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ELEMENTS OF VIBRATION ANALYSIS

(1986)

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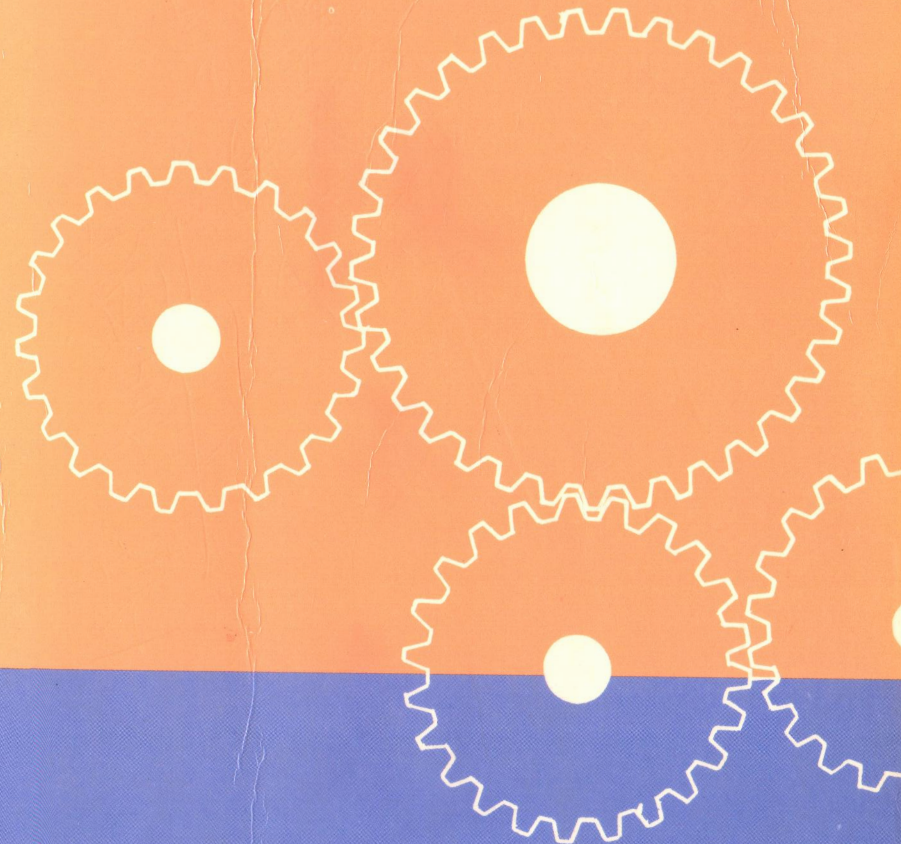
SECOND EDITION



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Mechanical Engineering Series

$$C_1 \sin\phi_1 = \frac{1}{\omega_1 \det[u]}(u_{22}v_{10} - u_{12}v_{20})$$

$$C_2 \sin\phi_2 = \frac{1}{\omega_2 \det[u]}(u_{11}v_{20} - u_{21}v_{10})$$

From Eqs. (3.57), we obtain

$$C_1 = \frac{1}{\det[u]}\sqrt{(u_{22}x_{10} - u_{12}x_{20})^2 + \frac{(u_{22}v_{10} - u_{12}v_{20})^2}{\omega_1^2}}$$

$$C_2 = \frac{1}{\det[u]}\sqrt{(u_{11}x_{20} - u_{21}x_{10})^2 + \frac{(u_{11}v_{20} - u_{21}v_{10})^2}{\omega_2^2}}$$

$$\phi_1 = \tan^{-1}\frac{u_{22}v_{10} - u_{12}v_{20}}{\omega_1(u_{22}x_{10} - u_{12}x_{20})} \qquad (3.58)$$

$$\phi_2 = \tan^{-1}\frac{u_{11}v_{20} - u_{21}v_{10}}{\omega_2(u_{11}x_{20} - u_{21}x_{10})}$$

Equations (3.55) and (3.58) define the response of a two-degree-of-freedom system to initial excitation completely. We shall see in Chap. 4 that the response can be obtained in matrix form in a more systematic way.

Example 3.3 Consider the system of Example 3.1 and obtain the response to the initial excitation $x_1(0) = x_{10} = 1.2$, $x_2(0) = x_{20} = 0$, $\dot{x}_1(0) = v_{10} = 0$, $\dot{x}_2(0) = v_{20} = 0$.

From Eqs. (*d*) of Example 3.1, we have $\omega_1 = \sqrt{k/m}$, $\omega_2 = 1.5811\sqrt{k/m}$. Moreover, choosing arbitrarily $u_{11} = 1$, $u_{12} = 1$ in Example 3.1, we obtained the modal matrix

Ref: Collection of problems in Classical mechanics by G. L. Kotkin and V. G. Serbo p.27–p.28

$$[u] = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -0.5 \end{bmatrix} \qquad (a)$$

which has the determinant

$$\det[u] = \begin{vmatrix} 1 & 1 \\ 1 & -0.5 \end{vmatrix} = -0.5 - 1 = -1.5 \qquad (b)$$

spring

Inserting the initial conditions listed above and the values given by Eqs. (*a*) and (*b*) into Eqs. (3.58), we obtain

$$C_1 = \frac{u_{22}x_{10}}{\det[u]} = \frac{-0.5 \times 1.2}{-1.5} = 0.4$$

$$C_2 = \frac{-u_{21}x_{10}}{\det[u]} = \frac{-1.2}{-1.5} = 0.8 \qquad (c)$$

$$\phi_1 = \phi_2 = 0$$

$m l \ddot{\theta} = -mg\theta$

$\ddot{\theta} + \frac{g}{l}\theta = 0$

$m\ddot{x} + Kx = 0$

$\theta_1 = \delta\theta \sin\omega_1 t$ $\quad \dot{\theta}_1 = \delta\theta\, \omega_1 \cos\omega_1 t$

K.E. $\frac{1}{2} m(\delta\theta)^2 \omega_1^2 L^2$ $\quad$ Diff. $\quad$ P.E. $\frac{1}{2} m(\delta\theta)^2 g L$

Hence, introducing Eqs. (*c*) into Eq. (3.55), we obtain the response

$$\begin{Bmatrix} x_1(t) \\ x_2(t) \end{Bmatrix} = 0.4\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}\cos\sqrt{\frac{k}{m}}\,t + 0.8\begin{Bmatrix} 1 \\ -0.5 \end{Bmatrix}\cos 1.5811\sqrt{\frac{k}{m}}\,t \qquad (d)$$

It must be pointed out that the arbitrary choice $u_{11} = 1$, $u_{12} = 1$ did not affect the final outcome. Indeed, any other choice would have resulted in such values for C_1 and C_2 as to keep Eq. (*d*) unchanged.

$\frac{1}{2}\phi_1^T M \phi_1 + \frac{1}{2}\phi_1^T K \phi_1$

3.7 BEAT PHENOMENON

$= \delta\theta^2 mL^2 + \boxed{mgL(\delta\theta)^2}$

A very interesting phenomenon is encountered when the natural frequencies of a two-degree-of-freedom system are very close in value. To illustrate the phenomenon, let us consider two identical pendulums connected by a spring, as shown in Fig. 3.7*a*. The corresponding free-body diagrams are shown in Fig. 3.7*b*, in which the assumption of small angles θ_1 and θ_2 is implied. The moment equations about the points O and O', respectively, yield the differential equations of motion

$$mL^2\ddot{\theta}_1 + mgL\theta_1 + ka^2(\theta_1 - \theta_2) = 0$$
$$mL^2\ddot{\theta}_2 + mgL\theta_2 - ka^2(\theta_1 - \theta_2) = 0 \qquad (3.59)$$

which can be arranged in the matrix form

$$\begin{bmatrix} mL_2 & 0 \\ 0 & mL^2 \end{bmatrix}\begin{Bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{Bmatrix} + \begin{bmatrix} mgL + ka^2 & -ka^2 \\ -ka^2 & mgL + ka^2 \end{bmatrix}\begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \qquad (3.60)$$

indicating that the system is coupled elastically. As expected, when the spring stiffness k reduces to zero the coupling disappears and the two pendulums reduce to independent simple pendulums with identical natural frequencies equal to $\sqrt{g/L}$. For $k \neq 0$, Eq. (3.60) yields the eigenvalue problem

$$-\omega^2\begin{bmatrix} mL^2 & 0 \\ 0 & mL^2 \end{bmatrix}\begin{Bmatrix} \Theta_1 \\ \Theta_2 \end{Bmatrix} + \begin{bmatrix} mgL + ka^2 & -ka^2 \\ -ka^2 & mgL + ka^2 \end{bmatrix}\begin{Bmatrix} \Theta_1 \\ \Theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \qquad (3.61)$$

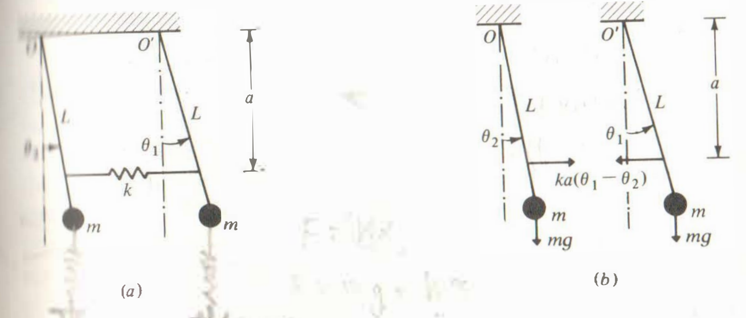


Figure 3.7

leading to the characteristic equation

$$\det\begin{bmatrix} mgL + ka^2 - \omega^2 mL^2 & -ka^2 \\ -ka^2 & mgL + ka^2 - \omega^2 mL^2 \end{bmatrix}$$
$$= (mgL + ka^2 - \omega^2 mL^2)^2 - (ka^2)^2 = 0 \quad (3.62)$$

which is equivalent to

$$mgL + ka^2 - \omega^2 mL^2 = \pm ka^2 \quad (3.63)$$

Hence, the two natural frequencies are

$$\omega_1 = \sqrt{\frac{g}{L}} \qquad \omega_2 = \sqrt{\frac{g}{L} + 2\frac{k}{m}\frac{a^2}{L^2}} \quad (3.64)$$

The natural modes are obtained from the equations

$$-\omega_i^2\begin{bmatrix} mL^2 & 0 \\ 0 & mL^2 \end{bmatrix}\begin{Bmatrix}\Theta_1 \\ \Theta_2\end{Bmatrix}_i + \begin{bmatrix} mgL + ka^2 & -ka^2 \\ -ka & mgL + ka^2 \end{bmatrix}\begin{Bmatrix}\Theta_1 \\ \Theta_2\end{Bmatrix}_i = \begin{Bmatrix}0 \\ 0\end{Bmatrix} \qquad i = 1, 2 \quad (3.65)$$

Inserting $\omega_1^2 = g/L$ and $\omega_2^2 = g/L + 2(k/m)(a^2/L^2)$ into Eqs. (3.65), and solving for the ratios Θ_{21}/Θ_{11} and Θ_{22}/Θ_{12}, we obtain

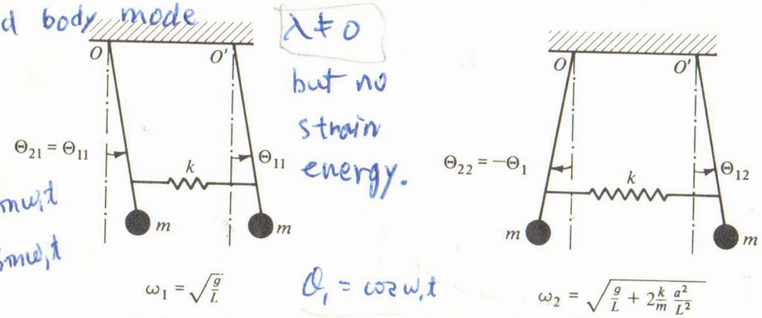
$$\frac{\Theta_{21}}{\Theta_{11}} = 1 \qquad \frac{\Theta_{22}}{\Theta_{12}} = -1 \quad (3.66)$$

so that in the first natural mode the two pendulums move like a single pendulum with the spring k unstretched, which can also be concluded from the fact that the first natural frequency of the system is that of the simple pendulum, $\omega_1 = \sqrt{g/L}$. On the other hand, in the second natural mode the two pendulums are 180° out of phase. The two modes are shown in Fig. 3.8.

As was pointed out in Sec. 3.5, the general motion of the system can be expressed as a superposition of the two natural modes multiplied by the associated

rigid body behavior Question
rigid body mode

$\lambda \neq 0$ but no strain energy.



$\theta_1 = \sin\omega_1 t$
$\theta_2 = \sin\omega_1 t$

Figure 3.8 (fixed) $\theta_1 = \cos\omega_1 t$ $\theta_2 = \cos\omega_1 t$

natural coordinates, or

$$\begin{Bmatrix}\theta_1(t) \\ \theta_2(t)\end{Bmatrix} = C_1\begin{Bmatrix}\Theta_1 \\ \Theta_2\end{Bmatrix}_1 \cos(\omega_1 t - \phi_1) + C_2\begin{Bmatrix}\Theta_1 \\ \Theta_2\end{Bmatrix}_2 \cos(\omega_2 t - \phi_2) \quad (3.67)$$

Choosing $\Theta_{11} = \Theta_{12} = 1$ and using Eqs. (3.66), Eqs. (3.67) can be rewritten in the scalar form

$$\begin{aligned} \theta_1(t) &= C_1\cos(\omega_1 t - \phi_1) + C_2\cos(\omega_2 t - \phi_2) \\ \theta_2(t) &= C_1\cos(\omega_1 t - \phi_1) - C_2\cos(\omega_2 t - \phi_2) \end{aligned} \quad (3.68)$$

Letting the initial conditions be $\theta_1(0) = \theta_0$, $\theta_2(0) = \dot\theta_1(0) = \dot\theta_2(0) = 0$, Eqs. (3.68) become

inertia

no external force

$$\begin{aligned} \theta_1(t) &= \tfrac{1}{2}\theta_0\cos\omega_1 t + \tfrac{1}{2}\theta_0\cos\omega_2 t \\ &= \theta_0\cos\frac{\omega_2 - \omega_1}{2}t\cos\frac{\omega_2 + \omega_1}{2}t \\ \theta_2(t) &= \tfrac{1}{2}\theta_0\cos\omega_1 t - \tfrac{1}{2}\theta_0\cos\omega_2 t \\ &= \theta_0\sin\frac{\omega_2 - \omega_1}{2}t\sin\frac{\omega_2 + \omega_1}{2}t \end{aligned} \quad (3.69)$$

Note that, in deriving Eqs. (3.69), we used the trigonometric relations $\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$, in which $\alpha = (\omega_2 - \omega_1)t/2$, $\beta = (\omega_2 + \omega_1)t/2$.

Next let us consider the case in which ka^2 is very small in value compared with mgL. Examining Eq. (3.60), we conclude that this statement is equivalent to saying that the coupling provided by the spring k is very weak. In this case, Eqs. (3.69) can be written in the form

$k \ll 0$

$$\begin{aligned} \theta_1(t) &\cong \theta_0\cos\tfrac{1}{2}\omega_B t\cos\omega_{ave}t \\ \theta_2(t) &\cong \theta_0\sin\tfrac{1}{2}\omega_B t\sin\omega_{ave}t \end{aligned} \quad (3.70)$$

where $\omega_B/2$ and ω_{ave} are approximated by

$$\frac{\omega_B}{2} = \frac{\omega_2 - \omega_1}{2} \cong \frac{1}{2}\frac{k}{m}\frac{a^2}{\sqrt{gL^3}} \qquad \omega_{ave} = \frac{\omega_2 + \omega_1}{2} \cong \sqrt{\frac{g}{L}} + \frac{1}{2}\frac{k}{m}\frac{a^2}{\sqrt{gL^3}} \quad (3.71)$$

Hence, $\theta_1(t)$ and $\theta_2(t)$ can be regarded as being harmonic functions with frequency ω_{ave} and with amplitudes varying slowly according to $\theta_0\cos\frac{1}{2}\omega_B t$ and $\theta_0\sin\frac{1}{2}\omega_B t$, respectively. The plots $\theta_1(t)$ versus t and $\theta_2(t)$ versus t are shown in Fig. 3.9, with the slowly varying amplitudes indicated by the dashed-line envelopes. Geometrically, Fig. 3.9*a* (or Fig. 3.9*b*) implies that if two harmonic functions possessing equal amplitudes and nearly equal frequencies are added, then the resulting function is an *amplitude-modulated* harmonic function with a frequency equal to the average frequency. At first, when the two harmonic waves reinforce each other, the amplitude is doubled, and later, as the two waves cancel each other

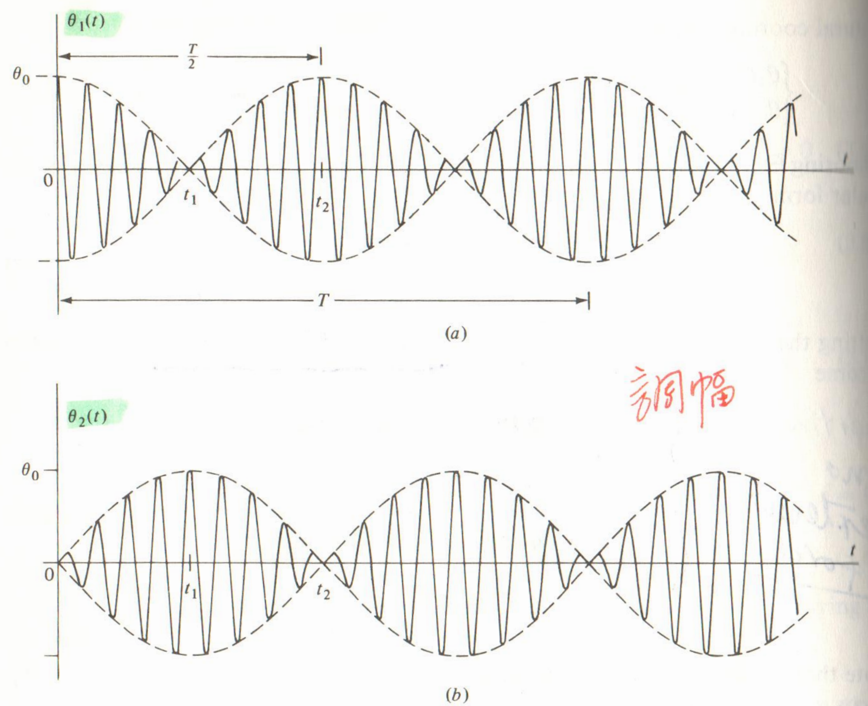


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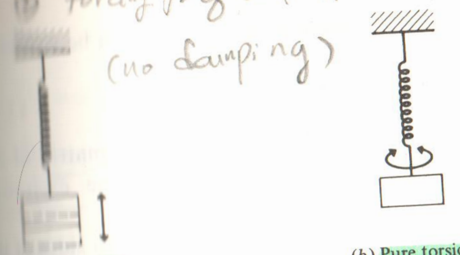
Figure 3.9

the amplitude reduces to zero. The phenomenon is known as the *beat phenomenon*, and the frequency of modulation ω_B, which in this particular case is equal to $ka^2/m\sqrt{gL^3}$, is called the *beat frequency*. From Fig. 3.9*a*, we conclude that the time between two maxima is $T/2 = 2\pi/\omega_B$, whereas the period of the amplitude-modulated envelope is $T = 4\pi/\omega_B$.

Although in our particular case the beat phenomenon resulted from the weak coupling of two pendulums, the phenomenon is not exclusively associated with two-degree-of-freedom systems. Indeed, the beat phenomenon is purely the result of adding two harmonic functions of equal amplitudes and nearly equal frequencies. For example, the phenomenon occurs in twin-engine propeller airplanes, in which the propeller noise grows and diminishes in intensity as the sound waves generated by the two propellers reinforce and cancel each other in turn.

We observe from Fig. 3.9 that there is a 90° phase angle between $\theta_1(t)$ and $\theta_2(t)$. At $t = 0$ the first pendulum (right pendulum in Fig. 3.7*a*) begins to swing with the amplitude θ_0 while the second pendulum is at rest. Soon thereafter the second pendulum is entrained, gaining amplitude while the amplitude of the first decreases. At $t_1 = \pi/\omega_B$ the amplitude of the first pendulum becomes zero, whereas the amplitude of the second pendulum reaches θ_0. At $t_2 = 2\pi/\omega_B$ the amplitude of the

Beat 发生的两种可能：
(1) internal system. (no damping)
(2) forcing freq $\omega \fallingdotseq \omega_n$ 時。(no damping)
→ 注意两者要产生的条件特别需同型
I.C. 條件如下型式
$\underline{A}(\sin \omega_1 t - \sin \omega_2 t)$



(a) Pure translation (b) Pure torsion **Figure 3.10**

first pendulum reaches θ_0 once again and that of the second pendulum reduces to zero. The motion keeps repeating itself, so that every interval of time $T/4 = \pi/\omega_B$ there is a complete transfer of energy from one pendulum to the other.

Another example of a system exhibiting the beat phenomenon is the "Wilberforce spring", consisting of a mass of finite dimensions suspended by a helical spring such that the frequency of vertical translation and the frequency of torsional motion are very close in value. In this case, the kinetic energy changes from pure translational in the vertical direction to pure rotational about the vertical axis, as shown in Fig. 3.10.

3.8 RESPONSE OF A TWO-DEGREE-OF-FREEDOM SYSTEM TO HARMONIC EXCITATION

Let us return to the damped system of Sec. 3.2 and write Eq. (3.5) in the expanded form

$$\begin{aligned} m_{11}\ddot{x}_1 + m_{12}\ddot{x}_2 + c_{11}\dot{x}_1 + c_{12}\dot{x}_2 + k_{11}x_1 + k_{12}x_2 &= F_1(t) \\ m_{12}\ddot{x}_1 + m_{22}\ddot{x}_2 + c_{12}\dot{x}_1 + c_{22}\dot{x}_2 + k_{12}x_1 + k_{22}x_2 &= F_2(t) \end{aligned} \tag{3.72}$$

where the diagonal mass matrix has been replaced by a more general nondiagonal but symmetric matrix. Next, let us consider the following harmonic excitation:

$$F_1(t) = F_1 e^{i\omega t} \qquad F_2(t) = F_2 e^{i\omega t} \tag{3.73}$$

and write the steady-state response as

$$x_1(t) = X_1 e^{i\omega t} \qquad x_2(t) = X_2 e^{i\omega t} \tag{3.74}$$

where X_1 and X_2 are in general complex quantities depending on the driving frequency ω and the system parameters. Inserting Eqs. (3.73) and (3.74) into (3.72), we obtain the two algebraic equations

$$\begin{aligned} (-\omega^2 m_{11} + i\omega c_{11} + k_{11})X_1 + (-\omega^2 m_{12} + i\omega c_{12} + k_{12})X_2 &= F_1 \\ (-\omega^2 m_{12} + i\omega c_{12} + k_{12})X_1 + (-\omega^2 m_{22} + i\omega c_{22} + k_{22})X_2 &= F_2 \end{aligned} \tag{3.75}$$

Introducing the notation

$$Z_{ij}(\omega) = -\omega^2 m_{ij} + i\omega c_{ij} + k_{ij} \qquad i, j = 1, 2 \tag{3.76}$$