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香港科技大學
THE HONG KONG UNIVERSITY OF SCIENCE AND TECHNOLOGY

Application of boundary integral quadrature method to torsion problems of the orthotropic bars and its treatment of degenerate scale problem

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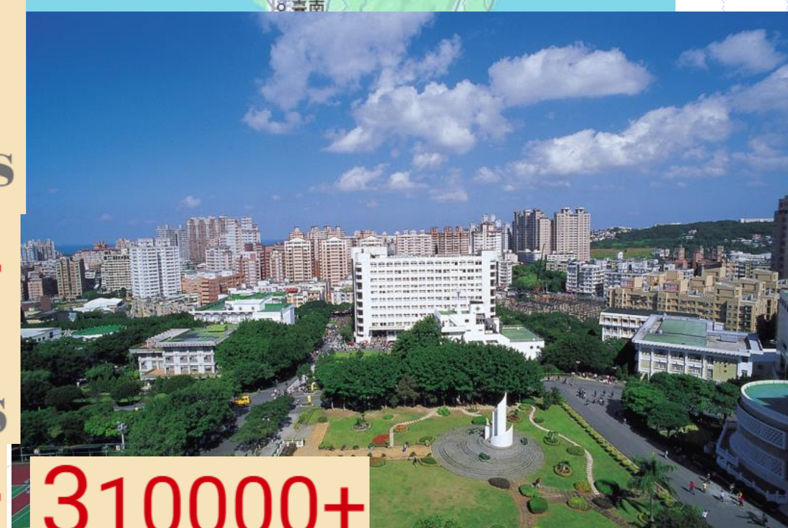
Date: Dec. 06, 2024
Time: 11:35~12:00
Place: Theater K, HKUST

Tamkang University

Private university



4
Campuses

22000+
Students

2000+
Staff

310000+
Alumni

Outline

- Introduction
- Problem statement
- Present approach
- Numerical results
- Conclusions

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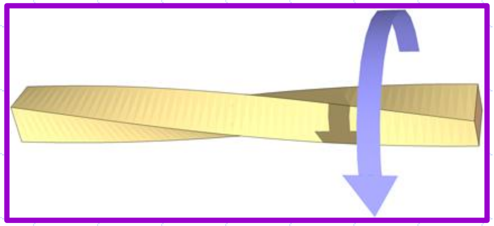
Motivation

The torsion problem is an important issue.



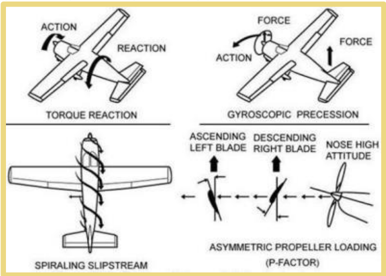
- https://upload.wikimedia.org/wikipedia/commons/4/4f/Twisted_bar.png

- https://www.abri.gov.tw/PeriodicalDetail.aspx?n=861&s=1786&key=70&isShowAll=false

Torsion Problem

the twisting of an object due to an applied torque

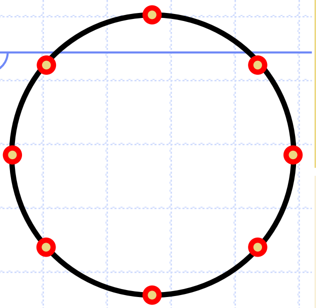
- **Civil engineering** → Architecture
  - → steel construction → torsion buckling
  - → Reinforced Concrete
- **Mechanical engineering** → Machine element
  - → screw
  - → shaft → automobile drive shaft
- **Aerospace engineering** → Aircraft
  - → flaps
  - → rotary wing



- https://www.apexflightacademy.com/post/hang-kong-xiao-jiao-shi-left-turning-tendency

The simulation of torsion problem by using the numerical method is our purpose.

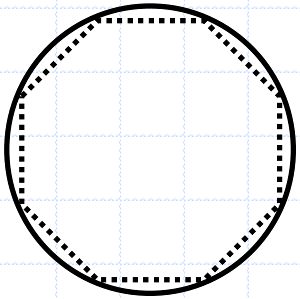
Motivation



Boundary Nodes

Degenerate kernel
Fourier series

	Null-field BIEM (2003 NTOU/MSV)	Conventional BEM
Pros	**1. Meshless** 2. Well-posed **3. Free of boundary layer effect** 4. Exponential rate of convergence 5. Free of principal values	**Arbitrary geometry**
Cons	Only special geometry (circle or ellipse)	Need mesh generation in model preparation



Boundary elements

Closed-form fundamental solution
Interpolation function

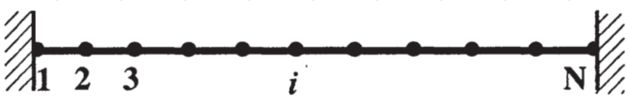
We proposed the **BIQM** that have both advantages of these two methods **since 2021**

Differential quadrature vs. Integral quadrature

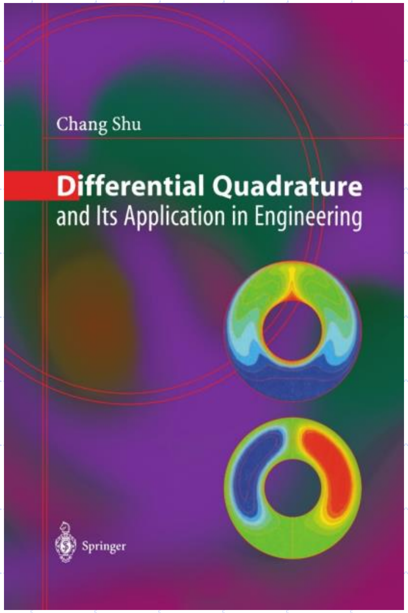
DQ IQ

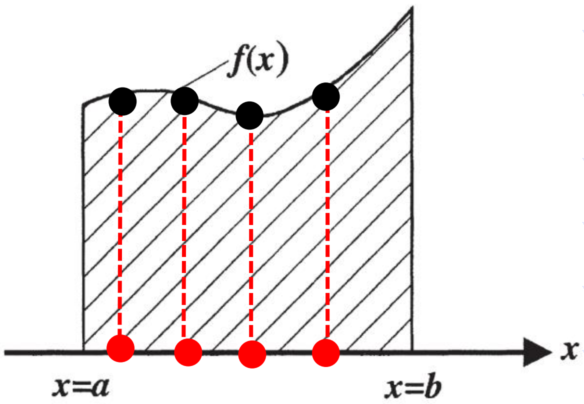
The differential quadrature (DQ) method was presented by R. E. Bellman and his associates in the early 1970's.

The DQ method was initiated from the idea of conventional integral quadrature.



$$f_x(x_i) = \left.\frac{df}{dx}\right|_{x_i} = \sum_{j=1}^{N} a_{ij} \cdot f(x_j), \quad \text{for } i = 1, 2, \ldots, N$$



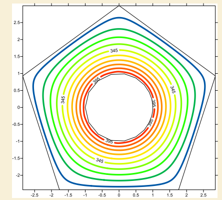


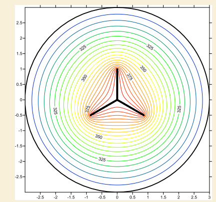
$$\int_a^b f(x)dx = w_1 f_1 + w_2 f_2 + \ldots + w_n f_n = \sum_{k=1}^{n} w_k f_k$$

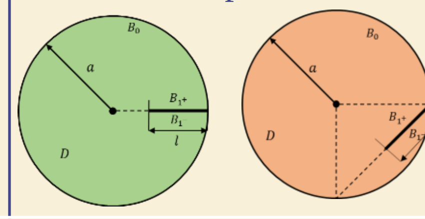
Differential → Linear sum ← Integral

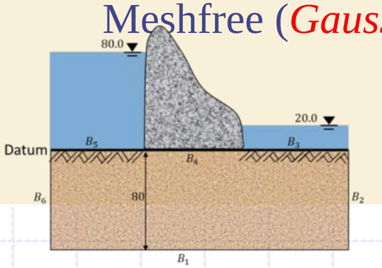
Development of BIQM

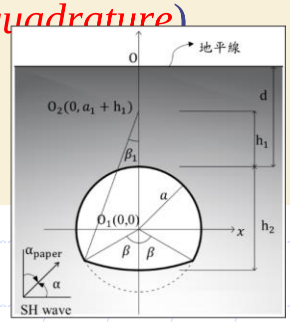
Since 2021		2023-2024	2024
Interior potential problem	Membrane free vibration problem	Steady state heat conduction problem	Torsion problem
Exterior anti-plane problem	SH-wave scattering problem		
Simply-connected domain		Multiply-connected domain	Degenerate boundary
Laplace equation	Helmholtz equation	Laplace equation	
Singular integral (*Adaptive exact solution*) Nearly singular integral (*Adaptive exact solution*) Meshfree (*Gaussian quadrature*)		Singular integral (*Adaptive exact solution*) Nearly singular integral (*Adaptive exact solution*) Meshfree (*Gaussian quadrature*) Degenerate scale (CLEEF) Degenerate boundary (Dual BIEM) Singular integral on slit (Adaptive exact solution for slit)	

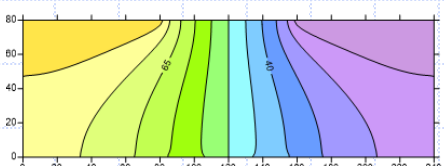


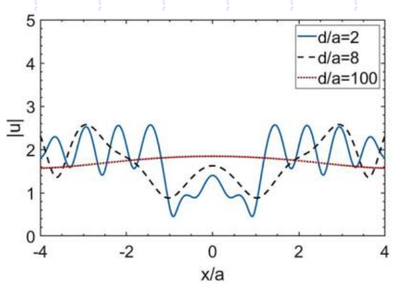


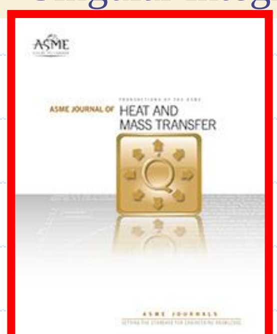


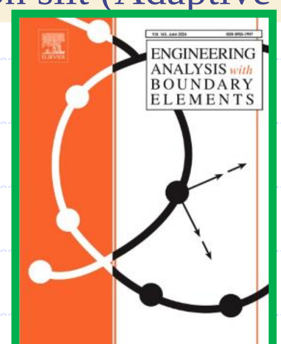












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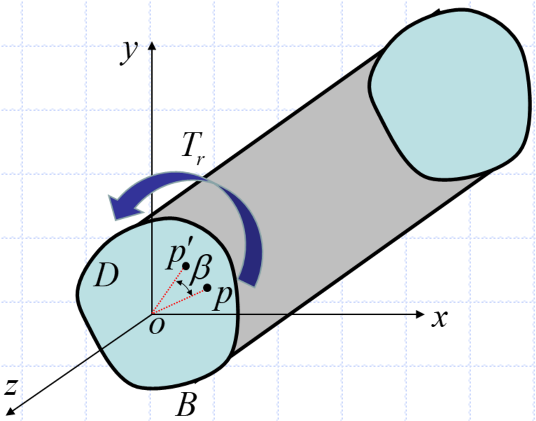
Saint-Venant torsion problem



Displacement fields

$\mathbf{x}$ is the field point (x, y)

$u = -\beta yz$

$v = \beta xz$

$w = w_a(\mathbf{x})$

where β is the angle of twist per unit length

Strain fields

$$\gamma_{xz} = \left(\frac{\partial w_a(\mathbf{x})}{\partial x} - \beta y\right)$$

$$\gamma_{yz} = \left(\frac{\partial w_a(\mathbf{x})}{\partial y} + \beta x\right)$$

Hook's Law →

Stress fields

$$\tau_{xz} = G_{xz}\gamma_{xz} = G_{xz}\left(\frac{\partial w_a(\mathbf{x})}{\partial x} - \beta y\right)$$

$$\tau_{yz} = G_{yz}\gamma_{yz} = G_{yz}\left(\frac{\partial w_a(\mathbf{x})}{\partial y} + \beta x\right)$$

Saint-Venant torsion problem

Warping function

The equilibrium equation

$$\frac{\partial \tau_{xz}}{\partial x}+\frac{\partial \tau_{yz}}{\partial y}+\frac{\partial \sigma_z}{\partial z}=0$$

of the z-direction

rewrite →

$$\frac{\partial}{\partial x}\left[G_{xz}\left(\frac{\partial w_a(\mathbf{x})}{\partial x}-\beta y\right)\right]+\frac{\partial}{\partial y}\left[G_{yz}\left(\frac{\partial w_a(\mathbf{x})}{\partial y}+\beta x\right)\right]=0$$

obtain →

$$G_{xz}\frac{\partial^2 w_a(\mathbf{x})}{\partial x^2}+G_{yz}\frac{\partial^2 w_a(\mathbf{x})}{\partial y^2}=0$$

G.E.

The traction boundary condition

$$t_z=\tau_{xz}n_x(\mathbf{x})+\tau_{yz}n_y(\mathbf{x})=0$$

→

$$\left[G_{xz}\left(\frac{\partial w_a(\mathbf{x})}{\partial x}-\beta y\right)\right]n_x(\mathbf{x})+\left[G_{yz}\left(\frac{\partial w_a(\mathbf{x})}{\partial y}+\beta x\right)\right]n_y(\mathbf{x})=0$$

obtain →

$$G_{xz}\frac{\partial w_a(\mathbf{x})}{\partial x}n_x(\mathbf{x})+G_{yz}\frac{\partial w_a(\mathbf{x})}{\partial y}n_y(\mathbf{x})=\beta\left(G_{xz}yn_x(\mathbf{x})-G_{yz}xn_y(\mathbf{x})\right)$$

rewrite →

$$t^o(\mathbf{x})=\beta\left(G_{xz}yn_x(\mathbf{x})-G_{yz}xn_y(\mathbf{x})\right)$$

B.C.

Saint-Venant torsion problem

Stress function

Beltrami–Michell compatibility equation

$$\frac{\partial}{\partial y}\left[\frac{\tau_{xz}}{G_{xz}}\right]-\frac{\partial}{\partial x}\left[\frac{\tau_{yz}}{G_{yz}}\right]=-2\beta$$

$$\tau_{xz}=G_{xz}G_{yz}\frac{\partial\Phi(\mathbf{x})}{\partial y}$$

$$\tau_{yz}=G_{yz}G_{xz}\,(-\frac{\partial\Phi(\mathbf{x})}{\partial x})$$

$$\frac{\partial}{\partial y}\left[G_{yz}\frac{\partial\Phi(\mathbf{x})}{\partial y}\right]-\frac{\partial}{\partial x}\left[G_{xz}\left(-\frac{\partial\Phi(\mathbf{x})}{\partial x}\right)\right]=-2\beta$$

$$G_{xz}\frac{\partial^2\Phi(\mathbf{x})}{\partial x^2}+G_{yz}\frac{\partial^2\Phi(\mathbf{x})}{\partial y^2}=-2\beta$$

G.E.

The traction boundary condition

$$\tau_{xz}n_x+\tau_{yz}n_y=0$$

$$G_{xz}G_{yz}\frac{\partial\Phi(\mathbf{x})}{\partial y}n_x+G_{yz}G_{xz}\left(-\frac{\partial\Phi(\mathbf{x})}{\partial x}\right)n_y=0$$

$$G_{xz}G_{yz}\left[\left(\frac{\partial\Phi(\mathbf{x})}{\partial x},\frac{\partial\Phi(\mathbf{x})}{\partial y}\right)\cdot(-n_y,n_x)\right]=0$$

tangential direction is free

$$\Phi(\mathbf{x})=\Phi_{\mathrm{h}}(\mathbf{x})+\Phi_{\mathrm{p}}(\mathbf{x})=0$$

B.C.

$\Phi_{\mathrm{p}}(\mathbf{x})$

$$\Phi_{\mathrm{p}}(\mathbf{x})=-\beta\left(\frac{x^2}{2G_{xz}}+\frac{y^2}{2G_{yz}}\right)$$

$\Phi_{\mathrm{h}}(\mathbf{x})$

BIQM

Saint-Venant torsion problem

Warping function vs. Stress function

	Warping Function	Stress Function
G.E.	$G_{xz}\dfrac{\partial^2 w_a(\mathbf{x})}{\partial x^2}+G_{yz}\dfrac{\partial^2 w_a(\mathbf{x})}{\partial y^2}=0$	$G_{xz}\dfrac{\partial^2 \Phi(\mathbf{x})}{\partial x^2}+G_{yz}\dfrac{\partial^2 \Phi(\mathbf{x})}{\partial y^2}=-2\beta$
B.C.	$t^o(\mathbf{x})=\beta(G_{xz}yn_x(\mathbf{x})-G_{yz}xn_y(\mathbf{x}))$	$\Phi(\mathbf{x})=\Phi_\mathrm{h}(\mathbf{x})+\Phi_\mathrm{p}(\mathbf{x})=0$
D_r	$\int_B \left[G_{yz}\dfrac{x^3}{3}n_x+G_{xz}\dfrac{y^3}{3}n_y\right]\mathrm{d}B(\mathbf{x})-\dfrac{1}{\beta}\int_B [w_a(\mathbf{x})\mathrm{t}^o(\boldsymbol{x})]\,\mathrm{d}B(\mathbf{x})$	$-\int_B G_{yz}\dfrac{x^3}{3}n_x+G_{xz}\dfrac{y^3}{3}n_y\,dB(\mathbf{x})-\dfrac{1}{\beta}\int_B \left(\dfrac{G_{yz}x^2+G_{xz}y^2}{2}\right)[t^o(\mathbf{x})]\,dB(\mathbf{x})$

$$
\begin{bmatrix}\varepsilon_x\\ \varepsilon_y\\ \varepsilon_z\\ \gamma_{yz}\\ \gamma_{xz}\\ \gamma_{xy}\end{bmatrix}=
\begin{bmatrix}
1/E_x & -\nu_{yx}/E_y & -\nu_{zx}/E_z & & & \\
-\nu_{xy}/E_x & 1/E_y & -\nu_{zy}/E_z & & & \\
-\nu_{xz}/E_x & -\nu_{yz}/E_y & 1/E_z & & & \\
 & & & 1/G_{yz} & & \\
 & & & & 1/G_{xz} & \\
 & & & & & 1/G_{xy}
\end{bmatrix}
\begin{bmatrix}\sigma_x\\ \sigma_y\\ \sigma_z\\ \tau_{yz}\\ \tau_{xz}\\ \tau_{xy}\end{bmatrix}
$$

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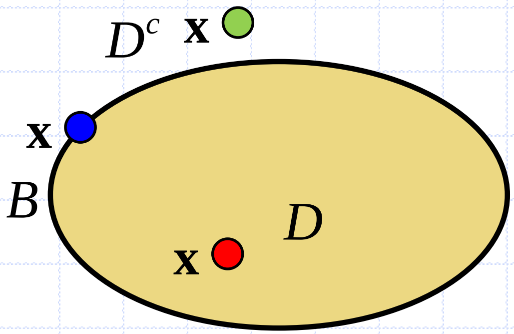
Boundary integral equation



$$U(\mathbf{s},\mathbf{x}) = \frac{1}{2\pi\sqrt{G_{xz}G_{yz}}} ln(r)$$

$$T(\mathbf{s},\mathbf{x}) = G_{xz}\frac{\partial U(\mathbf{s},\mathbf{x})}{\partial x} n_x(\mathbf{s}) + G_{yz}\frac{\partial U(\mathbf{s},\mathbf{x})}{\partial y} n_y(\mathbf{s})$$

$$t(\mathbf{s}) = G_{xz}\frac{\partial u(\mathbf{x})}{\partial x} n_x(\mathbf{s}) + G_{yz}\frac{\partial u(\mathbf{x})}{\partial y} n_y(\mathbf{s})$$

$$r = \sqrt{\frac{(s_x - x)^2}{G_{xz}} + \frac{(s_y - y)^2}{G_{yz}}}$$

in the domain

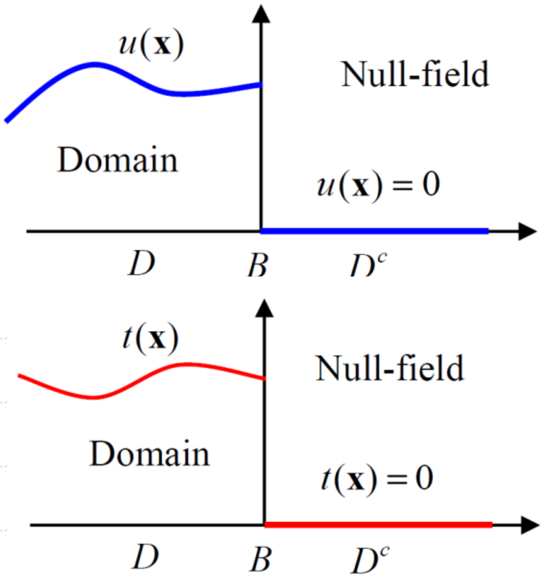
$$u(\mathbf{x}) = \int_B T(\mathbf{s},\mathbf{x})u(\mathbf{s})dB(\mathbf{s}) - \int_B U(\mathbf{s},\mathbf{x})t(\mathbf{s})dB(\mathbf{s}) \ ,\mathbf{x} \in \boldsymbol{D}$$

in the complementary domain

$$0 = \int_B T(\mathbf{s},\mathbf{x})u(\mathbf{s})dB(\mathbf{s}) - \int_B U(\mathbf{s},\mathbf{x})t(\mathbf{s})dB(\mathbf{s}) \ ,\mathbf{x} \in \boldsymbol{D}^c$$

on the boundary

$$\frac{\alpha}{2\pi}u(\mathbf{x}) = C.P.V.\int_B T(\mathbf{s},\mathbf{x})u(\mathbf{s})dB(\mathbf{s}) - R.P.V.\int_B U(\mathbf{s},\mathbf{x})t(\mathbf{s})dB(\mathbf{s}) ,\mathbf{x} \in \boldsymbol{B}$$



Who is responsible for the discontinuity of the double-layer potential? (boundary, kernel or density?)

邊界密度函數的解析延拓。

(Taylor expansion)

(Kisu, 1988)

$$u(\mathbf{s}) = u(\mathbf{x}) + \left.\frac{\partial u}{\partial s_1}\right|_{\mathbf{s}=\mathbf{x}} (s_1 - x_1) + \left.\frac{\partial u}{\partial s_2}\right|_{\mathbf{s}=\mathbf{x}} (s_2 - x_2) \Rightarrow p(\mathbf{x}) = u(\mathbf{x}) \int_D \nabla^2 U(\mathbf{x}, \mathbf{s}) dD(\mathbf{s})$$

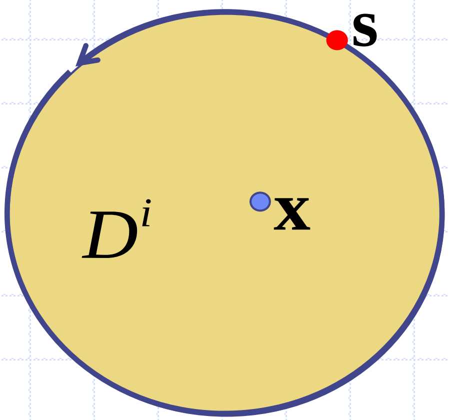
Gauss' theorem

$$\int_B T dB = \int_B \frac{\partial U}{\partial n} dB = \int_B \nabla \mathbf{U} \cdot \mathbf{n} dB = \int_D \nabla^2 \mathbf{U} dD = \begin{cases} 2\pi, & \mathbf{x} \in D^i \\ 0, & \mathbf{x} \in D^e \end{cases}$$

Residue Theorem

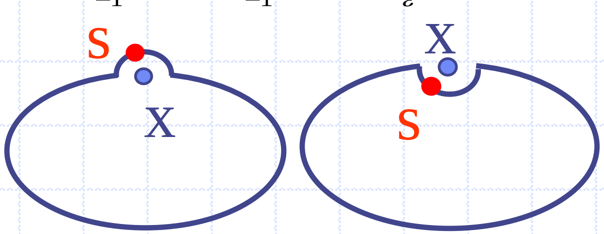
$$p(\mathbf{x}) = \int_B T(\mathbf{x}, \mathbf{s}) u(\mathbf{s}) dB(\mathbf{s}),$$ $p(x)$ jump from D^e to D^i.

$$T(\mathbf{x}, \mathbf{s}) = \frac{\partial U}{\partial n_s}$$

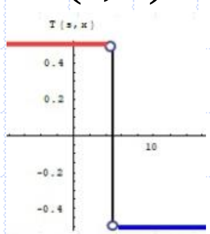


CPV (Cauchy)

$$\int_{-1}^{1} \frac{1}{x} dx = \int_{-1}^{\varepsilon} \frac{1}{x} dx + \int_{\varepsilon}^{1} \frac{1}{x} dx$$



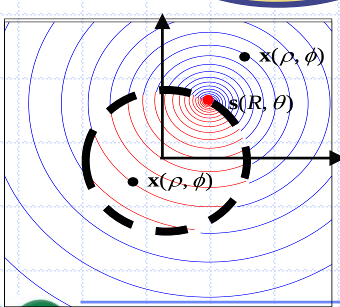
$$T(s, x) = \begin{cases} T^I(\mathbf{x}, \mathbf{s}), & \mathbf{x} < \mathbf{s} \\ T^E(\mathbf{x}, \mathbf{s}), & \mathbf{x} > \mathbf{s} \end{cases}$$



Degenerate kernel

discontinuity

D^e



山不轉。路轉。Bump contour

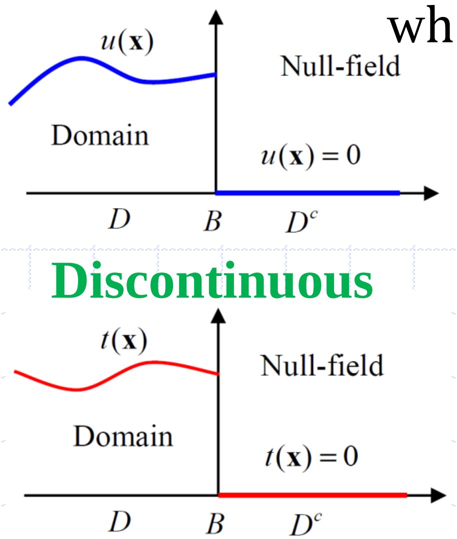
山不轉。路不轉。人轉。

Adaptive exact solution

To enforce the smoothness and continuity of potential and its normal flux across the boundary, we introduce the adaptive exact solution.

$$w(\mathbf{s}) = u(\mathbf{x}) + [\frac{(s_x - x)}{G_{xz}} n_x(\mathbf{x}) + \frac{(s_y - y)}{G_{yz}} n_y(\mathbf{x})] t(\mathbf{x})$$

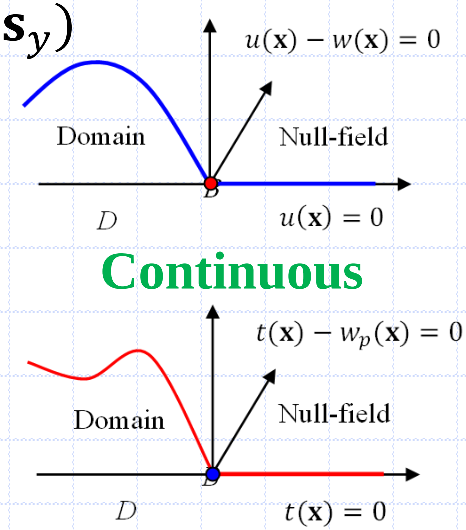
when $n(\mathbf{x})=(n_x(\mathbf{x}), n_y(\mathbf{x}))$ & $\mathbf{s} = (\mathbf{s}_x, \mathbf{s}_y)$



Discontinuous

$$(u(\mathbf{s}) - w(\mathbf{s}))|_{\mathbf{s}=\mathbf{x}} = 0$$

$$(t(\mathbf{s}) - w_p(\mathbf{s}))|_{\mathbf{s}=\mathbf{x}} = 0$$



Continuous

$$0 = \int_B T(\mathbf{s},\mathbf{x})[u(\mathbf{s}) - w(\mathbf{s})]dB(\mathbf{s}) - \int_B U(\mathbf{s},\mathbf{x})[t(\mathbf{s}) - w_p(\mathbf{s})]dB(\mathbf{s}), \mathbf{x} \in B$$

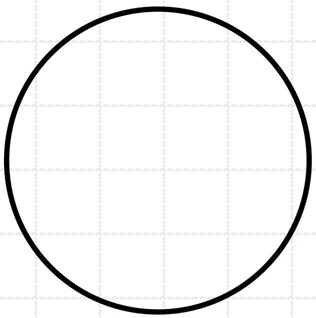
Skillful calculation of singular integral

computation of solid angle is free

Not required

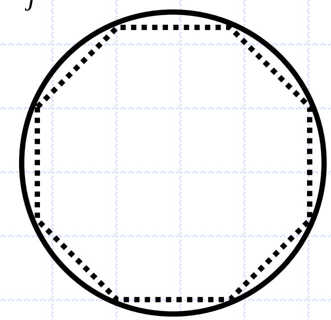
Integral equation to algebraic equation

$$\int_B K(\mathbf{s},\mathbf{x})y(\mathbf{s})dB(\mathbf{s})$$



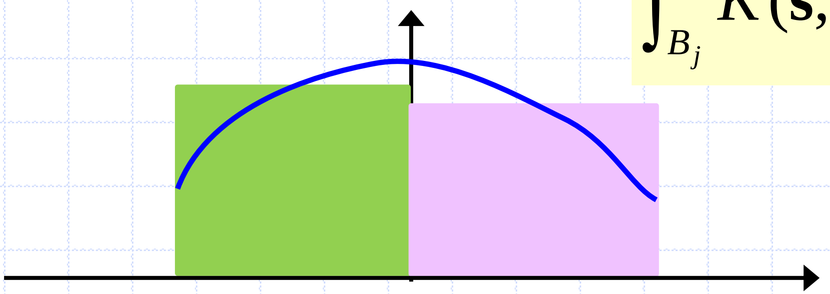
Discretization

$$\sum_{j=1}^{N}\int_{B_j} K(\mathbf{s},\mathbf{x})y(\mathbf{s})dB(\mathbf{s})$$



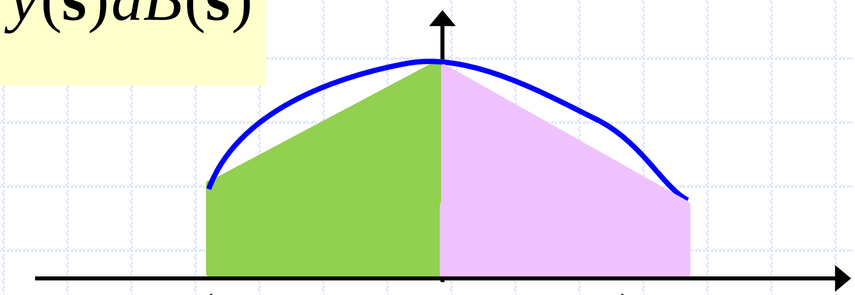
$$\int_{B_j} K(\mathbf{s},\mathbf{x})y(\mathbf{s})dB(\mathbf{s})$$

Constant element



$$\left(\int_{B_j} K(\mathbf{s},\mathbf{x})dB(\mathbf{s})\right)y_j$$

Linear element



$$\left(\int_{B_j} K(\mathbf{s},\mathbf{x})N_1(\mathbf{s})dB(\mathbf{s})\right)y_1 + \left(\int_{B_j} K(\mathbf{s},\mathbf{x})N_2(\mathbf{s})dB(\mathbf{s})\right)y_2$$

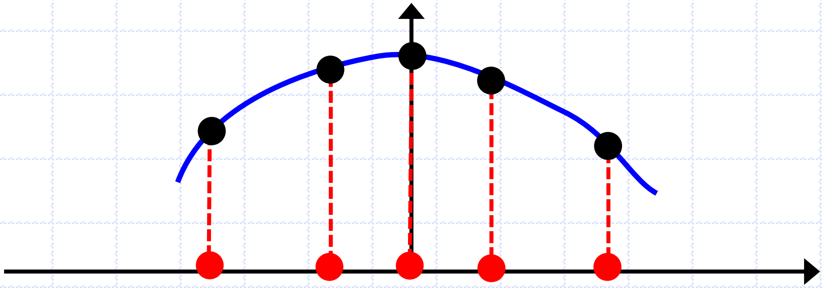
Integral equation to algebraic equation

Gaussian quadrature for an integral

$$I = \int_{-1}^{1} f(x)dx \Rightarrow I = \sum_{i=1}^{N} w_i f(x_i)$$

For an integral equation

$$I = \int_B K(\mathbf{s}, \mathbf{x}) y(\mathbf{s}) dB(\mathbf{s}) \Rightarrow I = \sum_{j=1}^{N} w_j K(\mathbf{s}_j, \mathbf{x}) y(\mathbf{s}_j) J(\mathbf{s}_j)$$



Gaussian quadrature points

$$\Rightarrow I = \sum_{j=1}^{N} y_j \left(w_j K(\mathbf{s}_j, \mathbf{x}) J(\mathbf{s}_j) \right)$$

Without the mesh distribution to calculate the boundary integral

Integral equation to algebraic equation

$$0=\int_B T(\mathbf{s},\mathbf{x})\left[u(\mathbf{s})-w(\mathbf{s})\right]dB(\mathbf{s})-\int_B U(\mathbf{s},\mathbf{x})\left[t(\mathbf{s})-w_p(\mathbf{s})\right]dB(\mathbf{s}),\ \mathbf{x}\in B$$

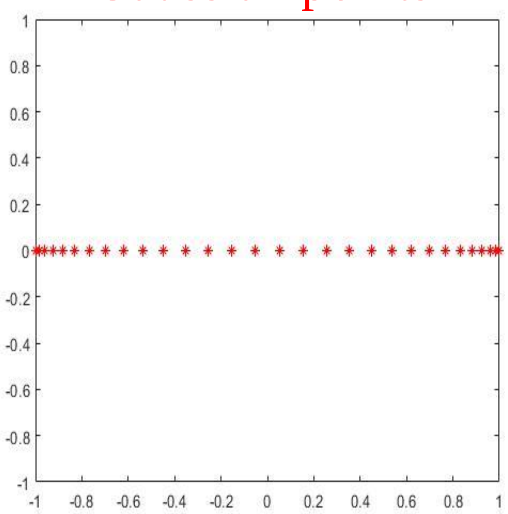
Parametric representation for each boundary

$$0=\int_{\tau_0}^{\tau_L} T(\mathbf{s},\mathbf{x})\left[u(\mathbf{s})-w(\mathbf{s})\right]\frac{dB(\mathbf{s})}{d\tau}d\tau-\int_{\tau_0}^{\tau_L} U(\mathbf{s},\mathbf{x})\left[t(\mathbf{s})-w_p(\mathbf{s})\right]\frac{dB(\mathbf{s})}{d\tau}d\tau$$

Gaussian quadrature → IE -> AE

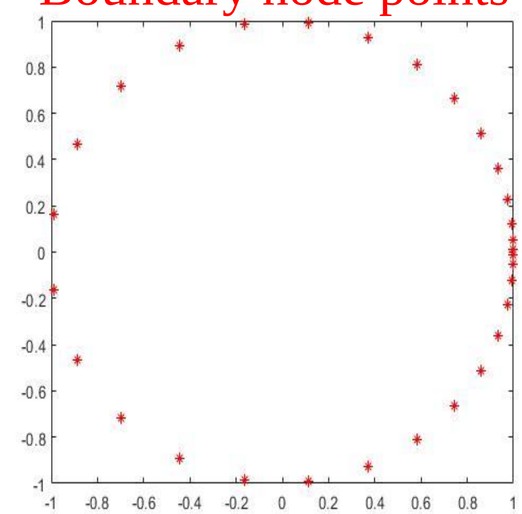
$$\sum_{j=1}^{N} w_j T(\mathbf{s}_j,\mathbf{x}_i)J(\mathbf{s}_j)\left[u_j-w_j^i\right]-\delta_{ij}u_j=\sum_{j=1}^{N} w_j U(\mathbf{s}_j,\mathbf{x}_i)J(\mathbf{s}_j)\left[t_j-w_{pj}^{\ i}\right]$$

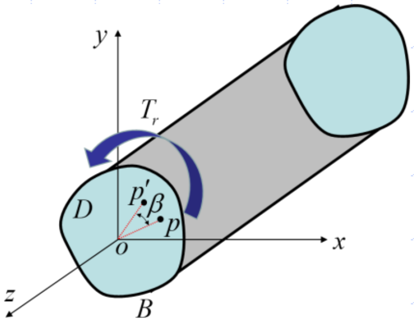
Gaussian points



$$[\mathbf{T}]\{\mathbf{u}\}=[\mathbf{U}]\{\mathbf{t}\}$$

Boundary node points





$$\nabla^2\Phi(\mathbf{x}) = -2G\beta, \mathbf{x} \in D$$
$$\Phi(\mathbf{x}) = \Phi_h(\mathbf{x}) + \Phi_p(\mathbf{x}) = 0, \mathbf{x} \in B$$

$$\tau_{xz} = \frac{\partial\Phi(\mathbf{x})}{\partial y}, \quad \tau_{yz} = -\frac{\partial\Phi(\mathbf{x})}{\partial x}$$

PDE model

$$\alpha u(\mathbf{x}) = \text{C.P.V.}\int_B T(\mathbf{s},\mathbf{x})u(\mathbf{s})dB(\mathbf{s}) - \text{R.P.V.}\int_B U(\mathbf{s},\mathbf{x})t(\mathbf{s})dB(\mathbf{s}), \mathbf{x} \in B$$

Adaptive exact solution

Singular integral can be regularized

Computation of solid angle is not required

Boundary integral equation

Analytical

$$0 = \int_B T(\mathbf{s},\mathbf{x})\left[u(\mathbf{s}) - w(\mathbf{s})\right]dB(\mathbf{s}) - \int_B U(\mathbf{s},\mathbf{x})\left[t(\mathbf{s}) - w_p(\mathbf{s})\right]dB(\mathbf{s}), \mathbf{x} \in B$$

Parametric representation

Gaussian quadrature

$$\sum_{j=1}^{N} w_j T(\mathbf{s}_j,\mathbf{x}_i)J(\mathbf{s}_j)\left[u_j - w_j^i\right] = \sum_{j=1}^{N} w_j U(\mathbf{s}_j,\mathbf{x}_i)J(\mathbf{s}_j)\left[t_j - w_{pj}^{\,i}\right]$$

Linear algebraic equation

Numerical

Collocation point

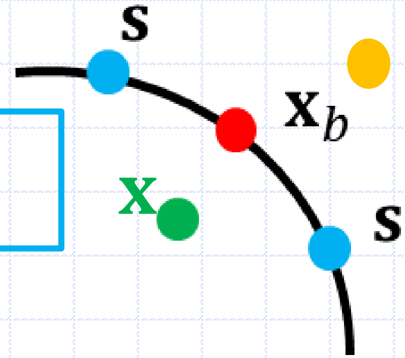
$$[\mathbf{T}]\{\mathbf{u}\} = [\mathbf{U}]\{\mathbf{t}\}$$

BIQM is a semi-analytical method

Deal with nearly singular integral by adaptive exact solution

$$U(\mathbf{s},\mathbf{x}) = \frac{1}{2\pi\sqrt{G_{xz}G_{yz}}} ln(r)$$

$$r = |\mathbf{x} - \mathbf{s}|$$

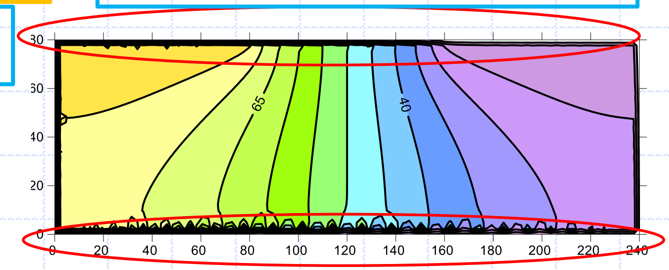


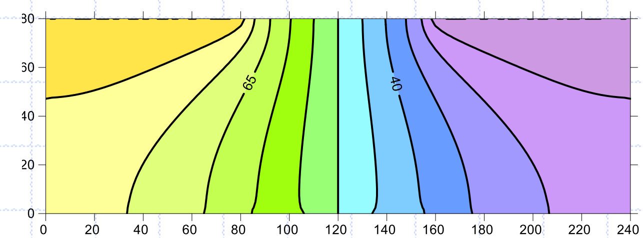
x in the domain

out of domain

$$\mathbf{x}_b = (x_b, y_b)$$

Boundary layer effect due to the nearly singular





$$u(\mathbf{x}) = \int_{B} T(\mathbf{s},\mathbf{x})u(\mathbf{s}) - U(\mathbf{s},\mathbf{x})t(\mathbf{s})dB(\mathbf{s}), \mathbf{x} \in \mathrm{D}$$

$$w(\mathbf{x}) = \int_{B} T(\mathbf{s},\mathbf{x})w(\mathbf{s}) - U(\mathbf{s},\mathbf{x})w_p(\mathbf{s})dB(\mathbf{s}), \mathbf{x} \in \mathrm{D}$$

Subtract

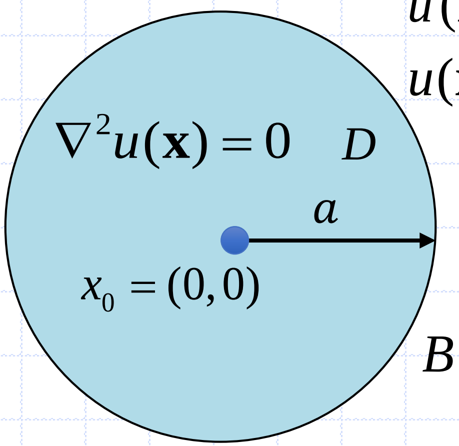
Adaptive exact solution

$$w(\mathbf{s}) = u(\mathbf{x}_b) + \left[\frac{(s_x - x_b)}{G_{xz}} n_x(\mathbf{x}_b) + \frac{(s_y - y_b)}{G_{yz}} n_y(\mathbf{x}_b)\right] t(\mathbf{x}_b), \ \mathbf{x}_b \in \mathrm{B}$$

$$u(\mathbf{x}) = \int_{B} T(\mathbf{s},\mathbf{x})[u(\mathbf{s}) - w(\mathbf{s})] - U(\mathbf{s},\mathbf{x})[t(\mathbf{s}) - w_p(\mathbf{s})]dB(\mathbf{s}) + w(\mathbf{s}), \quad \mathbf{x} \in \mathrm{D}$$

Degenerate scale

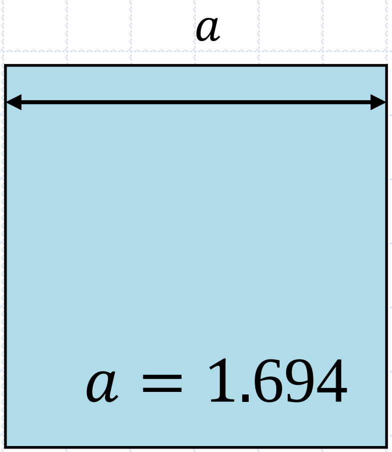
2D Laplace problem



$\bar{u}(\mathbf{x}) = 1, \mathbf{x} \in B$

$u(\mathbf{x}) \equiv 1, \mathbf{x} \in D$

$a = 1$



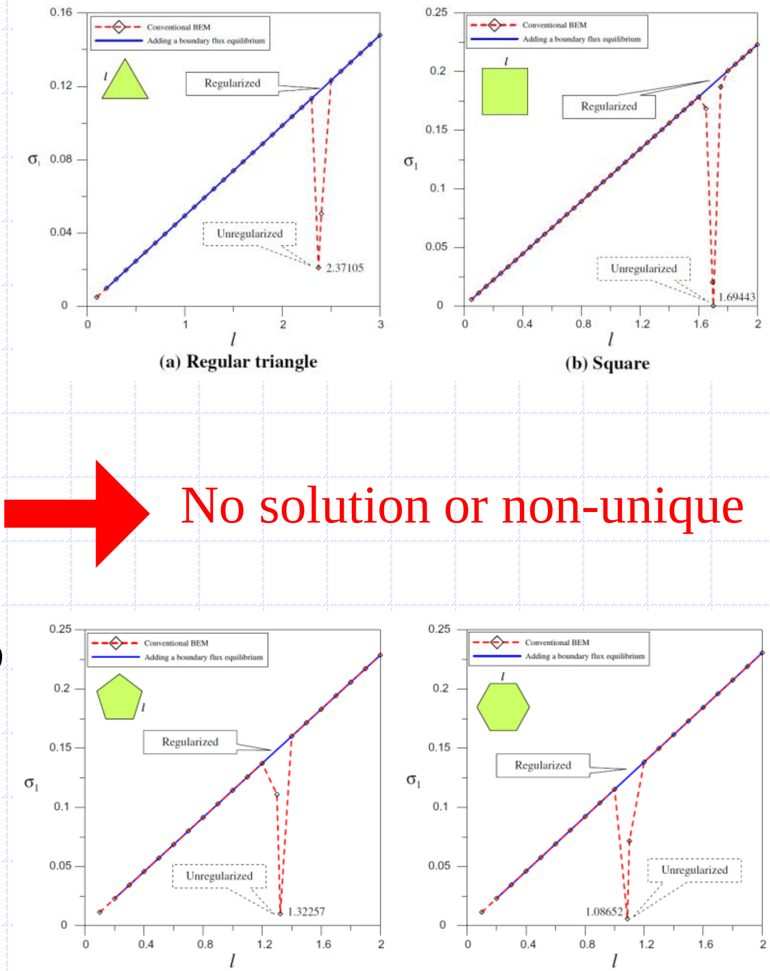
BEM/BIEM

$$u(\mathbf{x}) = \int_B T(\mathbf{s},\mathbf{x}) u(\mathbf{s}) dB(\mathbf{s}) - \int_B U(\mathbf{s},\mathbf{x}) t(\mathbf{s}) dB(\mathbf{s}) \ , \mathbf{x} \in D$$

$$U(\mathbf{s},\mathbf{x}) = \frac{1}{2\pi} ln|\mathbf{x}-\mathbf{s}|$$

$$T(\mathbf{s},\mathbf{x}) = \frac{\partial U(\mathbf{s},\mathbf{x})}{\partial n(\mathbf{s})}$$

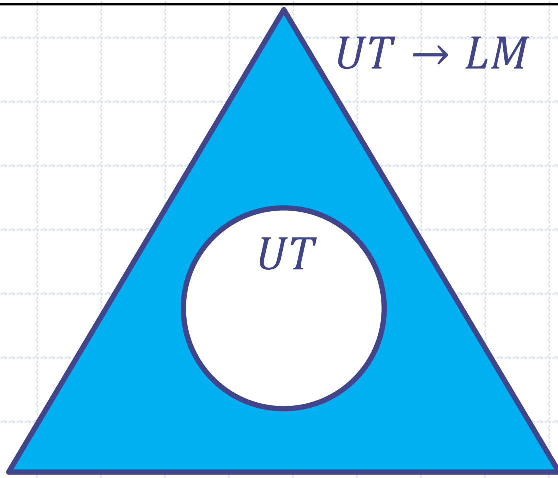
No solution or non-unique

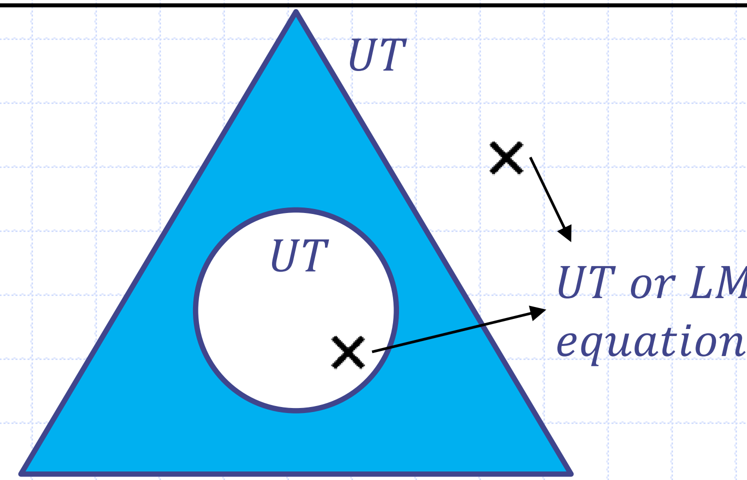


(a) Regular triangle (b) Square (c) Regular 5-gon (d) Regular 6-gon

Treatments for degenerate scale

Method 1	**Method 2 &3**
Hypersingular boundary integral equation	CLEEF (UT or LM) &CLIEF (UT or LM)





× =CLEEF or CLIEF point

$$[U_r]\{t_g\} = [T_r]\{u_g\}$$

$$[U_b^c]^T[U_b^c]\{t_g\} = [U_b^c]^T[T_b^c]\{u_g\}$$

$$\{t_g\} = ([U_b^c]^T[U_b^c])^{-1}([U_b^c]^T\,[T_b^c]\{u_g\})$$

$r = N_g \times N_g$

$g = N_g \times 1$

$b = (N_g + 1) \times N_g$

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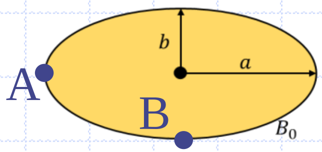
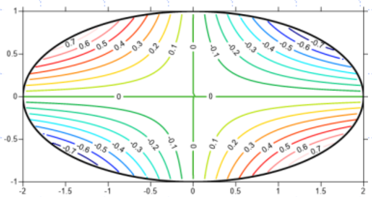
Case 1: An elliptical cross-section



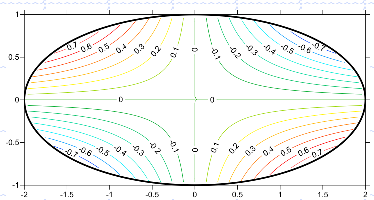
Ratio of torsional rigidity ($D_r/D_{r,ex}$), for the elliptical cross-section, ($D_{r,exact}=\frac{\pi a^3 b^3 G_{xz} R^2}{R^2 a^2 + b^2}$)			
$R=\sqrt{2}$		BIQM ($N_g = 200$)	
		Stress function $\Phi(\mathbf{x})$	Warping function $w_a(\mathbf{x})$
$D_r/D_{r,ex}$	$b/a = 1/2$	1.0000	1.0000
	$b/a = 1/3$	1.0000	1.0000

Comparison on the results of elliptical cross-section			
$a=3, b=2, R=\sqrt{2}$	$D_r/(a^4 G_{xz})$	$\left\|\tau_{xz}^{(B)}\right\| / G_{xz}\theta a$	$\left\|\tau_{yz}^{(A)}\right\| / G_{xz}\theta a$
BIQM ($N_g = 200$)	0.7616	1.0909	0.7273
LEM [Santoro]	0.761	1.0909	0.7273
CVBEM [Dumir and Kumar]	0.7618	1.0825	0.7170
Theory of elasticity [Lekhnitskii]	0.7616	1.0909	0.7273

Analytical solution

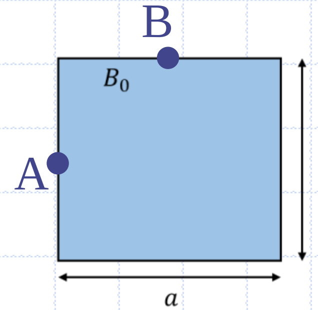


Present result



References：
- R. Santoro, The line element-less method analysis of orthotropic beam for the De Saint Venant torsion problem, International Journal of Mechanical Sciences, Vol. 52, pp.43-55, 2010.
- P. C. Dumir and R. Kumar, Complex variable boundary element method for torsion of anisotropic bars, Applied Mathematical Modelling, Vol. 17, pp. 80-88, 1993.
- S. G. Lekhnitskii, Theory of elasticity of an anisotropic body, Moscow, Mir Publishers, 1981.

Case 2: A rectangular cross-section



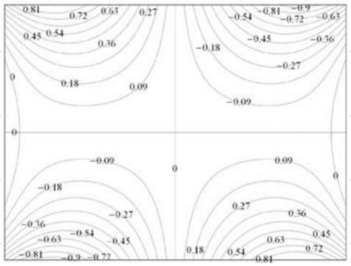
Comparison on the results of rectangle cross-section			
$a = 4, b = 3, R = 3$	$16D_r/(a^4 G_{xz})$	$2\left\|\tau_{xz}^{(B)}\right\|/G_{xz}\theta a$	$2\left\|\tau_{yz}^{(A)}\right\|/G_{xz}\theta a$
BIQM ($N_g = 200$)	1.895	1.4955	3.3410
LEM [Santoro]	1.906	1.4946	3.5901
CVBEM [Dumir and Kumar]	1.897	1.4851	3.2028
Theory of elasticity [Lekhnitskii]	1.897	1.4863	3.3433

Isotropic ($R = 1$)

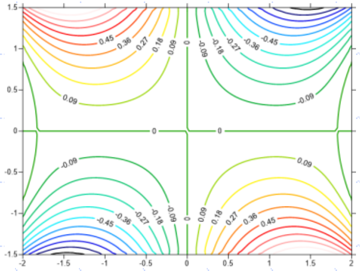
$N_g = 200$

Orthotropic ($R = 3$)

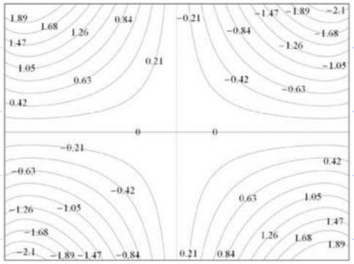
LEM [Santoro]



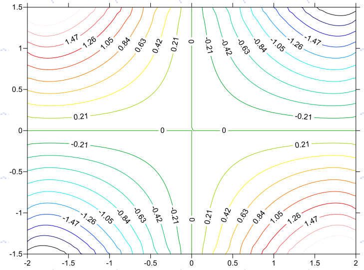
Present result



LEM [Santoro]



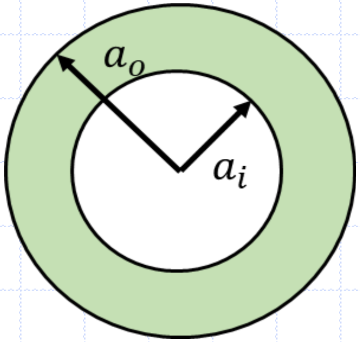
Present result



References：
- R. Santoro, The line element-less method analysis of orthotropic beam for the De Saint Venant torsion problem, International Journal of Mechanical Sciences, Vol. 52, pp.43-55, 2010.
- P. C. Dumir and R. Kumar, Complex variable boundary element method for torsion of anisotropic bars, Applied Mathematical Modelling, Vol. 17, pp. 80-88, 1993.
- S. G. Lekhnitskii, Theory of elasticity of an anisotropic body, Moscow, Mir Publishers, 1981.

Case 3: An annular cross-section

$(D_r/D_{r,ex})$, for annular cross-section, $(D_{r,ex} = \pi \frac{G_{xz}G_{yz}}{G_{xz}+G_{yz}} ({a_o}^2 + {a_i}^2)^2)$		
BIQM ($N_g = 150$)		$D_r/D_{r,ex}$
$R = \sqrt{2}$	$a_i/a_o = 1/2$	1.0000
	$a_i/a_o = 1/3$	1.0000
$R = \frac{1}{2}$	$a_i/a_o = 1/2$	1.0000
	$a_i/a_o = 1/3$	1.0000



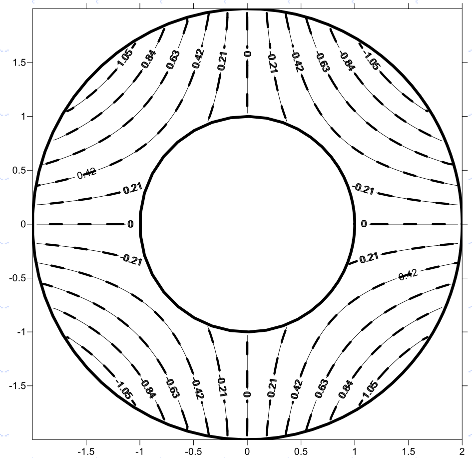
Analytical solution

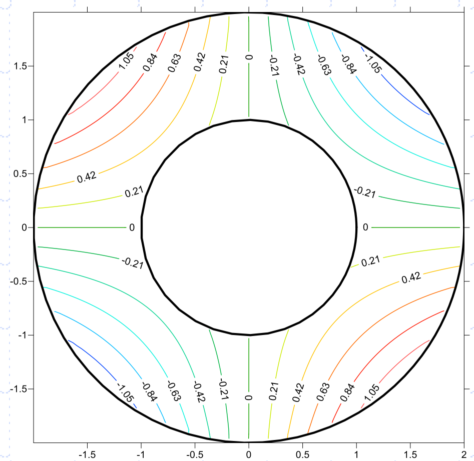
$$w_{a,ex} = \frac{R^2 - 1}{R^2 + 1} xy$$

$$R = \sqrt{\frac{G_{yz}}{G_{xz}}} = \sqrt{2}$$

Present result

$N_g = 150$



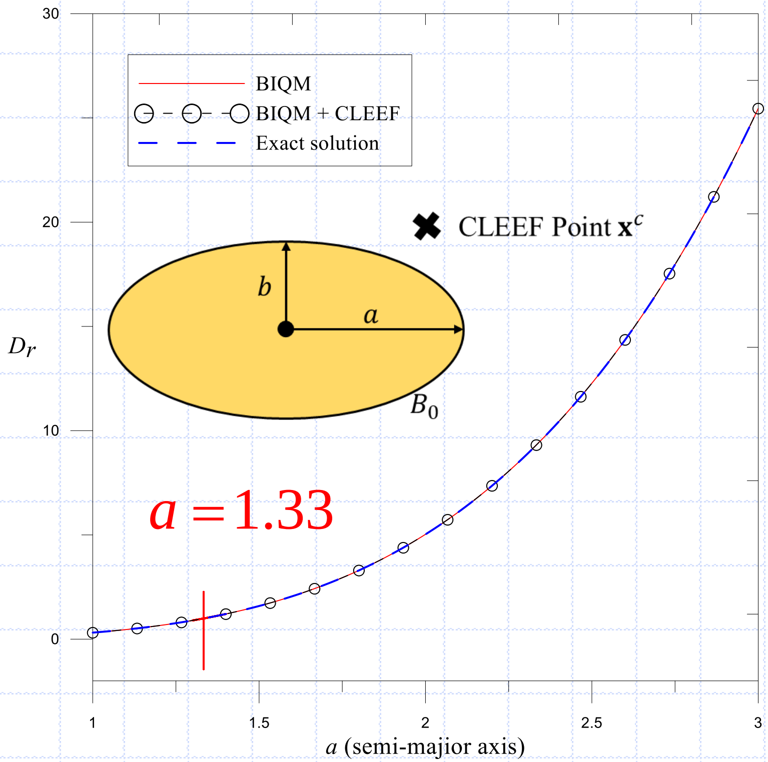


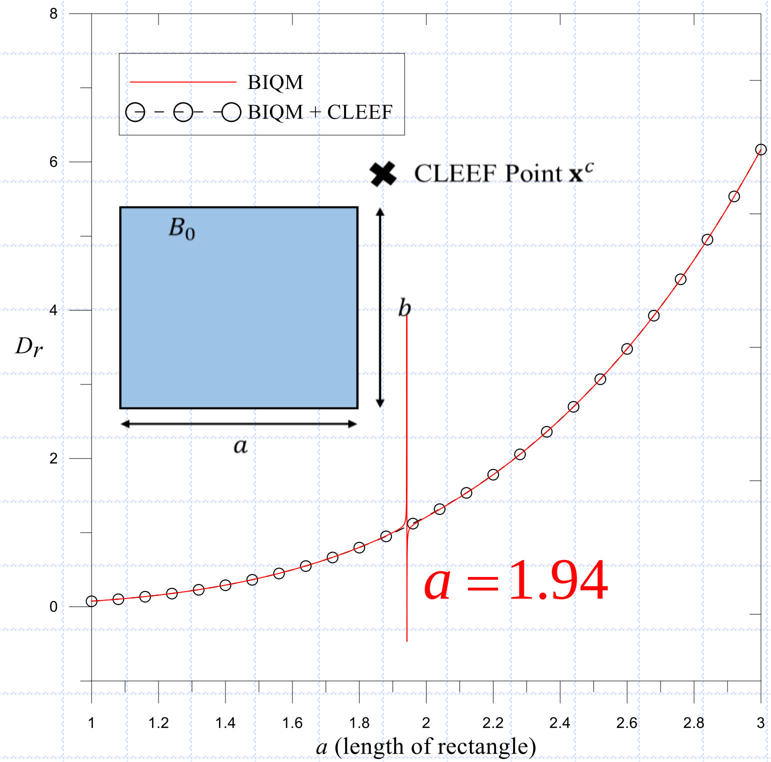
Degenerate scale and its treatment (isotropic case)

$$R=\sqrt{\frac{G_{yz}}{G_{xz}}}=1$$

$$b/a=\frac{1}{2}$$

$$a/b=\frac{4}{3}$$



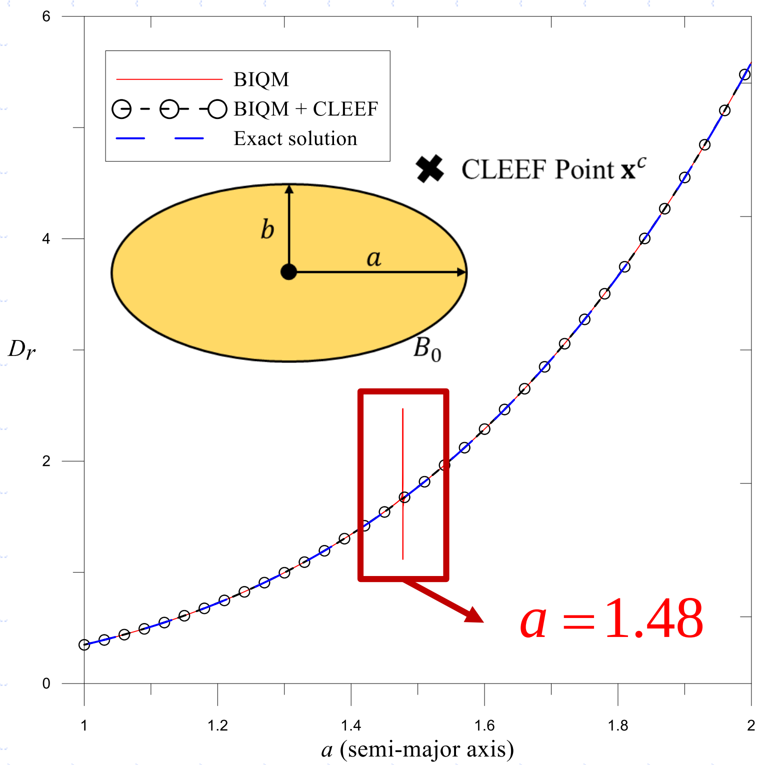


Degenerate scale and its treatment (orthotropic case)

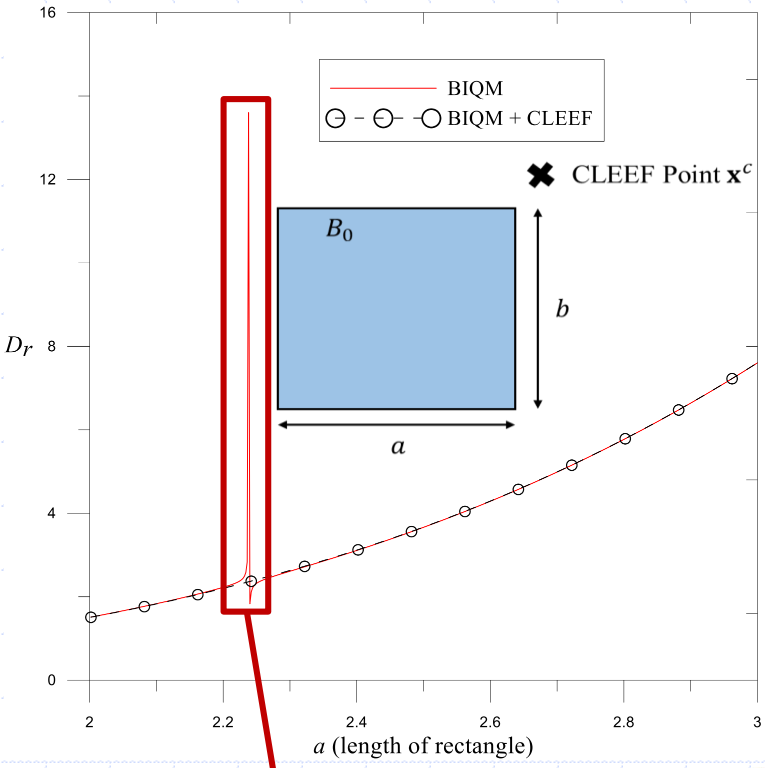
$$R=\sqrt{\frac{G_{yz}}{G_{xz}}}=2$$

$$b/a=\frac{1}{2}$$

$$a/b=\frac{4}{3}$$



$a = 1.48$

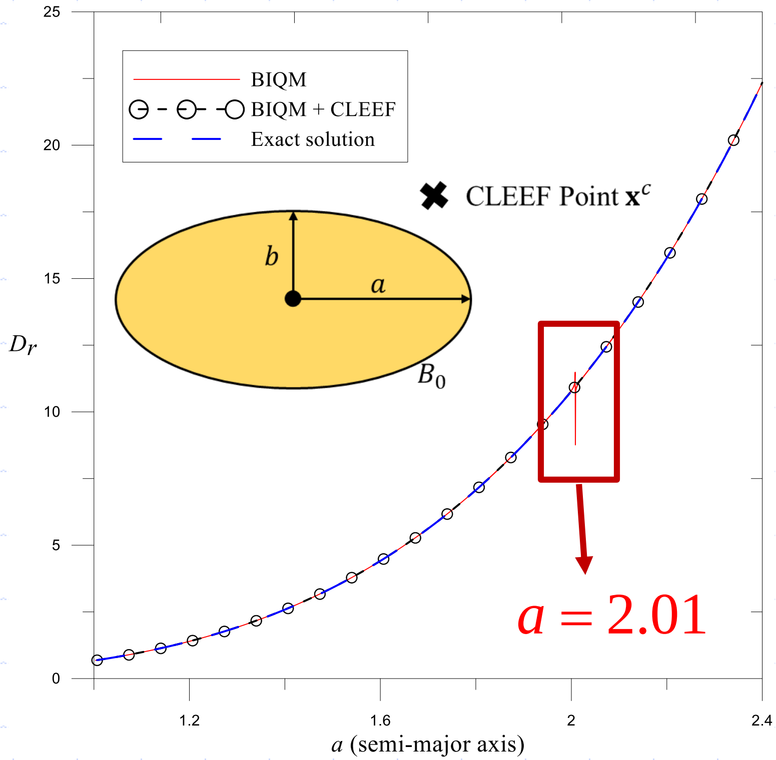


$a = 2.24$

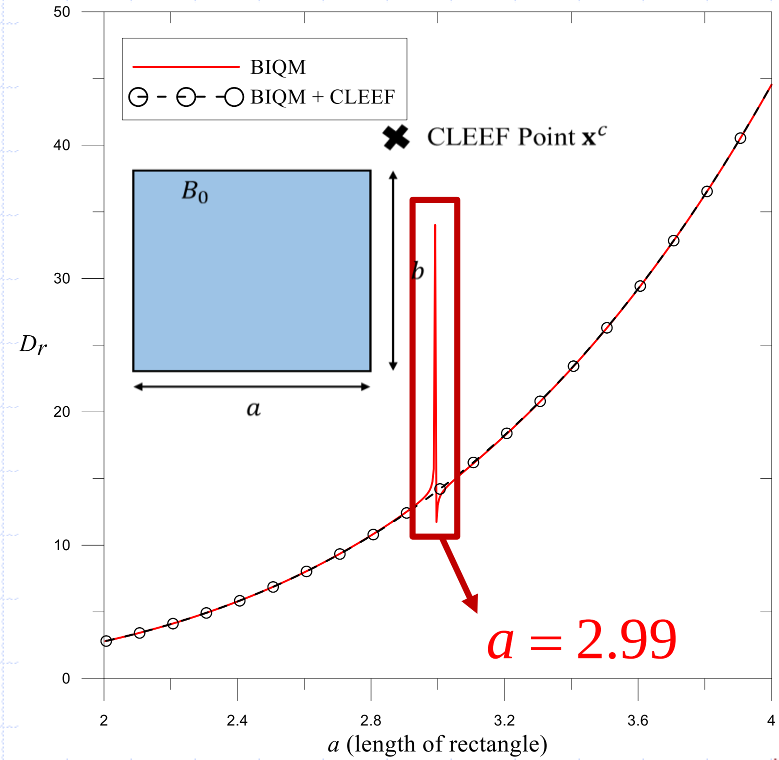
Degenerate scale and its treatment (orthotropic case)

$$R=\sqrt{\frac{G_{yz}}{G_{xz}}}=\sqrt{\frac{3}{2}}$$

$$b/a=\frac{1}{2}$$



$$a/b=\frac{4}{3}$$



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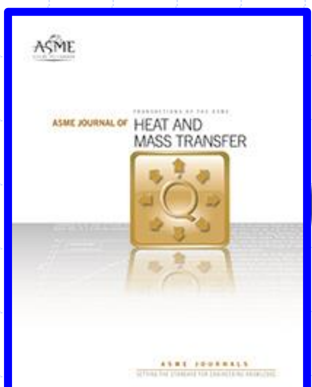
Conclusions

1. The BIQM is successfully extended to orthotropic cases.
2. By introducing the adaptive exact solution, the singular integral in the sense of Cauchy principal value was skillfully calculated.
3. The mesh generation on the boundary was not required in the mode preparation.
4. The algebraic equation was obtained by collocating only the boundary points through the use of the Gaussian point.
5. The BIQM is a semi-analytical method.
6. The CLEEF and dual ideas can be employed to effectively suppress the numerical instability due to a degenerate scale.

Related papers

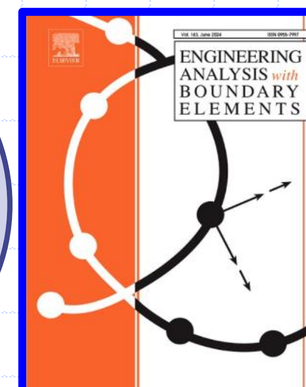
if anyone has interest on the BIQM, two papers are now available.
The pdf file is upon your request.

JOM
(Torsion)
2024



ASME
JHMT
(CSF)
2023

EABE
(CSF)
2024



The end

Thanks for your kind attentions