ELASTOPLASTIC COMPUTING SAINT-VENANT FLEXURE-TORSION AND WARPING TORSION IN THREE DIMENSIONS

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1. Motivation: computing by two cycles & Mutō Kiyoshi

2. Objective: Saint-Venant's flexure-torsion & warping torsion

3. Various prismatic cross sections (*e.g.*, ellipse, rectangle, thin-walled, singly-connected, multi-connected)

4. Different loading types

5. CM material & structural models of flow elastoplasticity

6. Conclusions

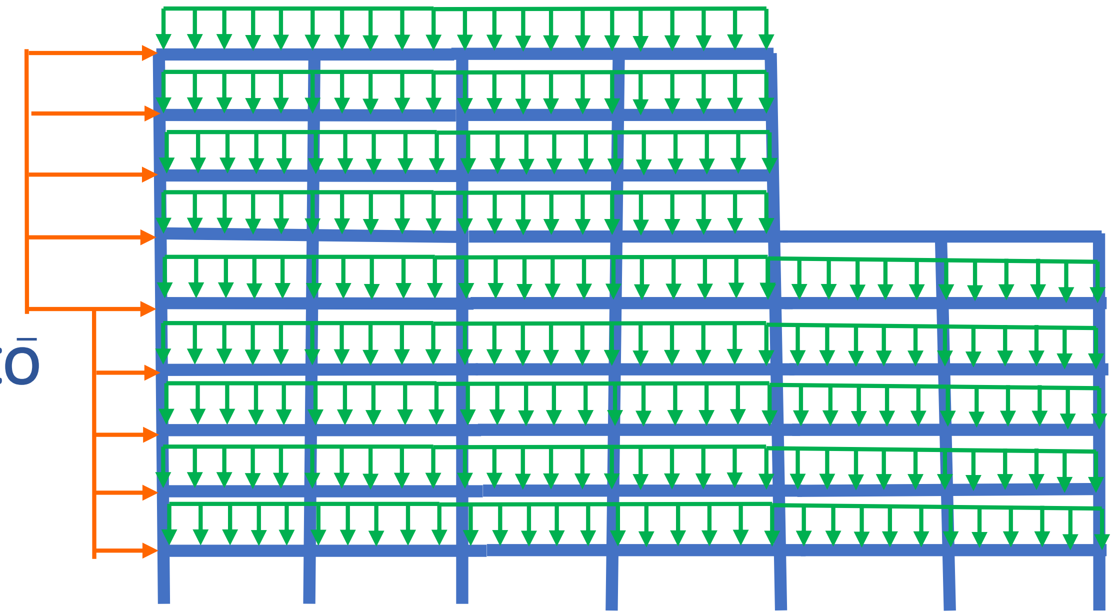
Motivation: Computing in Building Frame Design

Hardy Cross (1929) Statically indeterminate beam and frame

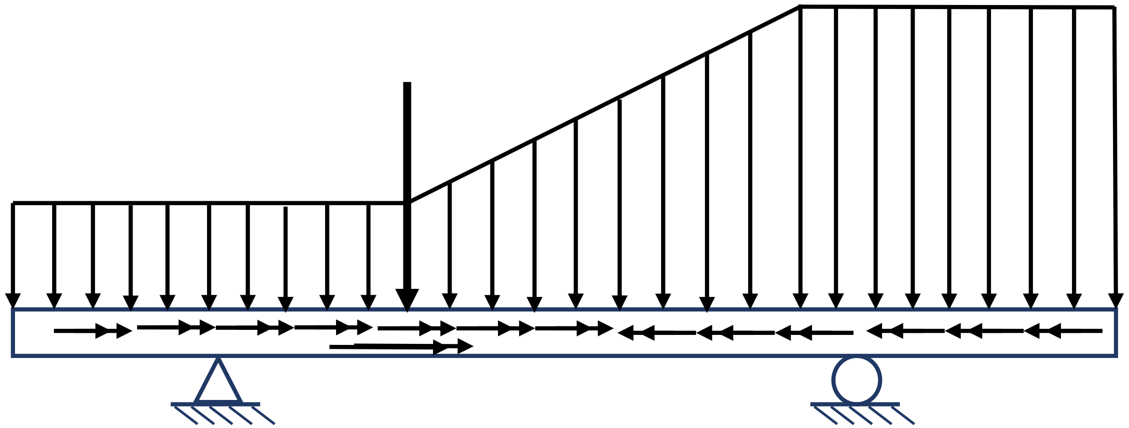
Vertical load (gravity): Two-Cycle Moment Distribution method

Horizontal load (earthquake, wind): 武藤清 Mutō Kiyoshi method

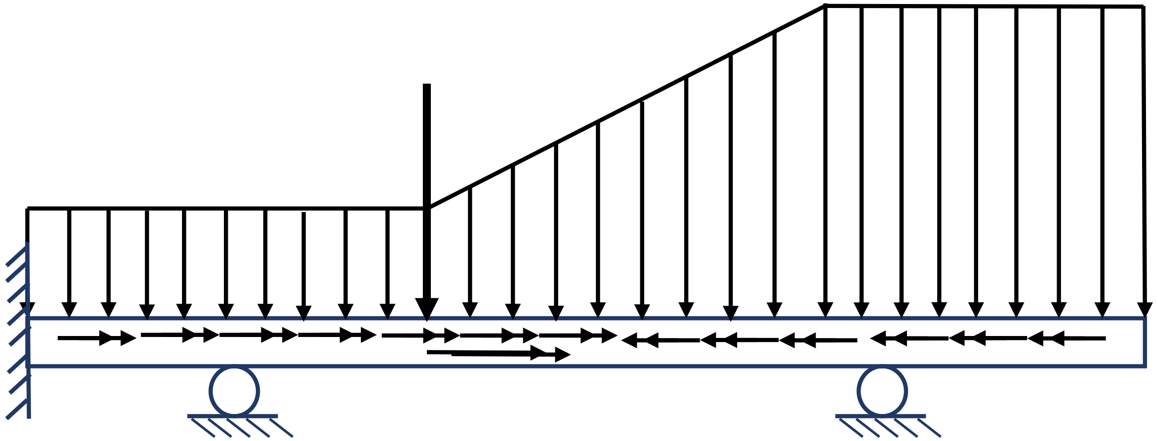
only flexure is considered



Statically determinate



Statically indeterminate



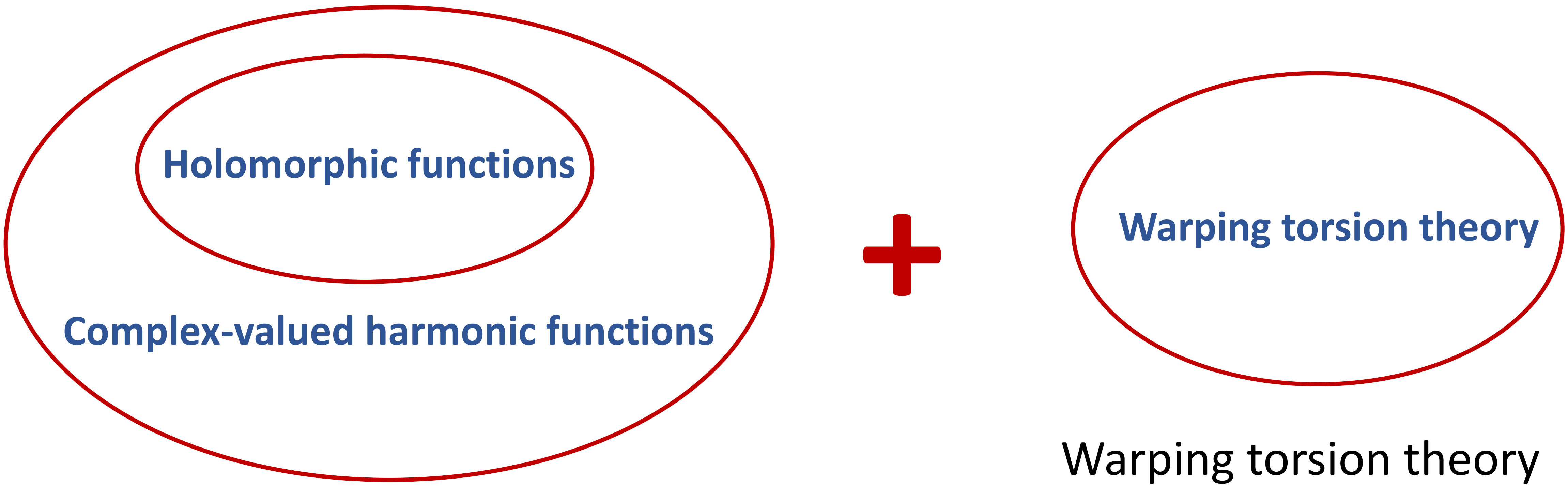
Distributed loads

Concentrated loads

Distributed torsional loads

Concentrated torsional loads

Flexure → Saint-Venant's **flexure**-torsion theory



Warping torsion theory

Twisting angle ϕ allowed to vary

Statically indeterminate

Holomorphic theory

Elegant but twisting rate ϕ' restrained to be constant

Statically determinate

Objective

The Saint-Venant's flexure-torsion theory can display stresses and warping displacements on prismatic cross sections. But it cannot demonstrate variations along the axial direction.

The warping torsion theory can display variations along the axial direction. But it cannot display stresses and warping displacements on sections.

Now we seek a approach which can display stresses and warping displacements on sections and display variations along the axial direction simultaneously.

Saint-Venant's semi-inverse method

Saint-Venant's flexure-torsion theory

GE $-\nabla^2 S = [\frac{vE}{1+v}\kappa_1' x_1 + \frac{\mathrm{d}f_2}{\mathrm{d}x_1}] + [\frac{vE}{1+v}\kappa_2' x_2 + \frac{\mathrm{d}f_1}{\mathrm{d}x_2}] + 2G\phi_3'$ for $R \subset \mathbb{R}^2$

BC $\frac{dS}{ds} = \frac{dx_1}{ds}[-E\kappa_1' \frac{x_2^2}{2} + f_2(x_1)] - \frac{dx_2}{ds}[E\kappa_2' \frac{x_1^2}{2} + f_1(x_2)]$ for ∂R

GE $\nabla^2 W = 0$ for $R \subset \mathbb{R}^2$

BC $\frac{dW}{ds} = \kappa_1'[(1+v)x_2^2 - vx_1^2]n_2 + \kappa_2'[vx_2^2 - (1+v)x_1^2]n_1 + \phi_3'(x_2 n_1 - x_1 n_2)$ for ∂R

E: Young's modulus
G: Shear modulus
v: Possion's ratio
S: stress function
W: warping function
F_1: concentrated transverse force in x_1 direction
F_2: concentrated transverse force in x_2 direction
T: concentrated torque

s: arc length coordinate
n: normal coordinate
κ_1: curvature of x_1 direction ; $\kappa_1' = \frac{F_2}{EI_{11}}$
κ_2: curvature of x_2 direction ; $\kappa_2' = \frac{F_1}{EI_{22}}$
f_1 & f_2: arbitrary function
ϕ_3: twist angle of x_3 direction
ϕ_3': rate of twist per unit length

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Notation

x_1 , x_2 , x_3 : location

F_1 , F_2 , T_D : Free end concentrated load & torsion

T_s : Saint-Venant torsion

T_w :warping torsion

K: section factor

w : displacement in z-direction

ϕ : angle of twist

I: second axial moment of area

$I_{ww} = \int w^2\, dF$: sectorial moment of inertia

f : warping factor

E: Young's modulus

G: Shear modulus of elasticity

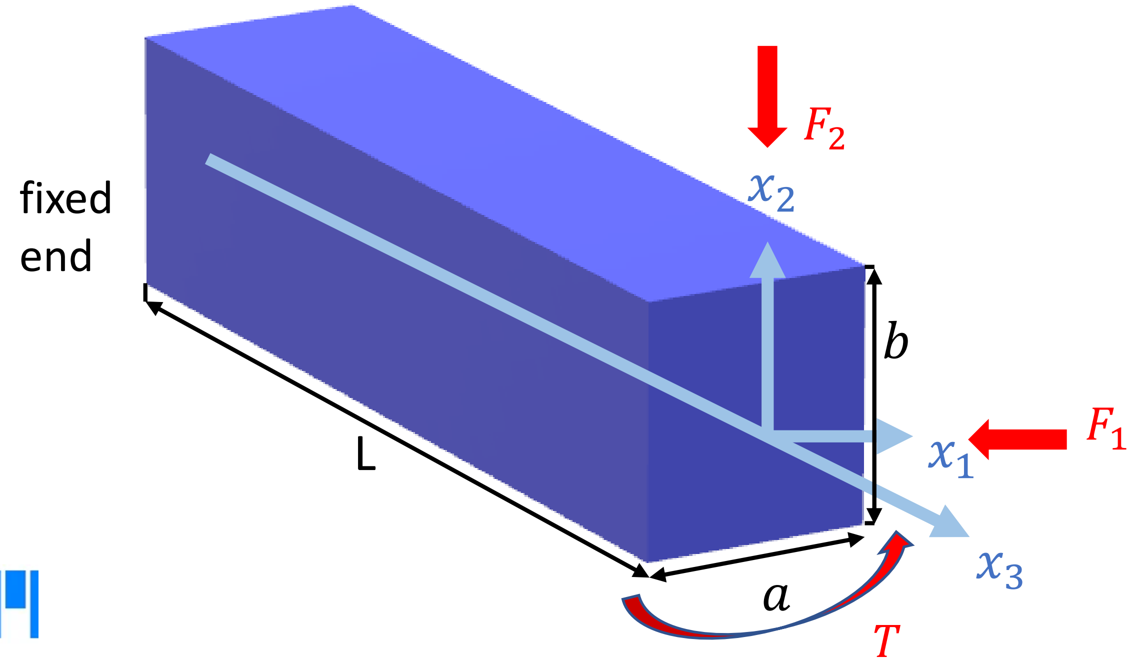
Saint-Venant's flexure-torsion theory

GE $-\nabla^2 S = [\frac{\nu E}{1+\nu}\kappa_1' x_1 + \frac{\mathrm{d}f_2}{\mathrm{d}x_1}] + [\frac{\nu E}{1+\nu}\kappa_2' x_2 + \frac{\mathrm{d}f_1}{\mathrm{d}x_2}] + 2G\phi_3'$ for $R \subset \mathbb{R}^2$

BC $\frac{dS}{ds} = \frac{dx_1}{ds}[-E\kappa_1' \frac{x_2^2}{2} + f_2(x_1)] - \frac{dx_2}{ds}[E\kappa_2' \frac{x_1^2}{2} + f_1(x_2)]$ for ∂R

GE $\nabla^2 W = 0$ for $R \subset \mathbb{R}^2$

BC $\frac{dW}{ds} = \kappa_1'[(1+\nu)x_2^2 - \nu x_1^2]n_2 + \kappa_2'[\nu x_2^2 - (1+\nu)x_1^2]n_1 + \phi_3'(x_2 n_1 - x_1 n_2)$ for ∂R



Warping torsion theory[1]

$$T = T_S + T_w$$

$$T_S = GK\phi_3' \quad T_w = -(EI_{ww}\,\phi_3'')'$$

$$EI_{ww}\,\phi_3'''' - GK\phi_3'' = t_D$$

$$\phi_3 = C_1 + C_2 x_3 f + C_3 \mathrm{e}^{-x_3 f} + + C_4 \mathrm{e}^{+x_3 f}$$

$$warping\ factor\ f = \sqrt{\frac{Gk}{EI_{ww}}}$$

BC $\phi_3(x_3 f = 0)=0$
$\phi_3'(x_3 f = 0)=0$
$\phi_3'(x_3 f = \infty)=0$
$T(x_3 f = \infty) = T_D$

$$\phi_3 = \frac{T_D}{GKf}\left(-1 + x_3 f + \mathrm{e}^{-x_3 f}\right)$$

[1]Torsion in Structures. C. F. Kollbrunner and K. Basler, 1969

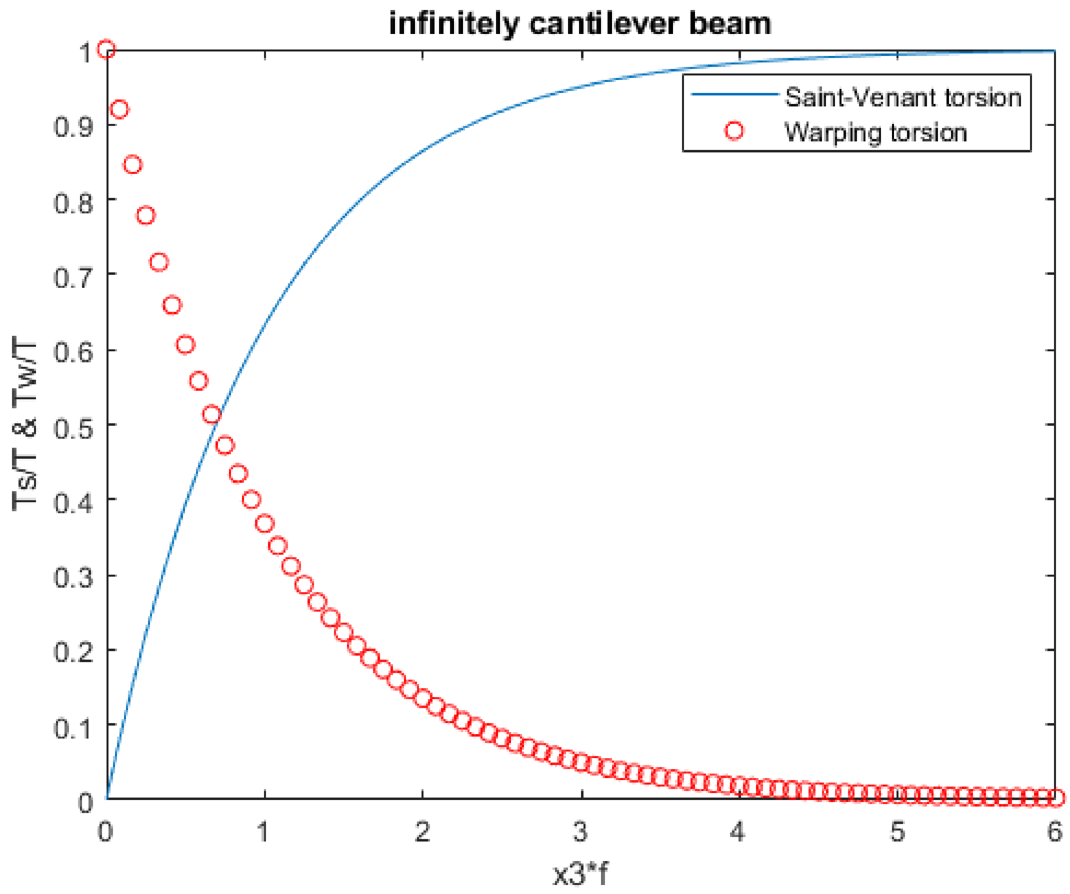
Warping torsion theory

Saint-Venant torsion vs. warping torsion

$$\phi_3 = \frac{T_D}{GKf}\left(-1 + x_3 f + e^{-x_3 f}\right)$$

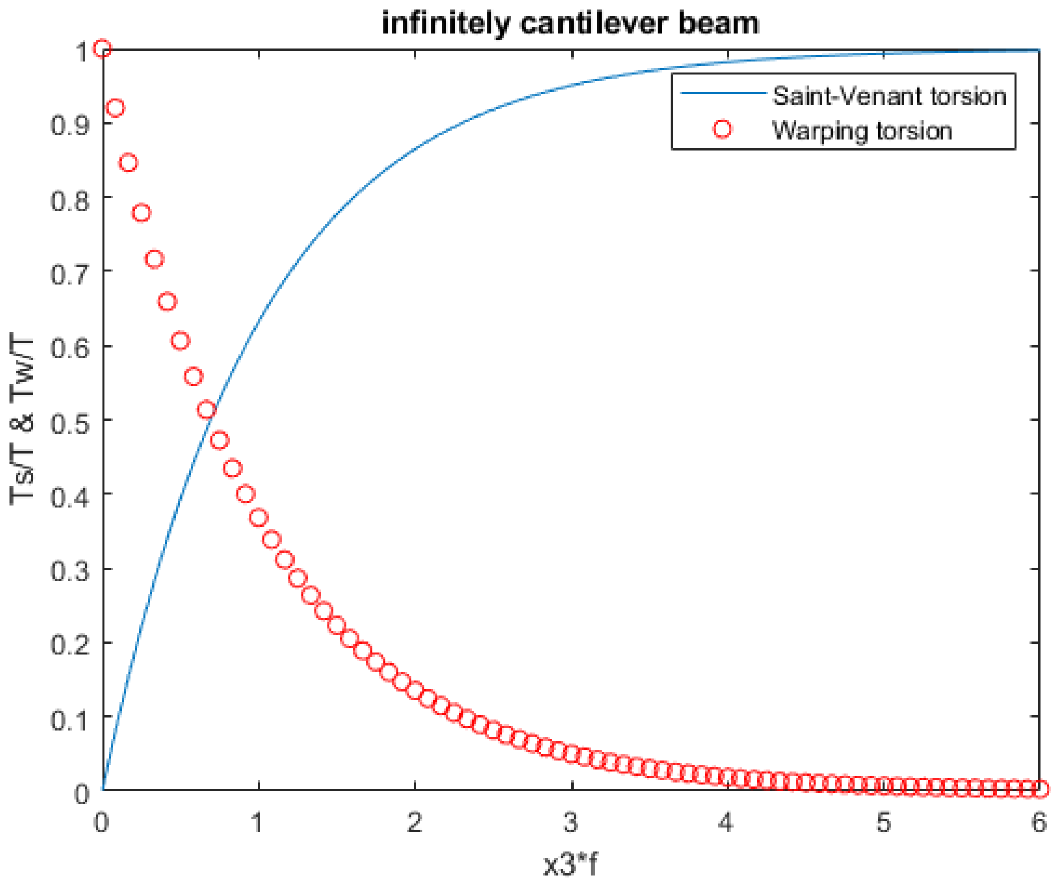
$$T_S = GK\,\phi_3' = T_D\left(1 - e^{-x_3 f}\right)$$

$$T_w = -\left(EI_{ww}\,\phi_3''\right)' = T_D\left(e^{-x_3 f}\right)$$

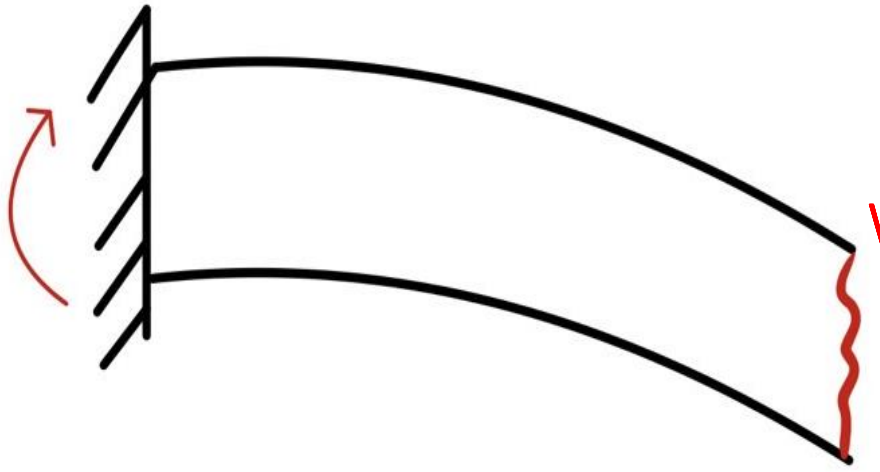


Warping torsion theory

Saint-Venant torsion vs. warping torsion



Warping moment



The warping torsion theory can display variations along the axial direction.

But it cannot display stresses and warping displacements on cross sections.

Methodology

1.According to the cross section, the stress function S *is assumed.*

2. $-\nabla^2 S = [\frac{\nu E}{1+\nu}\kappa_1' x_1 + \frac{\mathrm{d}f_2}{\mathrm{d}x_1}]+[\frac{\nu E}{1+\nu}\kappa_2' x_2 + \frac{\mathrm{d}f_1}{\mathrm{d}x_2}]+2G\phi_3'$

Getting the relationship between stress function S & curvature κ_1', κ_2' & rate of twist per unit length ϕ_3' .

3. Integrating the stress function S which equals T .Using the $T_S = GK\phi_3'$ get the cross section factor K .

4. Using Euler-Bernoulli beam formula .
$T_w = -(EI_{ww}\,\phi_3'')'$

5. Using formula $-EI_{ww}\theta'' + GK\theta = T$

to get $f = \sqrt{\frac{Gk}{EI_{ww}}}$ $\theta'' - f^2\theta = \frac{-T}{EI_{ww}}$

6. Getting the $\theta(x_3)$ & $\phi_3(x_3)$ according to the boundary condition

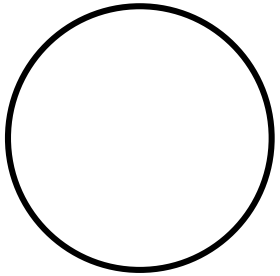
7. Using $EI_{ii}h_i\frac{\mathrm{d}\theta}{\mathrm{d}x_3} = M_\mathrm{i}$, $i = 1,2$

Getting normal stress σ_{33}

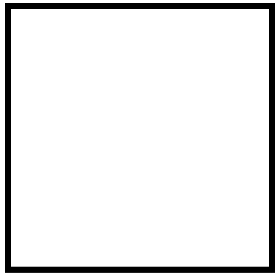
8. Getting stress function S & warping function W

Different cross sections

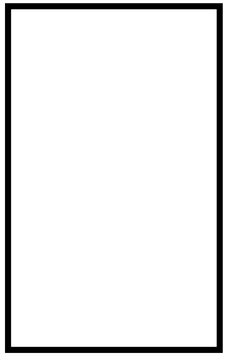
(*e.g.*, ellipse, rectangle, thin-walled, singly-connected, multi-connected)



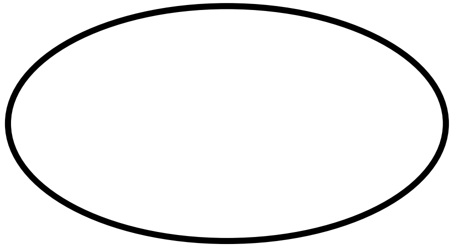
Circle



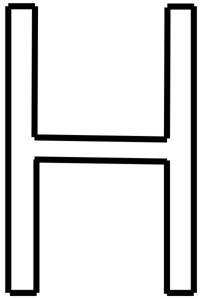
Square



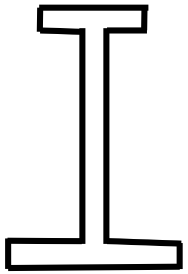
Rectangle



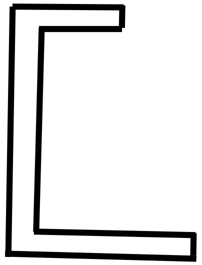
Ellipse



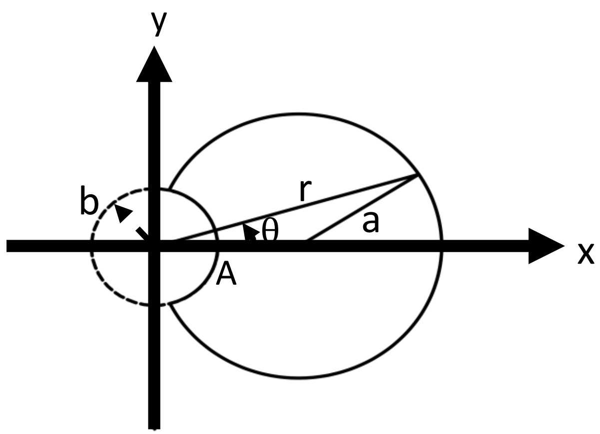
H-Section



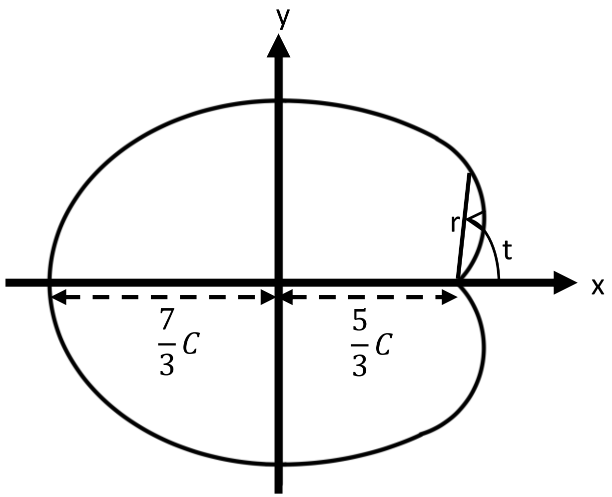
I-Section



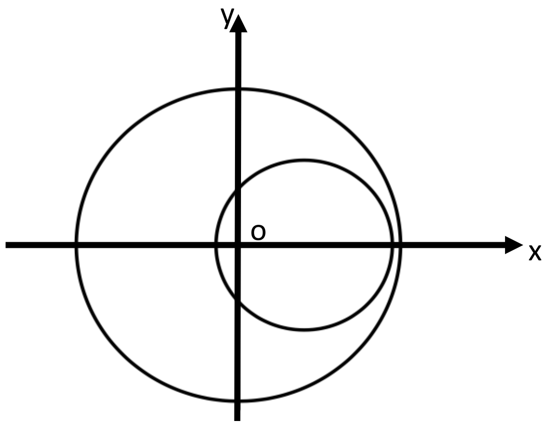
Channel



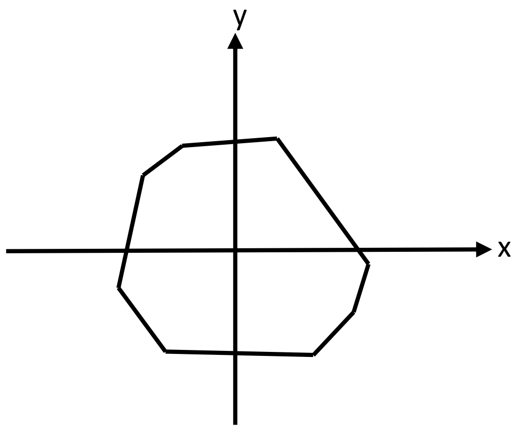
Keyway



Cardioid



Eccentric circles



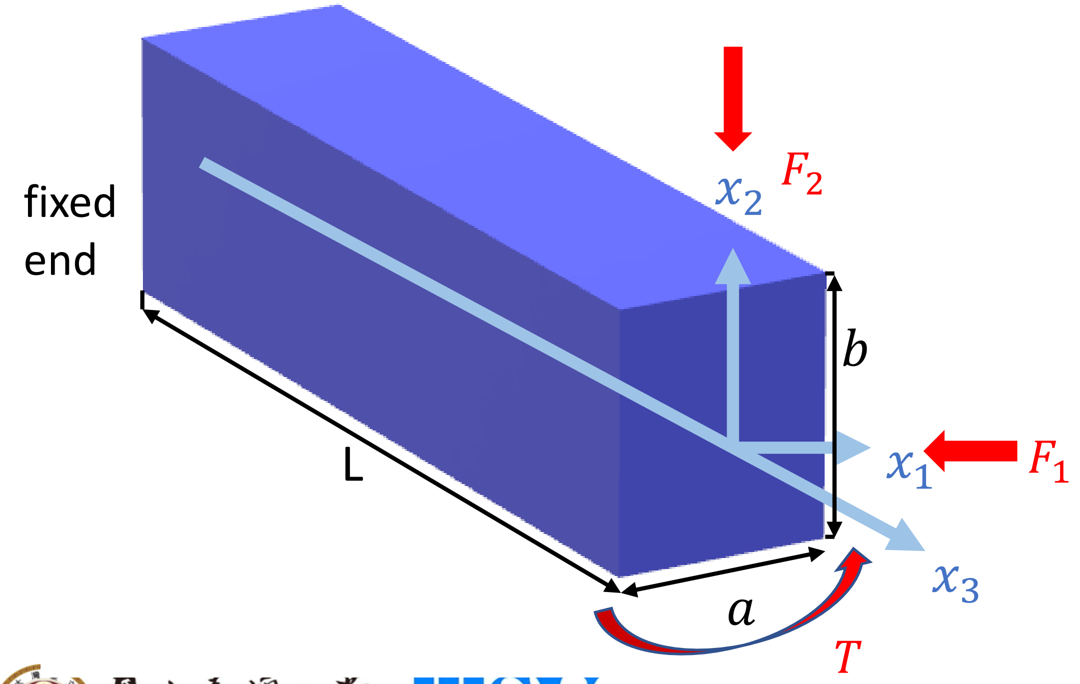
Polygon

Different load types

F_1&F_2&T

$$S_1 = \frac{v}{1+v}\frac{F_2}{6I_{11}}\left(\frac{a^2}{4}x_1 - x_1^3 + \sum_{n=1}^{\infty}\frac{3a^3}{2(n\pi)^3}\frac{(-1)^n}{\cosh\left(\frac{n\pi b}{a}\right)}\sin\left(\frac{2n\pi x_1}{a}\right)\cosh\left(\frac{2n\pi x_2}{a}\right)\right)$$

$$S_2 = \frac{v}{1+v}\frac{F_1}{6I_{22}}\left(\frac{b^2}{4}x_2 - x_2^3 + \sum_{n=1}^{\infty}\frac{3b^3}{2(n\pi)^3}\frac{(-1)^n}{\cosh\left(\frac{n\pi a}{b}\right)}\sin\left(\frac{2n\pi x_2}{b}\right)\cosh\left(\frac{2n\pi x_1}{b}\right)\right)$$

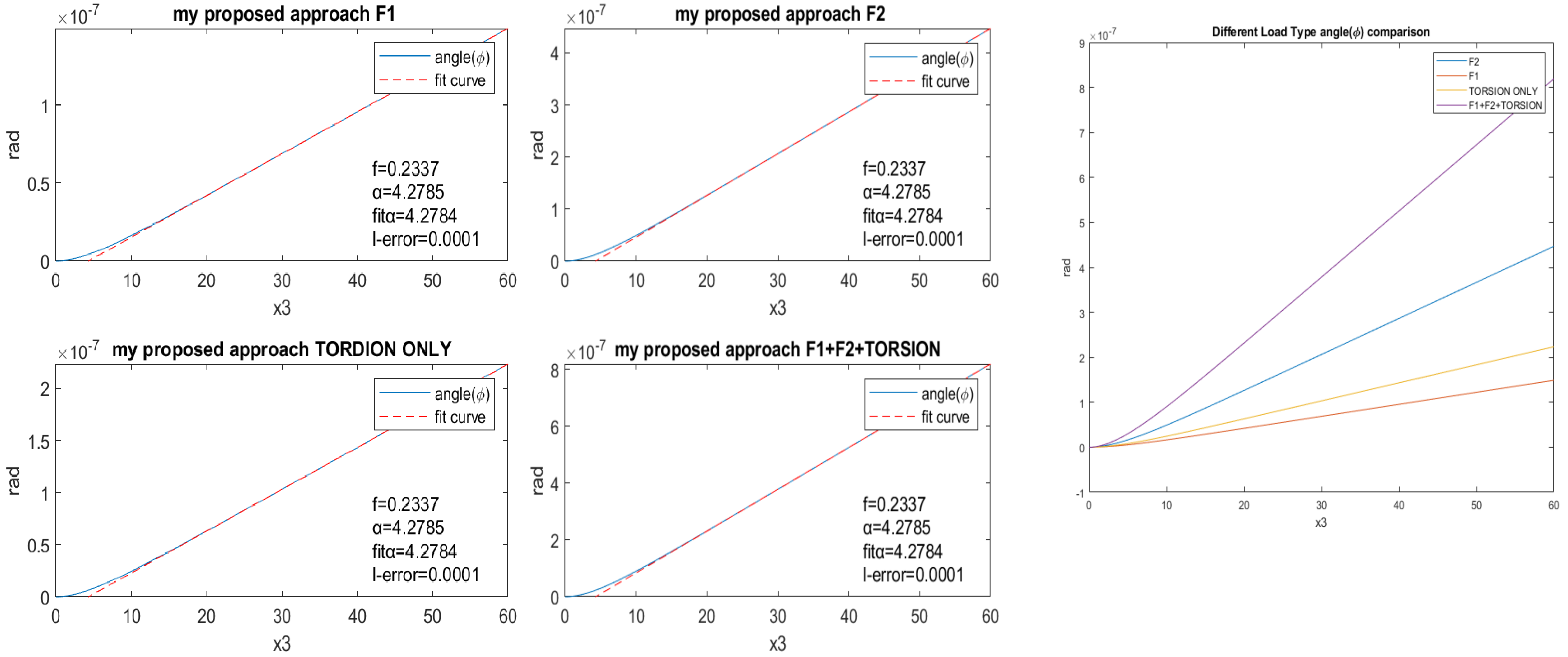


$$S_3 = G\phi_3'\left(\frac{a^2}{4} - x_1^2\right) + G\phi_3'\,v(x_1,x_2)$$

$$v(x_1,x_2) = \sum_{n=1}^{\infty} A_n\cos(\lambda_n x_1)\cos h(\lambda_n x_2)$$

$$\lambda_n = \frac{(2n-1)\pi}{a} \qquad A_n = \frac{8a^2(-1)^n}{(2n-1)^3\pi^3\cosh\left(\frac{\lambda_n b}{2}\right)}$$

Verification-rectangular F_1&F_2&T



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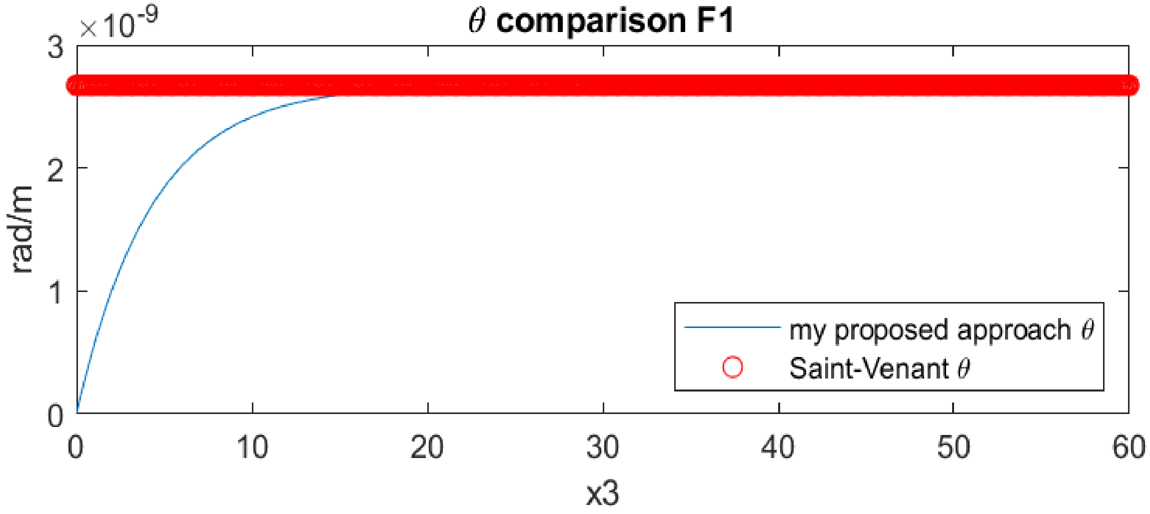
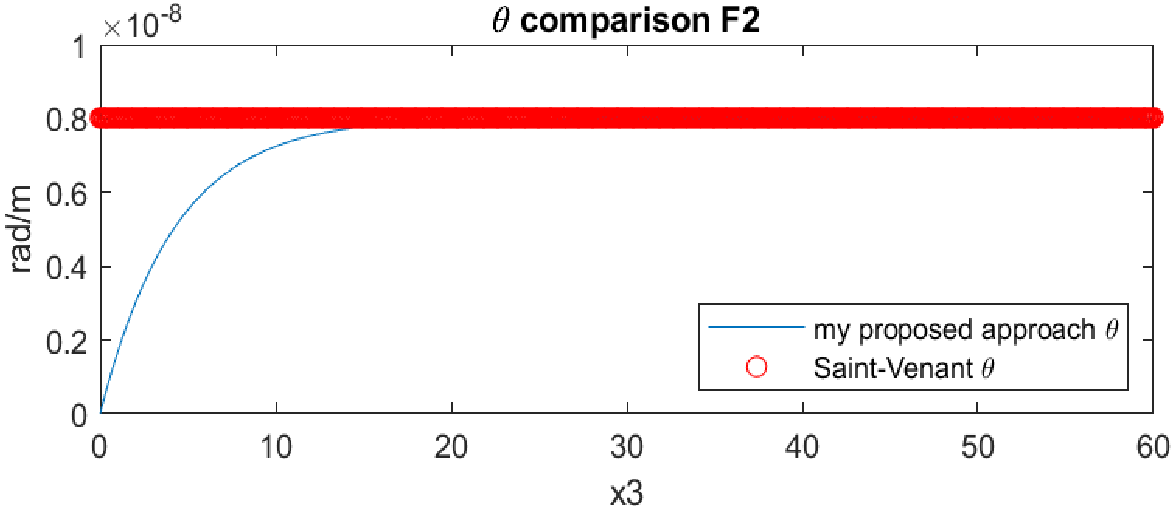
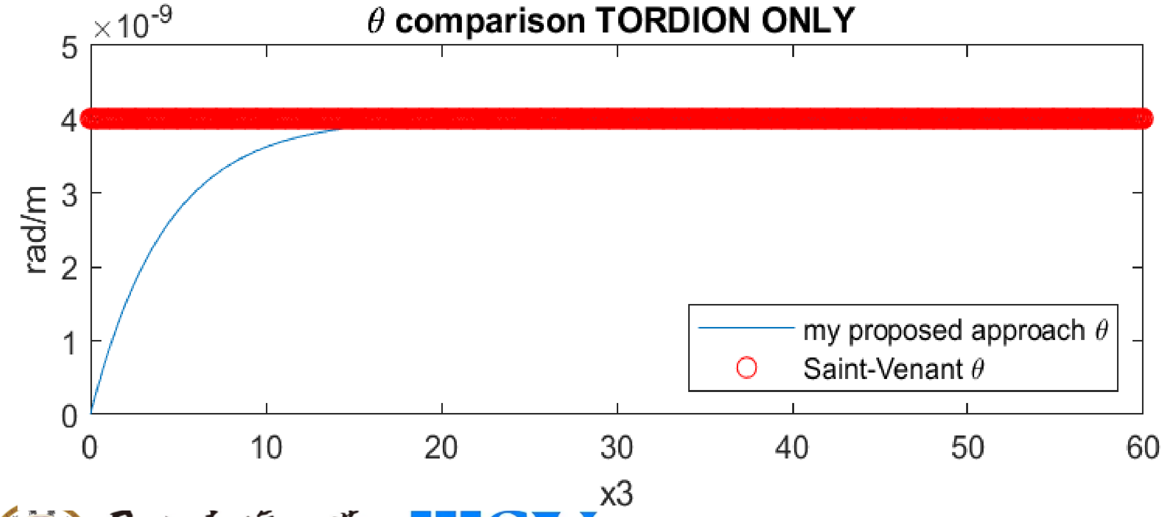
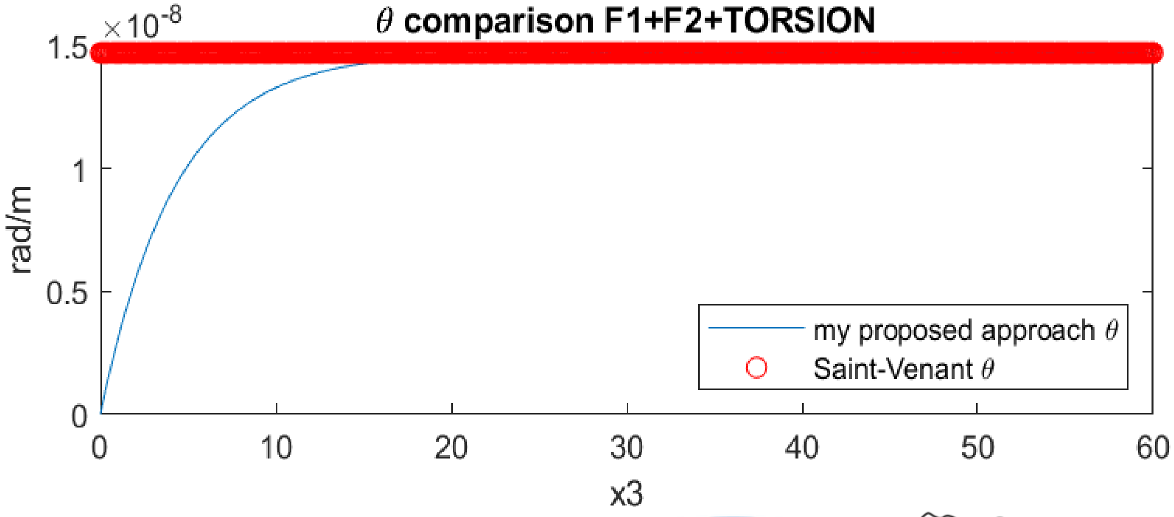
Verification-rectangular F_1&F_2&T

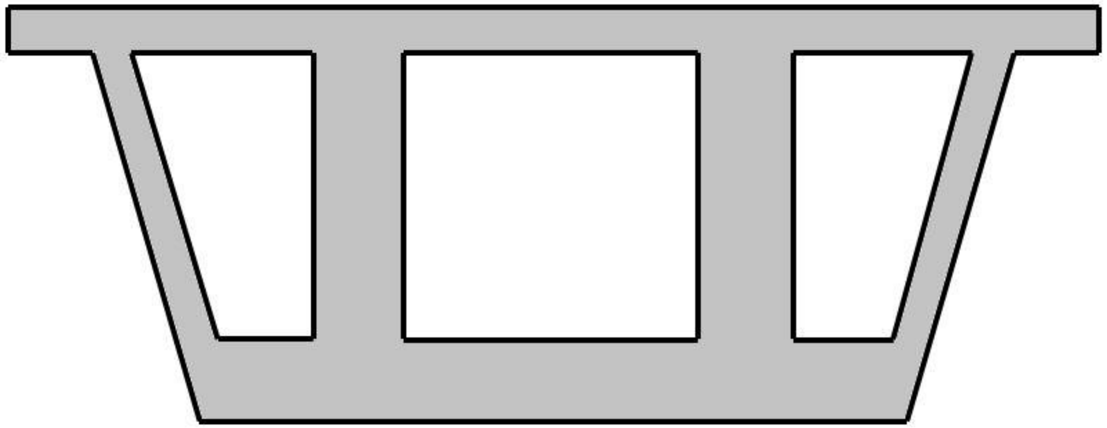








Thin-walled cross section
Thin-walled multi-cells



Thin-walled cross section

$$T_S = GK\phi_3' \quad T_w = -(EI_{ww}\,\phi_3'')'$$

Thin-walled cross section

$$T_S = GK\phi_3' \quad T_w = -(EI_{ww}\,\phi_3'')'$$

According to the cross section, the stress function *S is assumed.*

Assuming stress function directly.

Using $v(x_1, x_2)$ to get the stress function.

Using shear flow

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Shear center (x_1^{sc}, x_2^{sc})

$$x_1^{sc} = \frac{I_{22}J_1 - I_{12}J_2}{2(1+v)\left(I_{11}I_{22} - I_{12}^2\right)},$$
$$x_2^{sc} = \frac{I_{11}J_2 - I_{12}J_1}{2(1+v)\left(I_{11}I_{22} - I_{12}^2\right)}$$

where

$$J_1 = \int_R \left[\nu x_1^3 - (1+\nu)x_1 x_2^2 - x_2\frac{\partial \mathcal{W}_1}{\partial x_1} + x_1\frac{\partial \mathcal{W}_1}{\partial x_2}\right] dA,$$

$$J_2 = \int_R \left[\nu x_2^3 - (1+\nu)x_1^2 x_2 + x_1\frac{\partial \mathcal{W}_2}{\partial x_2} - x_2\frac{\partial \mathcal{W}_2}{\partial x_1}\right] dA$$

$$\kappa'_\alpha = \frac{1}{E}\left(I^{-1}\right)_{\alpha\beta} e_{3\beta\gamma} P_\gamma$$

$$\sigma_{ij}(x_1, x_2, x_3) = \begin{bmatrix} 0 & 0 & \frac{\partial S}{\partial x_2} + c_2 \frac{x_1^2}{2} + f_1(x_2) \\ 0 & 0 & -\frac{\partial S}{\partial x_1} - c_2 \frac{x_2^2}{2} + f_2(x_2) \\ \text{sym} & sym & \sigma_{33} \end{bmatrix}$$

Elasticity → Elastoplasticity

Continuum mechanics material model of flow elastoplasticity

Saint-Venant flow plasticity vs. deformation plasticity

Prandtl-Reuss flow elastoplastic model with von Mises yield surface

Flow elastoplastic model with cubic distortional yield surface [1]

- Active deviatoric stress closed-form solution: 閉合正解

$$\mathbf{s}_a^\alpha(t) = [\frac{\frac{\mathbf{s}_a^\alpha(t_1)}{R^\alpha(\lambda^\alpha(t_1))} + \frac{(a^\alpha-1)}{\|\dot{\mathbf{s}}^\alpha\|^2}\dot{\mathbf{s}}^\alpha\dot{\mathbf{s}}^{\alpha T}\frac{\mathbf{s}_a^\alpha(t_1)}{R^\alpha(\lambda^\alpha(t_1))} + b^\alpha\frac{\dot{\mathbf{s}}^\alpha}{\|\dot{\mathbf{s}}^\alpha\|}}{b^\alpha(\frac{\dot{\mathbf{s}}^\alpha}{\|\dot{\mathbf{s}}^\alpha\|})^T\frac{\mathbf{s}_a^\alpha(t_1)}{R^\alpha(\lambda^\alpha(t_1))} + a^\alpha}]R^\alpha(\lambda^\alpha(t))$$

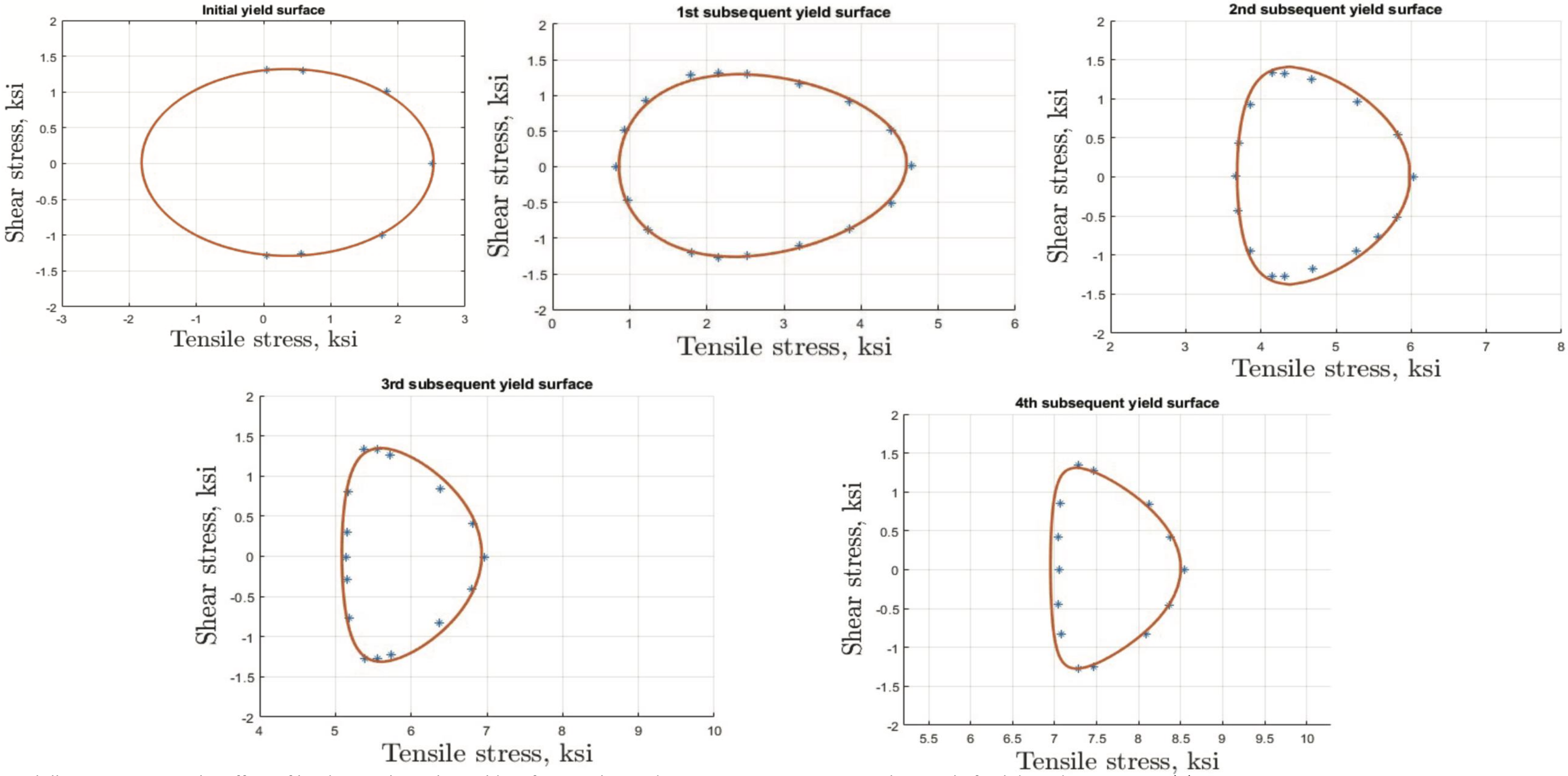
- Back deviatoric stress closed-form solution: 閉合正解

$$\mathbf{s}_b^\alpha(t) = exp(\frac{k_p^\alpha}{\eta^\alpha}[\lambda^\alpha(t_1) - \lambda^\alpha(t)])\mathbf{s}_b^\alpha(t_1) + \frac{\eta^\alpha}{R_\infty^\alpha}\left\{\mathbf{s}_a^\alpha(t) - exp(\frac{k_p^\alpha}{\eta^\alpha}[\lambda^\alpha(t_1) - \lambda^\alpha(t)])\mathbf{s}_a^\alpha(t_1)\right\}$$

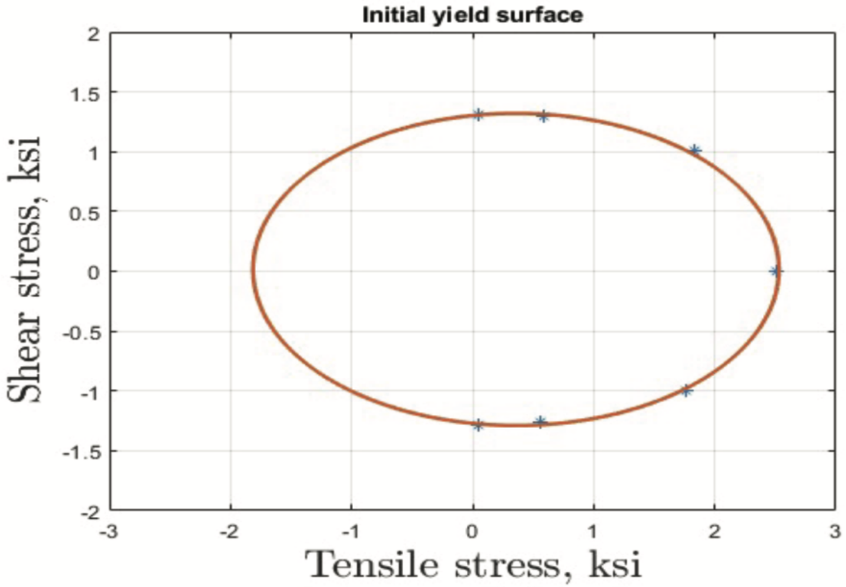
- Equivalent plastic strain closed-form solution: 閉合正解

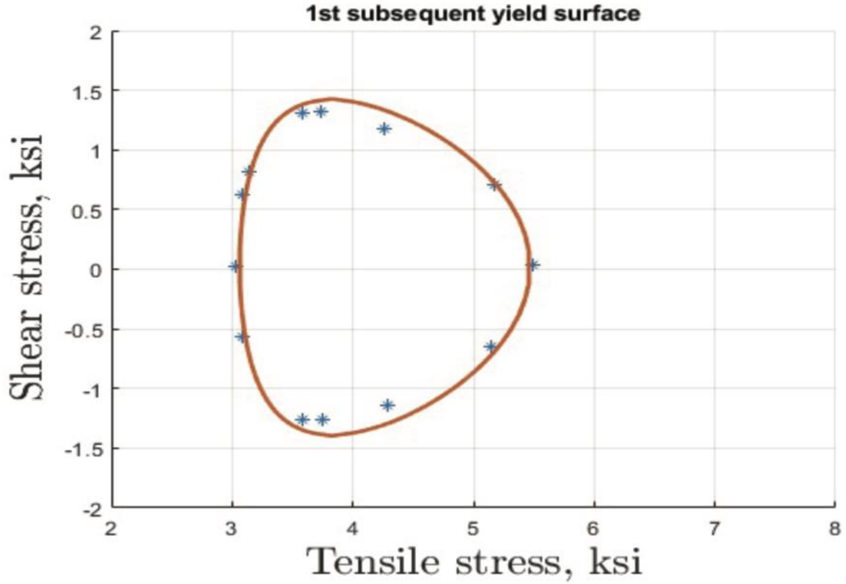
$$\lambda^\alpha(t) = \frac{R^\alpha(\lambda^\alpha(t_1))}{R^\alpha(\lambda^\alpha(t))}\lambda^\alpha(t_1) + \frac{R_\infty^\alpha R^\alpha(0)}{k_p^\alpha R^\alpha(\lambda^\alpha(t))}ln[b^\alpha(\frac{\dot{\mathbf{s}}^\alpha}{\|\dot{\mathbf{s}}^\alpha\|})^T\frac{\mathbf{s}_a^\alpha(t_1)}{R^\alpha(\lambda^\alpha(t_1))} + a^\alpha]$$

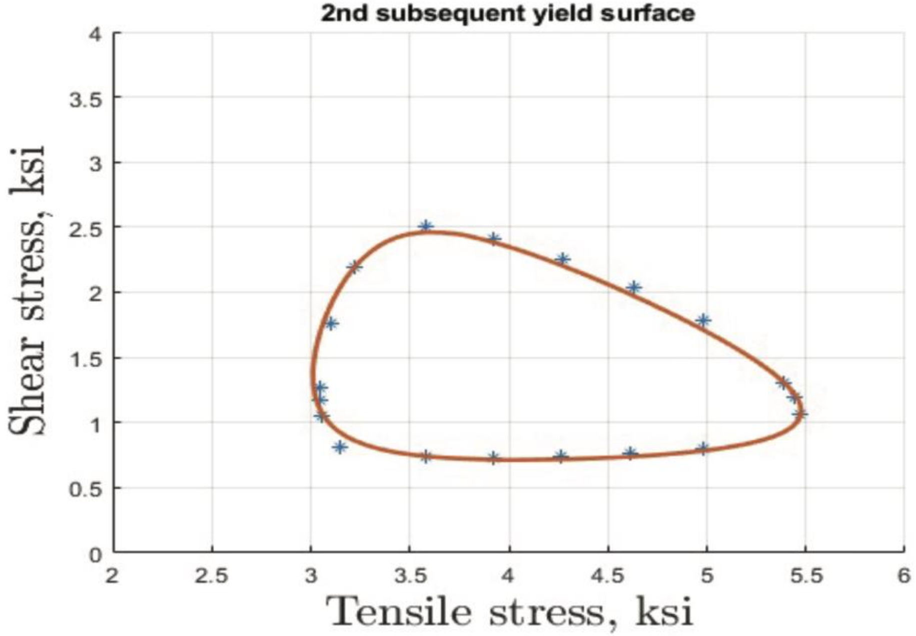
[1] Hong, H. K., Liu, L. W., Shiao, Y. P., & Yan, S. F. (2022). Yield Surface Evolution and Elastoplastic Model with Cubic Distortional Yield Surface. ASCE *Journal of Engineering Mechanics*, *148*(6), 04022027.

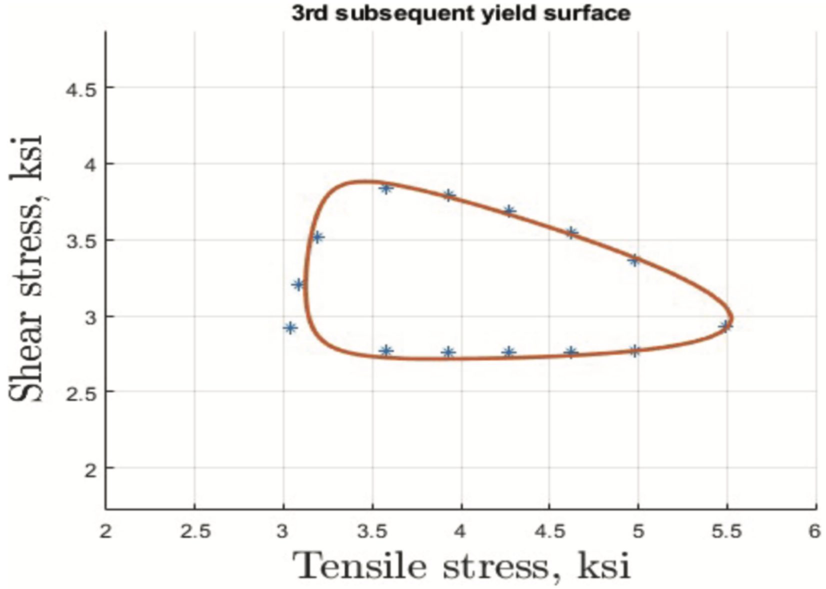


A. Phillips & J.-L. Tang, The effect of loading path on the yield surface at elevated temperatures. *International Journal of Solids and Structures*, 8(4) 463-474, 1972.
K.-M. Hou, The evolution of cubic distortional yield hypersurfaces in materials of flow elastoplasticity under prestress and at elevated temperatures. MS Thesis, Civil Engrg. Dept., National Taiwan University, 2023.









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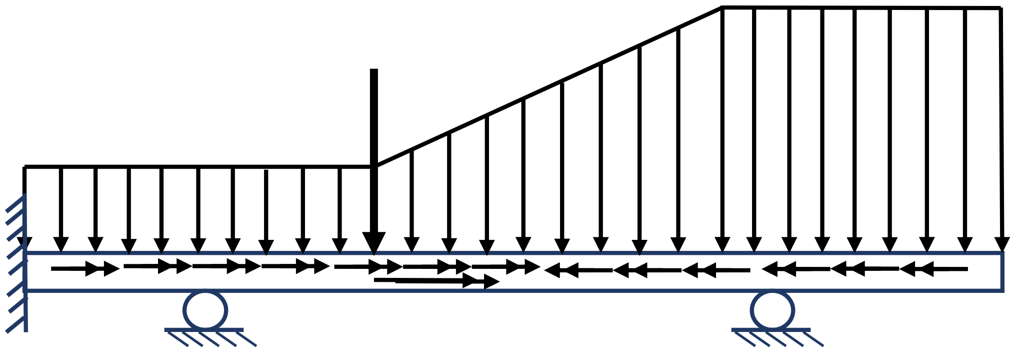
Structural model of flow elastoplasticity

Saint-Venant's "relaxed" conditions prescribing the stress resultants :

$$\left.\begin{array}{l}\int_R \sigma_{13} dA = N_1(x_3) = F_1, \\ \int_R \sigma_{23} dA = N_2(x_3) = F_2, \\ \int_R \sigma_{33} dA = N_3(x_3) = 0,\end{array}\right\}$$

$$\left.\begin{array}{l}\int_R x_2\sigma_{33} dA = M_1(x_3) = -(L - x_3)F_2, \\ -\int_R x_1\sigma_{33} dA = M_2(x_3) = (L - x_3)F_1, \\ \int_R (x_1\sigma_{23} - x_2\sigma_{13}) dA = T(x_3) = x_1^P F_2 - x_2^P F_1,\end{array}\right\}$$

Statically indeterminate



on the cross section planes $x_3 = 0, L,$ *with loads* F_1, F_2 *and point of application* (x_1^p, x_2^p) *prescribed.*

CM material model of flow elastoplasticity

Boundary/interior support condition

Force/moment load type

↓ Saint-Venant's "relaxed" condition

Structural model of flow elastoplasticity

i.e. Generalized plastic hinge

Conclusions

1. Computing stresses and warping displacements on cross sections and also their variations along the axial direction, *i.e.*, in three dimensions.

2. Valid for various prismatic cross sections (circle, ellipse, rectangle, polygon, I-section, H-section, channel, thin-walled multi-cells, cardioid, eccentric circles, grooved keyway, multi-connected). Formulae for cross section factor *K*, warping factor f, shear center location.

3. Valid for different loading types (concentrated and distributed force & moment loads) and different boundary/interior conditions on c_1, c_2, c_3, c_4.

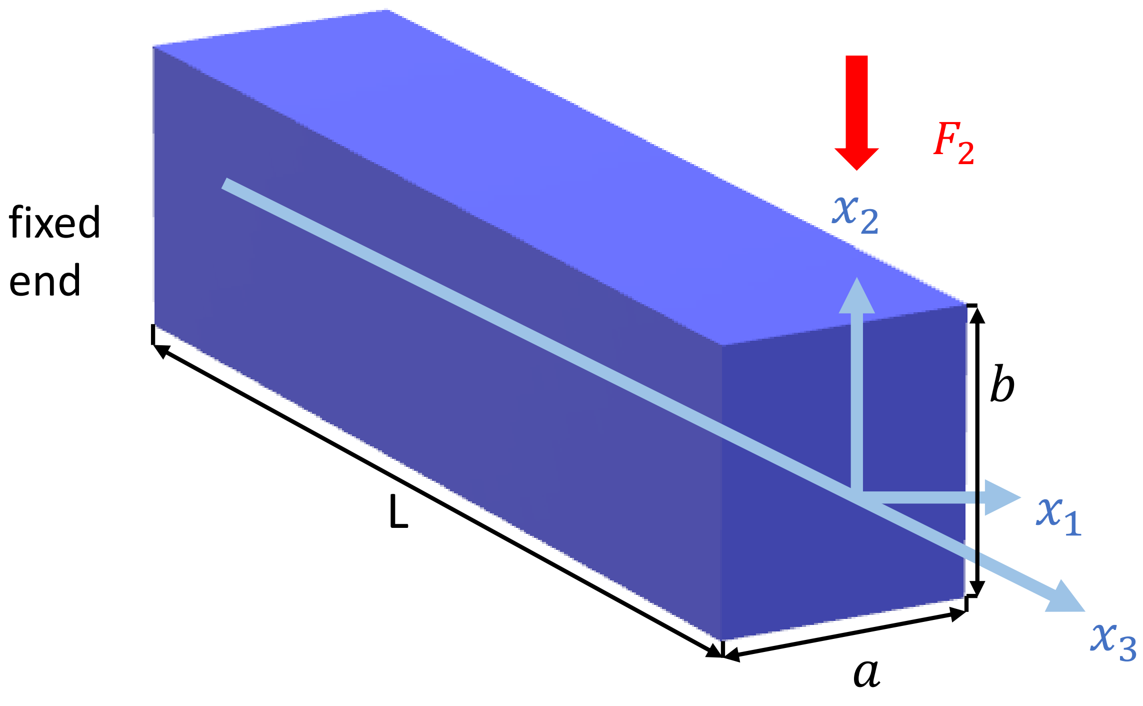
4. Elasticity to continuum mechanics material & structural models of flow elastoplastity.

Acknowledgment: H.-H. Huang, Three-dimensional Saint-Venant flexure-torsional and warping. MS Thesis, Civil Engrg. Dept., National Taiwan University, 2023.
National Science and Technology Council of Taiwan under Grant MOST 111-2221-E-002-055-MY2.

THANKS

Methodology-different load types

Methodology-different loading types



GE $\quad -\nabla^2 S = [\frac{vE}{1+v}\kappa_1' x_1 + \frac{df_2}{dx_1}] + [\frac{vE}{1+v}\kappa_2' x_2 + \frac{df_1}{dx_2}] + 2G\phi_3'$

BC $\quad \frac{dS}{ds} = \frac{dx_1}{ds}[-E\kappa_1' \frac{x_2^2}{2} + f_2(x_1)] - \frac{dx_2}{ds}[E\kappa_2' \frac{x_1^2}{2} + f_1(x_2)]$

$$\kappa_1' = \frac{c_1}{E} = \frac{M_1'}{EI_{11}} = \frac{F_2}{I_{11}}$$

When $x_1 = \frac{a}{2}$,

$$\frac{\partial S}{\partial s} = 0, \quad -E\kappa_1' \frac{x_2^2}{2} + f_2(x_1) = 0, \qquad f_2(x_1) = \frac{Fb^2}{8I_{11}}$$

$$\begin{cases} -\nabla^2 S_1 = \left[\frac{vE}{1+v} + \frac{F_2 x_1}{I_{11}}\right] \\ S_1 = 0 \end{cases}$$

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Methodology-F_2

When $x_1 = \frac{a}{2}$, $\begin{cases} -\nabla^2 S_1 = \left[\frac{vE}{1+v} + \frac{F_2 x_1}{I_{11}}\right] \\ S_1 = 0 \end{cases}$

$A_3 = \frac{v}{1+v}\frac{6}{I_{11}}\frac{a^2}{4}$, A_4=0

$H = S - P$

$P = A_1 x_1^3 + A_2 x_1^2 + A_3 x_1 + A_4$

$A_1 = \frac{v}{1+v}\frac{F_2}{6I_{11}}$, $A_2 = 0$

$H = S - P = S - \frac{v}{1+v}\frac{F_2}{6I_{11}} x_1^3 + A_3 x_1 + A_4$

$$\begin{cases} -\nabla^2 H_1 = 0 \\ H_1 = \frac{v}{1+v}\frac{F_2}{6I_{11}} x_1^3 + A_3 x_1 + A_4 \end{cases}$$

$$\begin{cases} \nabla^2 H_1 = 0 & (R) \\ H_1 = 0 & for\ x_1 = \pm\frac{a}{2} \\ H = \frac{v}{1+v}\frac{F_2}{6I_{11}}(x_1^3 - \frac{a^2}{4}x_1) & for\ x_2 = \pm\frac{b}{2} \end{cases}$$

$H(x_1, x_2) = \sum_{n=1}^{\infty} X(x_1)Y(x_2)$

$$= \sum_{n=1}^{\infty} B_n \sin\left(\frac{2n\pi x_1}{a}\right)\cosh\left(\frac{2n\pi x_2}{a}\right)$$

Methodology-$\boldsymbol{F_2}$

$$B_n = \frac{v}{1+v}\frac{F_2}{6I_{11}}\frac{(-1)^n}{\cosh\left(\frac{n\pi b}{a}\right)}\frac{3a^3}{2(n\pi)^3}$$

$$S_1 = \frac{v}{1+v}\frac{F_2}{6I_{11}}\left(\frac{a^2}{4}x_1 - x_1^3 + \sum_{n=1}^{\infty}\frac{3a^3}{2(\mathrm{n}\pi)^3}\frac{(-1)^n}{\cosh\left(\frac{n\pi b}{a}\right)}\sin\left(\frac{2n\pi x_1}{a}\right)\cosh\left(\frac{2n\pi x_2}{a}\right)\right)$$

$$S_2 = \frac{v}{1+v}\frac{F_1}{6I_{22}}\left(\frac{b^2}{4}x_2 - x_2^3 + \sum_{n=1}^{\infty}\frac{3b^3}{2(\mathrm{n}\pi)^3}\frac{(-1)^n}{\cosh\left(\frac{n\pi a}{b}\right)}\sin\left(\frac{2n\pi x_2}{b}\right)\cosh\left(\frac{2n\pi x_1}{b}\right)\right)$$

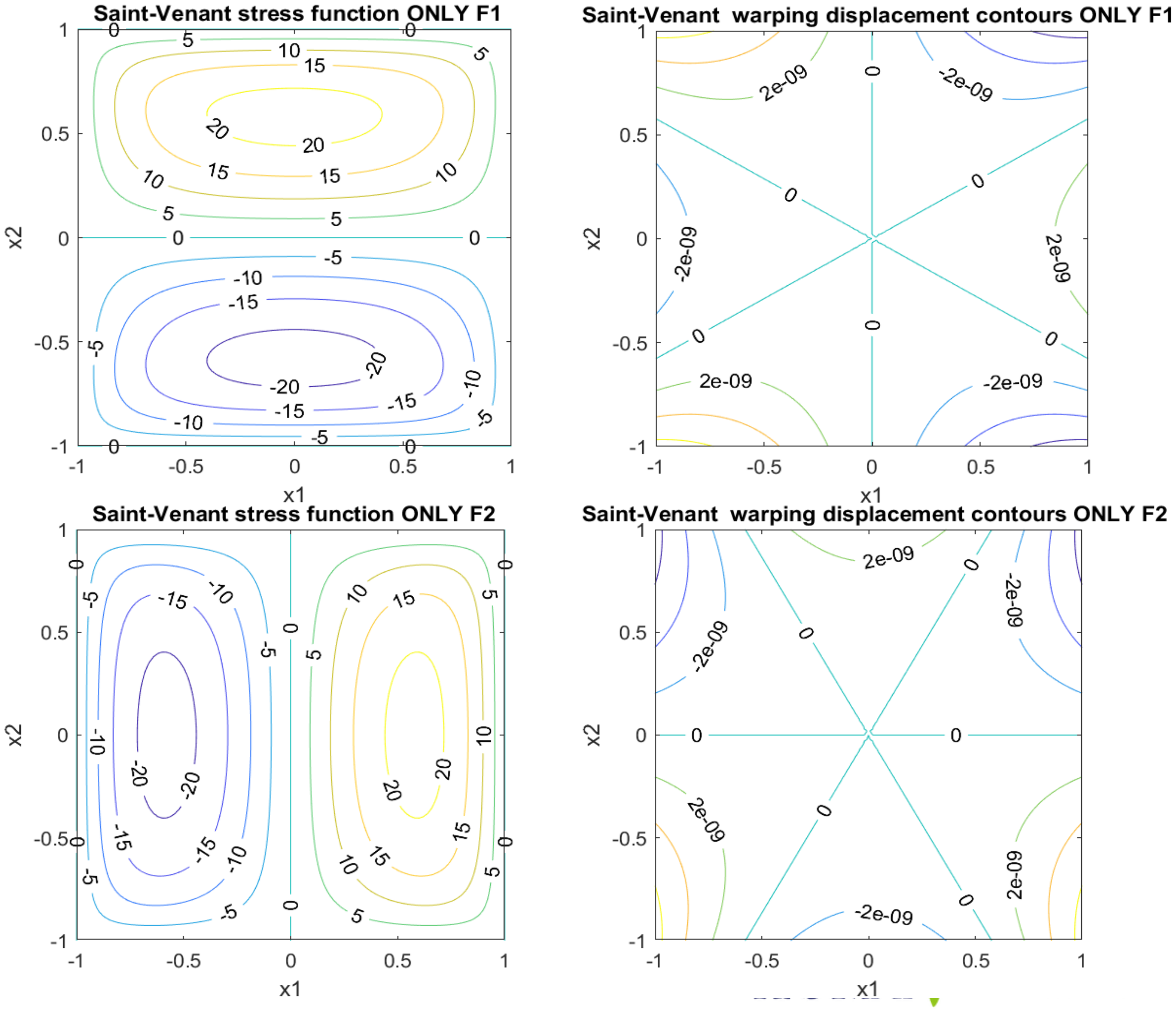
Comparison-F_1&F_2

Comparison-F_1&F_2

$$a = 2\ m, \qquad b = 2\ m$$

$$F_1 = 2000\ N, \quad F_2 = 2000\ N$$

$$xp_1 = 0, xp_2 = 0$$

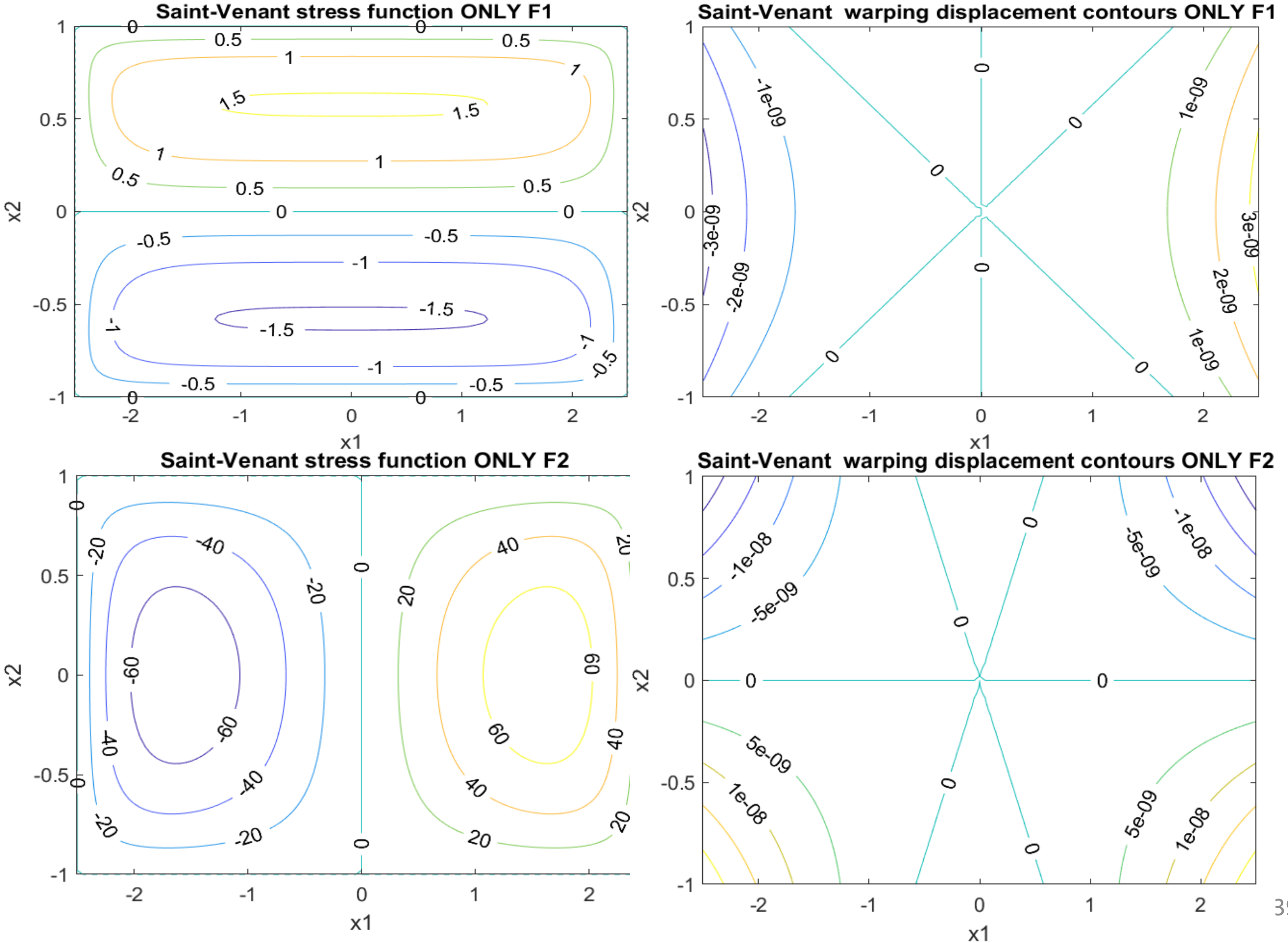


Comparison-F_1&F_2

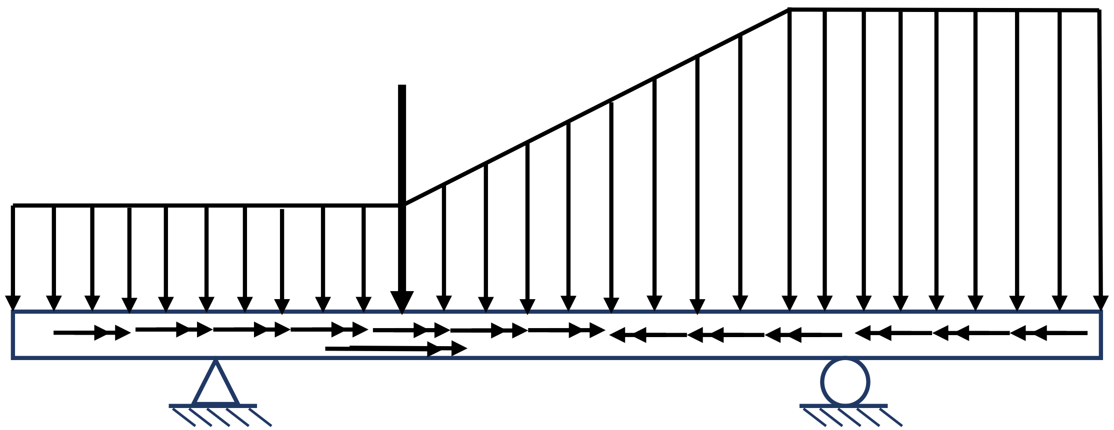
$a = 5\ m,\quad b = 2\ m$

$F_1 = 2000\ N,\quad F_2 = 2000\ N$

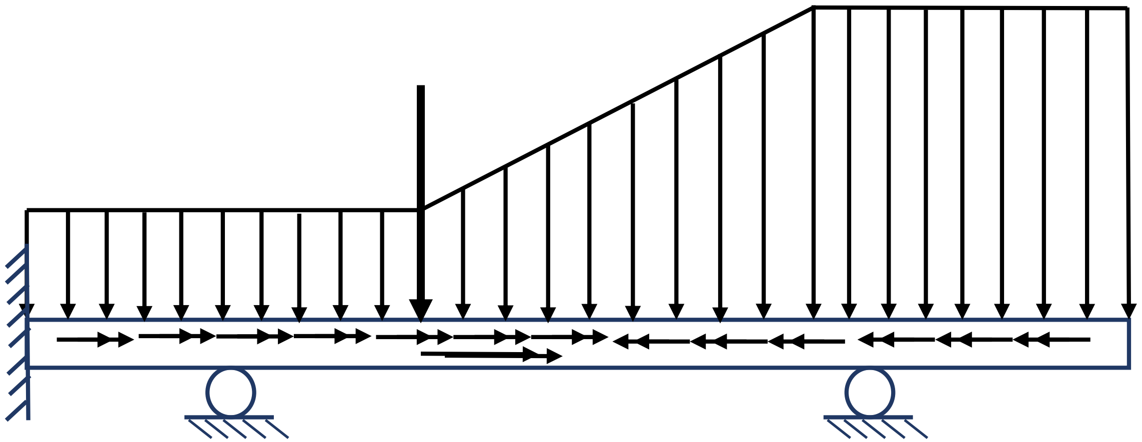
$xp_1 = 0, xp_2 = 0$



Statically determinate



Statically indeterminate

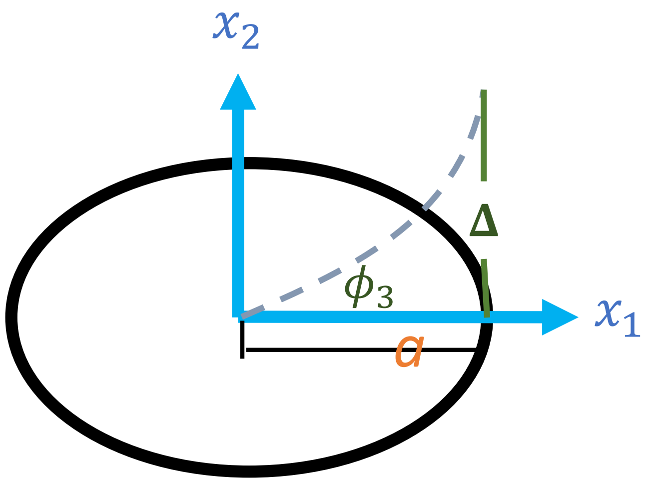


Distributed loads

Concentrated loads

Distributed torsional loads

Concentrated torsional loads



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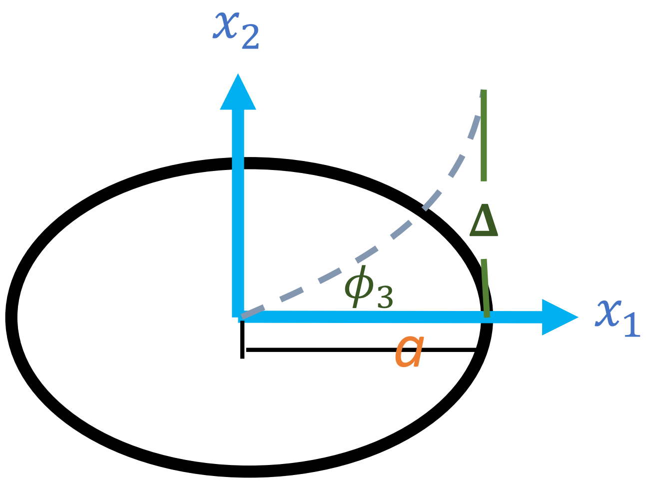


x_2
x_1
Δ
ϕ_3
a

$\boldsymbol{n_i}$

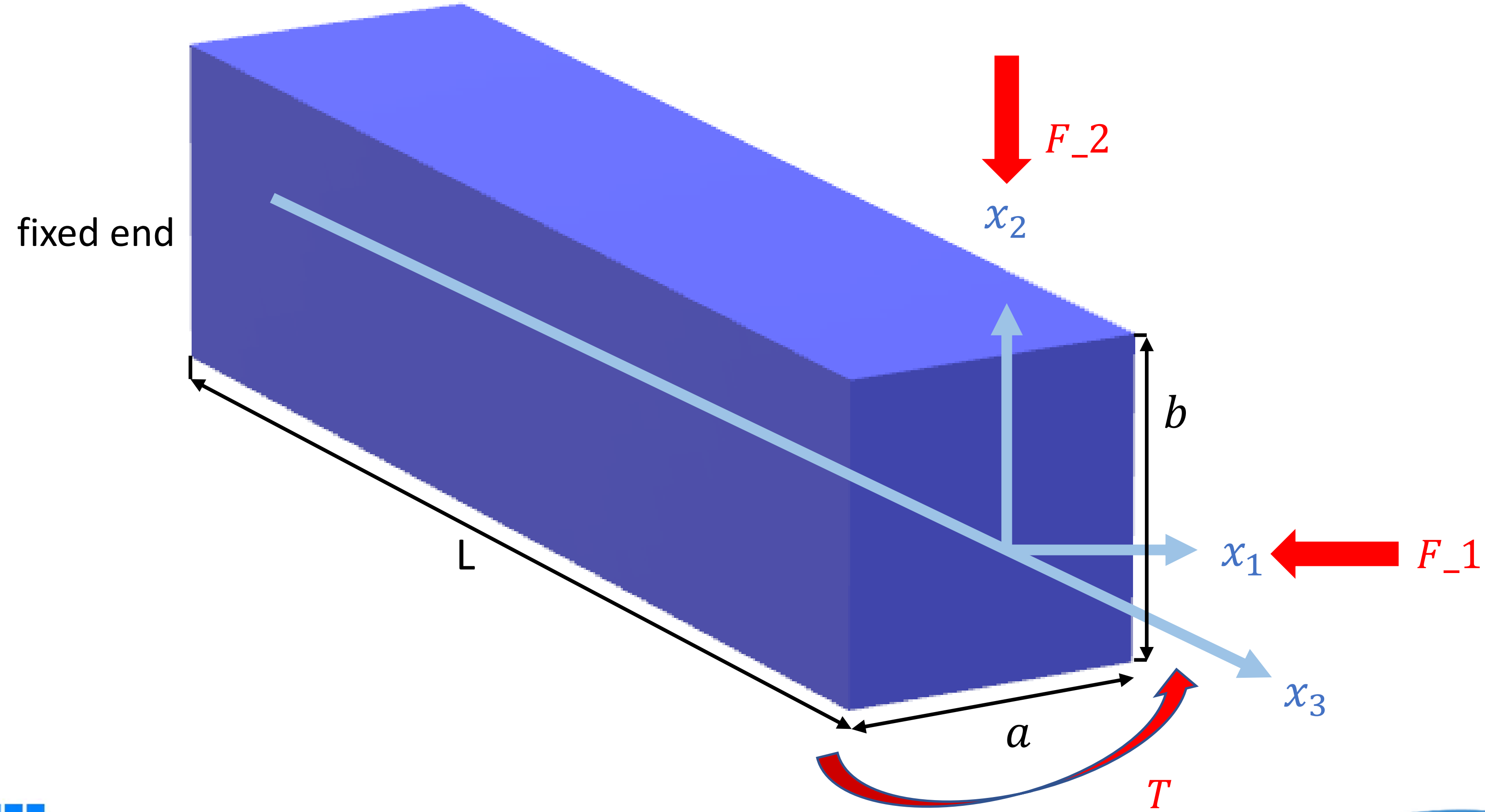
$$n_i = \begin{bmatrix} n_1 \\ n_2 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos(n, x_1) \\ \cos(n, x_2) \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{dx_1}{dn} \\ \frac{dx_1}{dn} \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{dx_2}{ds} \\ \frac{-dx_1}{ds} \\ 0 \end{bmatrix}$$

I_{ww}

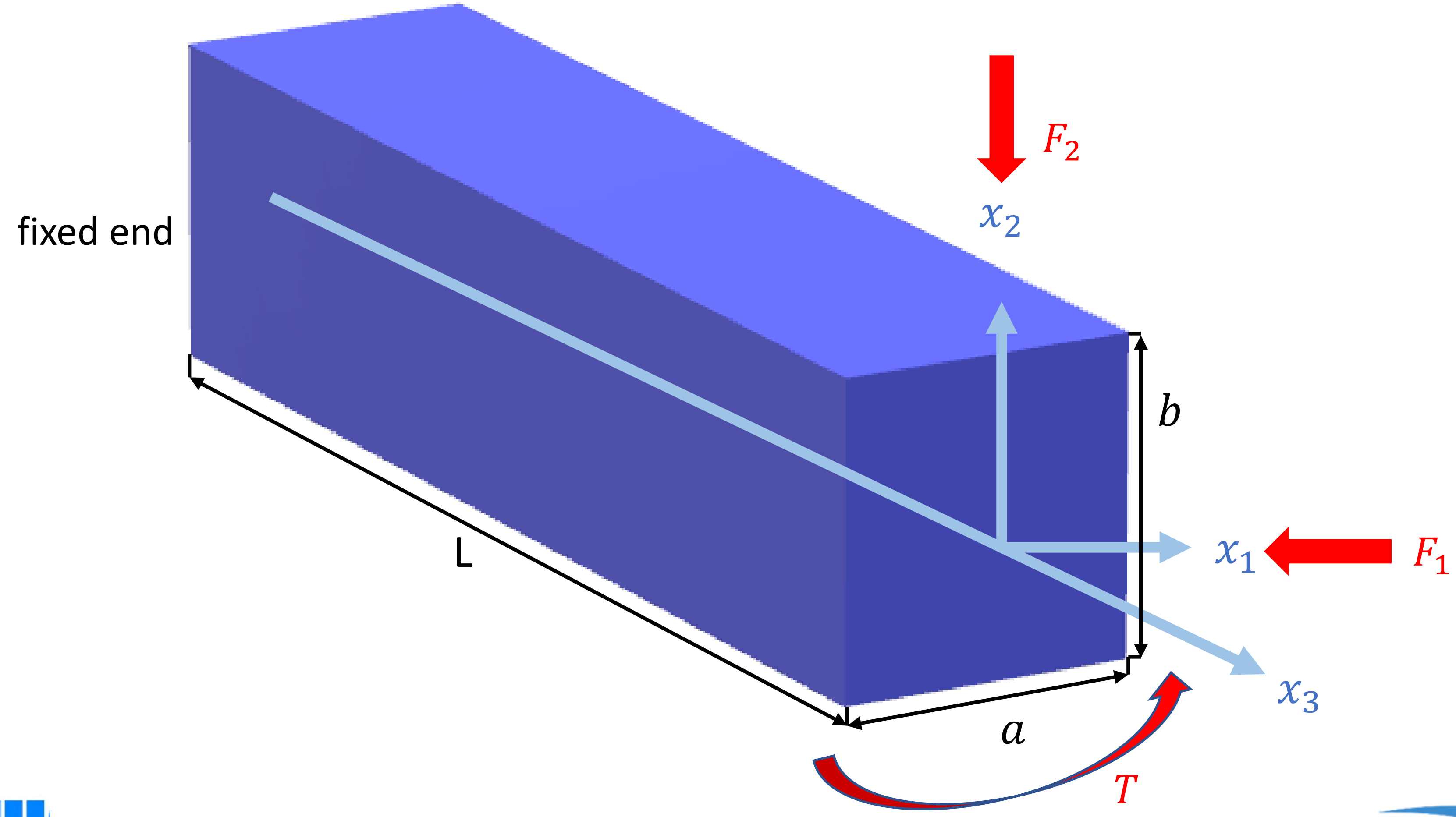
$I_{ww} = \int w^2 \, dF$: sectorial moment of inertia

w : displacement in z-direction

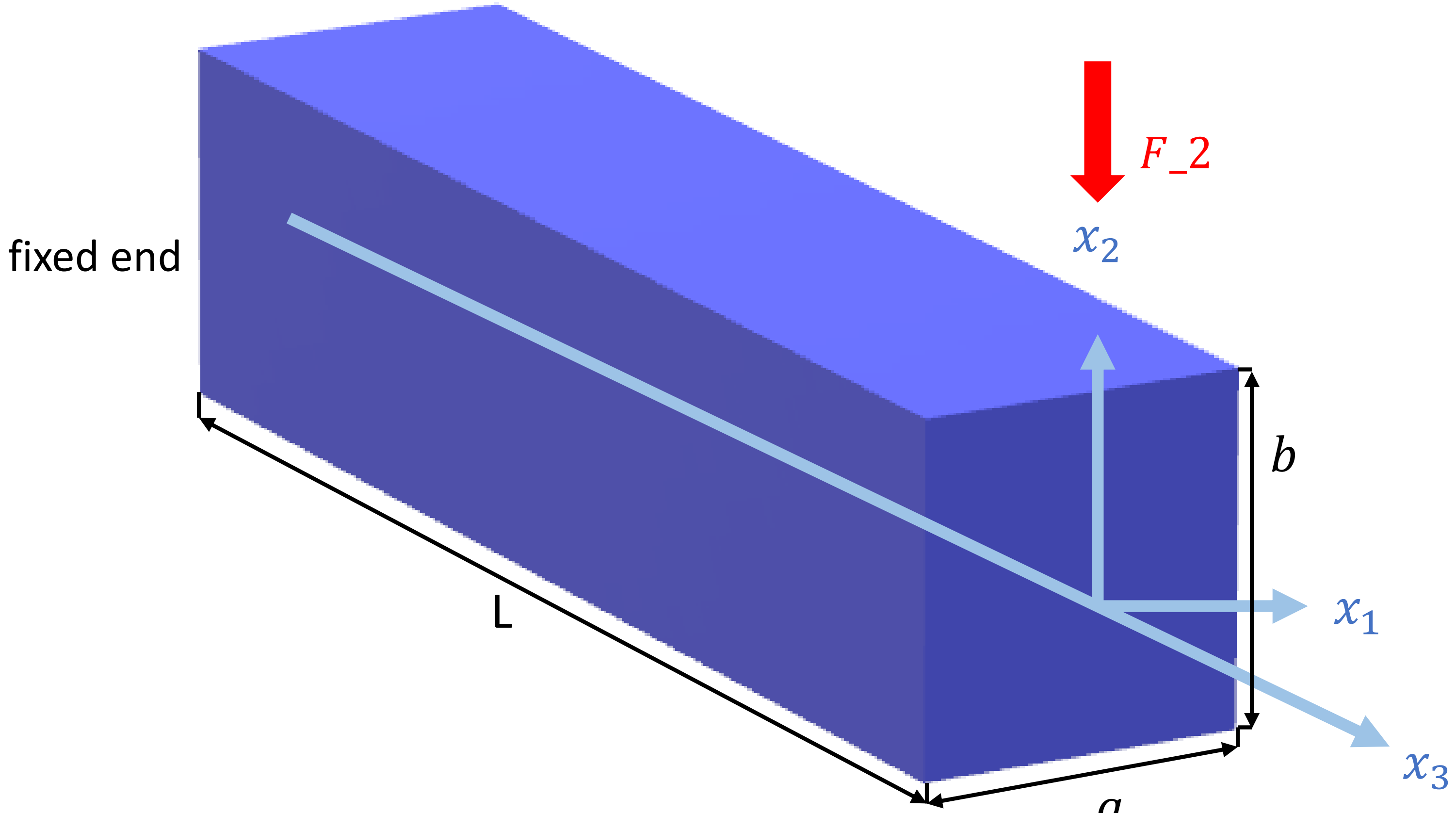
Recent research



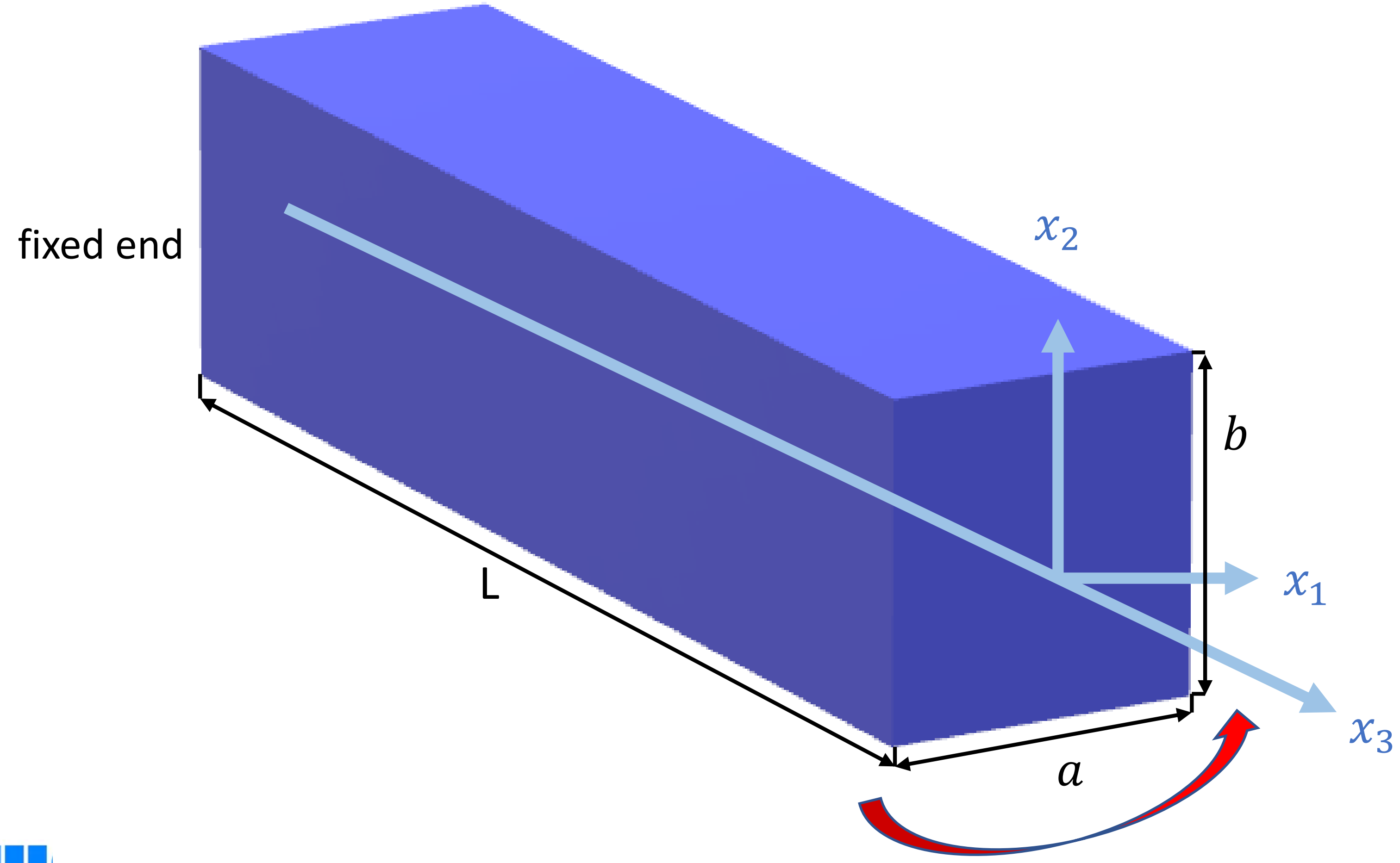
Recent research



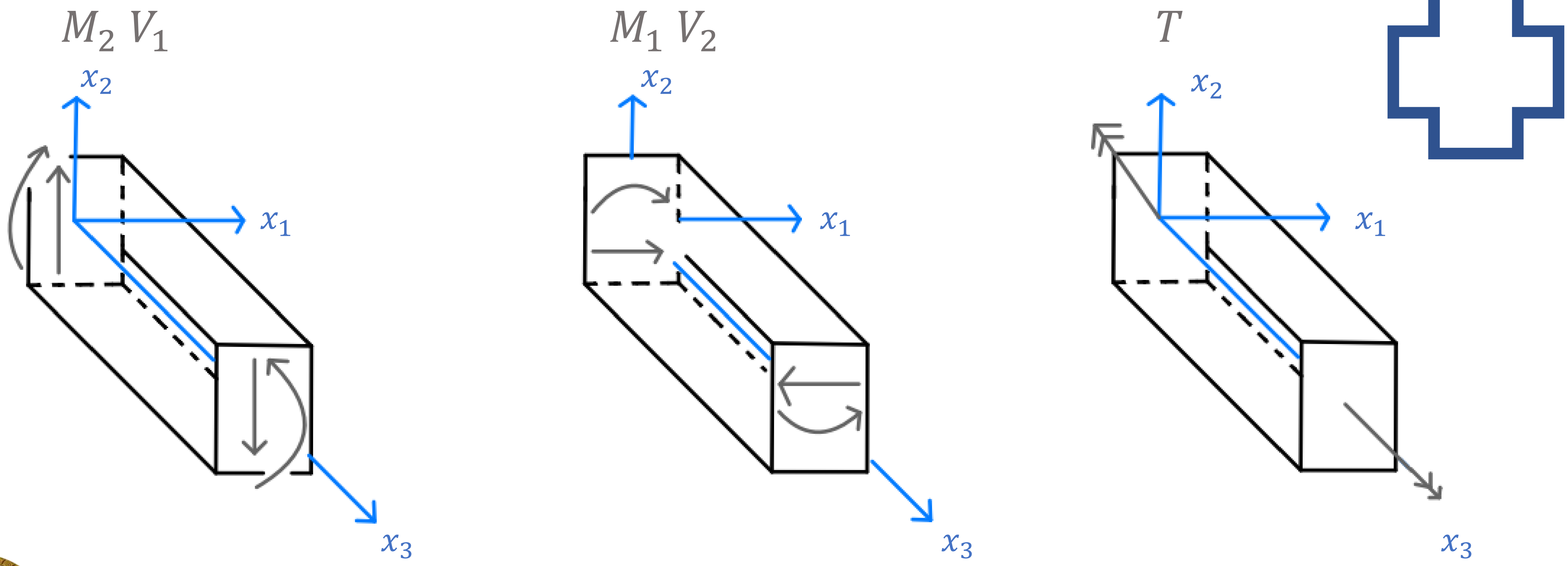
Recent research



Recent research



Notation



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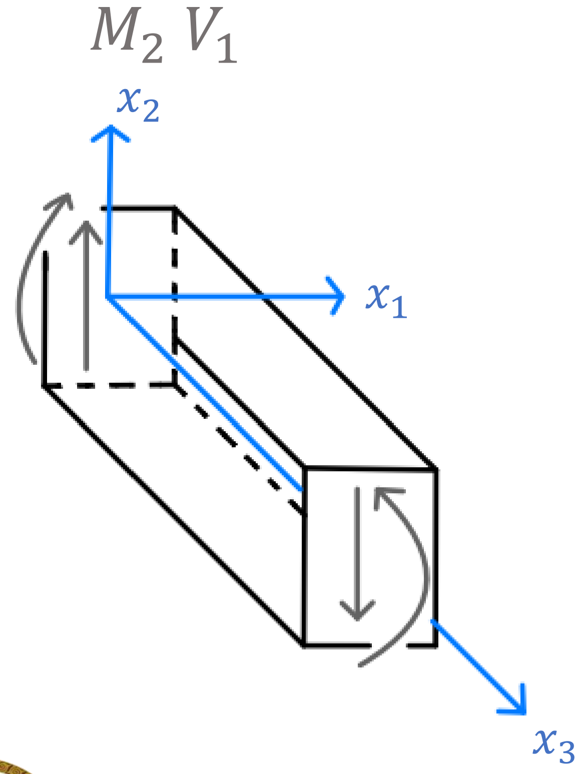

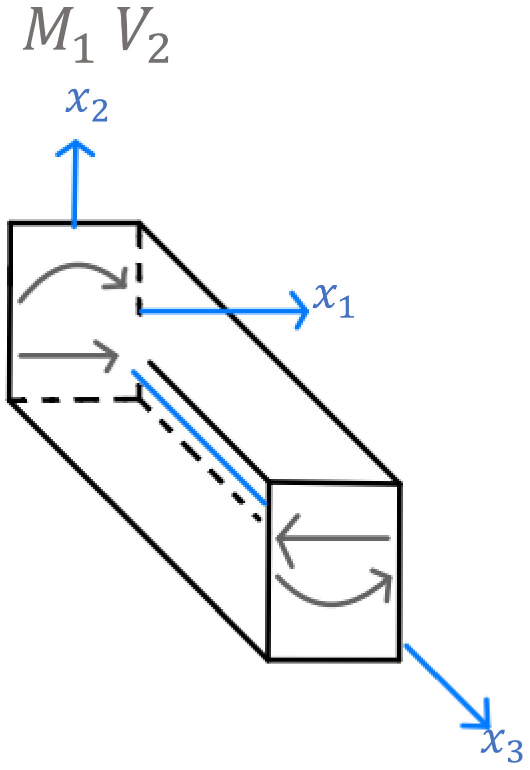

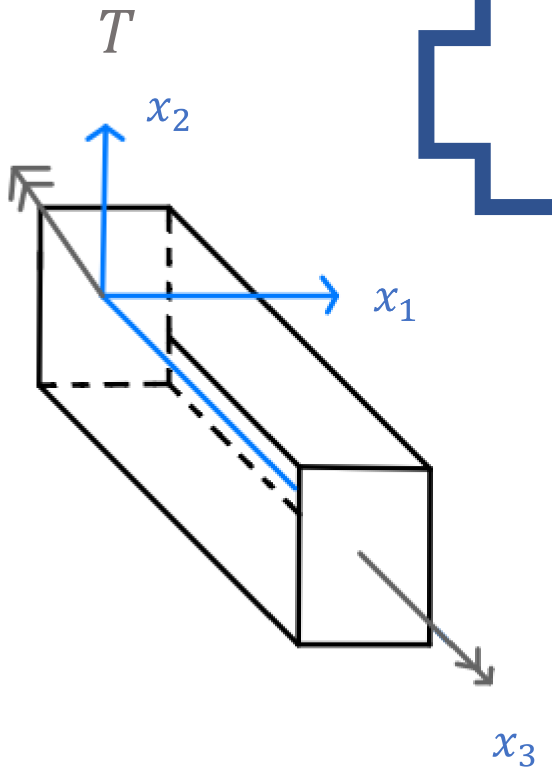

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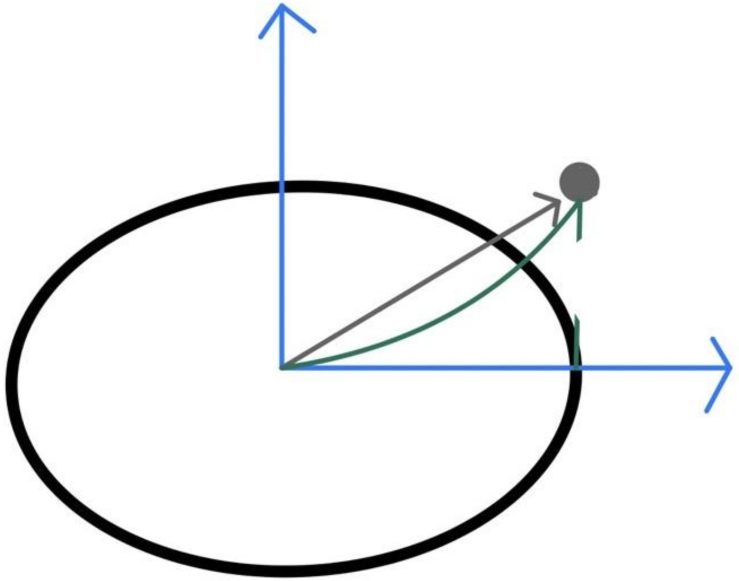
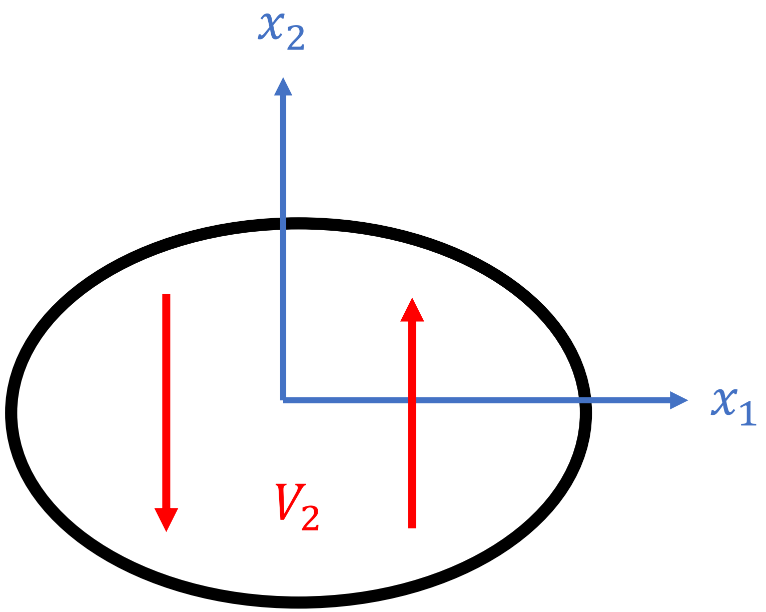
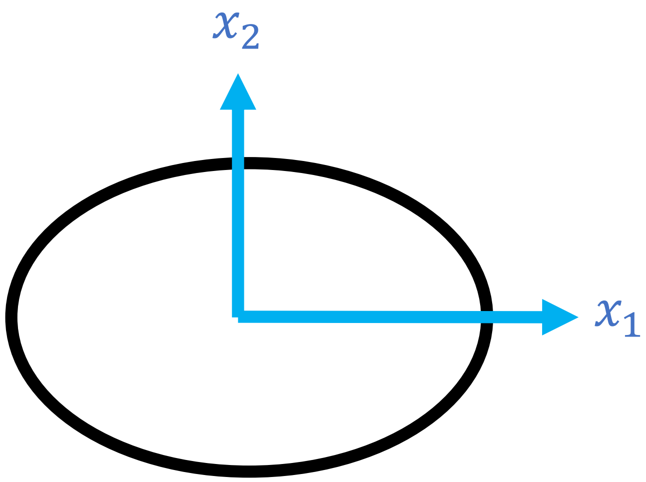



x_2
x_1
V_2

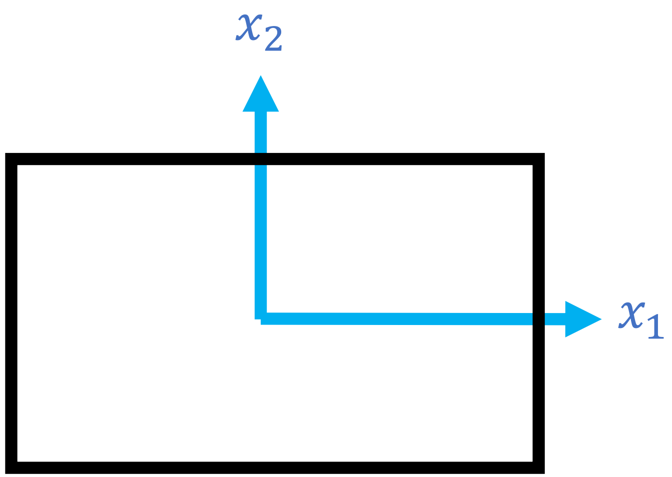
Methodology

Elliptical section



$$S = c\left(\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} - 1\right)$$

Rectangular section



$$S = ???$$

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Notation

x_1 , x_2 , x_3 : location

F_1 , F_2 , T_D : Free end concentrated load & torsion

T_s : Saint-Venant torsion

T_w :warping torsion

K: section factor

w : displacement in z-direction

ϕ : angle of twist

I: second axial moment of area

$I_{ww} = \int w^2 \, dF$: sectorial moment of inertia

f : warping factor

E: Young's modulus

G: Shear modulus of elasticity

Warping function: ellipse

$$\sigma_{13} = \frac{\partial S}{\partial x_2} = -\frac{2a^2}{a^2+b^2} G\phi_3' x_2$$

$$\sigma_{23} = \frac{\partial S}{\partial x_1} = \frac{2b^2}{a^2+b^2} G\phi_3' x_1$$

$$\sigma_{13} = G\phi_3' \left(\frac{\partial W_3}{\partial x_1} - x_2\right)$$

$$\sigma_{23} = G\phi_3' \left(\frac{\partial W_3}{\partial x_1} + x_1\right)$$

$$W_3 = \frac{-a^2+b^2}{a^2+b^2} x_1 x_2$$

Future work

1. Different boundary conditions

2. Different location of loads

3. Yield conditions

4. Thin-walled cross section & Shear center