ELEMENTARY APPLIED PARTIAL DIFFERENTIAL EQUATIONS:

WITH FOURIER SERIES AND BOUNDARY VALUE PROBLEMS,

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nonhomogeneous differential equations, such as the methods of undetermined coefficients, variation of parameters, or eigenfunction expansion (using sin nx).

As a more nontrivial example, we consider

$$\frac{d^2u}{dx^2} + \left(\frac{\pi}{L}\right)^2 u = \beta + x \quad \text{with} \quad u(0) = 0 \quad \text{and} \quad u(L) = 0.$$

Since $u = \sin \pi x/L$ is a solution of the homogeneous problem, the nonhomogeneous problem only has a solution if the right-hand side is orthogonal to $\sin \pi x/L$:

$$0 = \int_0^L (\beta + x) \sin \frac{\pi x}{L}\, dx.$$

This can be used to determine the only value of β for which there is a solution:

$$\beta = \frac{-\int_0^L x \sin \pi x/L\, dx}{\int_0^L \sin \pi x/L\, dx} = \frac{L}{2}.$$

However, again the Fredholm alternative cannot be used to actually obtain the solution, $u(x)$.

8.4.3 Modified Green's Functions

In this section, we will analyze

$$L(u) = f \tag{8.4.11}$$

subject to homogeneous boundary conditions when $\lambda = 0$ is an eigenvalue. If a solution to (8.4.11) exists, we will produce a particular solution of (8.4.11) by defining and constructing a modified Green's function.

If $\lambda = 0$ is not an eigenvalue, then there is a unique solution of the nonhomogeneous boundary value problem, (8.4.11), subject to homogeneous boundary conditions. In Sec. 8.3 we represented the solution using a Green's function $G(x, x_0)$ satisfying

$$L[G(x, x_0)] = \delta(x - x_0), \tag{8.4.12}$$

subject to the same homogeneous boundary conditions.

Here we analyze the case in which $\lambda = 0$ is an eigenvalue; there are nontrivial homogeneous solutions $\phi_h(x)$ of (8.4.11), $L(\phi_h) = 0$. We will assume that there are solutions of (8.4.11), that is,

$$\int_a^b f(x)\phi_h(x)\, dx = 0. \tag{8.4.13}$$

However, the Green's function defined by (8.4.12) does not exist for all x_0 since $\delta(x - x_0)$ is not orthogonal to solutions of the homogeneous problem for all x_0:

$$\int_a^b \delta(x - x_0)\phi_h(x)\, dx = \phi_h(x_0) \not\equiv 0.$$

We need to introduce a simple comparison problem which has a solution.

$\delta(x - x_0)$ is not orthogonal to $\phi_h(x)$ because it has a "component in the direction" $\phi_h(x)$. However, there is a solution for the forcing function

$$\delta(x - x_0) + c\phi_h(x),$$

if we pick c such that this function is orthogonal to $\phi_h(x)$:

$$0 = \int_a^b \phi_h(x)[\delta(x - x_0) + c\phi_h(x)]\, dx = \phi_h(x_0) + c \int_a^b \phi_h^2(x)\, dx.$$

Thus, we introduce the **modified Green's function** $G_m(x, x_0)$ which satisfies

$$\boxed{L[G_m(x, x_0)] = \delta(x - x_0) - \frac{\phi_h(x)\phi_h(x_0)}{\int_a^b \phi_h^2(x)\, dx},} \tag{8.4.14}$$

subject to the same homogeneous boundary conditions.

Since the right-hand side of (8.4.14) is orthogonal to $\phi_h(x)$, unfortunately there are an infinite number of solutions. In Exercise 8.4.9 it is shown that the modified Green's function can be chosen to be symmetric

$$\boxed{G_m(x, x_0) = G_m(x_0, x).} \tag{8.4.15}$$

If $g_m(x, x_0)$ is one symmetric modified Green's function, then the following is also a symmetric modified Green's function

$$G_m(x, x_0) = g_m(x, x_0) + \beta\phi_h(x_0)\phi_h(x)$$

for any *constant* β (independent of x and x_0). Thus, there are an infinite number of symmetric modified Green's functions. We can use any of these.

We use Green's formula to derive a representation formula for $u(x)$ using the modified Green's function. Letting $u = u(x)$ and $v = G_m(x, x_0)$, Green's formula states that

$$\int_a^b \{u(x)L[G_m(x, x_0)] - G_m(x, x_0)L[u(x)]\}\, dx = 0,$$

since both $u(x)$ and $G_m(x, x_0)$ satisfy the same homogeneous boundary conditions. The defining differential equations (8.4.11) and (8.4.14) imply that

$$\int_a^b \left\{ u(x)\left[\delta(x - x_0) - \frac{\phi_h(x)\phi_h(x_0)}{\int_a^b \phi_h^2(\bar{x})\, d\bar{x}}\right] - G_m(x, x_0)f(x)\right\} dx = 0.$$

Using the fundamental Dirac delta property (and reversing the roles of x and x_0) yields

$$u(x) = \int_a^b f(x_0)G_m(x, x_0)\, dx_0 + \frac{\phi_h(x)}{\int_a^b \phi_h^2(\bar{x})\, d\bar{x}} \int_a^b u(x_0)\phi_h(x_0)\, dx_0,$$

where the symmetry of $G_m(x, x_0)$ has also been utilized. The last expression is a multiple of the homogeneous solution, and thus a simple *particular* solution of (8.4.11) is

$$\boxed{u(x) = \int_a^b f(x_0)G_m(x, x_0)\, dx_0,} \tag{8.4.16}$$

the same form as occurs when $\lambda = 0$ is not an eigenvalue [see (8.3.31)].

Example. The simplest example of a problem with a nontrivial homogeneous solution is

$$\frac{d^2u}{dx^2} = f(x) \tag{8.4.17a}$$

$$\frac{du}{dx}(0) = 0 \qquad \text{and} \qquad \frac{du}{dx}(L) = 0. \tag{8.4.17b, c}$$

A constant is a homogeneous solution (eigenfunction corresponding to the zero eigenvalue). For a solution to exist, by the Fredholm alternative,* $\int_0^L f(x)\, dx = 0$. We assume $f(x)$ is of this type [e.g., $f(x) = x - L/2$]. The modified Green's function $G_m(x, x_0)$ satisfies

$$\frac{d^2G_m}{dx^2} = \delta(x - x_0) + c \tag{8.4.18a}$$

$$\frac{dG_m}{dx}(0) = 0 \qquad \text{and} \qquad \frac{dG_m}{dx}(L) = 0, \tag{8.4.18b, c}$$

since a constant is the eigenfunction. For there to be such a modified Green's function, the r.h.s. must be orthogonal to the homogeneous solutions:

$$\int_0^L [\delta(x - x_0) + c]\, dx = 0 \qquad \text{or} \qquad c = -\frac{1}{L}.$$

We use properties of the Dirac delta function to solve (8.4.18). For $x \neq x_0$,

$$\frac{d^2G_m}{dx^2} = -\frac{1}{L}.$$

By integration,

$$\frac{dG_m}{dx} = \begin{cases} -\dfrac{x}{L} & x < x_0 \\ -\dfrac{x}{L} + 1 & x > x_0, \end{cases} \tag{8.4.19}$$

where the constants of integration have been chosen to satisfy the boundary conditions at $x = 0$ and $x = L$. The jump condition for the derivative ($dG_m/dx|_{x_0-}^{x_0+} = 1$), obtained by integrating (8.4.18a), is already satisfied by (8.4.19).

* Physically with insulated boundaries there must be zero *net* thermal energy generated for equilibrium.

We integrate again to obtain $G_m(x, x_0)$. Assuming that $G_m(x, x_0)$ is continuous at $x = x_0$ yields

$$G_m(x, x_0) = \begin{cases} -\frac{1}{L}\frac{x^2}{2} + x_0 + c(x_0) & x < x_0 \\ -\frac{1}{L}\frac{x^2}{2} + x + c(x_0) & x > x_0. \end{cases}$$

$c(x_0)$ is an arbitrary additive constant that depends on x_0 and corresponds to an arbitrary multiple of the homogeneous solution. This is the representation of all possible modified Green's functions. Often we desire $G_m(x, x_0)$ to be symmetric. For example, $G_m(x, x_0) = G_m(x_0, x)$ for $x < x_0$ yields

$$-\frac{1}{L}\frac{x_0^2}{2} + x_0 + c(x) = -\frac{1}{L}\frac{x^2}{2} + x_0 + c(x_0)$$

or

$$c(x_0) = -\frac{1}{L}\frac{x_0^2}{2} + \beta$$

where β is an arbitrary *constant*. Thus, finally we obtain the modified Green's function,

$$G_m(x, x_0) = \begin{cases} -\frac{1}{L}\frac{(x^2 + x_0^2)}{2} + x_0 + \beta & x < x_0 \\ -\frac{1}{L}\frac{(x^2 + x_0^2)}{2} + x + \beta & x > x_0. \end{cases}$$

A solution of (8.4.17) is given by (8.4.16) with $G_m(x, x_0)$ given above.

An alternative modified Green's function. In order to solve problems with homogeneous solutions, we could introduce instead a comparison problem satisfying nonhomogeneous boundary conditions. For example, the Neumann function G_a is defined by

$$\frac{d^2G_a}{dx^2} = \delta(x - x_0) \tag{8.4.20a}$$

$$\frac{dG_a}{dx}(0) = -c \tag{8.4.20b}$$

$$\frac{dG_a}{dx}(L) = c. \tag{8.4.20c}$$

Physically, this represents a unit negative source $-\delta(x - x_0)$ of thermal energy with heat energy flowing into both ends at the rate of c per unit time. Thus, physically there will be a solution only if $2c = 1$. This can be verified by integrating (8.4.20a) from $x = 0$ to $x = L$ or by using Green's formula. This alternate modified Green's function can be obtained in a manner similar to the previous one. In terms of this Green's function, the representation of the solution of a nonhomogeneous problem can be obtained using Green's formula (see Exercise 8.4.12).

EXERCISES 8.4

8.4.1. Consider

$$L(u) = f(x) \qquad \text{with} \qquad L = \frac{d}{dx}\left(p\frac{d}{dx}\right) + q$$

subject to two homogeneous boundary conditions. *All* homogeneous solutions ϕ_h (if they exist) satisfy $L(\phi_h) = 0$ and the same two homogeneous boundary conditions. Apply Green's formula to prove that there are no solutions u if $f(x)$ is not orthogonal (weight 1) to all $\phi_h(x)$.

8.4.2. Modify Exercise 8.4.1 if

$$L(u) = f(x)$$

$$u(0) = \alpha \qquad \text{and} \qquad u(L) = \beta$$

***(a)** Determine the condition for a solution to exist.

(b) If this condition is satisfied, show that there are an infinite number of solutions.

8.4.3. Without determining $u(x)$ how many solutions are there of

$$\frac{d^2u}{dx^2} + \gamma u = \sin x$$

(a) If $\gamma = 1$ and $u(0) = u(\pi) = 0$?

***(b)** If $\gamma = 1$ and $\frac{du}{dx}(0) = \frac{du}{dx}(\pi) = 0$?

(c) If $\gamma = -1$ and $u(0) = u(\pi) = 0$?

(d) If $\gamma = 2$ and $u(0) = u(\pi) = 0$?

8.4.4. For the following examples, obtain the general solution of the differential equation using the method of undetermined coefficients. Attempt to solve the boundary conditions, and show that the result is consistent with the Fredholm alternative:

(a) Equation (8.4.7)

(b) Equation (8.4.10)

(c) Example after (8.4.10)

(d) Second example after (8.4.10)

8.4.5. Are there any values of β for which there are solutions of

$$\frac{d^2u}{dx^2} + u = \beta + x$$

$$u(-\pi) = u(\pi) \qquad \text{and} \qquad \frac{du}{dx}(-\pi) = \frac{du}{dx}(\pi)?$$

***8.4.6.** Consider

$$\frac{d^2u}{dx^2} + u = 1.$$

(a) Find the general solution of this differential equation. Determine all solutions with $u(0) = u(\pi) = 0$. Is the Fredholm alternative consistent with your result?

(b) Redo part (a) if $\frac{du}{dx}(0) = \frac{du}{dx}(\pi) = 0$.

(c) Redo part (a) if $\frac{du}{dx}(-\pi) = \frac{du}{dx}(\pi)$ and $u(-\pi) = u(\pi)$.