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On the Rank-deficiency Problems in Boundary Integral Formulation Using the Fredholm Alternative Theorem and Singular Value Decomposition Technique

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Abstract

We provide a perspective on the degenerate problems, including degenerate boundary, degenerate scale, spurious eigensolution and fictitious frequency, in the boundary integral formulation. All the degenerate problems originate from the rank deficiency in the influence matrix. Both the Fredholm alternative theorem and singular value decomposition (SVD) technique are employed to study the degenerate problems. Updating terms and updating documents of the SVD technique are utilized. The roles of right and left unitary vectors of SVD in BEM and their relations to true, spurious and fictitious modes are examined by using the Fredholm alternative theorem. A unified method for dealing with the degenerate problem in BEM is proposed. Several examples are demonstrated to check the validity of the unified formulation.

1 Introduction

The boundary integral equation method (BIEM) and the boundary element method (BEM) have received much attention since Rizzo [1] proposed a numerical treatment of the boundary integral equation for elastostatics. Most of the efforts have been focused on the singular boundary integral equation for primary fields (*e.g.* potential u or displacement $\mathbf{u}$). For most problems, the formulation of a singular boundary integral equation for the primary field provides sufficient conditions to ensure a unique solution. In some cases, *e.g.*, those with Hermite polynomial elements [2], degenerate boundaries [3, 4, 5, 6], corners [7], the construction of a symmetric matrix [8, 9, 10], the improvement of condition numbers [11], the construction of an image system [12], the tangent flux or hoop stress calculation on the boundary [13], an error indicator in the adaptive BEM [14], fictitious (irregular) frequencies in exterior acoustics [15, 16], spurious eigenvalues in the real-part BEM [17, 18, 19], the imaginary-part BEM [20, 21] and the multiple reciprocity method (MRM) [22, 23, 24, 25], degenerate scale [26, 27, 28, 29, 30] and the Tikhonov formulation for inverse problems, it is found that the integral representation for a primary field can not provide sufficient constraints. In another words, the influence matrices are rank deficient. It is well known that the hypersingular equation plays an important role in the aforementioned problems. Many researchers have paid attention to the hypersingular equation. A review article on hypersingularity can be found in Chen and Hong [31]. The hypersingular formulation provides the theoretical bases for degenerate boundary problems. Totally speaking, four degenerate problems in BEM, degenerate scale, degenerate boundary, spurious eigenvalues and fictitious frequency, are encountered. In the following, we will review the four sources which result in the rank deficiency.

1.1 Degenerate boundary in boundary value problems

For the problem with a degenerate boundary, the dual integral representation has been proposed for crack problems in elasticity by Hong and Chen [4, 5], and boundary element researchers [3, 6, 32, 33, 34, 35, 36] have increasingly paid attention to the second equation of the dual representation. The second equation, which is derived for the secondary field (*e.g.*, flux t or traction $\mathbf{t}$), is very popular now and is termed the hypersingular boundary integral equation. Hong and Chen [4] presented the theoretical bases of the dual integral equations in a general formulation which incorporates the displacement and traction boundary integral equations. Huang and So [37] extended the concept of the Hadamard principal value in the dual integral equations [4] to determine the dynamic stress intensity factors of multiple cracks. Gray [3, 32] also independently found the hypersingular integral representations for the Laplace equation and the Navier equation although he did not coin the formulation “dual”. Martin, Rizzo and Gonsalves [38] called the new kernel in the dual integral equations “hypersingular” while Kaya [39] earlier called the kernel “superstrong singularity”. Since the formulation was derived for the secondary field, by analogy with the term “natural boundary condition”, Feng and Yu [40, 41, 42] called the method “natural BEM” or “canonical integral equations”. Balas, Sladek and Sladek in their book [43] proposed a unified theory for crack problems by using the displacement boundary integral equation and another integro-differential equation for the traction field. Based on the dual integral representation for the degenerate boundary problems, Hong and Chen developed the dual BEM programs for crack [4] and potential flow problems with a cutoff wall [44]. Besides, Chen and his coworkers extended the dual BEM program for the Laplace equation and the Navier equation to three programs. One is for the Helmholtz equation by the dual MRM [45]. Another is for the Helmholtz equation by the complex-valued formulation [24, 46]. The other is for the modified Helmholtz equation [47]. A general purpose program, BEASY, was developed for crack problems by the Wessex Institute of Technology (WIT) and termed the “dual boundary element method (DBEM)” [6, 36]. This program has been extended to solve crack growth problems more efficiently by using the benefit of the single-domain approach [18, 36]. Chen and Hong [31], Mi and Aliabadi [33] extended two-dimensional cases to three-dimensional crack problems. A program implemented by Lutz *et al.* [48] was also reported. In the mathematical literature, the relationships between the boundary integral operators and various layer potentials are obtainable through the so-called Calderon projector [12]. Four identities to relate the four kernels have been constructed. The order of pseudo-differential operator for the integral equations on the circular case in the dual formulation was discussed by Amini [49], Chen and Chiu [50]. Detailed discussions can be found in [51, 52]. These mathematical problems were first studied by Hadamard [53] and Mangler [54]. The hypersingular integral equation was derived by Hadamard in solving the cylindrical wave equation by employing the spherical means of descent. The improper integral was then defined by Tuck [55] as the "Hadamard principal value". Almost at the same time of Hadamard's work, Mangler derived the same mathematical form in solving a thin airfoil problem. This is the reason why the improper integral of hypersingularity is called the "Mangler principal value" in theoretical aerodynamics [56]. This nonintegrable integral of hypersingularity [52] arises naturally in the dual boundary integral representations especially for problems with degenerate boundaries, *e.g.*, crack problems in elasticity [4, 5, 12], heat flow through a baffle [57], Darcy flow around a cutoff wall [58], a cracked bar under torsion [59], screen impinging in acoustics [22, 58, 60, 61, 62], antenna in electromagnetic wave [63], a thin breakwater [47] and aerodynamic problems of a thin airfoil [64]. Applications of the hypersingular integral equation in mechanics were discussed by Martin *et al.* [38] and by Chen and Hong [11]. Combining the singular integral equation, *e.g.*, Green's identity (scalar field) or Somigliana's identity (vector field), with the hypersingular integral equation, we can construct the dual integral equations according to the continuous and discontinuous properties of the potential as the field point moves across the boundary [44]. From the above point of view, the definition of the *dual (boundary) integral equations* is quite different from that of the *dual integral equations* given by Sneddon and Lowangrub [65] and Buecker [66], which, indeed, come from the same equation but different collocation points in crack problems of elastodynamics. The solution for the conventional dual integral equations was first studied by Beltrami [67]. The dual boundary integral equations for the primary and secondary fields defined and coined by Hong and Chen are generally independent of each other, and only for very special cases are they dependent [68]. To deal with the degenerate boundary problems, the hypersingular formulation is a powerful method in conjunction with the dual BEM. However, regularization for hypersingularity is required. To avoid hypersingularity, one alternative has been proposed by using the multi-domain approach of singular equation in sacrifice of introducing artificial boundary where the continuity and equilibrium conditions on the interface boundary are considered to condense the matrix. We may wonder whether it is possible to solve the degenerate problems by using only the singular equation in the single-domain approach. The SVD technique will be considered to achieve the goal.

1.2 Degenerate scale for 2-D Laplace and Navier problems

It is well known that rigid body motion test or so called use of simple solution can be employed to examine the singular matrices in BEM for the strongly singular and hypersingular kernels in the problems without degenerate boundaries. Zero eigenvalues associated with rigid body modes are imbedded in the corresponding influence matrices. In such a case, singular matrix occurs physically and mathematically. The nonunique solution for a singular matrix is found to include a rigid body term for the interior Neumann (traction) problem. However, for a certain geometry, the influence matrix of the weakly singular kernel may be singular for the Dirichlet problem [69]. In another words, the numerical results may be unstable when the used scale is changed or the considered domain is expanded to a special size. The nonunique solution is not physically realizable but results from the zero eigenvalue of the influence matrix in the BEM. The special geometry dimension which results in a nonunique solution for a potential problem is called a degenerate scale by He [30] and Chen *et al.* [27]. The term "scale" stems from the fact that degenerate mechanism depends on the geometry size used in the BEM implementation. Some mathematicians [29, 70] coined it a critical value (C.V.) since it is mathematically realizable. For several specific boundary conditions, some studies for potential problems (Laplace equations) [27], plate problems (biharmonic equations) [29] and plane elasticity problems [26, 30] have been done. The difficulties due to nonuniqueness of solutions were overcome by the necessary and sufficient boundary integral formulation [30] and boundary contour method [71]. The degenerate scale problems in the BEM have been studied analytically by Kuhn [72] and Constanda [28] and numerical experiments have been performed [27]. Degenerate kernels and circulant matrices were employed to determine the eigenvalues for the influence matrices analytically in a discrete system for circular and annular problems [27]. The singularity pattern distributed along a ring boundary resulting in a null field can be obtained when the ring boundary is a degenerate scale. An annular region has also been considered for the harmonic equation [4] and the biharmonic equation [73] and the possible degenerate scales were investigated. Hypersingular formulation is an alternative to study the degenerate scale problems for simply-connected problems [26], since eigenvalues are never zero. Another simple approach is to superimpose a rigid body motion in the fundamental solution so that the zero eigenvalue can be shifted to be nonzero. However, this treatment results in another degenerate scale. By employing the CHIEF concept [74, 75], a CHEEF approach was developed to obtain the independent constraint. A unified method will be proposed to study the problem by using the Fredholm alternative theorem and SVD updating technique. Both the spurious mode (mathematically realizable) and rigid body mode (physically realizable) can be determined. The roles of left and right unitary matrices in SVD for BEM will be examined. In addition, a direct treatment in the matrix operation instead of adding a rigid body term in the fundamental solution can be derived.

1.3 Spurious eigensolutions for interior eigenproblems

For interior problems, eigendata are very important informations in vibrations and acoustics. According to the complex-valued boundary element method [60, 76, 77], the eigenvalues and eigenmodes can be determined. Nevertheless, complex arithmetic is required. To avoid complex arithmetic, many approaches including the multiple reciprocity method (MRM) [78], the real-part [18, 19, 45] and the imaginary-part BEMs [20, 79] have been proposed. For example, Tai and Shaw [80] employed only real-part kernel in the integral formulation. A simplified method using only the real-part or imaginary-part kernel was also presented by De Mey [79] and Hutchinson [81]. Although De Mey found that the zeros for a real-part of the complex determinant may be different from the determinant using the real-part kernel, the spurious eigensolutions were not discovered analytically. Chen and Wong [23] and Yeih *et al.* [24, 46] found the spurious eigensolutions analytically in the MRM using simple examples of rod and beam, respectively. Later, Kamiya *et al.* [82] and Yeih *et al.* [25] independently claimed that MRM is no more than the real-part BEM. Kang *et al.* [83] employed the Nondimensional Dynamic Influence Function method (NDIF) to solve the eigenproblem. Chen *et al.* [21] commented that the NDIF method is a special case of imaginary-part BEM. Kang and Lee also found the spurious eigensolutions and filtered out the spurious eigenvalues by using the net approach [84]. Later, they extended to solve plate vibration problems [85]. Chen *et al.* [86] proposed a double-layer potential approach to filter out the spurious eigenmodes. The reason why spurious eigenvalues occur in the real-part BEM is the loss of the constraints, which was investigated by Yeih *et al.* [25]. The spurious eigensolutions and fictitious frequencies arise from an improper approximation of the null space operator [87]. The fewer number of constraint equations makes the solution space larger. Spurious eigensolutions were also found in the Maxwell equation [88]. The spurious eigensolutions can be filtered out by using many alternatives, *e.g.*, the complex-valued BEM [76], the domain partition technique [89], the dual formulation in conjunction with the SVD updating techniques [17, 45, 90] and the CHEEF (Combined Helmholtz Exterior integral Equation Formulation) method [74]. Besides, the spurious eigensolution for the multiply-connected problem was found even though the complex-valued kernel was used [91]. A unified formulation to study the phenomenon will be proposed by using the Fredholm alternative theorem and SVD technique. SVD updating techniques in conjunction with the dual formulation will be employed to sort out the true and spurious eigenvalues. In addition, the relation between the left unitary vector in SVD and the spurious mode will be discussed.

1.4 Fictitious frequency in exterior acoustics

For exterior acoustics, the solution to the boundary is perfectly unique for all wave numbers. This is not the case for the numerical treatment of integral equation formulation, which breaks down at certain frequency known as irregular frequency or fictitious frequency. This problem is completely nonphysical because there are no discrete eigenvalues for the exterior problems. It was found that the singular (*UT*) equation results in fictitious frequencies which are associated with the interior eigenfrequency of the Dirichlet problems while the hypersingular (*LM*) equation produces fictitious frequencies which are associated with the interior eigenfrequency of the Neumann problems [68]. The general derivation was provided in a continuous system [68], and a discrete system was analytically studied using the properties of circulant for a circular case [92, 93]. Schenck [94] proposed a CHIEF (Combined Helmholtz Interior integral Equation Formulation) method, which is easy to implement and is efficient but still has some drawbacks. Burton and Miller [95] proposed an integral equation that was valid for all wave numbers by forming a linear combination of the singular integral equation and its normal derivative through an imaginary constant. In case of a fictitious frequency, the resulting coefficient matrix for the exterior acoustic problems becomes ill-conditioned. This means that the boundary integral equations are not linearly independent and the resulted matrix is rank deficient. In the fictitious-frequency case, the rank of the coefficient matrix is less than the number of the boundary unknowns. The SVD updating technique can be employed to detect the possible fictitious frequencies and modes by checking whether the first minimum singular value, σ_1, is zero or not [75].

By employing the Fredholm alternative theorem and SVD updating technique, the degenerate mechanism for the four numerical problems, degenerate boundary, degenerate scale, spurious eigenvalues and fictitious frequencies, will be studied. A unified formulation will be constructed to solve for rank-deficiency problems. Illustrative examples will be illustrated to check the validity of the proposed method.

2 Mathematical tools

2.1 Degenerate kernels in the dual BEM

The kernel functions used in the dual BEM can be typically expressed in terms of degenerate kernels as follows [68]:

$$U(s,x) = \begin{cases} U^i(s,x) = \sum_{m=0}^{\infty} \frac{i}{\lambda_m} C_m(ks) R_m(kx), & x \in D^i \\ U^e(s,x) = \sum_{m=0}^{\infty} \frac{i}{\lambda_m} C_m(kx) R_m(ks), & x \in D^e \end{cases} \tag{1}$$

$$T(s,x) = \begin{cases} T^i(s,x) = \sum_{m=0}^{\infty} \frac{i}{\lambda_m} \{\nabla_s C_m(ks) \cdot n(s)\} R_m(kx), & x \in D^i \\ T^e(s,x) = \sum_{m=0}^{\infty} \frac{i}{\lambda_m} C_m(kx) \{\nabla_s R_m(ks) \cdot n(s)\}, & x \in D^e \end{cases} \tag{2}$$

$$L(s,x) = \begin{cases} L^i(s,x) = \sum_{m=0}^{\infty} \frac{i}{\lambda_m} C_m(ks) \{\nabla_x R_m(kx) \cdot n(x)\}, & x \in D^i \\ L^e(s,x) = \sum_{m=0}^{\infty} \frac{i}{\lambda_m} \{\nabla_x C_m(kx) \cdot n(x)\} R_m(ks), & x \in D^e \end{cases} \tag{3}$$

$$M(s,x) = \begin{cases} M^i(s,x) = \sum_{m=0}^{\infty} \frac{i}{\lambda_m} \{\nabla_s C_m(ks) \cdot n(s)\} \{\nabla_x R_m(kx) \cdot n(x)\}, & x \in D^i \\ M^e(s,x) = \sum_{m=0}^{\infty} \frac{i}{\lambda_m} \{\nabla_x C_m(kx) \cdot n(x)\} \{\nabla_s R_m(ks) \cdot n(s)\}, & x \in D^e \end{cases} \tag{4}$$

where D^i and D^e are the interior and exterior domains, respectively, C_m, R_m and λ_m are defined in Table 1 for one, two and three-dimensional Helmholtz problems. The bases of C_m and R_m are found to be the complete set functions in the Trefftz method. For the 2-D circular case, analytical study can be achieved by expanding the boundary density in terms of Fourier series. By considering the discrete Fourier series [96], the circulant property can be employed in analytical study for the discrete system [93].

2.2 The Fredholm alternative theorem

2.2.1 Discrete system

The linear algebraic equation $[K]\{u\} = \{p\}$ has a unique solution if and only if the only continuous solution to the homogeneous equation

$$[K]\{u\} = \{0\}, \tag{5}$$

is $\{u\} \equiv \{0\}$. Alternatively, the homogeneous equation has at least one solution if the homogeneous adjoint equation

$$[K]^H\{\phi\} = \{0\}, \tag{6}$$

has a nontrivial solution $\{\phi\}$, where $[K]^H$ is the transpose conjugate matrix of $[K]$ and $\{p\}$ must satisfy the constraint $(\{p\}^H\{\phi\} = 0)$. If the matrix $[K]$ is real, the transpose conjugate of a matrix is equal to its transpose only, *i.e.*, $[K]^H = [K]^T$.

2.2.2 Continuous system

The boundary integral equation $\int_B K(s,x)u(s)dB(s) = p(x)$ has a unique solution if and only if the only continuous solution to the homogeneous equation

$$\int_B K(s,x)u(s)dB(s) = 0, \tag{7}$$

is $u(s) \equiv 0$. Alternatively, the homogeneous equation has at least one solution if the homogeneous adjoint equation

$$\int_B K^H(s,x)\phi(s)dB(s) = 0, \tag{8}$$

has a nontrivial boundary solution $\phi(s)$, where $K^H(s,x)$ is the adjoint operator of $K(s,x)$ and $p(s)$ must satisfy the zero inner product between $p(s)$ and $\phi(s)$.

2.3 SVD technique

Employing the SVD technique for the $[K]$ matrix with dimension M by P, we have

$$[K]_{M\times P} = [\Phi]_{M\times M}[\Sigma]_{M\times P}[\Psi]^H_{P\times P}, \tag{9}$$

where $[\Phi]$ is a left unitary matrix constructed by the left singular vectors ($\{\phi_i\},\ \ i = 1,2,\cdots,M$), and $[\Sigma]$ is a diagonal matrix which has singular values $\sigma_1, \sigma_2, \cdots$, and σ_P allocated in a diagonal line as

$$[\Sigma] = \begin{bmatrix} \sigma_P & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sigma_1 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix}_{M\times P} , \tag{10}$$

in which $\sigma_P \geq \sigma_{P-1} \cdots \geq \sigma_1$ and $[\Psi]^H$ is the complex conjugate transpose of a right unitary matrix constructed by the right singular vectors ($\{\psi_i\},\ \ i = 1,2,\cdots,P$). As we can see in Eq.(10), there exists at most P nonzero singular values. By employing the SVD technique to determine the eigenvalue, we can obtain the boundary mode by extracting the right singular vector $\{\psi_i\}$ in the right unitary matrix $[\Psi]$ of SVD corresponding to the near zero or zero singular value. According to the properties of SVD, we have

$$[K]\{\psi_i\} = \sigma_i\{\phi_i\} \ \ i = 1,2,3\cdots P \ . \tag{11}$$

If the *q-th* singular value, σ_q, is zero, then we have the following equation from Eq.(11):

$$[K]\{\psi_q\} = 0\{\phi_q\} = \{0\},\ \ q \leq P \ , \tag{12}$$

According to Eq.(12), the nontrivial boundary mode is found to be the right singular vector, $\{\psi_q\}$, in the right unitary matrix.

3 Applications to the degenerate scale problem in the BEM

By using the conventional BEM (UT formulation) [96] for the potential problem, we have

$$[U]\ \{t\} = [T]\ \{u\} = \{p\}. \tag{13}$$

According to the Fredholm alternative theorem, Eq.(13) has at least one solution for $\{t\}$ if the homogeneous adjoint equation

$$[U]^T\ \{\phi_1\} = \{0\}, \tag{14}$$

has a nontrivial solution $\{\phi_1\}$, in which the constraint $(\{p\}^T\{\phi_1\} = 0)$ must be satisfied. By substituting Eq.(13) into $\{p\}^T\{\phi_1\} = 0$, we obtain

$$\{u\}^T\ [T]^T\ \{\phi_1\} = 0. \tag{15}$$

Since $\{u\}$ is an arbitrary vector for the Dirichlet problem, we have

$$[T]^T\ \{\phi_1\} = \{0\}, \tag{16}$$

where $\{\phi_1\}$ is the spurious mode. Combining Eq.(14) and Eq.(16) together, we have

$$\begin{bmatrix} [U]^T \\ [T]^T \end{bmatrix} \{\phi_1\} = \{0\} \ \text{ or } \ \{\phi_1\}^T \begin{bmatrix} [U] & [T] \end{bmatrix} = \{0\}. \tag{17}$$

Eq.(17) indicates that the two matrices have the same spurious mode $\{\phi_1\}$ corresponding to the same zero singular value when a degenerate scale occurs. The former one in Eq.(17) is a form of updating term. The latter one is a form of updating document. By using the SVD technique for the $[U]^T$ and $[T]^T$ matrices, we have

$$\begin{aligned} [U]^T &= [\Psi_U]\,[\Sigma_U]\,[\Phi_U]^T\,, \\ [T]^T &= [\Psi_T]\,[\Sigma_T]\,[\Phi_T]^T\,, \end{aligned} \tag{18}$$

where $\{\phi_1\}$ is imbedded in both the matrices, $[\Phi_U]$ and $[\Phi_T]$, with the corresponding zero singular value in the matrices, $[\Sigma_U]$ and $[\Sigma_T]$, respectively. Since $\{\phi_1\}$ is one of the left unitary vectors in $[\Phi_U]$ matrix with respect to the zero singular value, we have

$$[U]^T\{\phi_1\} = 0\,\{\psi_1\}, \tag{19}$$

where $\{\psi_1\}$ satisfies

$$[U]\{\psi_1\} = 0\,\{\phi_1\}. \tag{20}$$

To deal with the problem of degenerate scale in BEM, three approaches, method of adding a rigid body mode, hypersingular formulation (LM equation) and CHEEF method, can be employed. For an elliptical bar under torsion, the results are shown in Table 2. Degenerate scale occurs when the sum of the two axes are two $(\alpha+\beta = 2)$. It is found that the conventional BEM (UT method) can not obtain the correct torsional rigidity for the degenerate scale case. By employing the regularization techniques, the error of torsional rigidity can be reduced to be smaller than 10% after comparing with the exact solution.

4 Applications to the eigenproblem with a degenerate boundary

It is well known that the two methods, multi-domain BEM and dual BEM, can be applied to deal with the degenerate boundary problem. Here, we will propose a new approach to deal with the degenerate-boundary eigenproblem by using SVD. In the Dirichlet eigenproblem for a membrane with a stringer, the influence matrix $[U(k)]$ is rank deficient due to two sources, the degeneracy of stringers and the nontrivial mode for eigensolution. Since N_d constant elements locate on the stringer in Fig.1(a), the matrix $[U(k)]$ results in N_d zero singular values

$(\sigma_1 = \sigma_2 \cdots = \sigma_{N_d} = 0)$. The next $N_d + 1$ zero singular value $\sigma_{N_d+1} = 0$ originates from the nontrivial eigensolution. To detect the eigenvalues, the $(N_d + 1)^{th}$ zero singular value versus k is plotted to find the drop where nontrivial eigensolution occurs in Fig.1(a). Good agreement for the eigenvalues is obtained as shown in Table 3 and Fig.1(a) after comparing with those of the multi-domain BEM in Fig.1(c) and the dual BEM in Fig.1(b).

5 Applications to the spurious mode for interior problems

By using the real-part BEM (UT formulation), the spurious eigenvalue k_s satisfies

$$\begin{bmatrix} [U_R(k_s)]^T \\ [T_R(k_s)]^T \end{bmatrix} \{\phi_R^{(UT)}\} = \{0\}, \tag{21}$$

where the subscript R denotes the real part. In the hypersingular formulation (LM method), the spurious eigenvalue satisfies

$$\begin{bmatrix} [L_R(k_s)]^T \\ [M_R(k_s)]^T \end{bmatrix} \{\phi_R^{(LM)}\} = \{0\}. \tag{22}$$

By using the imaginary-part BEM, the spurious eigenvalue satisfies

$$\begin{bmatrix} [U_I(k_s)]^T \\ [T_I(k_s)]^T \end{bmatrix} \{\phi_I^{(UT)}\} = \{0\}, \tag{23}$$

where the subscript I denotes the imaginary part. In the hypersingular formulation of imaginary-part BEM, the spurious eigenvalue satisfies

$$\begin{bmatrix} [L_I(k_s)]^T \\ [M_I(k_s)]^T \end{bmatrix} \{\phi_I^{(LM)}\} = \{0\}. \tag{24}$$

For the Dirichlet problem, the true eigenvalue k_t satisfies

$$\begin{bmatrix} [U_R(k_t)] \\ [L_R(k_t)] \end{bmatrix} \{\psi_R^{(UL)}\} = \{0\}, \tag{25}$$

and

$$\begin{bmatrix} [U_I(k_t)] \\ [L_I(k_t)] \end{bmatrix} \{\psi_I^{(UL)}\} = \{0\}, \tag{26}$$

by using the real-part and imaginary-part BEMs, respectively. For the Neumann problem, the true eigenvalue can be sorted out by using

$$\begin{bmatrix} [T_R(k_t)] \\ [M_R(k_t)] \end{bmatrix} \{\psi_R^{(TM)}\} = \{0\}, \tag{27}$$

and

$$\begin{bmatrix} [T_I(k_t)] \\ [M_I(k_t)] \end{bmatrix} \{\psi_I^{(TM)}\} = \{0\}, \tag{28}$$

by using the real-part and imaginary-part BEMs, respectively. By demonstrating a circular case, the true and spurious eigenvalues are shown in Tables 4 and 5 by using the real-part and imaginary-part BEMs, respectively. It is found that the figures drop at the positions as predicted in Eqs.(21)∼(28).

6 Applications to the fictitious frequency for exterior acoustics

For exterior acoustics using the BEM, the fictitious wave number, k_f, satisfies

$$\begin{bmatrix} [U(k_f)]^H \\ [T(k_f)]^H \end{bmatrix} \{\phi_1\} = \{0\}, \tag{29}$$

$$\begin{bmatrix} [L(k_f)]^H \\ [M(k_f)]^H \end{bmatrix} \{\phi_1\} = \{0\}, \tag{30}$$

by using the singular and hypersingular formulations, respectively.

According to the unitary vectors, we can express boundary data into

$$\{u\} = \sum_{i=1}^{N} \beta_i \{\psi_i^{(T)}\}\,, \tag{31}$$

$$\{t\} = \sum_{i=1}^{N} \alpha_i \{\psi_i^{(U)}\}\,, \tag{32}$$

where N is the number of unknowns, α_i and β_i are the generalized coordinates. By multiplying $\{\phi_j\}^H$ (regular mode) into

$$[T]\,\{u\} = [U]\,\{t\}, \tag{33}$$

we can determine β_j easily by

$$\beta_j = \frac{1}{\sigma_j} \{\phi_j\}^H \, [U]\,\{t\}. \tag{34}$$

By multiplying $\{\phi_j\}^H$ (fictitious mode) into Eq.(34), we have

$$\sigma_i^{(T)}\,\beta_j = \sigma_i^{(U)}\,\alpha_j. \tag{35}$$

We can determine the value of $\alpha_j(\beta_j)$ with respect to the Dirichlet (Neumann) problem by

$$\beta_j = \frac{\sigma_i^{(U)}}{\sigma_i^{(T)}}\,\alpha_j, \quad \text{(Neumann problem)} \tag{36}$$

or

$$\alpha_j = \frac{\sigma_i^{(T)}}{\sigma_i^{(U)}}\,\beta_j, \quad \text{(Dirichlet problem)} \tag{37}$$

where

$$[U]^H\,\{\phi_j^{(U)}\} = \sigma_j^{(U)}\,\{\psi_j^{(U)}\}, \quad [U]\,\{\psi_j^{(U)}\} = \sigma_j^{(U)}\,\{\phi_j^{(U)}\}, \tag{38}$$

$$[T]^H\,\{\phi_j^{(T)}\} = \sigma_j^{(T)}\,\{\psi_j^{(T)}\}, \quad [T]\,\{\psi_j^{(T)}\} = \sigma_j^{(T)}\,\{\phi_j^{(T)}\}. \tag{39}$$

By using the same spurious mode $\{\phi_j\}$, we can determine

$$\sigma_j^{(U)} = \langle\,[U]^H\,\{\phi_j^{(U)}\},\,\{\psi_j^{(U)}\}\rangle, \tag{40}$$

$$\sigma_j^{(T)} = \langle\,[T]^H\,\{\phi_j^{(T)}\},\,\{\psi_j^{(T)}\}\rangle. \tag{41}$$

Therefore, the zero division by zero may be determined in the principal axis, *i.e.*, the L'Hospital rule is implemented for the single degree of freedom using the generalized coordinate.

By demonstrating a cylinder radiator, the fictitious frequency occurs at the eigenvalue of corresponding interior problem as shown in Fig.2. Numerical results using Eqs.(29) and (30) match well with the analytical data by using the singular (UT) and hypersingular (LM) formulations, respectively.

7 Conclusions

Four degenerate problems in the BEM were reviewed. A unified formulation to study degenerate problems in the BEM was proposed. Mathematically speaking, the numerical problems originate from the rank deficiency of the influence matrix. By decomposing the matrix using the SVD updating techniques, spurious mode and true mode were separated to be imbedded in the left and right unitary vectors, respectively. Fredholm alternative theorem was adopted to obtain the updating documents in SVD. Numerical examples were demonstrated to check the validity of the unified formulation.

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Table 1: Degenerate kernels for one, two and three-dimensional problems.

Helmholtz Equation	**Cartesian coordinate (1-D)**	**Cylindrical coordinate (2-D)**	**Spherical coordinate (3-D)**
$R_m(ks)$	$\cos(ks)$	$J_m(k\bar{\rho})e^{im\bar{\theta}}$	$j_m(k\bar{\rho})P_m^l(\cos\bar{\theta})\cos(l\bar{\phi})$
$C_m(ks)$	e^{-iks}	$H_m^{(1)}(k\bar{\rho})e^{-im\bar{\theta}}$	$h_m^{(1)}(k\bar{\rho})P_m^l(\cos\bar{\theta})\cos(l\bar{\phi})$
$I_m(ks)$	$\sin(ks)$	$Y_m(k\bar{\rho})e^{-im\bar{\theta}}$	$y_m(k\bar{\rho})P_m^l(\cos\bar{\theta})\cos(l\bar{\phi})$
λ_m	k	4	$4\pi/k$

where $s=(\bar{\rho},\bar{\theta})$ for the cylindrical coordinate, $s=(\bar{\rho},\bar{\theta},\bar{\phi})$ for the spherical coordinate, P_m^l is the Legendre polynomial, and J_m, Y_m, j_m and y_m are the *m-th* order cylindrical and spherical Bessel functions, respectively.

Table 2: Degenerate scale and torsion rigidity for an elliptical bar under torsion.

<table>
<tr><th>Torsion rigidity \ Scale
Method</th><th>Normal scale
$(\alpha=3.0,\ \beta=1.0)$</th><th>Normal scale</th><th>Degenerate scale
$(\alpha=1.5,\ \beta=0.5)$</th></tr>
<tr><td>Analytical method</td><td>8.4823</td><td rowspan="5">$\Gamma=\int_B\psi_1(s)dB(s)=1.4509$
$d=e^{-\frac{1}{\Gamma}}=0.5019$
(Expansion ratio)
$\alpha+\beta=2.0058$
(Degenerate scale)</td><td>0.5301</td></tr>
<tr><td>Direct BEM (UT)</td><td>8.7623 (error=3.30%)</td><td>-0.8911 (error=268.10%)</td></tr>
<tr><td>Direct BEM (LM)</td><td rowspan="3">Regularization techniques are not necessary</td><td>0.4812 (error=9.22%)</td></tr>
<tr><td>Adding rigid body term (c=1.0)</td><td>0.5181 (error=2.26%)</td></tr>
<tr><td>CHEEF technique (2.0, 2.0)</td><td>0.5647 (error=6.53%)</td></tr>
</table>

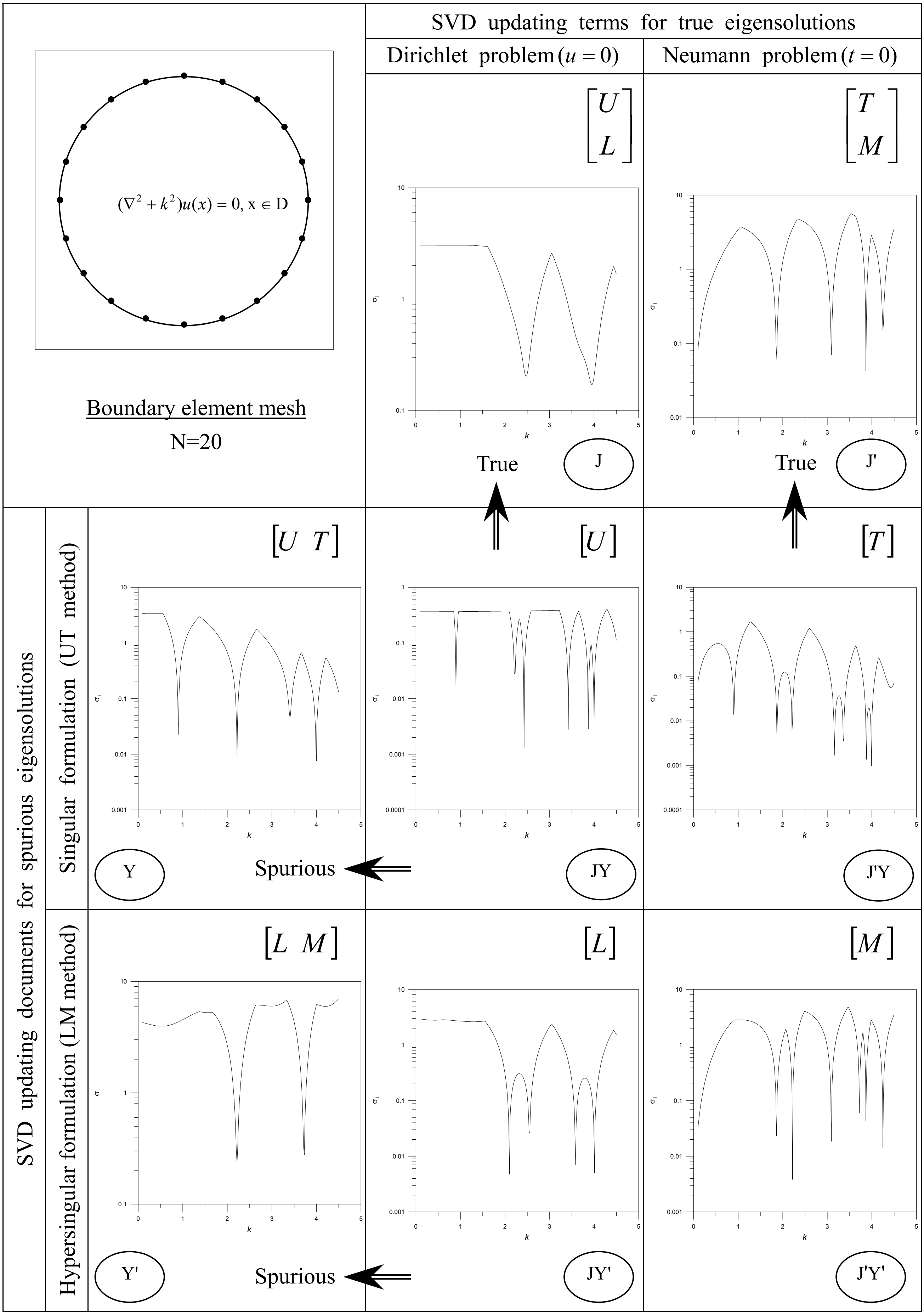
The exact solution for torsional rigidity is $T_r=G\dfrac{\pi\alpha^3\beta^3}{\alpha^2+\beta^2}$, where G is the shear modulus.

Table 3: The former eight eigenvalues for membranes with a single-edge stringer (a=1.0).

Eigenvalues / **Method**	k_1	k_2	k_3	k_4	k_5	k_6	k_7	k_8
FEM	3.14	3.82	4.48	5.12	5.74	6.27	6.35	6.95
DBEM	3.13	3.83	4.49	5.14	5.75	6.29	6.36	6.96
UT BEM+SVD	3.09	3.84	4.50	5.14	5.77	6.17	6.39	6.99
Multi-domain BEM	3.21	3.76	4.51	5.14	5.80	6.27	6.49	6.78
Exact solution*	π	3.83	4.50	5.14	5.76	2π	6.38	6.92

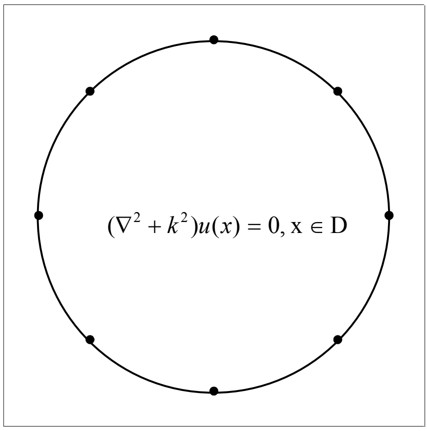
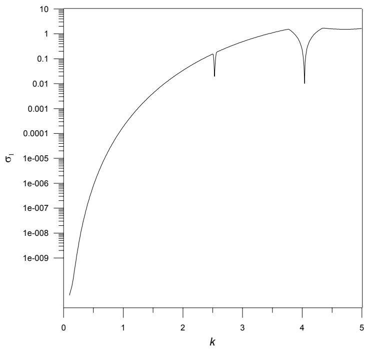
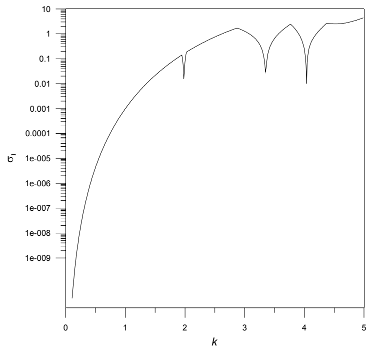
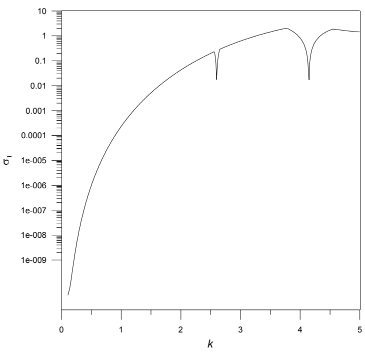
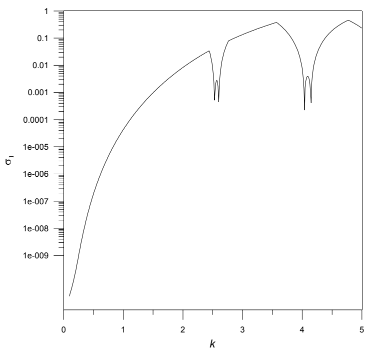
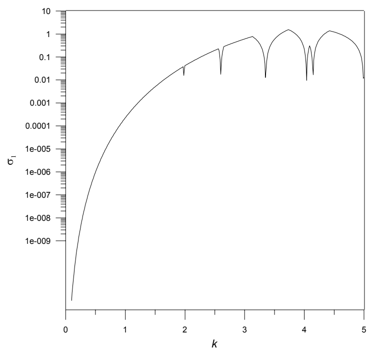
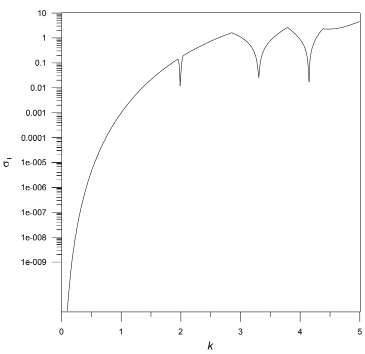
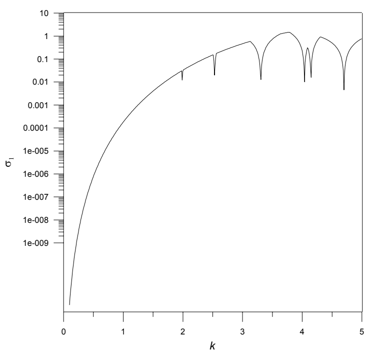
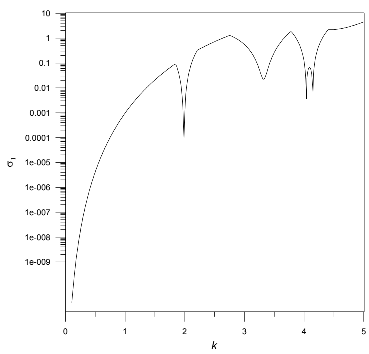
* $J_{n/2}(k)=0,\quad n=1,2,3\cdots$.

Table 4: True and spurious eigensolutions using the real-part dual BEM



where J, Y and J', Y' are the Bessel functions and their derivatives.

Table 5: True and spurious eigensolutions using the imaginary-part dual BEM

		SVD updating terms for true eigensolutions	
		Dirichlet problem ($u = 0$)	Neumann problem ($t = 0$)
 Boundary element mesh N=8		$\begin{bmatrix} U \\ L \end{bmatrix}$  True J	$\begin{bmatrix} T \\ M \end{bmatrix}$  True J'
SVD updating documents for spurious eigensolutions	Singular formulation (UT method): $\begin{bmatrix} U & T \end{bmatrix}$  J Spurious	$\begin{bmatrix} U \end{bmatrix}$  JJ	$\begin{bmatrix} T \end{bmatrix}$  J'J
	Hypersingular formulation (LM method): $\begin{bmatrix} L & M \end{bmatrix}$  J' Spurious	$\begin{bmatrix} L \end{bmatrix}$  JJ'	$\begin{bmatrix} M \end{bmatrix}$  J'J'

where J and J' are the Bessel functions and their derivatives.

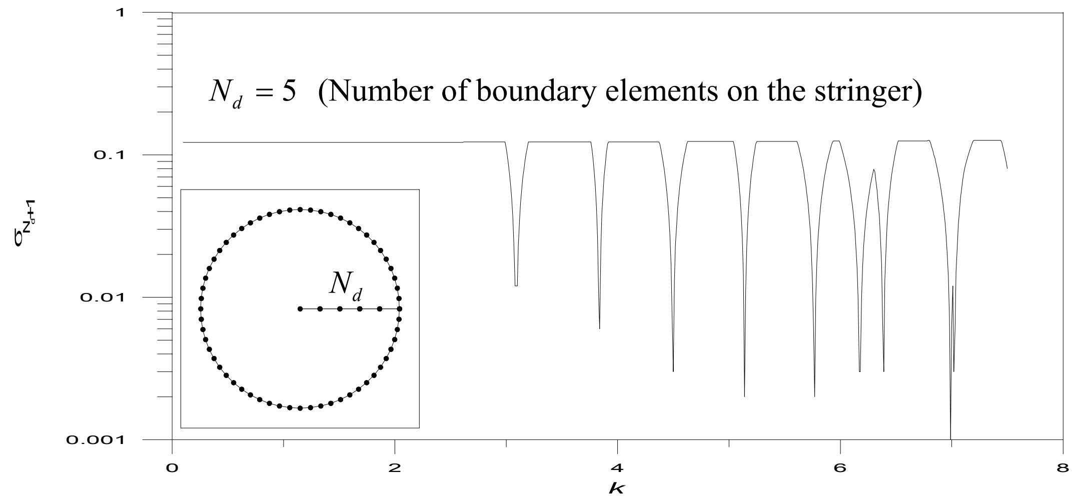


Figure 1(a): The $(\sigma_{N_d+1})^{th}$ zero singular value versus the wave number using the *UT* BEM+SVD.

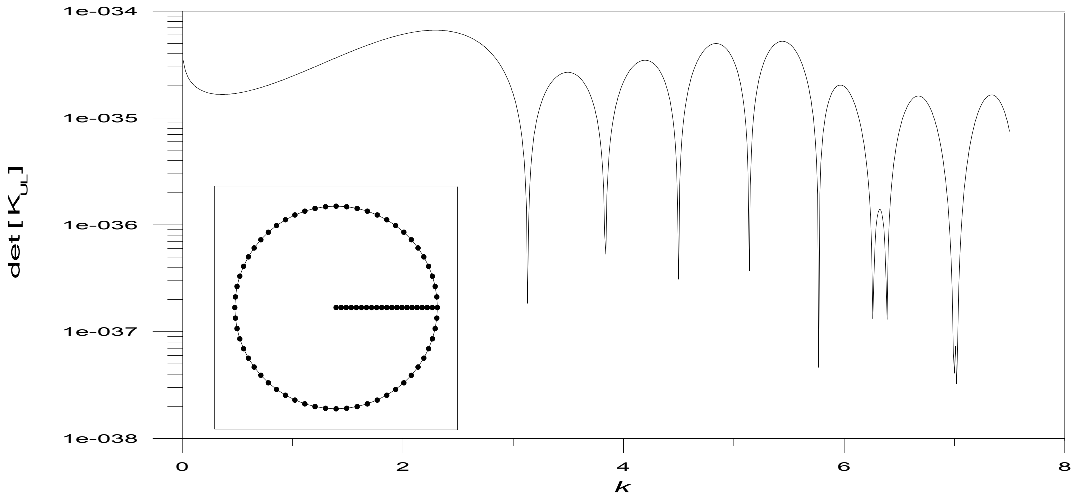


Figure 1(b): The determinant versus the wave number using the dual BEM.

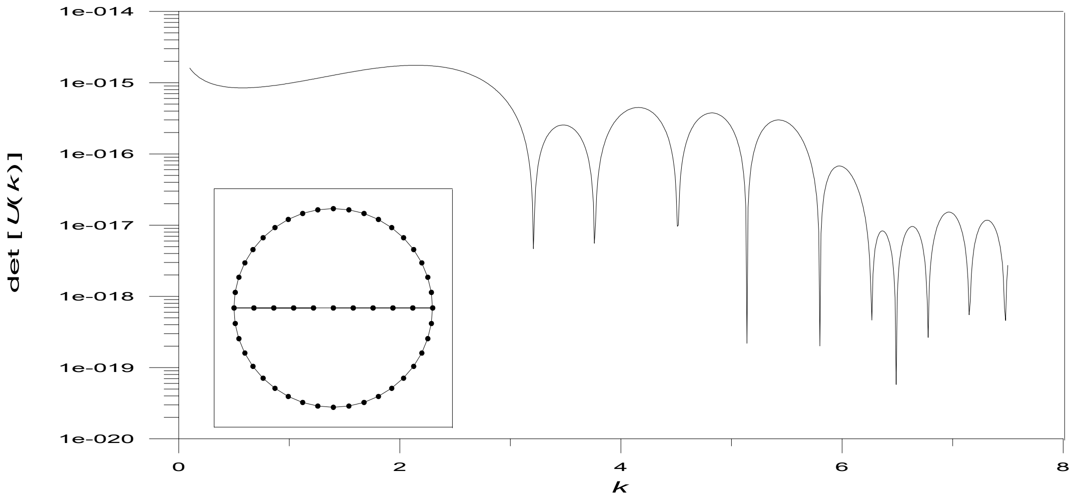


Figure 1(c): The determinant versus the wave number using the multi-domain BEM.

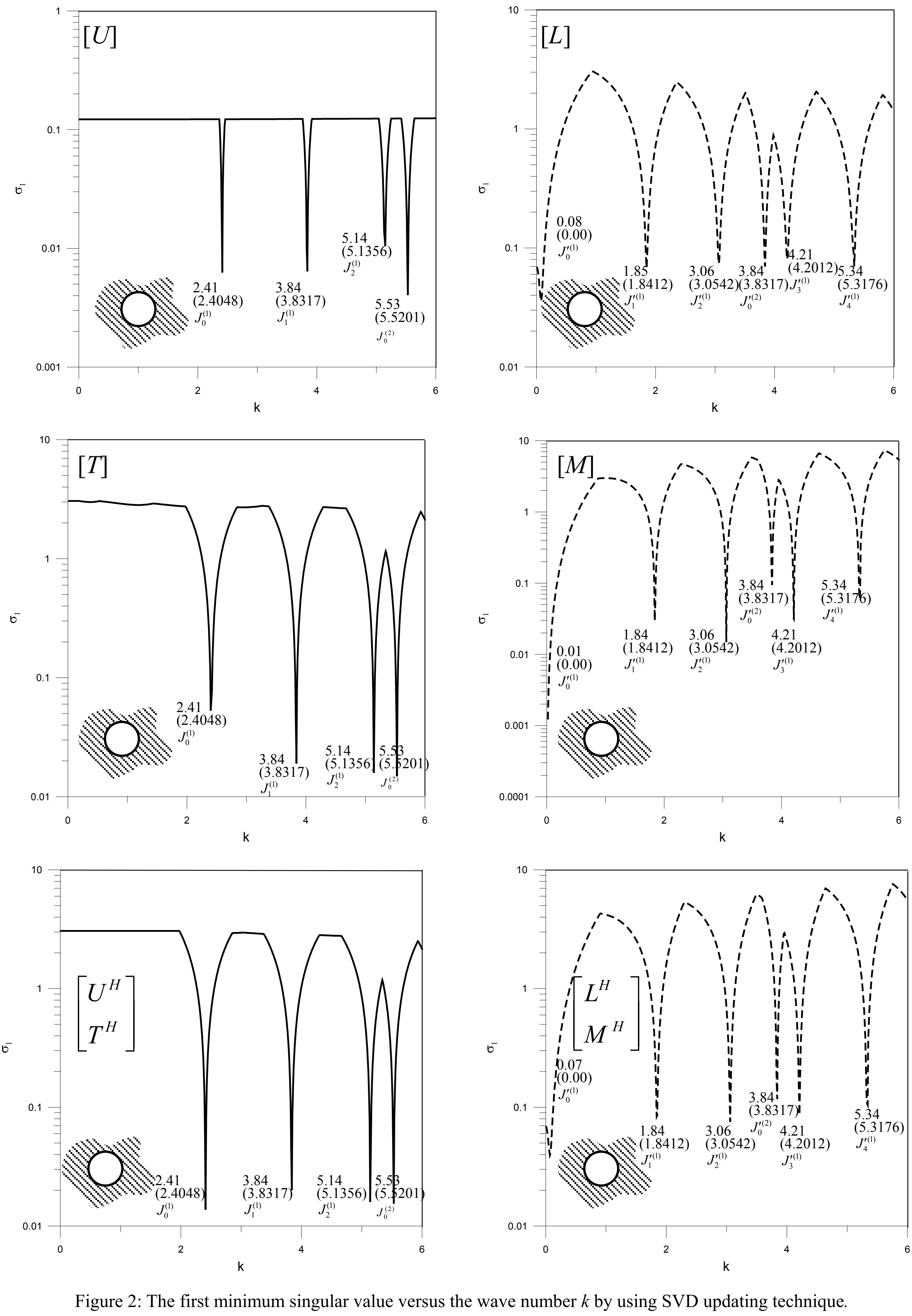


Figure 2: The first minimum singular value versus the wave number *k* by using SVD updating technique.