Elevated temperatures and prestresses on evolving yield surfaces for modeling experimental data

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Abstract. Phillips and his associates obtained a series of experimental data from thin-walled tubular specimens of commercially pure aluminium 1100-0 at room and at elevated temperatures. These data were neatly recoded in many aspects, executed faithfully in stress-controlled experiments and well known already for five decades; however, modeling these experimental data encountered either tremendous difficulties or over complications regardless of various attempts. In this paper selecting prominent test evidences based upon experiences on aluminium alloy Al6061 gained in our lab [ASCE Journal of Engineering Mechanics 148(6):04022027, 2022] over the course of the years, we created a three-dimensional tensor model of flow elastoplasticity, grasping all axial-torsional experimental features reported in the literature; in particular, in Phillips *et al.* at room and at elevated temperatures. For each temperature the model needs a total of 8 material constants in addition to Young's modulus and shear modulus and presents an evolving cubic distortion yield hypersurface, which is articulated with two Mises hyperspheres, characteristic of internal symmetry of two elements of the projective proper orthochronous Poincare group in the plastic phase. Associated with each Mises hypersphere in stress space is a normality plastic flow rule of mixed-exp-AF, referring to a combined isotropic-kinematic rule of hardening-softening, which combines the isotropic exponential rule of degree 2 and the kinematic rule of Armstrong-Frederick. By using the model and employing Lie group theory, closed-form exact solutions are derived and used to identify a unique set of parameters for fitting successfully evolving shapes of yield surfaces with clear physical meaning.

Keywords: elevated temperatures, modeling experimental data, flow elastoplasticity, Lie groups, closed-form solutions, cubic distortion yield surface, axial-torsional stress-controlled experiment.

1. Introduction

The paper presents a flow elastoplastic model and investigates the temperature effect in thermo-elasto-plasticity, utilizing the theory of Lie groups (Lorentz group, Poincare group, projective groups and their Lie algebras) to derive closed-form exact solutions in the plastic state and elastic solutions in the elastic state. By incorporating flow elastoplastic models in the place of linear elasticity, we provides the automatic on-off shift capability of plasticity and elastic unloading solution [1].

The flow elstoplastic model we established is an evolving cubic distortion yield surface together with a combined isotropic-kinematic rule of hardening-softening, which combines the isotropic exponential rule of degree 2 and the kinematic rule of Armstrong-Frederick. With a total of 8 material constants in addition to Young's modulus and shear modulus our model successfully predicts axial-torsional strain-controlled experimental data obtained in our NTU lab [2] and also the well-known stress-controlled prestress experimental data at room temperature (70°F = 21°C) of Phillips and Tang [5, 4]. The present paper further investigates the experimental data at room (70°F) and at elevated temperatures (151°F, 227°F, 267°F) [5, 6, 7].

The paper in fact briefly reports a long journey through experiments, identification, and development of modeling experimental data.

2. Flow elastoplastic model

Let us propose the following model :

$$\boldsymbol{\epsilon} = \frac{1}{3} tr\boldsymbol{\epsilon}\mathbf{1} + \mathbf{e}, \qquad \boldsymbol{\sigma} = \frac{1}{3} tr\boldsymbol{\sigma}\mathbf{1} + \mathbf{s} \tag{1}$$

$$tr\boldsymbol{\epsilon} = \frac{1}{3K} tr\boldsymbol{\sigma}, \qquad 3K = GE/(3G - E) \tag{2}$$

$$\boldsymbol{\epsilon} = \boldsymbol{\epsilon}^{e\alpha} + \boldsymbol{\epsilon}^{p\alpha}, \qquad \mathbf{e} = \mathbf{e}^{e\alpha} + \mathbf{e}^{p\alpha} \tag{3}$$

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}_a + \boldsymbol{\sigma}_b \tag{4}$$

$$\dot{\boldsymbol{\sigma}}^\alpha = \frac{R^\alpha(0)}{R^\alpha} \dot{\boldsymbol{\sigma}}_a^\alpha + \dot{\boldsymbol{\sigma}}_b^\alpha + \frac{k_p^\alpha \dot{\lambda}^\alpha}{\eta^\alpha} (\boldsymbol{\sigma}_{\boldsymbol{b}}^{\boldsymbol{\alpha}} - Ktr\boldsymbol{\epsilon}\mathbf{1}) \tag{5}$$

$$\dot{\mathbf{s}}^\alpha = \frac{R^\alpha(0)}{R^\alpha} \dot{\mathbf{s}}_a^\alpha + \dot{\mathbf{s}}_b^\alpha + \frac{k_p^\alpha \dot{\lambda}^\alpha}{\eta^\alpha} \mathbf{s}_b^\alpha \tag{6}$$

$$\boldsymbol{\sigma}^\alpha = \mathbf{s}^\alpha + Ktr\boldsymbol{\epsilon}\mathbf{1}, \qquad \boldsymbol{\sigma}_a^\alpha = \mathbf{s}_a^\alpha, \qquad \boldsymbol{\sigma}_b^\alpha = \mathbf{s}_b^\alpha + Ktr\boldsymbol{\epsilon}\mathbf{1} \tag{7}$$

$$\dot{\mathbf{s}}_a^\alpha = 2G\dot{\mathbf{e}}^{e\alpha}, \qquad R_\infty^\alpha \dot{e}^{p\alpha} = \mathbf{s}_a^\alpha \dot{\lambda}^\alpha \tag{8}$$

$$\dot{\mathbf{e}}^{p\alpha} = \frac{1}{k_p^\alpha} \dot{\mathbf{s}}_b^\alpha + \frac{\dot{\lambda}^\alpha}{\eta^\alpha} \mathbf{s}_b^\alpha, \qquad \|\mathbf{s}_a^\alpha\| \dot{\lambda}^\alpha = R^\alpha(\lambda^\alpha) \dot{\lambda}^\alpha, \qquad \Phi\dot{\Lambda} = \dot{\Lambda} \tag{9}$$

$$0 \leq \Phi = \Phi(\sigma_{\mathbf{a}}) = \frac{\|\boldsymbol{\sigma}_a^\alpha\|}{C} \leq 1 \tag{10}$$

$$\dot{\Lambda} = (\frac{1}{R_\infty^s} \|\mathbf{s}_a^s\|^2 + \frac{1}{2\eta^s} \|\mathbf{s}_b^s\|^2) \dot{\lambda}^s + (\frac{1}{R_\infty^l} \|\mathbf{s}_a^l\|^2 + \frac{1}{2\eta^l} \|\mathbf{s}_b^l\|^2) \dot{\lambda}^l \geq 0 \tag{11}$$

$$R^\alpha(\lambda^\alpha) = R_\infty^\alpha \sqrt{1 - r^\alpha exp(\frac{-2\lambda^\alpha}{\lambda_u^\alpha})}. \tag{12}$$

Here $\boldsymbol{\epsilon}$ is the strain tensor and $\boldsymbol{\sigma}$ is the stress tensor; $\mathbf{e}$ and $\mathbf{s}$ are their respective deviatoric parts. tr stands for trace. **1** is the identity tensor whereas **0** denotes the zero tensor. The positive real numbers K, G, and E are the bulk modulus, shear modulus, and Youngs modulus, respectively. Let the superscript α represent two copies l, s; namely, the superscript l signifies large hypersphere or longitudinal, long range, whereas the superscript s is small hypersphere or shear, short range. The superscripts e and p = elastic and plastic, respectively; and the subscripts a and b = active and back, respectively.

The proposed model slightly adjusts and extends the validity of our 2022 model [2], which proves fitting well the strain-controlled axial-torsional experimental data performed in our lab and supported by (NSC, MOST) NSTC for more than a decade.

We first combine Eqs. (7), (9), and (10) to obtain the key equation for the deviatoric stress:

$$\dot{\mathbf{s}}^\alpha = \frac{R^\alpha(0)}{R^\alpha} \dot{\mathbf{s}}_a^\alpha + \frac{k_p^\alpha \dot{\lambda}^\alpha}{R_\infty^\alpha} \mathbf{s}_a^\alpha, \tag{13}$$

which on multiplying by $\mathbf{s}_a^{\alpha T}$ and substituting (10) renders

$$\mathbf{s}_a^{\alpha T}\dot{\mathbf{s}}^\alpha = \frac{k_p^\alpha \dot{\lambda}^\alpha}{R_\infty^\alpha}(R^\alpha)^2, \tag{14}$$

so that the formula for $\dot{\lambda}^\alpha$ is

$$\dot{\lambda}^\alpha = \frac{\mathbf{s}_a^{\alpha T}\dot{\mathbf{s}}^\alpha R_\infty^\alpha}{k_p^\alpha (R^\alpha)^2}. \tag{15}$$

The sufficient and necessary condition for using the formula can be proved rigorously. The slight adjustment occurs on the innocent equations (6) and (7); however, the outcome becomes not only strain-controlled but also stress-controlled are all applicable. The delicate logical details will be described elsewhere.

The flow elstoplastic model we established is an evolving cubic distortion yield surface together with a combined isotropic-kinematic rule of hardening-softening, which combines the isotropic exponential rule of degree 2 and the kinematic rule of Armstrong-Frederick. With a total of 8 material constants in addition to Young's modulus and shear modulus our model successfully predicts axial-torsional strain-controlled experimental data obtained in our NTU lab [2] and also successfully predicts the well-known axial-torsional stress-controlled prestress experimental data at room temperature by Phillips and Tang [5, 4] obtained five decades ago. It cannot be overemphasized that the data were neatly and carefully recoded but has long been lacking a model (before our effort). The present paper further investigates the experimental data at elevated temperatures [5, 6, 7].

3. Closed-form exact solutions by Lie group theory

As pointed out by Hong and Liu [1] there are two and only two phases: either the off phase (the elastic state) or the on phase (the elastoplastic state). Logically it is expressed as follows.

$$\begin{cases} \dot{\lambda}^\alpha = \dfrac{\mathbf{s}_a^{\alpha T}\dot{\mathbf{s}}^\alpha R_\infty^\alpha}{k_p^\alpha (R^\alpha)^2}, & if \quad \Phi(\sigma_{\mathbf{a}}) = 1 \quad and \quad \|\mathbf{s}_a^\alpha\| = R^\alpha, \\ \dot{\lambda}^\alpha = 0, \quad if \quad 0 \le \Phi(\sigma_{\mathbf{a}}) < 1 \quad or \quad \|\mathbf{s}_a^\alpha\| < R^\alpha. \end{cases} \tag{16}$$

In the following we convert Euclidean space $\mathbb{R}^n$ to Minkowski spacetime $\mathbb{R}^{n,1}$. As a result, non-homogeneous coordinates turn out to be homogeneous coordinates and non-linear equations become linear equations, illustrating proper orthochronous Lorentz group $SO_o(n,1)$ and its Lie algebra.

Convert (14) to be

$$\dot{\mathbf{s}}^\alpha exp(\frac{R^\alpha k_p^\alpha \lambda^\alpha}{R^\alpha(0)R_\infty^\alpha}) = \frac{d}{dt}(\frac{R^\alpha(0)}{R^\alpha}\mathbf{s}_a^\alpha exp(\frac{R^\alpha k_p^\alpha \lambda^\alpha}{R^\alpha(0)R_\infty^\alpha})), \tag{17}$$

and multiply (17) by $\mathbf{s}_a^{\alpha T}$

$$\dot{\mathbf{s}}^{\alpha T}\mathbf{s}_a^\alpha exp(\frac{R^\alpha k_p^\alpha \lambda^\alpha}{R_\infty^\alpha R^\alpha(0)}) = \frac{d}{dt}(exp(\frac{R^\alpha k_p^\alpha \lambda^\alpha}{R_\infty^\alpha R^\alpha(0)}))R^\alpha R^\alpha(0). \tag{18}$$

Once $\dot{\lambda}^\alpha$ is found, we combine (17) and (18) to define Minkowski spacetime :

$$\mathbf{X}^\alpha = \begin{bmatrix} \mathbf{X}_s^\alpha \\ X_0^\alpha \end{bmatrix} = \begin{bmatrix} \frac{\mathbf{s_a}^\alpha}{R^\alpha(\lambda^\alpha)} X_0^\alpha \\ exp(\frac{R^\alpha k_p^\alpha \lambda^\alpha}{R_\infty^\alpha R^\alpha(0)}) \end{bmatrix}. \tag{19}$$

Hence,

$$\begin{bmatrix} \dot{\mathbf{X}}_s^\alpha \\ \dot{X}_0^\alpha \end{bmatrix} = \begin{bmatrix} 0 & \frac{\dot{\mathbf{s}}^\alpha}{R^\alpha(0)} \\ \frac{\dot{\mathbf{s}}^{\alpha T}}{R^\alpha(0)} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{X}_s^\alpha \\ X_0^\alpha \end{bmatrix} \ i.e. \ \ \dot{\mathbf{X}}^\alpha = \mathbf{A}\mathbf{X}^\alpha, \tag{20}$$

where the square matrix is called the control matrix $\mathbf{A}$.

Therefore $\mathbf{X}^\alpha(t) = \mathbf{G}_{on}\mathbf{X}^\alpha(t_1)$ for $t_1 \leq t$ with the state transition matrix $\mathbf{G}_{on}^\alpha$ being

$$\mathbf{G}_{on}^\alpha = \begin{bmatrix} \mathbf{I} + \frac{(a^\alpha - 1)}{\|\dot{\mathbf{s}}^\alpha\|^2}\dot{\mathbf{s}}^\alpha\dot{\mathbf{s}}^{\alpha T} & b^\alpha \frac{\dot{\mathbf{s}}^\alpha}{\|\dot{\mathbf{s}}^\alpha\|} \\ b^\alpha (\frac{\dot{\mathbf{s}}^\alpha}{\|\dot{\mathbf{s}}^\alpha\|})^T & a^\alpha \end{bmatrix}, \tag{21}$$

where $a^\alpha = \cosh(\frac{(t-t_1)\|\mathbf{s}_a^\alpha\|}{R^\alpha(0)})$ and $b^\alpha = \sinh(\frac{(t-t_1)\|\mathbf{s}_a^\alpha\|}{R^\alpha(0)})$.

Thus, we obtain the closed-form exact solutions as follows.

For the active deviatoric stress tensors,

$$\mathbf{s}_a^\alpha(t) = [\frac{\frac{\mathbf{s}_a^\alpha(t_1)}{R^\alpha(\lambda^\alpha(t_1))} + \frac{(a^\alpha-1)}{\|\dot{\mathbf{s}}^\alpha\|^2}\dot{\mathbf{s}}^\alpha\dot{\mathbf{s}}^{\alpha T}\frac{\mathbf{s}_a^\alpha(t_1)}{R^\alpha(\lambda^\alpha(t_1))} + b^\alpha\frac{\dot{\mathbf{s}}^\alpha}{\|\dot{\mathbf{s}}^\alpha\|}}{b^\alpha(\frac{\dot{\mathbf{s}}^\alpha}{\|\dot{\mathbf{s}}^\alpha\|})^T\frac{\mathbf{s}_a^\alpha(t_1)}{R^\alpha(\lambda^\alpha(t_1))} + a^\alpha}]R^\alpha(\lambda^\alpha(t)). \tag{22}$$

For the back deviatoric stress tensors,

$$\mathbf{s}_b^\alpha(t) = exp(\frac{k_p^\alpha}{\eta^\alpha}[\lambda^\alpha(t_1) - \lambda^\alpha(t)])\mathbf{s}_b^\alpha(t_1) + \frac{\eta^\alpha}{R_\infty^\alpha}\left\{\mathbf{s}_a^\alpha(t) - exp(\frac{k_p^\alpha}{\eta^\alpha}[\lambda^\alpha(t_1) - \lambda^\alpha(t)])\mathbf{s}_a^\alpha(t_1)\right\}. \tag{23}$$

For the equivalent plastic strain,

$$\lambda^\alpha(t) = \frac{R^\alpha(\lambda^\alpha(t_1))}{R^\alpha(\lambda^\alpha(t))}\lambda^\alpha(t_1) + \frac{R_\infty^\alpha R^\alpha(0)}{k_p^\alpha R^\alpha(\lambda^\alpha(t))} ln[b^\alpha(\frac{\dot{\mathbf{s}}^\alpha}{\|\dot{\mathbf{s}}^\alpha\|})^T\frac{\mathbf{s}_a^\alpha(t_1)}{R^\alpha(\lambda^\alpha(t_1))} + a^\alpha]. \tag{24}$$

4. Temperature effect

Table 1. For each temperature a unique set of material constants along an entire path however complicated in stress space.

Temp.	E	R_∞^l	R_∞^s	r_∞^l	r_∞^s	G	k_p^l	k_p^s	η^l	η^s
70°F	10845	0.7521	0.4177	-8.5995	-15.6683	4077	65268	559724	11.6377	26.4252
151°F	10526	0.5146	0.3503	-11.6802	-18.8306	3957	53885	407189	10.2215	16.6686
227°F	10227	0.3615	0.2381	-17.9214	-27.5025	3844	48834	398506	7.8491	13.1810
267°F	10069	0.2771	0.1535	-20.9339	-27.5025	3785	41003	353641	5.3040	10.5451

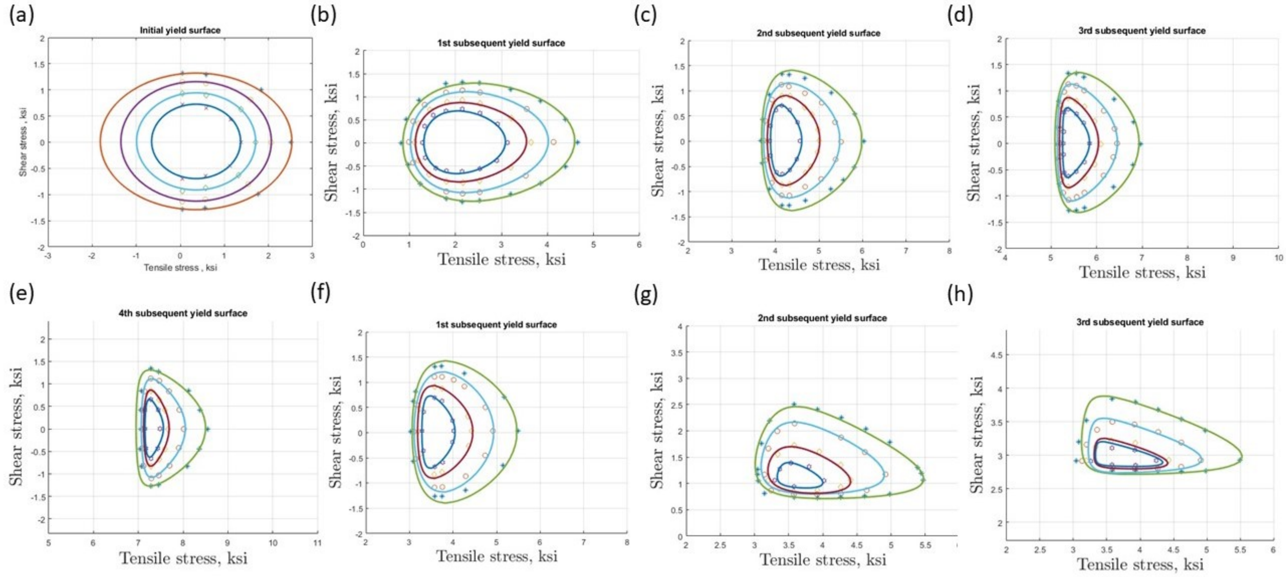

Figure 1. First pull/apply axial stress from (a) initial yield surface (b) 1st subsequent yield surface (c) 2nd s.y.s. (d) 3rd s.y.s. to (e) 4th s.y.s. and then twist/apply *additional* torsional stress from (f) axial-torsional subsequent yield surface (g) axial-torsional 2nd s.y.s. to (h) axial-torsional 3rd s.y.s.; temperatures increase, sizes decrease.

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