$



Numerical resonance



$$u = \lim_{\bar{\omega} \rightarrow \omega} \frac{0}{0} \rightarrow finite, \text{ if } \bar{\omega} \rightarrow \omega$$



Exterior Acoustics Governing Equation

$$(\nabla^2 + k^2)\vec{u}(x_1, x_2) = 0, (x_{1,} x_2) \in D$$

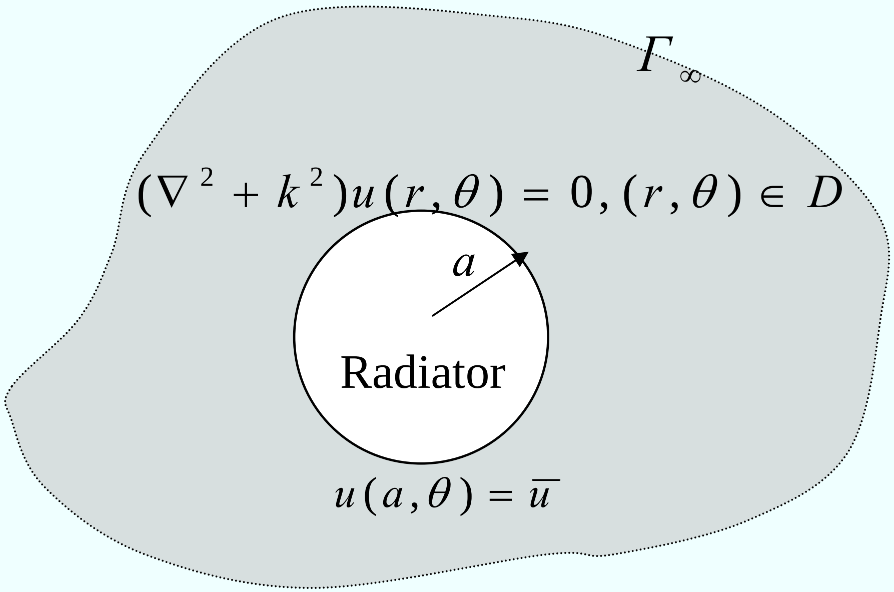
∇^2 : Laplacian operator

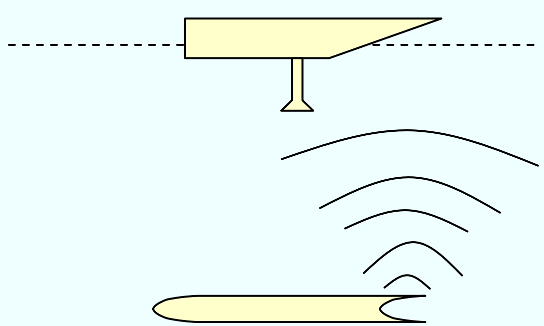
D : domain

k : wave number

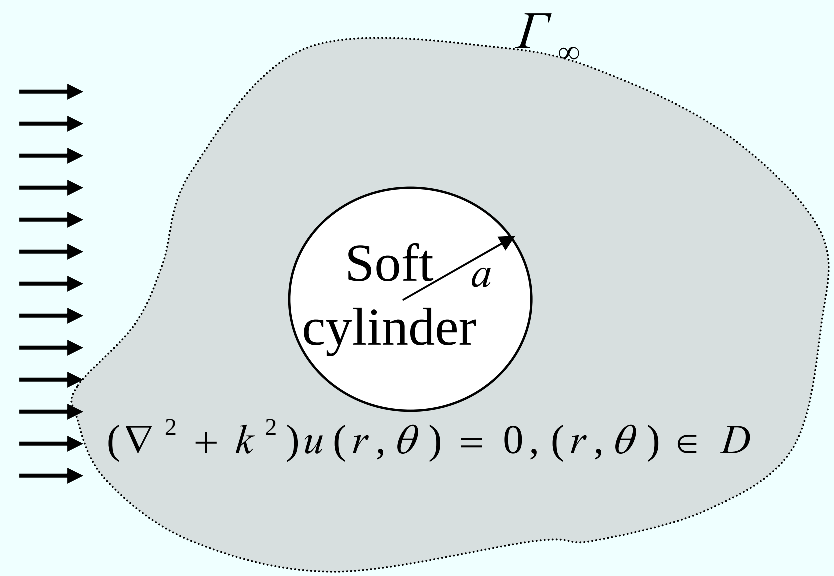
Problem definition

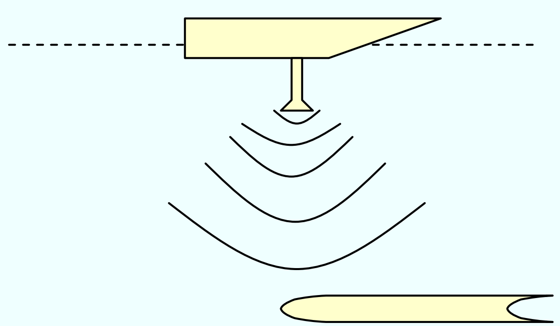
Radiation problem





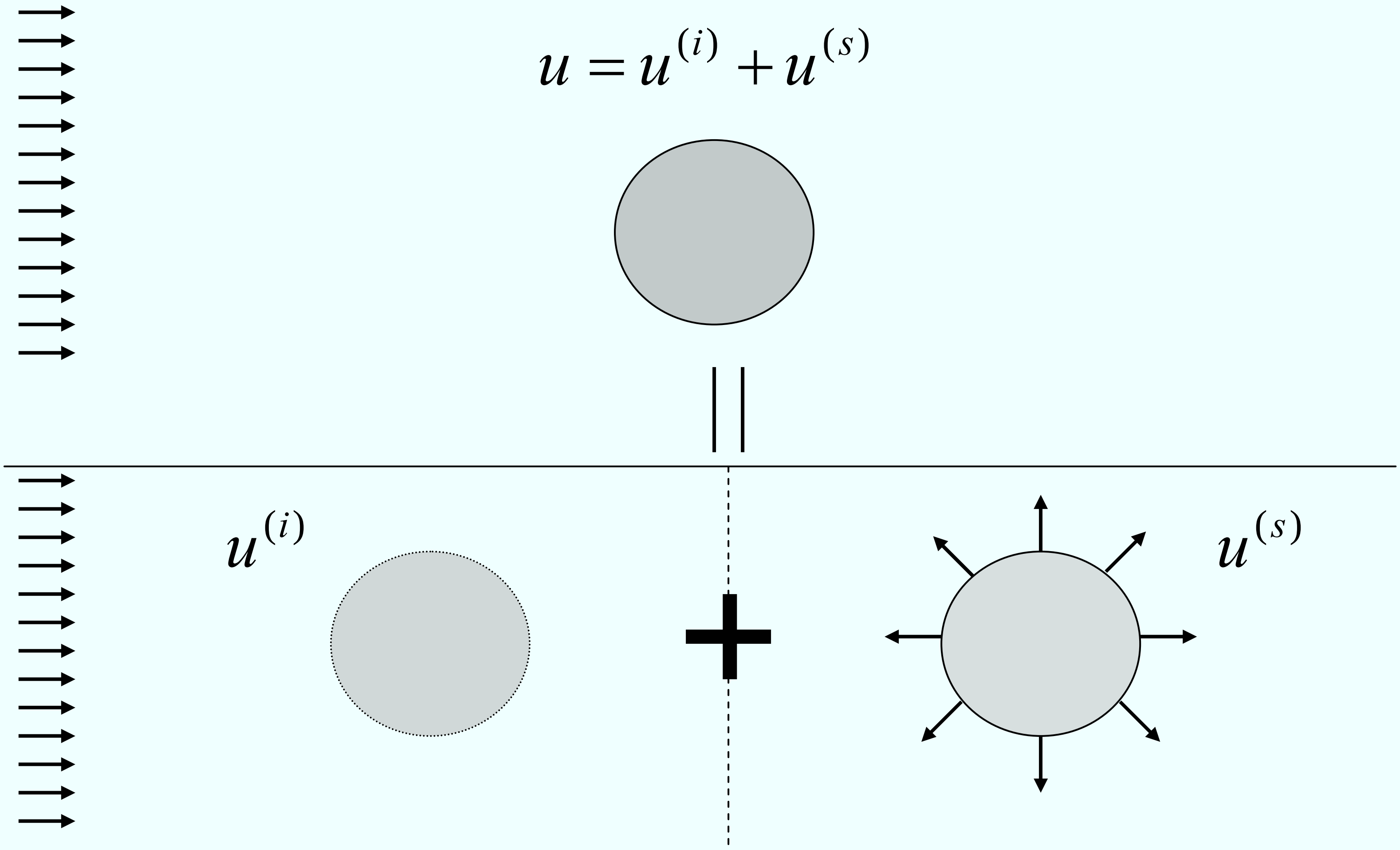
Scattering problem





Principal of scattering superposition

$$u = u^{(i)} + u^{(s)}$$



The numerical instability in dual BEM

$$\pi u(x) = C.P.V.\int_B T(s,x)u(s)dB(s) - R.P.V.\int_B U(s,x)t(s)dB(s), x \in B$$

$$\pi u(x) = H.P.V.\int_B M(s,x)u(s)dB(s) - C.P.V.\int_B L(s,x)t(s)dB(s), x \in B$$

Discretizing type

$$[T_{pq}]\{u_q\} = [U_{pq}]\{t_q\}$$
$$[M_{pq}]\{u_q\} = [L_{pq}]\{t_q\}$$

given t (Neumann) →

$$[T_{pq}]\{u_q\} = \{b_1\}$$
$$[M_{pq}]\{u_q\} = \{b_2\}$$

$$\begin{cases} \det\left|[T_{pq}]\right| \approx 0 \\ \det\left|[M_{pq}]\right| \approx 0 \end{cases}$$

→ Numerical instability

(fictitious frequency)

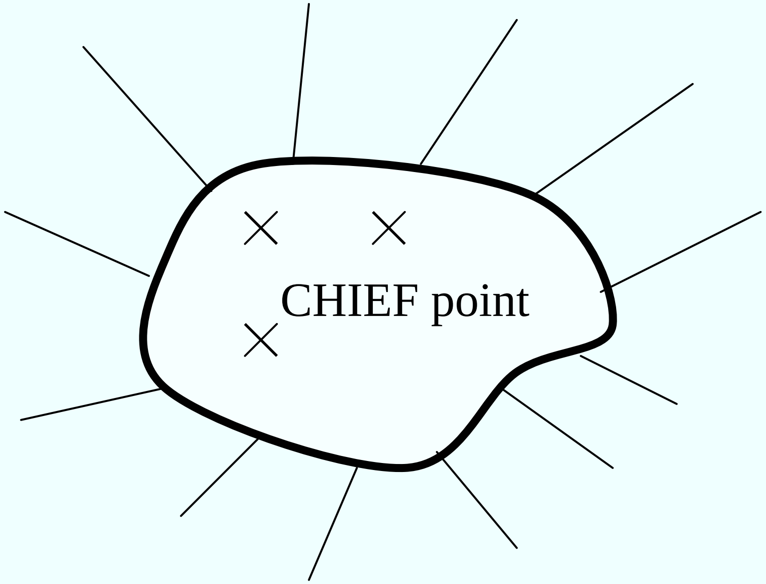
How to avoid the problem of fictitious frequency

- Burton and Miller method

$$\left\{\left[T_{pq}\right]+\frac{i}{k}\left[M_{pq}\right]\right\}\left\{u_q\right\}=\left\{\left[U_{pq}\right]+\frac{i}{k}\left[L_{pq}\right]\right\}\left\{t_q\right\}$$

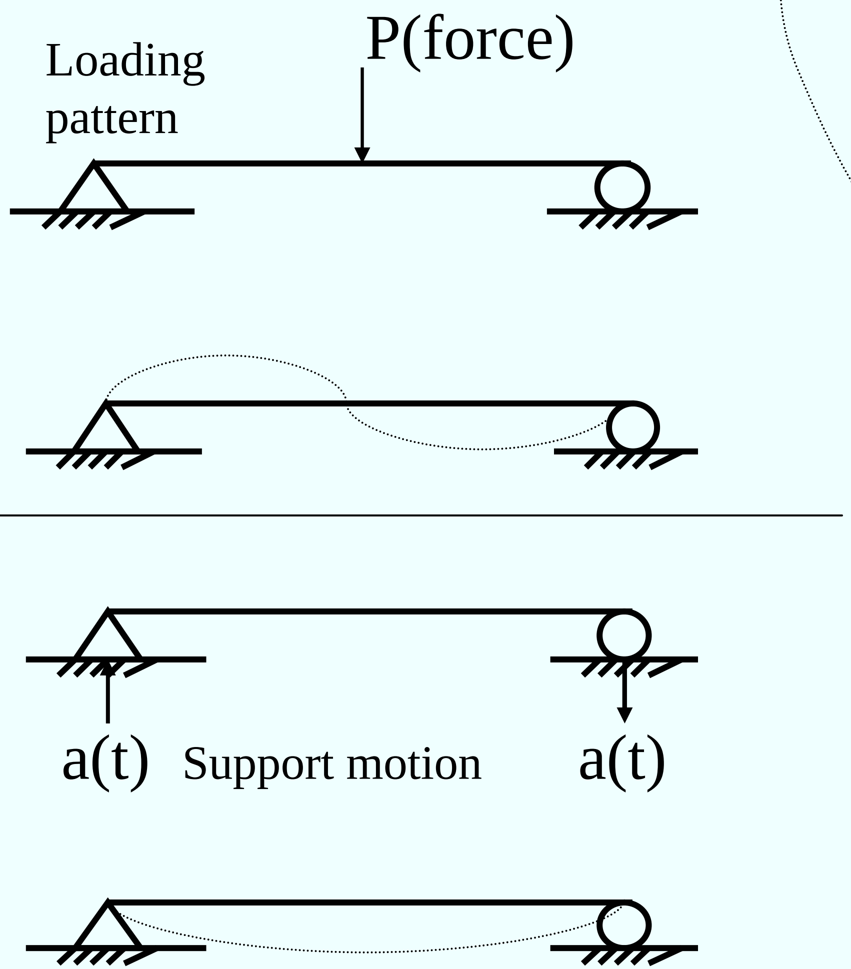
- CHIEF method

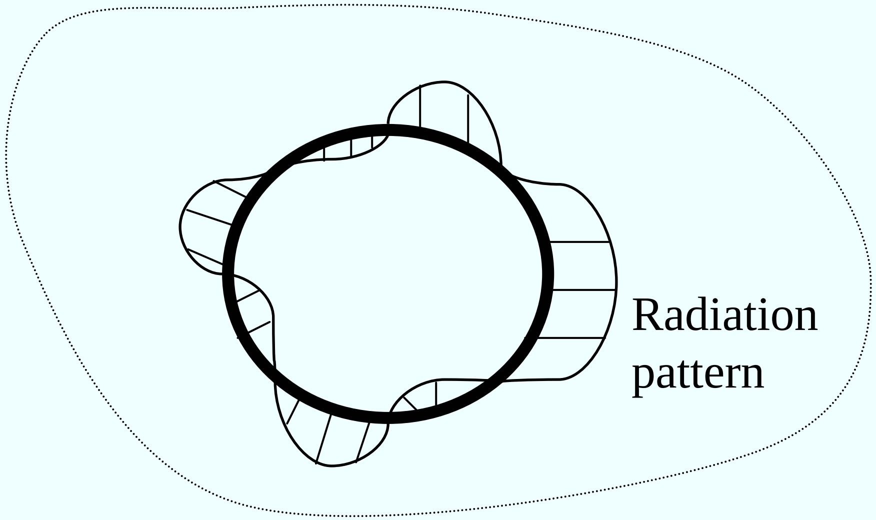
$$\left[C(k)\right]=\begin{bmatrix} U^B(k) \\ U^I(k) \end{bmatrix}_{(2N+a)\times 2N}$$

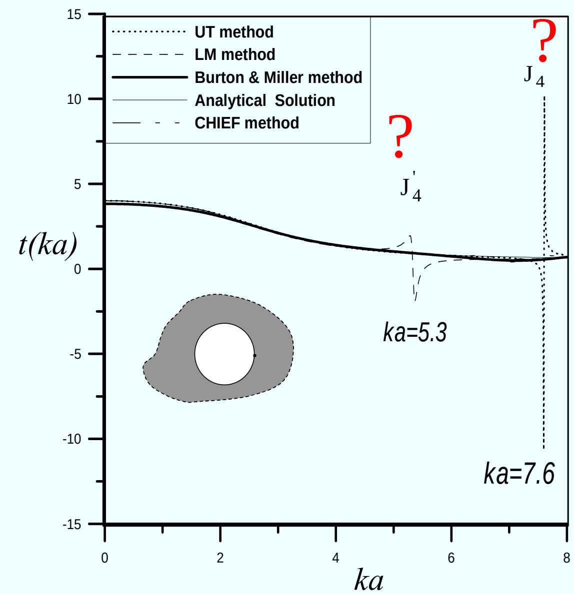


What is modal participation factor ?

Modal participation factor







Modal participation factor for numerical instability

- **Continuous system**

$$u(\theta) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta), 0 < \theta < 2\pi$$

$$t(\theta) = p_0 + \sum_{n=1}^{\infty} (p_n \cos n\theta + q_n \sin n\theta), 0 < \theta < 2\pi$$

where a_0, a_n, b_n, p_0, p_n and q_n are the Fourier coefficients.

1. ***UT*** **method**

Null-field equation:

$$0 = \int_B T^i(s,x)u(s)dB(s) - \int_B U^i(s,x)t(s)dB(s), x \in \overline{D}$$

where $\overline{D}$ is outside the domain of interest.

Modal participation factor for numerical instability

$$U(s,x)=\begin{cases} U^{i}(R,\theta;\rho,\phi)=\sum_{m=-\infty}^{\infty}\frac{\pi}{2}\left[-iJ_m(kR)+Y_m(kR)\right]J_m(k\rho)\cos(m(\theta-\phi)), R>\rho \\ U^{e}(R,\theta;\rho,\phi)=\sum_{m=-\infty}^{\infty}\frac{\pi}{2}\left[-iJ_m(k\rho)+Y_m(k\rho)\right]J_m(kR)\cos(m(\theta-\phi)), R<\rho \end{cases}$$

$$T(s,x)=\begin{cases} T^{i}(R,\theta;\rho,\phi)=\sum_{m=-\infty}^{\infty}\frac{\pi}{2}\left[-iJ'_m(kR)+Y'_m(kR)\right]J_m(k\rho)\cos(m(\theta-\phi)), R>\rho \\ T^{e}(R,\theta;\rho,\phi)=\sum_{m=-\infty}^{\infty}\frac{\pi}{2}\left[-iJ_m(k\rho)+Y_m(k\rho)\right]J'_m(kR)\cos(m(\theta-\phi)), R<\rho \end{cases}$$

$$u(\rho,\phi)=\sum_{m=0}^{\infty}\left(\frac{H_m^{(1)}(k\rho)}{H_m^{(1)}(ka)}\right)\left(\frac{J_m(ka)}{J_m(ka)}\right)\sqrt{a_m^2+b_m^2}\cos(m\phi-\phi_0),\ \ \rho\geq a, 0\leq\phi\leq 2\pi$$

Modal participation factor:

$$\left(\frac{H_m^{(1)}(k\rho)}{H_m^{(1)}(ka)}\right)\left(\frac{J_m(ka)}{J_m(ka)}\right)\sqrt{a_m^2+b_m^2}\cdots\cdots\text{with mode}\cos(m\phi-\phi_0)$$

Modal participation factor for numerical instability

2. *LM* method

Null-field equation:

$$0=\int_B M^i(s,x)u(s)dB(s)-\int_B L^i(s,x)t(s)dB(s),x\in\overline{D}$$

$$L(s,x)=\begin{cases}L^i(R,\theta;\rho,\phi)=\sum_{m=-\infty}^{\infty}\frac{\pi}{2}\left[-iJ_m(kR)+Y_m(kR)\right]J'_m(k\rho)\cos(m(\theta-\phi)),R>\rho\\ L^e(R,\theta;\rho,\phi)=\sum_{m=-\infty}^{\infty}\frac{\pi}{2}\left[-iJ'_m(k\rho)+Y'_m(k\rho)\right]J_m(kR)\cos(m(\theta-\phi)),R<\rho\end{cases}$$

$$M(s,x)=\begin{cases}M^i(R,\theta;\rho,\phi)=\sum_{m=-\infty}^{\infty}\frac{\pi}{2}\left[-iJ'_m(kR)+Y'_m(kR)\right]J'_m(k\rho)\cos(m(\theta-\phi)),R>\rho\\ M^e(R,\theta;\rho,\phi)=\sum_{m=-\infty}^{\infty}\frac{\pi}{2}\left[-iJ'_m(k\rho)+Y'_m(k\rho)\right]J'_m(kR)\cos(m(\theta-\phi)),R<\rho\end{cases}$$

$$u(\rho,\phi)=\sum_{m=0}^{\infty}\left(\frac{H_m^{(1)}(k\rho)}{H_m^{(1)}(ka)}\right)\left(\frac{J'_m(ka)}{J'_m(ka)}\right)\sqrt{a_m^2+b_m^2}\cos(m\phi-\phi_0),\ \ \rho\geq a,0\leq\phi\leq 2\pi$$

Modal participation factor:

$$\left(\frac{H_m^{(1)}(k\rho)}{H_m^{(1)}(ka)}\right)\left(\frac{J'_m(ka)}{J'_m(ka)}\right)\sqrt{a_m^2+b_m^2}\cdots\cdots\text{with mode}\cos(m\phi-\phi_0)$$

Modal participation factor for numerical instability

- **Discrete system**

$$\lfloor T_{pq} \rfloor \{ u_q \} = \lfloor U_{pq} \rfloor \{ t_q \} \qquad \lfloor M_{pq} \rfloor \{ u_q \} = \lfloor L_{pq} \rfloor \{ t_q \}$$

Using the SVD technique

$$\Phi_U \Sigma_U \Psi_U^+ t = \Phi_T \Sigma_T \Psi_T^+ u \qquad \Phi_M \Sigma_M \Psi_M^+ t = \Phi_L \Sigma_L \Psi_L^+ u$$

$+$: the transpose conjugate

Φ, Ψ : the unitary matrix

Σ : the digonal matrix composed

by the singular value σ

Modal participation factor for numerical instability

1. UT method

$$u = \Psi_T \beta = \sum_{n=-(N-1)}^{N} \beta_n \psi_n^{(T)}$$

$$t = \Psi_U \beta = \sum_{n=-(N-1)}^{N} \alpha_n \psi_n^{(U)}$$

$$\Rightarrow \quad \Phi_U \Sigma_U \alpha = \Phi_T \Sigma_T \beta$$

where α and β are the generalized coordinates.

Using the Fredholm alternative theorem

$$\begin{bmatrix} U^+(k_f) \\ T^+(k_f) \end{bmatrix} \phi_i = 0 \quad \xrightarrow{\text{Transpose conjugate}} \quad \phi_i^+ \begin{bmatrix} U(k_f) & T(k_f) \end{bmatrix} = 0$$

SVD updating terms SVD updating documents

where k_f is the fictitious frequency, ϕ_i is one of column vector in Φ_U and Φ_T

Modal participation factor for numerical instability

$$\phi_i^{+}\Phi_U\Sigma_U\alpha = \phi_i^{+}\Phi_T\Sigma_T\beta$$

For the Dirichlet problem $\Rightarrow$ $\alpha_i = \dfrac{\sigma_i^{(T)}}{\sigma_i^{(U)}}\beta_i$

$\dfrac{\sigma_i^{(T)}}{\sigma_i^{(U)}} = \dfrac{0}{0}$ when k is k_f which satisfies $\sigma_i^{(T)} = \sigma_i^{(U)} = 0$

$\Rightarrow$ **Fictitious frequency**

Modal participation factor: $\dfrac{\sigma_i^{(T)}}{\sigma_i^{(U)}}\beta_i \cdots\cdots$ with mode $\psi_i^{(T)}$

Modal participation factor for numerical instability

•The modal participation factor in the special case of a circular radiation

mode	ψ_0	ψ_{-1}	ψ_1
participation factor	$\frac{H_0^{(1)'}(ka)}{H_0^{(1)}(ka)} \frac{J_0(ka)}{J_0(ka)} \beta_0$	$\frac{H_{-1}^{(1)'}(ka)}{H_{-1}^{(1)}(ka)} \frac{J_{-1}(ka)}{J_{-1}(ka)} \beta_{-1}$	$\frac{H_1^{(1)'}(ka)}{H_1^{(1)}(ka)} \frac{J_1(ka)}{J_1(ka)} \beta_1$
mode	$\psi_{-(N-1)}$	$\psi_{(N-1)}$	ψ_N
participation factor	$\frac{H_{-(N-1)}^{(1)'}(ka)}{H_{-(N-1)}^{(1)}(ka)} \frac{J_{-(N-1)}(ka)}{J_{-(N-1)}(ka)} \beta_{-(N-1)}$	$\frac{H_{(N-1)}^{(1)'}(ka)}{H_{(N-1)}^{(1)}(ka)} \frac{J_{(N-1)}(ka)}{J_{(N-1)}(ka)} \beta_{(N-1)}$	$\frac{H_N^{(1)'}(ka)}{H_N^{(1)}(ka)} \frac{J_N(ka)}{J_N(ka)} \beta_N$

(Using *UT* formulation)

Modal participation factor for numerical instability

2. LM method

$$u = \Psi_M \beta = \sum_{n=-(N-1)}^{N} \beta_n \psi_n^{(M)}$$

$$t = \Psi_L \beta = \sum_{n=-(N-1)}^{N} \alpha_n \psi_n^{(L)}$$

⇨

$$\Phi_L \Sigma_L \alpha = \Phi_M \Sigma_M \beta$$

Modal participation factor: $\dfrac{\sigma_i^{(M)}}{\sigma_i^{(L)}} \beta_i \cdots\cdots \text{with mode } \psi_{\mathrm{i}}^{(M)}$

Modal participation factor for numerical instability

•The modal participation factor in the special case of a circular radiation

mode	ψ_0	ψ_{-1}	ψ_1
participation factor	$\frac{H_0^{(1)'}(ka)}{H_0^{(1)}(ka)}\frac{J'_0(ka)}{J'_0(ka)}\beta_0$	$\frac{H_{-1}^{(1)'}(ka)}{H_{-1}^{(1)}(ka)}\frac{J'_{-1}(ka)}{J'_{-1}(ka)}\beta_{-1}$	$\frac{H_1^{(1)'}(ka)}{H_1^{(1)}(ka)}\frac{J'_1(ka)}{J'_1(ka)}\beta_1$
mode	$\psi_{-(N-1)}$	$\psi_{(N-1)}$	ψ_N
participation factor	$\frac{H_{-(N-1)}^{(1)'}(ka)}{H_{-(N-1)}^{(1)}(ka)}\frac{J'_{-(N-1)}(ka)}{J'_{-(N-1)}(ka)}\beta_{-(N-1)}$	$\frac{H_{(N-1)}^{(1)'}(ka)}{H_{(N-1)}^{(1)}(ka)}\frac{J'_{(N-1)}(ka)}{J'_{(N-1)}(ka)}\beta_{(N-1)}$	$\frac{H_N^{(1)'}(ka)}{H_N^{(1)}(ka)}\frac{J'_N(ka)}{J'_N(ka)}\beta_N$

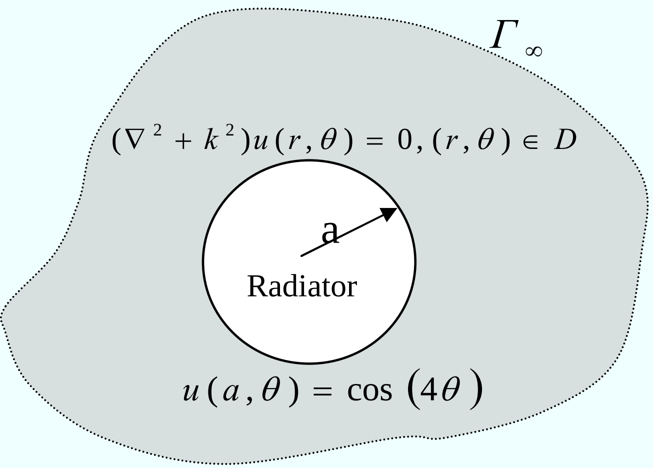
(Using *LM* formulation)

Numerical examples

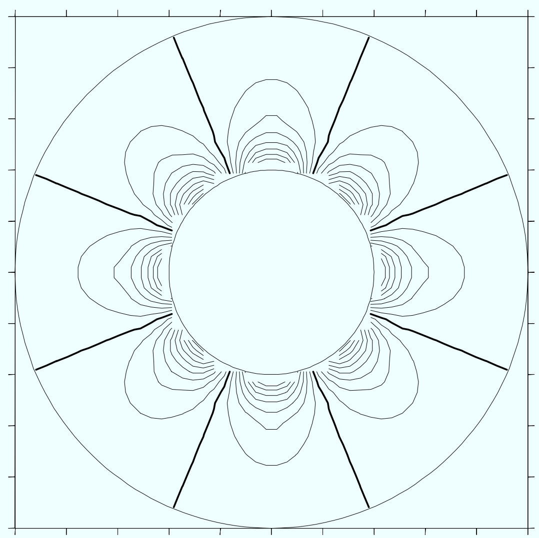
- Radiation problem

1. The radiation problem

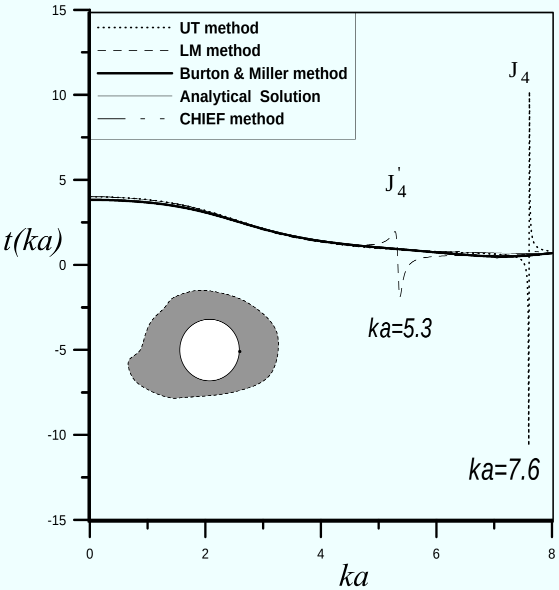
Analytical solution: $u(\rho,\phi)=\dfrac{H_4(k\rho)}{H_4^{(1)}(ka)}\cos(4\phi), \rho\geq a, 0\leq\phi<2\pi$



The radiation problem (Dirichlet type) for a cylinder



The contour plot for the real-part solution.



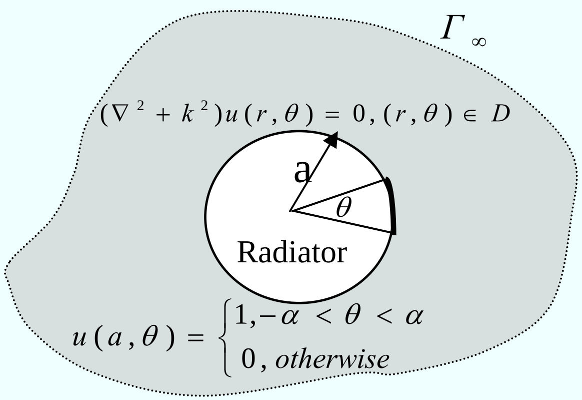
The positions of irregular values using different methods.

Numerical examples

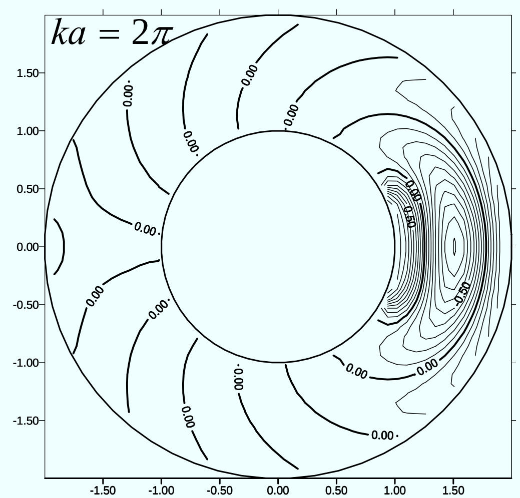
2. The nonuniform radiation problem

Analytical solution:

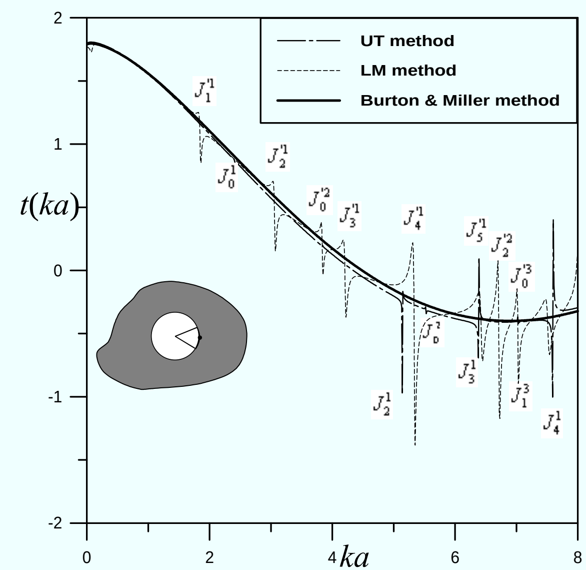
$$u(\rho,\phi)=\frac{1}{\pi}\frac{-\alpha}{k}\frac{H_0^{(1)}(k\rho)}{H_0^{(1)'}(ka)}H_0^{(1)}(k\rho)+\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{-1}{k}\frac{\sin(n\alpha)}{n}\frac{H_n^{(1)}(k\rho)}{H_n^{(1)'}(ka)}\cos(n\phi),\ \rho\geq a, 0\leq\phi<2\pi$$



The radiation problem (Dirichlet type) for a cylinder



The contour plot for the real-part solutions.



The positions of irregular values using different methods.

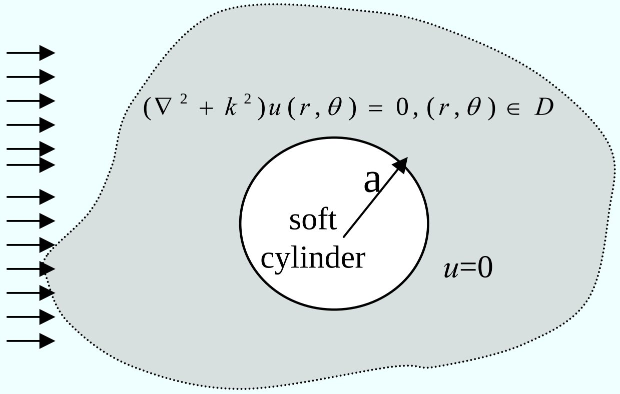
Numerical examples

- Scattering problem

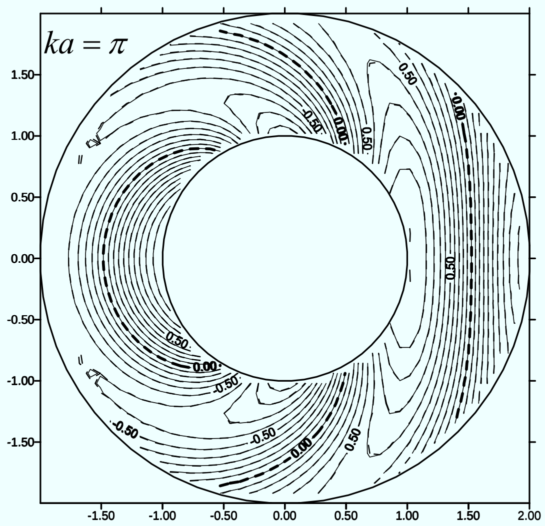
1. Dirichlet condition

Analytical solution:

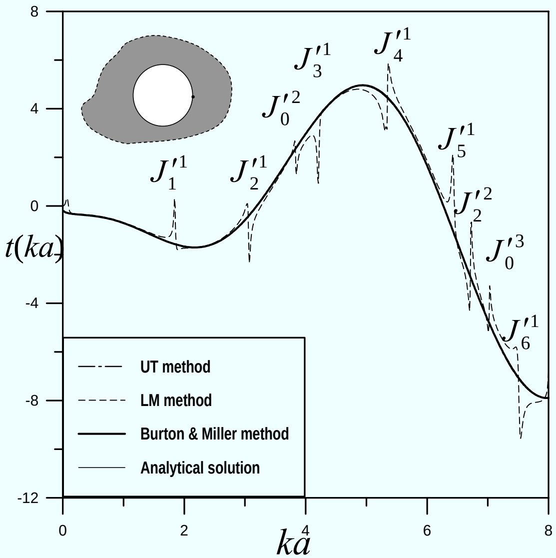
$$u(\rho,\phi) = -\frac{J_0(ka)}{H_0^{(1)}(ka)} H_0^{(1)}(k\rho) - 2\sum_{n=1}^{\infty} i^n \frac{J_n(ka)}{H_n^{(1)}(ka)} H_n^{(1)}(k\rho)\cos(n\phi),\ \rho \geq a, 0 \leq \phi < 2\pi$$



The scattering problem (Dirichlet type) for a cylinder



The contour plot for the real-part solutions

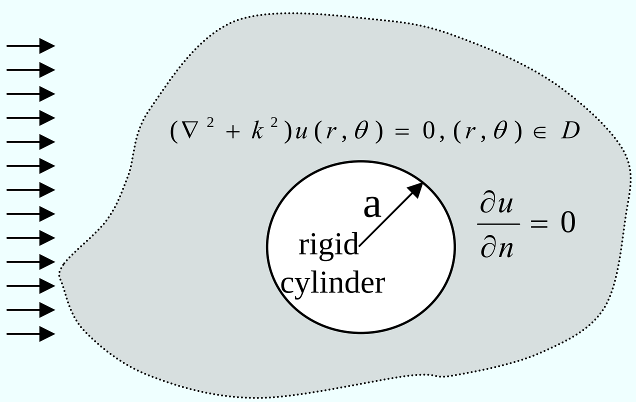


The positions of irregular values using different methods

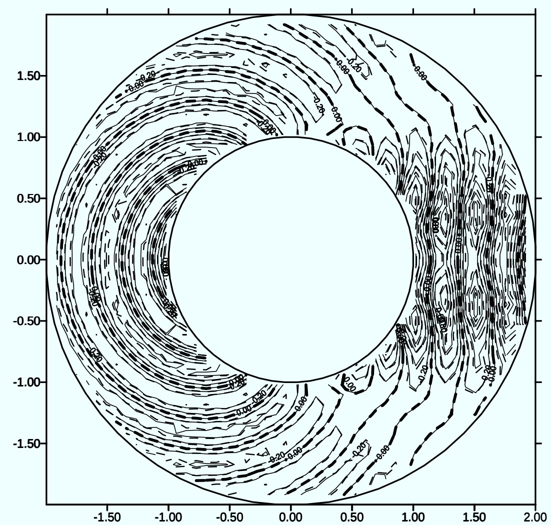
2. Neumann condition

Analytical solution:

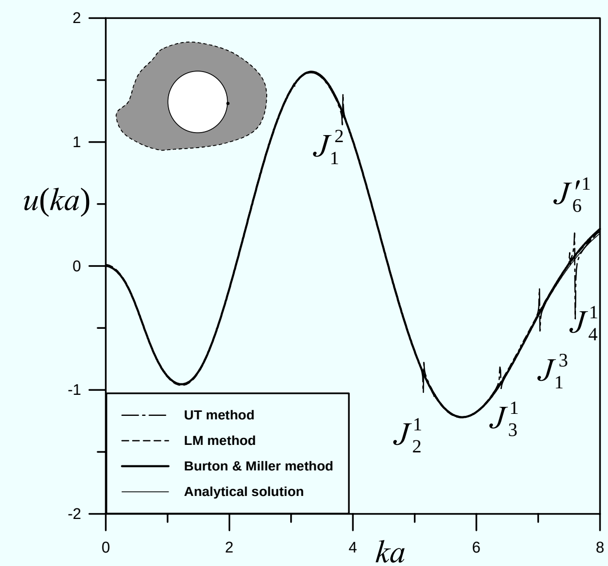
$$u(\rho,\phi) = -\frac{J'_0(ka)}{H_0^{(1)'}(ka)} H_0^{(1)}(k\rho) - 2\sum_{n=1}^{\infty} i^n \frac{J'_n(ka)}{H_n^{(1)'}(ka)} H_n^{(1)}(k\rho)\cos(n\phi), \rho \geq a, 0 \leq \phi < 2\pi$$



The scattering problem (Neumann type) for a cylinder



The contour plot for the real-part solutions



The positions of irregular values using different methods

Conclusion

1. We have proposed the concept of modal participation factor for numerical instability in the dual BEM for exterior acoustics.
2. The modal participation factor for both the continuous and discrete system were derived.
3. The irregular values depend on the integral formulation (*UT*-$J_n(ka)=0$ *or LM* -$J'_n(ka)=0$) instead of B.C.(Dirichlet or Neumann).
4. The numerical results using the dual BEM program agree very well with the analytical solution and the DtN results except at k_f.
5. Burton and Miller approach and the CHIEF method were successfully to deal with numerical instability.