Journal of the Chinese Institute of Engineers

ISSN: (Print) (Online) Journal homepage: https://www.tandfonline.com/loi/tcie20

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To cite this article: Jeng-Tzong Chen, Chia-Ying Yang, Yen-Ting Chou & Chi-Ning Tsang (2023) Animation of cycloid and spiral curves in companion with instantaneous center of rotation and radius of curvature, Journal of the Chinese Institute of Engineers, 46:7, 693-702, DOI: 10.1080/02533839.2023.2238768

To link to this article: https://doi.org/10.1080/02533839.2023.2238768

Published online: 02 Aug 2023.

JOURNAL OF THE CHINESE INSTITUTE OF ENGINEERS
2023, VOL. 46, NO. 7, 693–702
https://doi.org/10.1080/02533839.2023.2238768

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Animation of cycloid and spiral curves in companion with instantaneous center of rotation and radius of curvature

Jeng-Tzong Chen[a,b,c,d,e], Chia-Ying Yang[a], Yen-Ting Chou[a] and Chi-Ning Tsang[a]

[a]Department of Harbor and River Engineering, National Taiwan Ocean University, Keelung, Taiwan; [b]Department of Mechanical and Mechatronic Engineering, National Taiwan Ocean University, Keelung, Taiwan; [c]Center of Excellence for Ocean Engineering, National Taiwan Ocean University, Keelung, Taiwan; [d]Department of Civil Engineering, National Cheng Kung University, Tainan, Taiwan; [e]Department of Civil Engineering, National Taiwan University, Taipei, Taiwan

ABSTRACT
In this paper, the animations for the 2D cycloid and the 3D spiral curves are done. The trajectories of instantaneous rotation center and the corresponding radius of curvature are given. We prove that the trajectory of the instantaneous center of rotation is also a cycloid. For a 3D spiral curve, the two radii and the two instantaneous centers of rotation for the spiral curve are also given. It is interesting to find that the two parameters in the Frenet equation have the same meaning of radius of curvature but in different planes. In a similar way of the 2D experience, we also confirm that the trajectory of the instantaneous center of rotation for a spiral curve is also a spiral curve. An example is also given to discuss the Puyuma express incident, a major accident in 2018. The curve of rail is interpolated and the radius of curvature is determined. Discussions on the radius of rail curve and the speed of train for the failure are done. Finally, the animation is implemented by using the MATLAB and the Mathematica software. Not only theoretical derivation for the curvature of a curve but also its real application to rail engineering is proposed.

ARTICLE HISTORY
Received 27 December 2022
Accepted 25 May 2023

CO EDITOR-IN-CHIEF
Ou, Yu-Chen

ASSOCIATE EDITOR
Ou, Yu-Chen

KEYWORDS
Frenet formula; radius of curvature; animation

1. Introduction

Radius of curvature for a curve is an important parameter in many branches of science and engineering. For example, it always appears in the statics, dynamics, mechanics of material, rail, and highway engineering. The available five formulae for the curvature of a plane curve were reviewed by Chen et al. (2021) in the literature not only for the time-like parameter curve but also for the space-like parameter curve. Two simpler and easier formulae were derived in Chen et al. (2021). Two alternative ways to geometrically describe the curve are available. One is the straight way to obtain the Frenet formula according to the given curve of parameter either for the time-like parameter curve or for the space-like parameter curve. The other is that we can reconstruct the curve by solving the state equation of Frenet formula subject to the initial position, the initial tangent, normal, and binormal vectors and the given radius of curvature and the torsion constant. The remainder theorem of the matrix and the Cayley-Hamilton theorem were both employed to solve the Frenet equation by Chen et al. (2021). Frenet formula can be seen as a tensorial equation under a given spin history. Liu (2014) developed a novel method to solve the equation by a single parameter. Ricardo (2005) showed explicitly the relations of the Frenet frame and the Euclidean curvature. How to derive the Frenet formula can be found in the book of Riley and Hobson (2008). The Frenet model was also required to simulate the dynamic behavior of an extensible string in 3D space by Nishinari (1999) and Okuyucu (2022) revisited the Frenet curve by using Lie groups for 3D problems.

Animation tools, Mathematica and Matlab, are now very popular in engineering education. Chen, Chou, and Kao (2009) presented how one-dimensional wave phenomenon was animated. Several methods including the D'Alembert solution, the diamond rule, the Laplace transform and the convolution integral, were employed in the Mathematica animation. All the analytical derivations were also carried out by using the symbolic software. Therefore, animation for the curve is also our concern.

Zhou and Zhu (2007) discussed the relationship between natural coordinates in fluid mechanics and orthogonal curvilinear coordinates. Since orthogonal curvilinear coordinates may have some excellent mathematical properties, natural coordinates can be applied more widely if they can be transformed to the orthogonal curvilinear coordinates. Frenet formula, which describes the differential property of natural coordinates, was compared with the derivative formulae of orthogonal curvilinear coordinates to show that natural coordinates are not generally orthogonal curvilinear coordinates. The geometry of a space curve can be completely defined in terms of two parameters: the horizontal and vertical curvatures, or equivalently, the curvature and torsion in the literature of Riley and Hobson (2008). Distinction is made between the track angle and space-curve bank angle, referred to as the Frenet bank angle by Ling and Shabana (2021). In railroad vehicle systems, the track bank angle measures the track super-elevation required to define a balance speed and achieve a safe vehicle operation by Ling and Shabana (2021). Frenet formula is the governing equation of a curve in space. Two-

CONTACT Jeng-Tzong Chen jtchen@mail.ntou.edu.tw Department of Harbor and River Engineering, National Taiwan Ocean University, Keelung, Taiwan

dimensional curve is a special case. Once the radius of curvature is determined, the location of instantaneous center of rotation can be found. We may wonder what is the curve constructed by the instantaneous center of rotation. Regarding the two parameters in the Frenet formulae, the radius of curvature and the torsion constant are referred in the literature. We intend to explore the geometric meaning of the two parameters in the 2D cycloid and 3D spiral curve by way of animation. Both are demonstrated to see the trajectories of instantaneous center of rotation.

There are numerous causative factors for the train derailment. A study by Ju and Hung (2019) examined the reduction in soil strength due to liquefaction, which potentially led to the derailment of trains during seismic events. Another investigation by Ju (2023) explored the impact of centrifugal force and dampers on the train derailment coefficient on curved tracks against earthquakes. Ju (2020) further investigated the influence of lead rubber bearings on bridge stability and the resulting derailment coefficient. Liao (2021) suggested to place a camera on the train's front-end and took recording the track to assess the safety condition of the train during operation. According to the image, the curvature can be measured.

In this paper, we focus on the centrifugal force's effect caused by the radius of curvature on the train derailment. In addition, a case study of Puyuma expresses derailment due to over speed of the train and smaller radius of curvature of the rail path was proposed for students and engineers to understand the mechanism.

2. Derivation of the trajectories of the two instantaneous centers of rotation for spiral curves in 3D

To describe a curve in space, the Frenet formula is a typical way (Peter 2003). Two planes, the τ-ν plane and the β-ν plane, respectively, are shown in Figure 1. On the τ-ν plane, $\tilde{\boldsymbol{\tau}}$, $\tilde{\boldsymbol{\nu}}$ and $\tilde{\boldsymbol{\beta}}$ are the tangent, the normal and the binormal vectors as shown in Figure 1, respectively, while $\tilde{\boldsymbol{\beta}}$, $\tilde{\boldsymbol{\nu}}$ and $\tilde{\boldsymbol{\tau}}$ are the tangent, the normal and the binormal vectors for the β-ν plane, respectively. If the time-like parameter representation is transformed to the space-like parameter (arc length) representation, we have

$$\tilde{\boldsymbol{r}} = (X(t), Y(t), Z(t))^T = (x(s), y(s), z(s))^T, \quad (1)$$

where t and s are parameters of the time and the arc length, respectively, $\tilde{\boldsymbol{r}}(s)$ is the position vector, and T is the transpose. The unit tangent vector $\tilde{\boldsymbol{\tau}}(s)$ is defined as

$$\tilde{\boldsymbol{\tau}}(s) = \frac{d\tilde{\boldsymbol{r}}(s)}{ds}. \quad (2)$$

The unit normal vector $\tilde{\boldsymbol{\nu}}(s)$ is orthogonal to $\tilde{\boldsymbol{\tau}}(s)$, i.e.

$$\tilde{\boldsymbol{\nu}}(s) = \frac{\tilde{\boldsymbol{\tau}}'(s)}{|\tilde{\boldsymbol{\tau}}'(s)|}. \quad (3)$$

According to Equation (2), the unit tangent vector can be expressed as

$$\boldsymbol{\tau}(s) = \boldsymbol{r}'(s), \quad (4)$$

due to

$$|\tilde{\boldsymbol{r}}'(s)| = 1. \quad (5)$$

The unit binormal vector can be obtained by the outer product of $\tilde{\boldsymbol{\tau}}(s)and\tilde{\boldsymbol{\nu}}(s)$,

$$\tilde{\boldsymbol{\beta}}(s) = \tilde{\boldsymbol{\tau}}(s) \times \tilde{\boldsymbol{\nu}}(s), \quad (6)$$

where the two parameters can be determined by

$$\rho(s) = \frac{|\tilde{\boldsymbol{\nu}}(s)|}{|\tilde{\boldsymbol{\tau}}'(s)|}, \quad (7)$$

and

$$\sigma(s) = \frac{|\tilde{\boldsymbol{\nu}}(s)|}{|\tilde{\boldsymbol{\beta}}'(s)|}. \quad (8)$$

The Frenet formula for the 3D curve is given below:

$$\begin{pmatrix} \tilde{\boldsymbol{\tau}}'(s) \\ \tilde{\boldsymbol{\nu}}'(s) \\ \tilde{\boldsymbol{\beta}}'(s) \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{\rho} & 0 \\ -\frac{1}{\rho} & 0 & \frac{1}{\sigma} \\ 0 & -\frac{1}{\sigma} & 0 \end{pmatrix} \begin{pmatrix} \tilde{\boldsymbol{\tau}}(s) \\ \tilde{\boldsymbol{\nu}}(s) \\ \tilde{\boldsymbol{\beta}}(s) \end{pmatrix}, 0 < s < \infty, \quad (9)$$

subject to initial vectors of $\tilde{\boldsymbol{\tau}}(0)$, $\tilde{\boldsymbol{\nu}}(0)$ and $\tilde{\boldsymbol{\beta}}(0)$ for $s = 0$.

3. On the trajectory of the instantaneous center of rotation for a cycloid example (2D case)

A space-like parameter curve is shown in Figure 2. The radius of curvature can be derived from the calculus as follows:

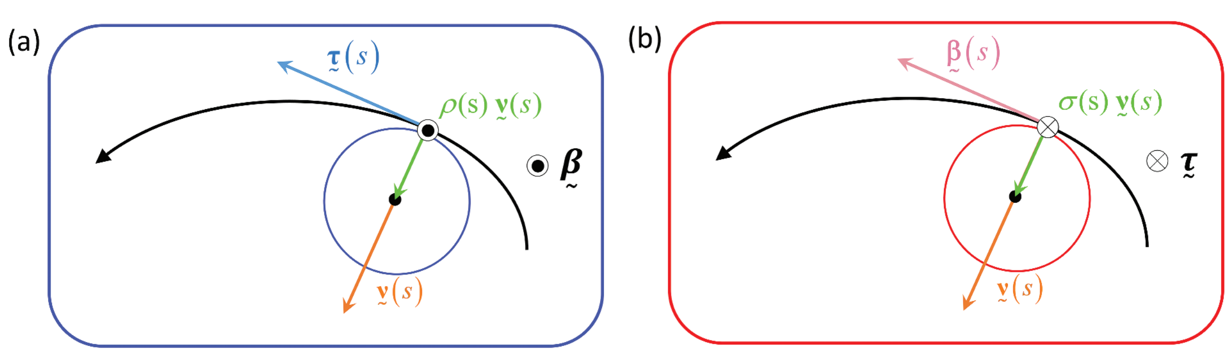

Figure 1. The two instantaneous centers of rotation and the corresponding radii of the curvature (a) τ-ν plane; (b) β-ν plane.

$$\rho = \frac{ds}{d\theta}, \tag{10}$$

where ρ is the radius of curvature, $d\theta$ is the infinitesimal angle, and ds is an infinitesimal arc length as shown in Figure 2, which can be written by the arc–length relationship of

$$(ds)^2 = (dX)^2 + (dY)^2. \tag{11}$$

The time-like parameter curve of the cycloid is given by

$$\begin{cases} X(t) = t - sint, \\ Y(t) = 1 - cost. \end{cases} \tag{12}$$

By using Equation (11), we have

$$s(t) = 4 - 4\cos\left(\frac{t}{2}\right). \tag{13}$$

Equation (12) can be transformed to the space-like parameter curve of a cycloid,

$$x(s) = 2cos^{-1}\left(\frac{4-s}{4}\right) - \frac{(4-s)\sqrt{8s-s^2}}{8}, y(s) = \frac{8s-s^2}{8}. \tag{14}$$

The initial tangent, normal and binormal vectors are obtained as follows:

$$\widetilde{\boldsymbol{\tau}}(s) = \left(\frac{\sqrt{8s-s^2}}{4}, \frac{4-s}{4}\right), \widetilde{\boldsymbol{\nu}}(s) = (v_x(s), v_y(s)) = \left(\frac{4-s}{4}, -\frac{\sqrt{8s-s^2}}{4}\right), \widetilde{\boldsymbol{\beta}}(s) = (0,0,1). \tag{15}$$

Following the Frenet formula in Equation (9), we obtain the radius of curvature as

$$\rho(s) = \sqrt{8s-s^2}, \sigma(s) = \infty, \tag{16}$$

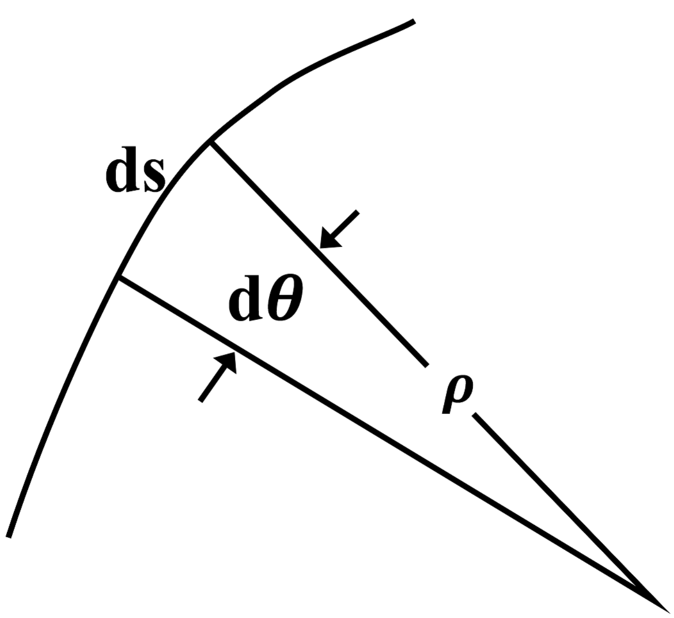

Figure 2. Schematic diagram of a plane curve.

where the infinite radius of curvature in Equation (16) indicates the 2D curve only. The coordinates of instantaneous rotation center of cycloid are

$$x_c(s) = x(s) + \rho(s)v_x(s) = \frac{(4-s)\sqrt{8s-s^2}}{8} + 2cos^{-1}\left(\frac{4-s}{4}\right), \quad y_c(s) = y(s) + \rho(s)v_y(s) = -\left(\frac{8s-s^2}{8}\right). \tag{17}$$

The instantaneous center of rotation is shown in Figure 3. After a translation of $(-\pi, 2)$, we have

$$X_c(s) = x(s) + \rho(s)v_x(s) - \pi = \frac{(4-s)\sqrt{8s-s^2}}{8} + 2cos^{-1}\left(\frac{4-s}{4}\right) - \pi, \quad Y_c(s) + 2 = y(s) + \rho(s)v_y(s) + 2 = -\left(\frac{8s-s^2}{8}\right) + 2. \tag{18}$$

The instantaneous center of rotation after translation is shown in Figure 4. By changing variable $s = s_1 + 4$, $X_c(s)$ and $Y_c(s)$ become $\bar{x}(s_1)$ and $\bar{y}(s_1)$, respectively,

$$x(s_1) = \frac{(-s_1)\sqrt{16s_1 - s_1{}^2}}{8} + 2cos^{-1}\frac{(-s_1)}{4} - \pi, \bar{y}(s_1) = 2 - \frac{(16 - s_1{}^2)}{8} = \frac{s_1{}^2}{8}. \tag{19}$$

By setting $t = 2cos^{-1}\left(\frac{-s_1}{4}\right) - \pi$, Equation (18) reduces to the original time-like parameter curve of Equation (12) as shown below:

$$x(s_1) = \frac{(-s_1)\sqrt{16s_1 - s_1{}^2}}{8} + 2cos^{-1}\frac{(-s_1)}{4} - \pi = -sint + t = X(t), \tag{20}$$

and

$$y(s_1) = \frac{s_1{}^2}{8} = \frac{16sin^2\left(\frac{t}{2}\right)}{8} = 1 - cost = Y(t). \tag{21}$$

It is proved that the trajectory of the instantaneous center of rotation for a cycloid is also a cycloid.

4. On the trajectory of the instantaneous center of rotation for a spiral curve (3D case)

The space-like parameter curve of the spiral curve is defined by

$$x(s) = 3\cos\left(\frac{s}{5}\right), y(s) = 3\sin\left(\frac{s}{5}\right), z(s) = \frac{4s}{5}. \tag{22}$$

Based on Equation (22), we find the radius of curvature and the torsion constant (Zhou and Zhu 2007),

$$\rho(s) = \frac{25}{3}, \sigma(s) = \frac{25}{4}. \tag{23}$$

The $\widetilde{\boldsymbol{\tau}}, \widetilde{\boldsymbol{\nu}}$ and $\widetilde{\boldsymbol{\beta}}$ vectors are also obtained as follows,

$$\widetilde{\boldsymbol{\tau}}(s) = \left(-\frac{3}{5}\sin\left(\frac{s}{5}\right), \frac{3}{5}\cos\left(\frac{s}{5}\right), \frac{4}{5}\right), \tag{24a}$$

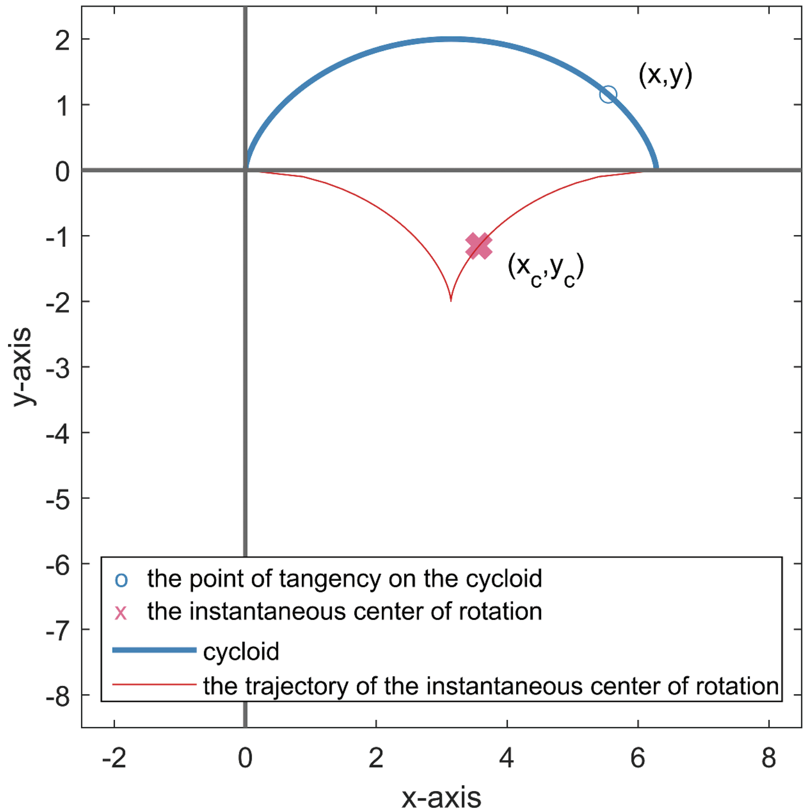

Figure 3. The trajectory of a cycloid (x, y) and the instantaneous center of rotation (x_c, y_c).

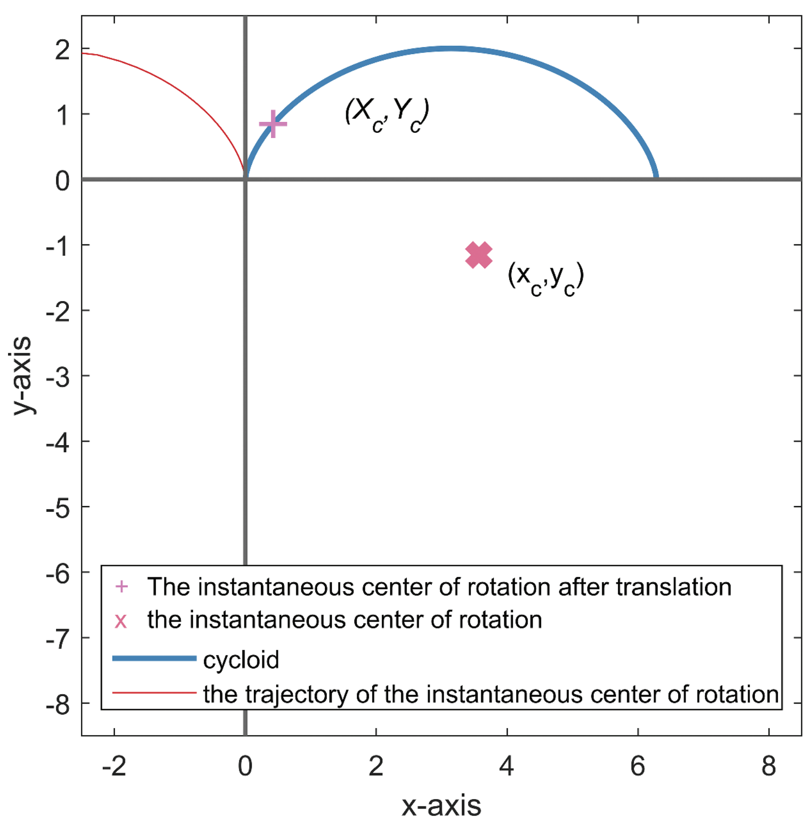

Figure 4. The instantaneous center of rotation after translation (x_c, y_c) to (X_c, Y_c).

$$\boldsymbol{v}(s) = \left(-\cos\left(\frac{s}{5}\right), -\sin\left(\frac{s}{5}\right), 0\right), \quad (24b)$$

$$\widetilde{\boldsymbol{\beta}}(s) = \left(\frac{4}{5}\sin\left(\frac{s}{5}\right), -\frac{4}{5}\cos\left(\frac{s}{5}\right), \frac{3}{5}\right). \quad (24c)$$

Therefore, the Frenet formula in Equation (9) is given below:

$$\begin{pmatrix} \widetilde{\boldsymbol{\tau}}'(s) \\ \widetilde{\boldsymbol{v}}'(s) \\ \widetilde{\boldsymbol{\beta}}'(s) \end{pmatrix} = \begin{pmatrix} 0 & \frac{3}{25} & 0 \\ -\frac{3}{25} & 0 & \frac{4}{25} \\ 0 & -\frac{4}{25} & 0 \end{pmatrix} \begin{pmatrix} \widetilde{\boldsymbol{\tau}}(s) \\ \widetilde{\boldsymbol{v}}(s) \\ \widetilde{\boldsymbol{\beta}}(s) \end{pmatrix}. \quad (25)$$

By setting $s = 0$, Equations (22) and (24) reduce to

$$x(0) = 3, \quad (26a)$$

$$y(0) = 0, \quad (26b)$$

$$z(0) = 0, \quad (26c)$$

and

$$\widetilde{\boldsymbol{\tau}}(0) = \left(0, \frac{3}{5}, \frac{4}{5}\right), \quad (27a)$$

$$\widetilde{\boldsymbol{v}}(0) = (-1, 0, 0), \quad (27b)$$

$$\widetilde{\boldsymbol{\beta}}(0) = \left(0, -\frac{4}{5}, \frac{3}{5}\right), \quad (27c)$$

respectively. We can reconstruct the space-like parameter curve of Equation (22) by following the same procedure of Chen et al. (2021). From Equation (9) to (23), it is a direct way to describe the curve. On the other hand, Equation (22) can be reconstructed by solving the Frenet formula of Equation (25) subject to the initial position of Equation (26), the initial tangent, normal, and binormal vectors of Equation (27). This is another way to describe the curve.

The coordinates of instantaneous centers of rotation for the spiral curve, $\widetilde{\boldsymbol{c}_1}$ and $\widetilde{\boldsymbol{c}_2}$ can be expressed as

$$\widetilde{\boldsymbol{c}_1} = (x, y, z) + \rho(s)\,\widetilde{\boldsymbol{v}}(s), \quad (28)$$

and

$$\widetilde{\boldsymbol{c}_2} = (x, y, z) + \sigma(s)\,\widetilde{\boldsymbol{v}}(s). \quad (29)$$

The coordinates of instantaneous rotation centers of spiral curves for the τ-ν plane and the β-ν plane are

$$\widetilde{\boldsymbol{c}_1} = \begin{pmatrix} x_{c1}(s) \\ y_{c1}(s) \\ z_{c1}(s) \end{pmatrix} = \begin{pmatrix} \left(3 - \frac{25}{3}\right)cos\left(\frac{s}{5}\right) \\ \left(3 - \frac{25}{3}\right)sin\left(\frac{s}{5}\right) \\ \frac{4s}{5} \end{pmatrix}, \quad (30)$$

and

$$\widetilde{\boldsymbol{c}_2} = \begin{pmatrix} x_{c2}(s) \\ y_{c2}(s) \\ z_{c2}(s) \end{pmatrix} = \begin{pmatrix} \left(3 - \frac{25}{4}\right)cos\left(\frac{s}{5}\right) \\ \left(3 - \frac{25}{4}\right)sin\left(\frac{s}{5}\right) \\ \frac{4s}{5} \end{pmatrix}, \quad (31)$$

respectively. The general formulae of spiral curves are

$$x^2(s) + y^2(s) = a^2, z(s) = bs, \quad (32)$$

where a and b are constants. It is proved that the trajectory of the instantaneous center of rotation for spiral curves, $\widetilde{\boldsymbol{c}_1}$ and $\widetilde{\boldsymbol{c}_2}$ of Equations (30) and (31) satisfying Equation (32), is also a spiral curve, but with different values of ρ and σ.

For a real example as a case study, we consider a Puyuma express accident as shown in Figure 5 (Newtalk, 2018). From the report on facts (TTSB, 2008), causes and suggestions for improving problems of the derailment of the main line of Taiwan railway 6432 train at Xinma Station, we obtain the track plan before and after the derailment of Puyuma Express (Xinma Station) from the Taiwan Railway Administration. First, we used Grapher to construct the (*x,y*) coordinates of each point according to the track point map. The number of points are fewer in the straight part of the track, and denser in the front and back of the Puyuma express derailed position in order to make the curve more close to the real track curve. The resulting data were input into the MATLAB, and the curve fitting toolbox was used to fit the polynomial function of order seven, $F(x)$, as shown in Figure 6.

$$F(x) = p_1x^7 + p_2x^6 + p_3x^5 + p_4x^4 + p_5x^3 + p_6x^2 + p_7x + p_8, \quad (33)$$

where $p_1 = -2.112 \times 10^{-16}$, $p_2 = 5.927 \times 10^{-13}$, $p_3 = 6.011 \times 10^{-10}$, $p_4 = 2.778 \times 10^{-7}$, $p_5 = 6.383 \times 10^{-5}$, $p_6 = 8.813 \times 10^{-3}$, $p_7 = -0.65$, and $p_8 = 70.37$. By changing Equation (9) from ρ(s) to ρ(x), we have the following results according to the chain rule,

Figure 5. Puyuma express accident (Oct. 21, 2018, Taiwan).

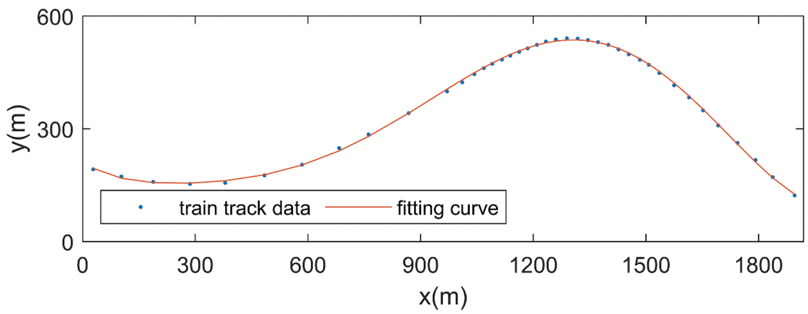

Figure 6. Curve fitting by using the MATLAB.

$$\rho(x) = \frac{\left(1 + F'(x)^2\right)^{\frac{3}{2}}}{|F''(x)|}. \tag{34}$$

By differentiating Equation (33) with respect to x, we have the tangential direction vector $\boldsymbol{\tau}(x)$,

$$\boldsymbol{\tau}(x) = (1, F'(x)). \tag{35}$$

The normalized vector is obtained as

$$\hat{\boldsymbol{\tau}}(x) = \left(\frac{1}{\sqrt{1 + F'(x)^2}}, \frac{F'(x)}{\sqrt{1 + F'(x)^2}}\right). \tag{36}$$

By substituting Equation (36) into Equation (3), we obtain the normal vector,

$$\hat{\boldsymbol{v}}(x) = \left(\frac{F'(x)}{\sqrt{1 + F'(x)^2}}, \frac{-1}{\sqrt{1 + F'(x)^2}}\right). \tag{37}$$

The instantaneous centers of rotation trajectory, $F_c(x)$, is obtained from Equations (33), (34), and (37), as shown below:

$$F_c(x) = F(x) + \hat{v}(x)\rho(x) \tag{38}$$

The minimum radius of curvature of the track at the Xinma Station is 306 m. It is clear from Figure 7 that Puyuma Express derailed before the position that reached the minimum radius of curvature (blue circle). The actual speed at derailment and the speed limit at the Xinma Station are shown in Table 1. According to dynamics, we have the centripetal force F,

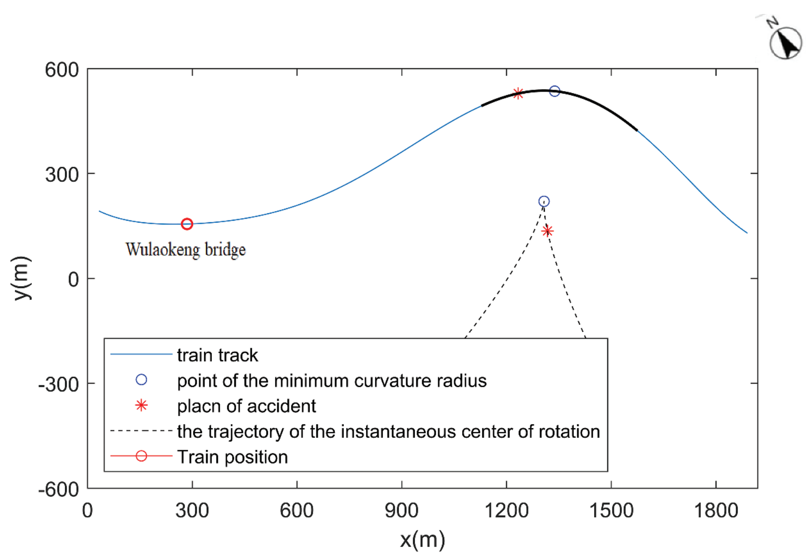

Figure 7. Track solid curve and dotted curve of instantaneous rotation center.

Table 1. Comparison of the speed limit and the actual speed.

	km/h	m/s
Speed limit	75 km/h	20.80 m/s
Actual speed	141 km/h	39.17 m/s

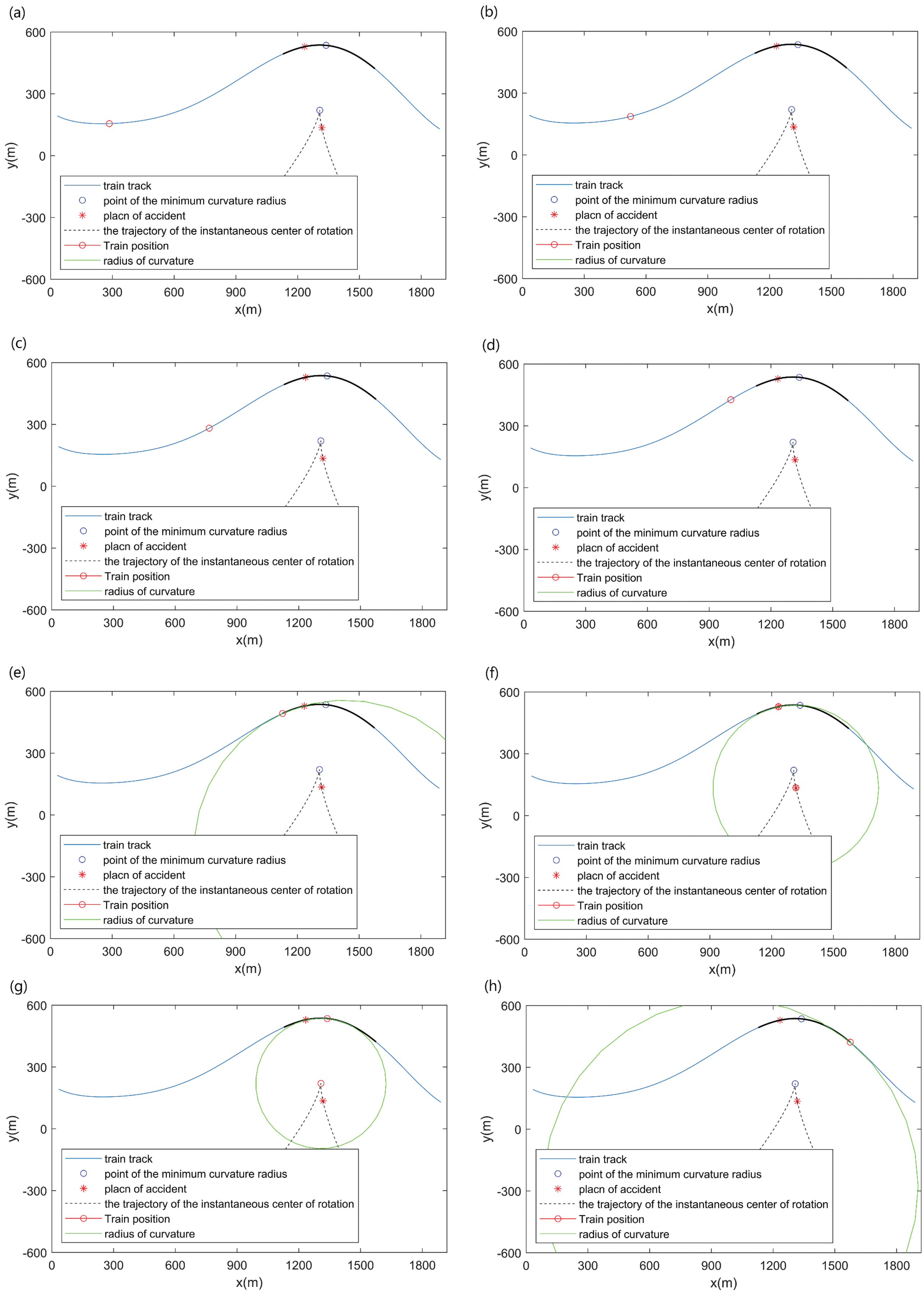

Figure 8. Animation of the trajectory of instantaneous centers of rotation for the track in each step by using the path parameter, x. (x = 325 ~ 525) (a) x = 255; (b) x = 495; (c) x = 735; (d) x = 975; (e) x = 1128; (f) x = 1233; (g) x = 1338; (h) x = 1575.

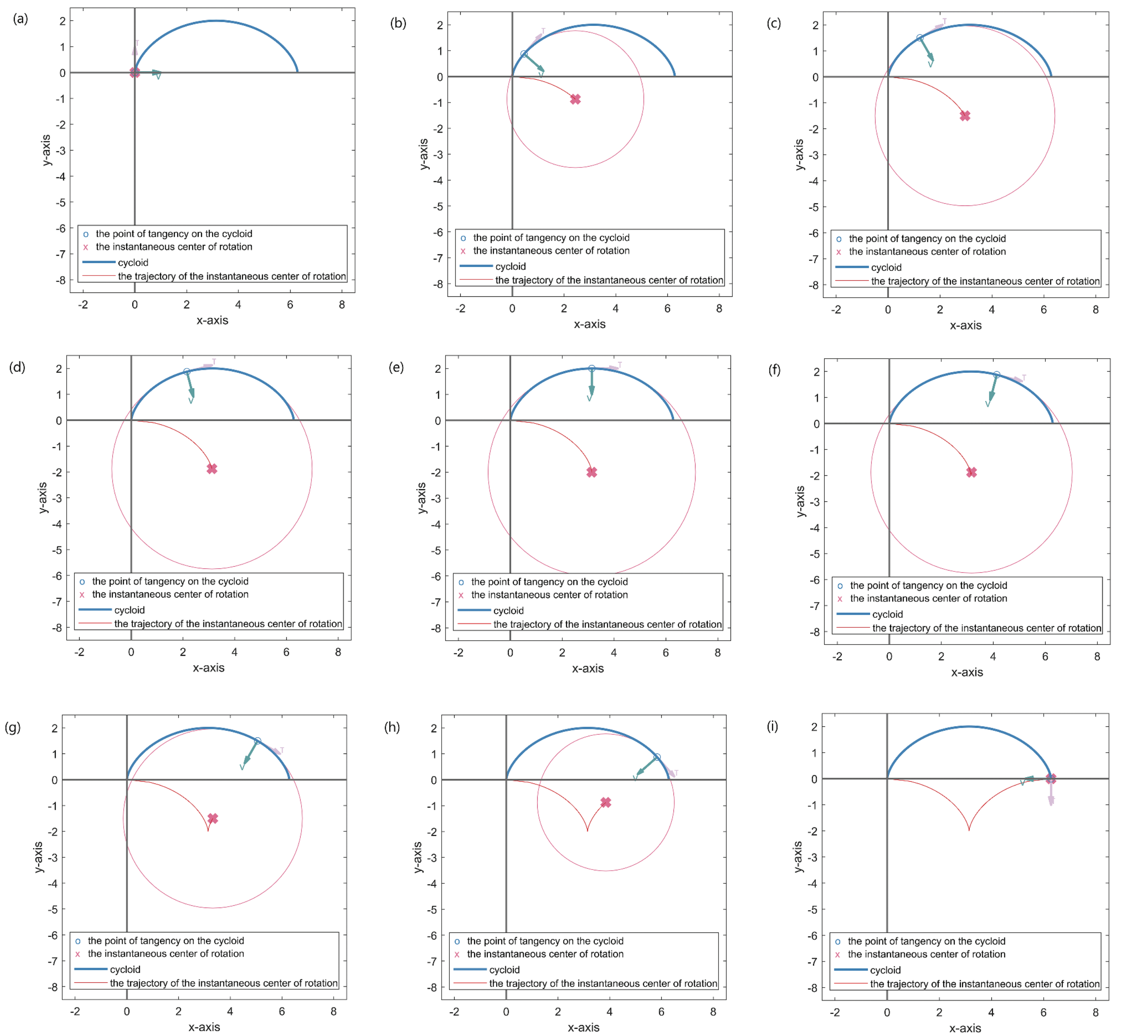

Figure 9. Animation of the cycloid curve in each step by the space-like parameter, s. ($s = 0 \sim 8, \Delta s = 1$)(a)$s = 1$; (b)$s = 2$; (c)$s = 3$; (d)$s = 4$; (e)$s = 5$; (f)$s = 6$; (g)$s = 7$; (h)$s = 8$; (i)$s = 9$.

Table 2. Frenet formula for cycloid and spiral curves.

Equation Curve	Cycloid	Spiral curve
Time-like parameter $(X(t), Y(t))$	$X(t) = t - \sin t,\ Y(t) = 1 - \cos t$	$X(t) = 3\cos t,\ Y(t) = 3\sin t, z(t) = 4t$
Arc length, $s(t)$	$s = 4 - 4\cos\frac{t}{2}$	$s = 5t$
Space-like parameter$(x(s), y(s))$	$x(s) = 2\cos^{-1}\left(\frac{4-s}{4}\right) - \frac{(4-s)\sqrt{8s-s^2}}{8}$ $y(s) = \frac{8s-s^2}{8}$	$x(s) = 3\cos\left(\frac{s}{5}\right)$ $y(s) = 3\sin\left(\frac{s}{5}\right)$ $z(s) = \frac{4s}{5}$
Tangent, normal and binormal vectors $\widetilde{\tau}(s), \widetilde{\nu}(s)$ and $\widetilde{\beta}(s)$	$\widetilde{\tau}(s) = \left(\frac{\sqrt{8s-s^2}}{4}, \frac{4-s}{4}\right)$ $\widetilde{\nu}(s) = \left(\frac{4-s}{4}, -\frac{\sqrt{8s-s^2}}{4}\right)$	$\widetilde{\tau}(s) = \left(-\frac{3}{5}\sin\left(\frac{s}{5}\right), \frac{3}{5}\cos\left(\frac{s}{5}\right), \frac{4}{5}\right)$ $\widetilde{\nu}(s) = \left(-\cos\left(\frac{s}{5}\right), -\sin\left(\frac{s}{5}\right), 0\right)$ $\widetilde{\beta}(s) = \left(\frac{4}{5}\sin\left(\frac{s}{5}\right), -\frac{4}{5}\cos\left(\frac{s}{5}\right), \frac{3}{5}\right)$
Frenet formula $\begin{pmatrix} \widetilde{\tau}'(s) \\ \widetilde{\nu}'(s) \\ \widetilde{\beta}'(s) \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{\rho} & 0 \\ -\frac{1}{\rho} & 0 & \frac{1}{\sigma} \\ 0 & -\frac{1}{\sigma} & 0 \end{pmatrix} \begin{pmatrix} \widetilde{\tau}(s) \\ \widetilde{\nu}(s) \end{pmatrix} \widetilde{\beta}(s)$	$\rho(s) = \frac{\left\lvert\widetilde{\nu}(s)\right\rvert}{\left\lvert\widetilde{\tau}'(s)\right\rvert}, \sigma(s) = \infty$	$\rho(s) = \frac{\lvert\nu(s)\rvert}{\lvert\tau'(s)\rvert}, \sigma(s) = \frac{\lvert\nu(s)\rvert}{\lvert\beta'(s)\rvert}$
Radius of curvature$\rho(s), \sigma(s)$	$\rho(s) = \sqrt{8s - s^2}, \sigma(s) = \infty$	$\rho(s) = \frac{25}{3},\ \sigma(s) = \frac{25}{4}$
Instantaneous rotation center $x_c(s), y_c(s)$	$x_c(s) = \frac{(4-s)\sqrt{8s-s^2}}{8} + 2\cos^{-1}\left(\frac{4-s}{4}\right)$ $y_c(s) = -\left(\frac{8s-s^2}{8}\right)$	τ-ν plane: $x_{c1}(s) = \left(\frac{-16}{3}\right)\cos\left(\frac{s}{5}\right)$, $y_{c1}(s) = \left(\frac{-16}{3}\right)\sin\left(\frac{s}{5}\right)$, $z_{c1}(s) = \frac{4s}{5}$ β-ν plane: $x_{c2}(s) = \left(\frac{-13}{3}\right)\cos\left(\frac{s}{5}\right)$, $y_{c2}(s) = \left(\frac{-13}{3}\right)\sin\left(\frac{s}{5}\right)$, $z_{c2}(s) = \frac{4s}{5}$

$$F = ma_c = m\frac{V^2}{\rho} \quad (39)$$

where m is the mass, a_c is the acceleration, V is the velocity, and ρ is the radius of curvature. The theoretical maximum centrifugal acceleration can be deduced from Equation (39), namely,

$$a_{max} = \frac{20.8^2}{306} = 1.414m/s^2, \quad (40)$$

where 306 m is the minimum radius of curvature along the trajectory. The radius of curvature of the orbital point is 394.348 m in the system of this paper. The centripetal acceleration at the location of the orbit is obtained by substituting into Equation (39).

$$a = \frac{39.17^2}{394.348} = 3.89m/s^2. \quad (41)$$

It is found that the centrifugal acceleration at the orbit is $3.89m/s^2$ in Equation (41), which is much higher than the designed maximum centripetal acceleration of $1.414m/s^2$ in Equation (40). The accident occurred as a nature outcome appearing in Figure 5.

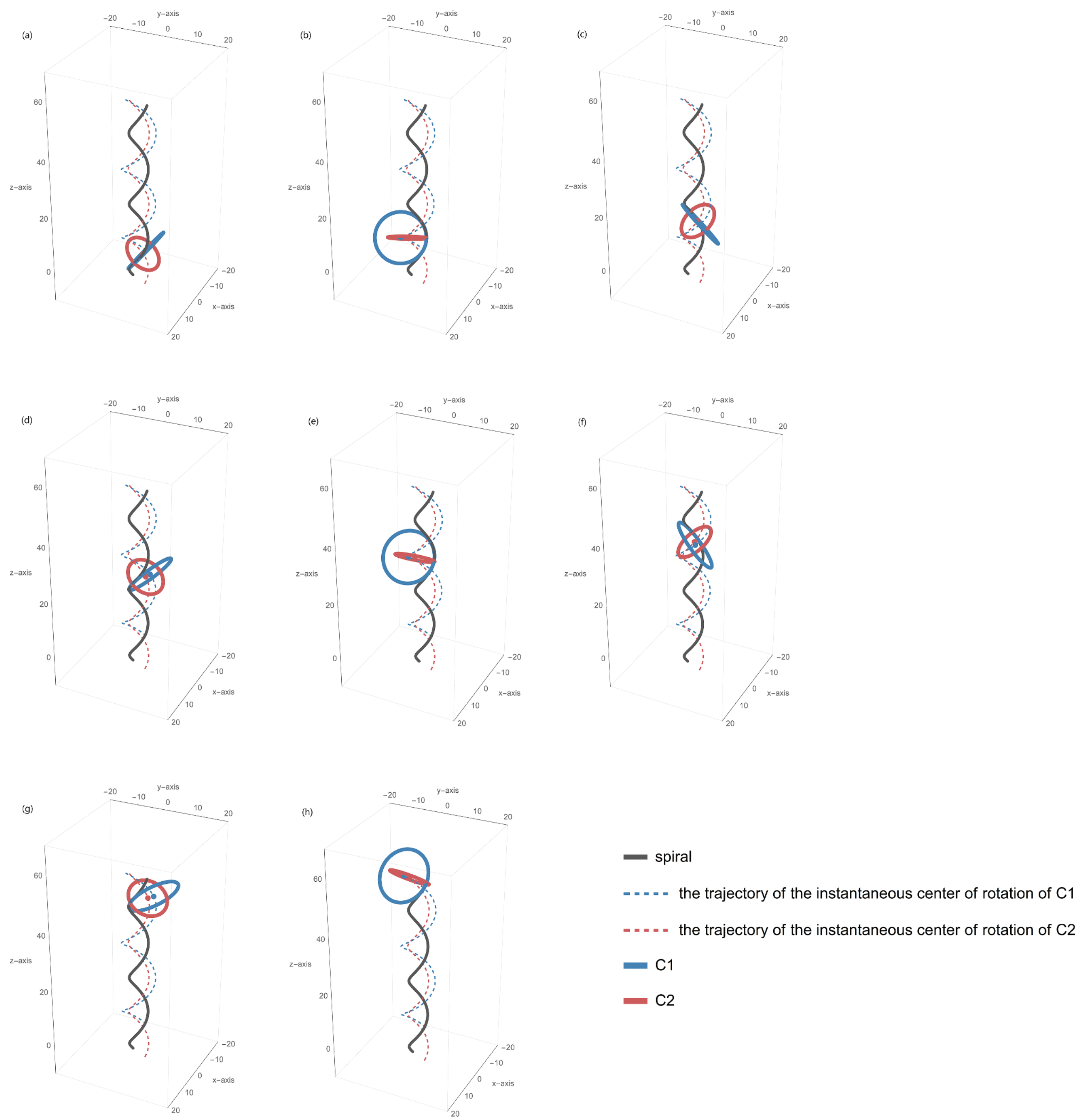

Figure 10. Animation of the spiral curve in each step by the space-like parameter, $s.(s = 0{\sim}70, \Delta s = 10)(a)s = 0$; (b)$s = 10$; (c)$s = 20$; (d)$s = 30$; (e)$s = 40$; (f)$s = 50$; (g)$s = 60$; (h)$s = 70$.

5. Animation

To view the trajectory of curves, the animation for a cycloid (2D) and a spiral curve (3D) are performed by using the MATLAB (2D) and the Mathematica (3D) software as shown in the web site. Besides, the trajectories of instantaneous center of rotation are also shown in Figure 8 for the Puyuma express case. Since the article is presented in a paper form, the animation action is disassembled as piecewisely shown in Figure 9 for the 2D cycloid $(s = 0 \sim 8, \Delta s = 1)$. Figure 10 shows the 3D spiral curve step by step $(s = 0 \sim 70, \Delta s = 10)$. The animation for the cycloid and spiral curves can be found in the following web site, https://pse.is/4dw4av and https://pse.is/4ebahj, respectively. Both VCR videos indicate that the instantaneous center of rotation for the cycloid generates another cycloid, while the instantaneous center of rotation of the spiral curve generates another spiral curve. Table 2 summarizes the results of the cycloid and the spiral curves. The animation for the Puyuma Express track simulation can be found in the following web site, https://reurl.cc/pMEo9Q.

6. Conclusions

Not only a cycloid (2D) but also a spiral (3D) curve was animated by using MATLAB (2D) and Mathematica (3D) software. The trajectories of the instantaneous center of rotation and the corresponding radius were also shown. The trajectory of the instantaneous center of rotation for the cycloid also belonged to the same form of cycloid after translation. It is demonstrated that the two parameters in the Frenet formula have the same geometric meaning of radius of curvature but in different planes. We also found the instantaneous center of rotation also reacts in the same type of curve, cycloid to cycloid, and spiral to spiral. In the case study, it was found that the centripetal acceleration at the orbit was $3.89 m/s^2$, which was much higher than the designed maximum centripetal acceleration of $1.414 m/s^2$ derived from the design specification, which was consistent with the record. The Puyuma express was derailed before reaching the minimum curvature radius.

Disclosure statement

No potential conflict of interest was reported by the author(s).

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