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A self-regularized approach for deriving the free–free flexibility and stiffness matrices

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ABSTRACT

Motivated by Fichera's idea for regularizing the rank-deficiency model, we derive the free–free flexibility matrices by inverting the bordered stiffness matrix. The singular stiffness matrix of a free–free structure is expanded to a bordered matrix by adding n slack variables, where n is the nullity of the singular stiffness matrix. Besides, the corresponding n constraints are accompanied to result in a nonsingular matrix. The constraints filter out the homogeneous solution for the regularized solution. By inverting the nonsingular matrix, we can obtain the free–free flexibility matrix from the submatrices. The value of the extra degree of freedom shows the role of no solution (nonzero case) or infinite solution (zero case) with respect to the loading vector. After constructing the bordered system, the equilibrium of the specified force and the compatibility of the specified displacement can be tested according the zero slack variable. Similarly, the free–free flexibility matrix is obtained from the free–free stiffness matrix. Finally, four examples, a rod with symmetric stiffness, a plane truss, a beam and a bar with unsymmetric stiffness, were demonstrated to see the validity of the present formulation.

1. Introduction

There are two kinds of rank-deficiency problems in the boundary element method (BEM) or finite element method (FEM). Physically speaking, a rigid body mode exists in a free–free structure for structural mechanics. This means that the free–free stiffness matrix is singular in companion with zero eigenvalues (singular values). No matter which numerical method, BEM or FEM, is employed, the obtained influence matrix (stiffness matrix) is rank deficient. Such outcome occurs naturally in the Neumann problem or the traction problem for potential and elasticity problems, respectively [1–7].

Regarding the Dirichlet problem in potential theory or constrained structure in elasticity, the solution is mathematically unique. However, this yields rank-deficiency problem if a single-layer potential approach (indirect BEM) is employed to solve it for a critical scale (degenerate scale). To avoid this unreasonable model, Fichera proposed a well-posed model to make it full rank [8]. Two steps are utilized at the same time. One is to introduce a free constant field. The other is to provide a corresponding constraint. After discretization, the singular system is transformed to a nonsingular bordered system. It is interesting to find that the discretization system to promote the full rank is the same as the self-regularized linear algebraic system for deriving the flexibility of a free–free body. Following this finding, we will drive the free–free flexibility in the way of inverting the full-rank bordered matrix. On the contrary, finding the free–free stiffness from the free–free flexibility is also discussed. Physical rigid-body modes for the displacement as well as nonphysical spurious force modes corresponding to zero singular values are found. The spurious mode also appears in the finite element method. For example, hourglass mode occurs in the reduced integration to soften the shear locking. This zero-energy mode is not physically realizable but due to mathematics. The nonphysical outcome due to mathematics (rank deficiency) needs regularization in the mathematical model. Table 1 indicates the relation between mathematics and structural mechanics. Zero eigenvalues imply the rigid body mode (physics) and spurious mode (mathematics). Bordered matrix introduces an extra degree of freedom and transforms a singular matrix to be a nonsingular one. Free–free structure yields a rank-deficiency matrix.

Regarding the inverse of a singular matrix, Felippa et al. [1] have introduced the dual of free–free stiffness $\mathbf{K}$ and flexibility $\mathbf{F}$. They also emphasized the potential applications of free–free flexibility for substructure-based solution algorithm in the direct flexibility matrix. Construction of free–free flexibility matrices can be derived by using the generalized inverse of stiffness. Derivation

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Table 1
Relation between mathematics and structural mechanics.

Mathematics	Structural mechanics
Null space	Spurious mode
	Rigid body mode
Rank deficiency	Free–free structure
Bordered matrix	Adding an extra degree of freedom
Generalized inverse	Free–free flexibility matrix
Moore–Penrose	Free–free stiffness matrix
Influence matrix	Stiffness or flexibility matrix

of flexibility and stiffness matrices of rod and beam was also investigated by using the dual BEM [6]. Generalized inverse has been studied by Fredholm, Moore and Penrose in the twenty century. Generalized inverses was mathematically studied by using the bordered matrix [9]. However, its engineering applications in structural mechanics were not noticed in that book. In this paper, the proposed self-regularized approach is similar to the Moore–Penrose/Singular Value Decomposition (SVD) approach for computing pseudo-inverses of rank-deficient matrices. But the main difference is that we add a slack variable and a corresponding constraint in the present method. This idea was similarly used in the optimization theory by adding n slack variables. Besides, the flexibility matrix is more efficient in the substructure method, especially in the case of replacement of failure element. The idea was addressed in the Felippa's paper [10].

In non-linear geometry analysis, the arc length method was introduced in the analysis which is similar to the present slack variable. Introducing a slack variable is very popular to transform an inequality to an equation in the optimization theory. The same algorithm in this article is the addition of one degree of freedom in accompany with an extra equation. An incremental force parallel to the critical eigenvectors of the tangential stiffness matrix is separately treated. Using eigenvector projections, we can improve convergence in non-linear finite element equilibrium iterations [11,12]. Although with other objectives in [13], very similar analysis and derivation methods were used in the context of 'eigenvector projections', in the stabilization of non-linear equilibrium iteration methods already in the 1980s. In those, the parts of an incremental force parallel to the critical eigenvectors of the tangential stiffness matrix are separately treated, which is a very similar idea as in this paper. The treatment distinguishes and separately handles two components of the 'load vector', and thereby also the 'displacement response': one parallel to the critical/singular directions, and one orthogonal part. This is a fundamental fact of structural response, which is not related to the need to invert the structural stiffness matrix or parts of it. It also gives a way to explain the introduced unknown coefficients. It corresponds to the value c in this paper. Eigenvector projection provides an efficient way to improve the stability in the iterations for the choice of the optimal corrections. Checking eigenvectors corresponding to near-zero eigenvalues is very important for selecting the damping. In our approach, the singular vector of corresponding to zero singular value provides us the row and column vectors in the bordered matrices, where the unknown coefficient c is introduced.

Based on the structures with symmetry, group-theoretical insight and graph theory can decompose the system to a small one and bypass intrinsic singularities. Related works can be found in the four references [14–17]. However, our approach introduces a slack variable as well as a corresponding constraint to deal with rank-deficient matrices.

In this paper, we derive the free–free flexibility matrix directly from the physical concept as well as the mathematical technique of bordered matrix in the linear algebra. Four examples, a rod with symmetric stiffness, a plane truss, a beam and a bar with unsymmetric stiffness, were demonstrated to see the validity of the present formulation.

2. Formulation

In potential theory, the single-layer representation model is often used to solve the boundary value problem as shown below:

$$u(x) = \int_B U(x,s)\phi(s)dB(s), \quad x \in D, \tag{1}$$

where $u(x)$ is the potential field, $\phi(s)$ is the unknown boundary density, $U(x,s)$ is the fundamental solution and B is the boundary of the domain D.

However, Eq. (1) may fail for the Dirichlet problem with a specific scale (degenerate scale). To overcome this ill-posed (rank-deficiency) model, Fichera proposed a regularized formulation by simultaneously adding a constant and an extra constraint as shown below:

$$u(x) = \int_B U(x,s)\phi_r(s)dB(s) + c, \quad x \in D, \tag{2}$$

$$\int_B \phi_r(s)dB(s) = 0, \quad s \in B. \tag{3}$$

After discretizing the boundary by using the constant element, Eq. (1) reduces to

$$\mathbf{U}\underset{\sim}{\phi} = \underset{\sim}{b}. \tag{4}$$

By employing the boundary element implementation, Eqs. (2) and (3) together yield

$$\begin{bmatrix} \mathbf{U} & \{1\} \\ \{l\} & 0 \end{bmatrix} \begin{Bmatrix} \underset{\sim}{\phi_r} \\ c \end{Bmatrix} = \begin{Bmatrix} \underset{\sim}{b} \\ 0 \end{Bmatrix}, \tag{5}$$

where $\mathbf{U}$ is the influence matrix and $\{l\}$ is the vector of length for boundary elements. It is noted that $\underset{\sim}{\phi}$ in Eq. (4) is the unregularized unknown vector, while ϕ_r in Eq. (5) is the regularized unknown vector.

By using analogy between the singular stiffness matrix for structural mechanics and the influence matrix for the indirect BEM as shown in Fig. 1, a regularized (bordered) matrix provides an alternative way to construct the free–free flexibility matrix.

The linear algebraic system is

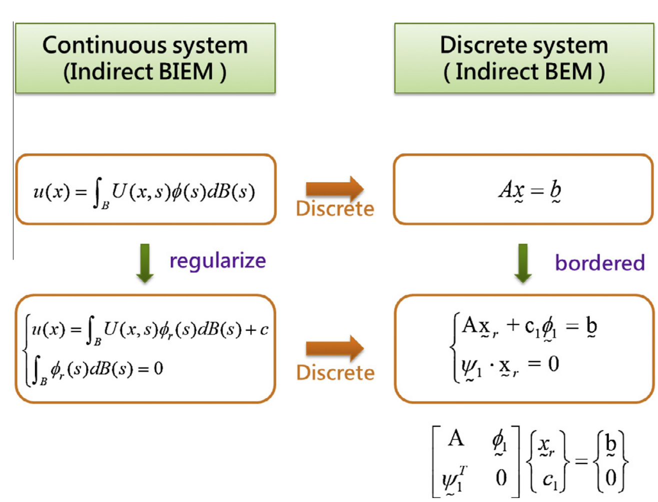


Fig. 1. The self-regularized linear algebraic system from the continuous BIE system.

$$\mathbf{A}\underset{\sim}{x} = \underset{\sim}{b}, \tag{6}$$

where $\mathbf{A}$ is obtained by using either the BEM or FEM. The matrix $\mathbf{A}$ may be a singular matrix which needs special care for the inversion. By employing the singular value decomposition (SVD) to the matrix $\mathbf{A}$, we have

$$\mathbf{A} = \Phi\Sigma\Psi^T = \Phi\begin{bmatrix}\sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_N\end{bmatrix}\Psi^T, \tag{7}$$

where the left singular matrix $\Phi = \left\{\underset{\sim}{\phi}_1, \underset{\sim}{\phi}_2, \ldots, \underset{\sim}{\phi}_N\right\}$, the right singular matrix $\Psi = \{\underset{\sim}{\psi}_1, \underset{\sim}{\psi}_2, \ldots, \underset{\sim}{\psi}_N\}$ and the singular values $0 \leqslant \sigma_1 \leqslant \sigma_2 \leqslant \cdots \leqslant \sigma_N$. If the matrix $\mathbf{A}$ is symmetric, the right and left singular matrices are the same due to the symmetric property. Therefore, we have $\mathbf{A} = \Phi\Sigma\Phi^T = \Psi\Sigma\Psi^T$. No matter which Φ or Ψ is considered, we can represent the unknown vector $\underset{\sim}{x}$ by using the right singular vector $\underset{\sim}{\psi}_i$ as

$$\underset{\sim}{x} = \sum_{i=1}^{N}\alpha_i\underset{\sim}{\psi}_i. \tag{8}$$

Similarly, we can expand the forcing vector $\underset{\sim}{b}$ by the superposition of the left singular vector $\underset{\sim}{\phi}_i$ as follows:

$$\underset{\sim}{b} = \sum_{i=1}^{N}\beta_i\underset{\sim}{\phi}_i. \tag{9}$$

If the singular value, σ_1, is zero, α_1 cannot be determined. By suppressing α_1 to be zero in Eq. (8), we have

$$\underset{\sim}{\psi}_1 \cdot \underset{\sim}{x}_r = 0, \tag{10}$$

where the regularized solution $\underset{\sim}{x}_r$ can be understood as the pure particular solution without containing any component of the complementary solution (rigid body mode) $\underset{\sim}{x}_c$ such that $\mathbf{A}\underset{\sim}{x}_c = \underset{\sim}{0}$.

Since the range of $\mathbf{A}\underset{\sim}{x}$ is deficient by $\underset{\sim}{\phi}_1$, we can regularize Eq. (6) into

$$\mathbf{A}\underset{\sim}{x}_r + c_1\underset{\sim}{\phi}_1 = \underset{\sim}{b}, \tag{11}$$

where c_1 is a free constant to be determined.

By combining Eqs. (10) and (11), we have a regularized linear algebraic system

$$\begin{bmatrix}\mathbf{A} & \underset{\sim}{\phi}_1 \\ \underset{\sim}{\psi}_1^T & 0\end{bmatrix}\begin{Bmatrix}\underset{\sim}{x}_r \\ c_1\end{Bmatrix} = \begin{Bmatrix}\underset{\sim}{b} \\ 0\end{Bmatrix}. \tag{12}$$

Then, the bordered matrix $\mathbf{A}_B$ is defined by

$$\mathbf{A}_B = \begin{bmatrix}\mathbf{A} & \underset{\sim}{\phi}_1 \\ \underset{\sim}{\psi}_1^T & 0\end{bmatrix}. \tag{13}$$

Since $\mathbf{A}_B$ is nonsingular, its inverse yields

$$\mathbf{A}_B^{-1} = \begin{bmatrix}\mathbf{A}^{\dagger} & \underset{\sim}{\psi}_1 \\ \underset{\sim}{\phi}_1^T & 0\end{bmatrix} \tag{14}$$

where $\mathbf{A}_B^{-1}$ is the generalized inverse of $\mathbf{A}$. Premultiplying Eq. (14) by $\mathbf{A}_B$ in Eq. (13) gives

$$\mathbf{A}_B\mathbf{A}_B^{-1} = \begin{bmatrix}\mathbf{A}\mathbf{A}^{\dagger} + \underset{\sim}{\phi}_1\underset{\sim}{\phi}_1^T & \mathbf{A}\underset{\sim}{\psi}_1 \\ \underset{\sim}{\psi}_1^T\mathbf{A}^{\dagger} & \underset{\sim}{\psi}_1^T\underset{\sim}{\psi}_1\end{bmatrix} = \begin{bmatrix}\mathbf{I}_{N\times N} & 0 \\ 0 & 1\end{bmatrix}. \tag{15}$$

Since $\mathbf{A}^{\dagger}$ is the pseudo-inverse of $\mathbf{A}$, we obtain

$$\mathbf{A}\mathbf{A}^{\dagger} + \underset{\sim}{\phi}_1\underset{\sim}{\phi}_1^T = \mathbf{I}_{N\times N}. \tag{16}$$

Postmultiplying Eq. (16) by $\mathbf{A}$, we have

$$\mathbf{A}\mathbf{A}^{\dagger}\mathbf{A} = \mathbf{A}, \tag{17}$$

since $\underset{\sim}{\phi}_1^T\mathbf{A} = \{0\}^T$.

The element of $\mathbf{A}^{\dagger}$ in Eq. (17) satisfies the definition of Moore–Penrose pseudo-inverse [18]. Therefore, the generalized inverse of the singular matrix can be obtained from the inversion of the nonsingular bordered matrix.

By taking inner product for Eq. (11) with respect to $\underset{\sim}{\phi}_1$ to both sides, we have

$$\mathbf{A}\underset{\sim}{x}_r \cdot \underset{\sim}{\phi}_1 + c_1\underset{\sim}{\phi}_1 \cdot \underset{\sim}{\phi}_1 = \underset{\sim}{b} \cdot \underset{\sim}{\phi}_1. \tag{18}$$

Since $\underset{\sim}{\phi}_1 \cdot \underset{\sim}{\phi}_1 = 1$, Eq. (18) reduces to

$$\mathbf{A}\underset{\sim}{x}_r \cdot \underset{\sim}{\phi}_1 + c_1 = \underset{\sim}{b} \cdot \underset{\sim}{\phi}_1. \tag{19}$$

Following the property of $\left(\mathbf{A}\underset{\sim}{x}_r \cdot \underset{\sim}{\phi}_1\right) = \left(\underset{\sim}{x}_r \cdot \mathbf{A}^T\underset{\sim}{\phi}_1\right)$, we have

$$(\underset{\sim}{x}_r \cdot \mathbf{A}^T\underset{\sim}{\phi}_1) + c_1 = \underset{\sim}{b} \cdot \underset{\sim}{\phi}_1. \tag{20}$$

Based on SVD structure $\mathbf{A}\underset{\sim}{\psi}_1 = \sigma_1\underset{\sim}{\phi}_1$ and $\mathbf{A}^T\underset{\sim}{\phi}_1 = \sigma_1\underset{\sim}{\psi}_1 = \{0\}$ (now $\sigma_1 = 0$), Eq. (20) yields $c_1 = \underset{\sim}{b} \cdot \underset{\sim}{\phi}_1$.

3. Numerical examples

Four examples to determine the free–free flexibility and stiffness matrices by using the self-regularized technique are given in Fig. 2. In addition, the rigid body mode and spurious force mode would also be discussed.

Example 1: A linear rod element

For a one-dimensional and 2-nodes rod element as shown in Fig. 3, the free–free stiffness matrix $\mathbf{K}$ is shown below

$$\mathbf{K} = k\begin{bmatrix}1 & -1 \\ -1 & 1\end{bmatrix}, \tag{21}$$

where $k = EA/L$ is the axial stiffness. The flexibility matrix cannot be directly obtained, because the stiffness matrix $\mathbf{K}$ is singular. By employing the SVD, we have

$$\Sigma = k\begin{bmatrix}0 & \\ & 2\end{bmatrix}, \tag{22}$$

$$\Phi = \Psi = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & -1 \\ 1 & 1\end{bmatrix}, \tag{23}$$

where Φ and Ψ are the left and right singular matrix, respectively, and Σ is a diagonal matrix composed of singular value of the stiffness matrix $\mathbf{K}$. Since the stiffness matrix is a symmetric matrix, the right and left singular matrices are the same due to the symmetric stiffness matrix. According to Eq. (12), a linear algebraic system $\mathbf{K}\underset{\sim}{u} = \underset{\sim}{p}$ can be bordered as shown below:

$$\begin{bmatrix}\mathbf{K} & \underset{\sim}{\phi}_1 \\ \underset{\sim}{\psi}_1^T & 0\end{bmatrix}\begin{Bmatrix}\underset{\sim}{u}_r \\ c_1\end{Bmatrix} = \begin{Bmatrix}\underset{\sim}{p} \\ 0\end{Bmatrix}, \tag{24}$$

where $\underset{\sim}{\phi}_1$ and $\underset{\sim}{\psi}_1$ are the left and right singular vectors corresponding to the zero singular value of the stiffness matrix $\mathbf{K}$, respectively, $\underset{\sim}{u}_r$ is a regularized vector for the original vector $\underset{\sim}{u}$, and c_1 is an extra constant. The stiffness matrix $\mathbf{K}$ can be bordered as follows:

$$\mathbf{K}_B = \begin{bmatrix}\mathbf{K} & \underset{\sim}{\phi}_1 \\ \underset{\sim}{\psi}_1^T & 0\end{bmatrix} = \begin{bmatrix}k & -k & \frac{1}{\sqrt{2}} \\ -k & k & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0\end{bmatrix}. \tag{25}$$

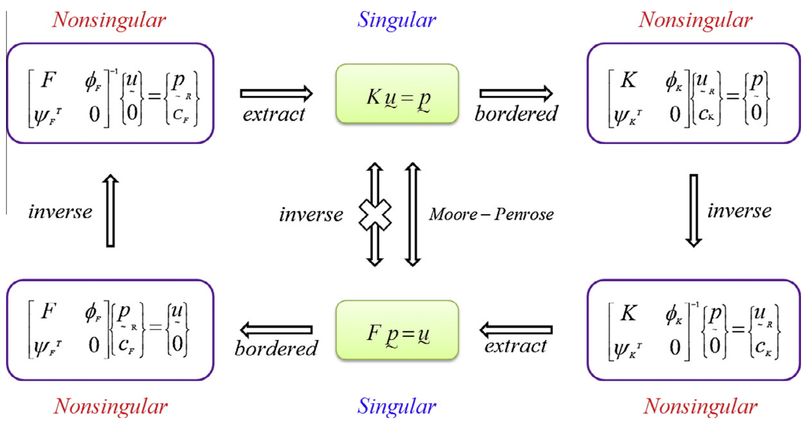


Fig. 2. The self-regularized system for flexibility and stiffness of the free–free structure.

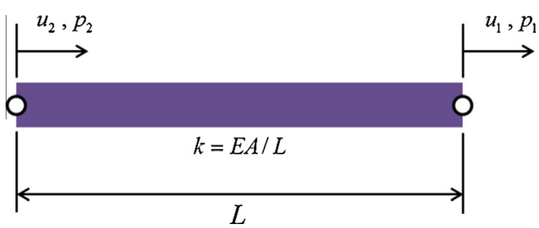


Fig. 3. The rod element for the example 1.

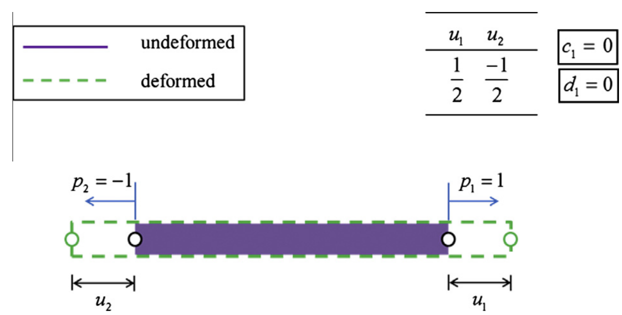


Fig. 4. The regularized displacement $\underset{\sim}{u}_r$ and the regularized force $\underset{\sim}{p}_r$ for a rod by using the self-regularized approach.

It is interesting to find that the bordered matrix $\mathbf{K}_B$ is nonsingular. Moreover, its inverse, $\mathbf{K}_B^{-1}$ yields

$$\begin{bmatrix} \frac{1}{4k} & -\frac{1}{4k} & \frac{1}{\sqrt{2}} \\ -\frac{1}{4k} & \frac{1}{4k} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix}. \tag{26}$$

By extracting the free–free flexibility matrix $\mathbf{F}$ from the matrix $\mathbf{K}_B^{-1}$, we obtain

$$\mathbf{F} = \frac{1}{4k}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}. \tag{27}$$

Since the singular value σ_1 is zero, the stiffness matrix $\mathbf{K}$ results in a null-space mode (rigid body mode) $\underset{\sim}{\psi}_1$ as shown below

$$\underset{\sim}{\psi}_1 = \frac{1}{\sqrt{2}}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}. \tag{28}$$

Without loss of generality, we set k equal to 1 for numerically calculating the inverse of the free–free stiffness matrix and axial loading are $p_1 = 1$ and $p_2 = -1$. According to Eq. (24), the displacement can be obtained

$$\underset{\sim}{u}_r = \frac{1}{2}\begin{Bmatrix} 1 \\ -1 \end{Bmatrix}, \tag{29}$$

and the constant c_1 is zero. The regularized displacement $\underset{\sim}{u}_r$ for a rod is shown in Fig. 4. The solution $\underset{\sim}{u}_r$ does not contain any homogeneous part (rigid body motion). Following the result of above section, the free–free flexibility matrix $\mathbf{F}$ can be obtained from the free–free stiffness matrix $\mathbf{K}$, even though the stiffness matrix $\mathbf{K}$ is singular. On the contrary, according to Eqs. (12) and (27), the linear algebraic system $\mathbf{F}\underset{\sim}{p} = \underset{\sim}{u}$ can be bordered to

$$\begin{bmatrix} \mathbf{F} & \underset{\sim}{\phi}_1 \\ \underset{\sim}{\psi}_1^T & 0 \end{bmatrix}\begin{Bmatrix} \underset{\sim}{p}_r \\ d_1 \end{Bmatrix} = \begin{Bmatrix} \underset{\sim}{u} \\ 0 \end{Bmatrix}. \tag{30}$$

where $\underset{\sim}{\phi}_1$ and $\underset{\sim}{\psi}_1$ are the left and right singular vectors corresponding to the zero singular value of the flexibility matrix $\mathbf{F}$, respectively, and $\underset{\sim}{p}_r$ is the regularized vector for the original vector $\underset{\sim}{p}$, and d_1 is a slack variable. Similarly, we can obtain the free–free stiffness matrix $\mathbf{K}$ to be the same as Eq. (21).

By specifying $u_1 = \frac{1}{2}$ and $u_2 = \frac{-1}{2}$, the spurious force mode $\underset{\sim}{\psi}_1$ and the regularized force $\underset{\sim}{p}_r$ can be determined as

$$\underset{\sim}{\psi}_1 = \frac{1}{\sqrt{2}}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}, \tag{31}$$

$$\underset{\sim}{p}_r = \begin{Bmatrix} 1 \\ -1 \end{Bmatrix}, \tag{32}$$

and the constant d_1 is zero. The nodal reaction force is shown in Fig. 4.

It is interesting to find the zero constant c_1. We will examine what the situation is if the constant c is not zero in the following example 2.

Example 2: A triangular truss constructed by three linear rods

A two-dimensional, 3-node and 6-dof, this triangular truss is shown in Fig. 5 and the free–free stiffness matrix $\mathbf{K}$ of the truss is shown below

$$\mathbf{K} = k\begin{bmatrix} \frac{5}{4} & -\frac{\sqrt{3}}{4} & -\frac{1}{4} & \frac{\sqrt{3}}{4} & -1 & 0 \\ -\frac{\sqrt{3}}{4} & \frac{3}{4} & \frac{\sqrt{3}}{4} & -\frac{3}{4} & 0 & 0 \\ -\frac{1}{4} & \frac{\sqrt{3}}{4} & \frac{1}{2} & 0 & -\frac{1}{4} & -\frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & -\frac{3}{4} & 0 & \frac{3}{2} & -\frac{\sqrt{3}}{4} & -\frac{3}{4} \\ -1 & 0 & -\frac{1}{4} & -\frac{\sqrt{3}}{4} & \frac{5}{4} & \frac{\sqrt{3}}{4} \\ 0 & 0 & -\frac{\sqrt{3}}{4} & -\frac{3}{4} & \frac{\sqrt{3}}{4} & \frac{3}{4} \end{bmatrix}, \tag{33}$$

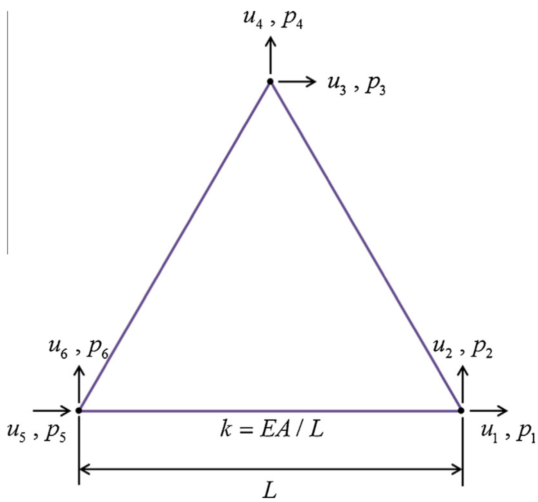


Fig. 5. The triangular truss for the example 2.

where $k = EA/L$ is the axial stiffness of the two-force member. The flexibility matrix **F** cannot be directly determined by inversing the singular stiffness matrix **K**. Theoretically speaking, the symbolic form of free–free flexibility matrix can be obtained. But the result is awkward and the derivation is tedious. We obtain the numerical result for the case of $k = 1$ for simplicity. By employing the SVD with respect to **K** and setting $k = 1$, we obtain

$$\Sigma = \begin{bmatrix} 0 & & & & & \\ & 0 & & & & \\ & & 0 & & & \\ & & & \frac{3}{2} & & \\ & & & & \frac{3}{2} & \\ & & & & & 3 \end{bmatrix}, \tag{34}$$

$$\Phi = \Psi = \begin{bmatrix} \frac{1}{2\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 & -\frac{1}{2\sqrt{3}} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{\sqrt{3}} & -\frac{1}{2} & -\frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 & -\frac{1}{2\sqrt{3}} & -\frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{\sqrt{3}} & \frac{1}{2} & -\frac{1}{2\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ \frac{1}{2\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{3}} & 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 & \frac{1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{3}} & \frac{1}{2\sqrt{3}} \end{bmatrix}. \tag{35}$$

The right and left singular matrices are the same due to the symmetric property. According to Eq. (12), the linear algebraic system $\mathbf{K}\underset{\sim}{u} = \underset{\sim}{p}$ can be bordered to

$$\mathbf{K}_B \begin{Bmatrix} \underset{\sim}{u}_r \\ c_1 \\ c_2 \\ c_3 \end{Bmatrix} = \begin{bmatrix} \mathbf{K} & \underset{\sim}{\phi}_1 & \underset{\sim}{\phi}_2 & \underset{\sim}{\phi}_3 \\ \underset{\sim}{\psi}_1^T & 0 & 0 & 0 \\ \underset{\sim}{\psi}_2^T & 0 & 0 & 0 \\ \underset{\sim}{\psi}_3^T & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \underset{\sim}{u}_r \\ c_1 \\ c_2 \\ c_3 \end{Bmatrix} = \begin{Bmatrix} \underset{\sim}{p} \\ 0 \\ 0 \\ 0 \end{Bmatrix}, \tag{36}$$

where $\underset{\sim}{\phi}_1, \underset{\sim}{\phi}_2, \underset{\sim}{\phi}_3$ and $\underset{\sim}{\psi}_1, \underset{\sim}{\psi}_2, \underset{\sim}{\psi}_3$ are the left singular vectors and the right singular vectors corresponding to three zero singular values of the free–free stiffness matrix **K**, respectively.

It is fantastic to find that the bordered matrix $\mathbf{K}_B$ is nonsingular even if the bordered matrix $\mathbf{K}_B$ consist of many zero entries. Therefore, the free–free flexibility matrix **F** can be obtained by inversing the regularized nonsingular matrix as shown below:

$$\mathbf{F} = \begin{bmatrix} \frac{11}{36} & -\frac{1}{12\sqrt{3}} & -\frac{1}{9} & -\frac{1}{6\sqrt{3}} & -\frac{7}{36} & \frac{1}{4\sqrt{3}} \\ -\frac{1}{12\sqrt{3}} & \frac{1}{4} & \frac{1}{3\sqrt{3}} & -\frac{1}{6} & -\frac{1}{4\sqrt{3}} & -\frac{1}{12} \\ -\frac{1}{9} & \frac{1}{3\sqrt{3}} & \frac{2}{9} & 0 & -\frac{1}{9} & -\frac{1}{3\sqrt{3}} \\ -\frac{1}{6\sqrt{3}} & -\frac{1}{6} & 0 & \frac{1}{3} & \frac{1}{6\sqrt{3}} & -\frac{1}{6} \\ -\frac{7}{36} & -\frac{1}{4\sqrt{3}} & -\frac{1}{9} & \frac{1}{6\sqrt{3}} & \frac{11}{36} & \frac{1}{12\sqrt{3}} \\ \frac{1}{4\sqrt{3}} & -\frac{1}{12} & -\frac{1}{3\sqrt{3}} & -\frac{1}{6} & \frac{1}{12\sqrt{3}} & \frac{1}{4} \end{bmatrix}. \tag{37}$$

Since three singular values σ_1, σ_2 and σ_3 are zeros, the rigid body mode can be found in the right singular vectors $\underset{\sim}{\psi}_1$, $\underset{\sim}{\psi}_2$ and $\underset{\sim}{\psi}_3$, respectively. In this paper, we employed the Mathematica software to calculate the singular vectors. The numerical results of singular vectors obtained from the Mathematica software may be different from those of using Fortran or Matlab. In order to easily understand the rigid body mode, we combined the singular vectors of zero singular values by using the linear superposition to obtain the easy-view rigid-body mode. The rigid body modes are shown in Fig. 6 and are given below:

$$\underset{\sim}{\psi}_1^T = \left\{ \frac{1}{2\sqrt{3}} \quad \frac{1}{2} \quad \frac{-1}{\sqrt{3}} \quad 0 \quad \frac{1}{2\sqrt{3}} \quad \frac{-1}{2} \right\}, \tag{38}$$

$$\underset{\sim}{\psi}_2^T = \left\{ \frac{1}{\sqrt{3}} \quad 0 \quad \frac{1}{\sqrt{3}} \quad 0 \quad \frac{1}{\sqrt{3}} \quad 0 \right\}, \tag{39}$$

$$\underset{\sim}{\psi}_3^T = \left\{ 0 \quad \frac{1}{\sqrt{3}} \quad 0 \quad \frac{1}{\sqrt{3}} \quad 0 \quad \frac{1}{\sqrt{3}} \right\}, \tag{40}$$

For the numerical implementation, we choose the vector of the external force

$$\underset{\sim}{p}^T = \{ 1 \quad 0 \quad 0 \quad 0 \quad -1 \quad 0 \}. \tag{41}$$

According to Eq. (36), the regularized solution $\underset{\sim}{u}_r$ can be obtained as

$$\underset{\sim}{u}_r^T = \left\{ \frac{1}{2} \quad \frac{1}{6\sqrt{3}} \quad 0 \quad \frac{-1}{3\sqrt{3}} \quad \frac{-1}{2} \quad \frac{1}{6\sqrt{3}} \right\} \tag{42}$$

and constants c_1, c_2 and c_3 are all zeros. The regularized displacement $\underset{\sim}{u}_r$ for the triangular truss is shown in Fig. 7.

Following the result of above section, the free–free flexibility matrix **F** can be obtained from the free–free stiffness matrix **K**, even though the stiffness matrix **K** is singular. According to Eqs. (12) and (37), the linear singular algebraic system $\mathbf{F}\underset{\sim}{p} = \underset{\sim}{u}$ can be similarly rewritten as a nonsingular bordered system,

$$\begin{bmatrix} \mathbf{F} & \underset{\sim}{\phi}_1 & \underset{\sim}{\phi}_2 & \underset{\sim}{\phi}_3 \\ \underset{\sim}{\psi}_1^T & 0 & 0 & 0 \\ \underset{\sim}{\psi}_2^T & 0 & 0 & 0 \\ \underset{\sim}{\psi}_3^T & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \underset{\sim}{p}_r \\ d_1 \\ d_2 \\ d_3 \end{Bmatrix} = \begin{Bmatrix} \underset{\sim}{u} \\ 0 \\ 0 \\ 0 \end{Bmatrix}, \tag{43}$$

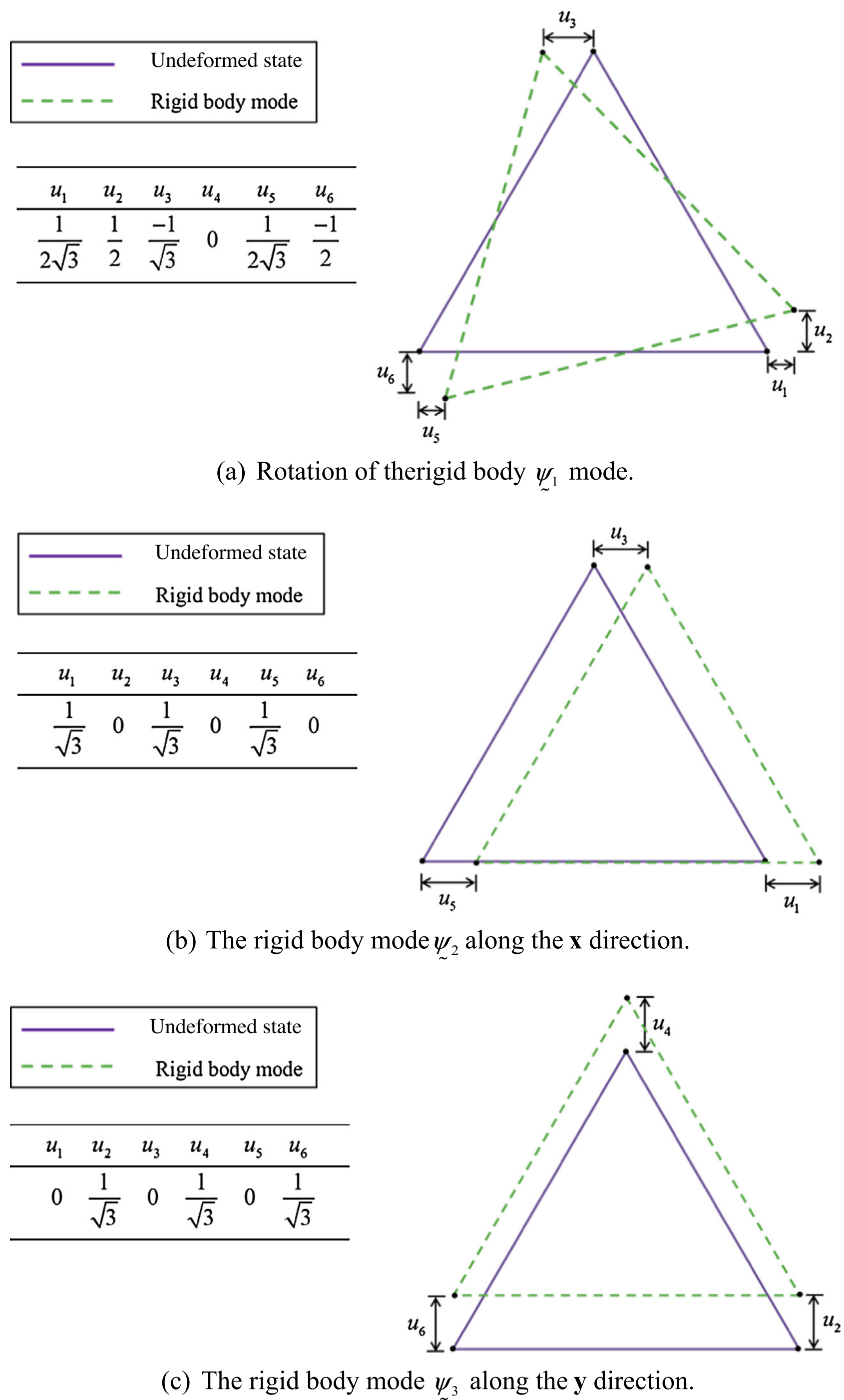
where $\underset{\sim}{\phi}_1, \underset{\sim}{\phi}_2, \underset{\sim}{\phi}_3$ and $\underset{\sim}{\psi}_1, \underset{\sim}{\psi}_2, \underset{\sim}{\psi}_3$ are the left singular vectors and the right singular vectors corresponding to zero singular values of the flexibility matrix **F**, respectively.

According to Eq. (43) for the bordered system, we obtain the stiffness matrix **K** with Eq. (33) and the right singular vectors corresponding to zero singular values (spurious force mode) as shown below:

$$\underset{\sim}{\psi}_1^T = \left\{ \frac{1}{2\sqrt{3}} \quad \frac{1}{2} \quad \frac{-1}{\sqrt{3}} \quad 0 \quad \frac{1}{2\sqrt{3}} \quad \frac{-1}{2} \right\} \tag{44}$$

$$\underset{\sim}{\psi}_2^T = \left\{ \frac{1}{\sqrt{3}} \quad 0 \quad \frac{1}{\sqrt{3}} \quad 0 \quad \frac{1}{\sqrt{3}} \quad 0 \right\}, \tag{45}$$

$$\underset{\sim}{\psi}_3^T = \left\{ 0 \quad \frac{1}{\sqrt{3}} \quad 0 \quad \frac{1}{\sqrt{3}} \quad 0 \quad \frac{1}{\sqrt{3}} \right\}. \tag{46}$$



u_1	u_2	u_3	u_4	u_5	u_6
$\frac{1}{2\sqrt{3}}$	$\frac{1}{2}$	$\frac{-1}{\sqrt{3}}$	0	$\frac{1}{2\sqrt{3}}$	$\frac{-1}{2}$

(a) Rotation of therigid body $\underset{\sim}{\psi}_1$ mode.

u_1	u_2	u_3	u_4	u_5	u_6
$\frac{1}{\sqrt{3}}$	0	$\frac{1}{\sqrt{3}}$	0	$\frac{1}{\sqrt{3}}$	0

(b) The rigid body mode $\underset{\sim}{\psi}_2$ along the **x** direction.

u_1	u_2	u_3	u_4	u_5	u_6
0	$\frac{1}{\sqrt{3}}$	0	$\frac{1}{\sqrt{3}}$	0	$\frac{1}{\sqrt{3}}$

(c) The rigid body mode $\underset{\sim}{\psi}_3$ along the **y** direction.

Fig. 6. The rigid body modes for a truss by using the SVD technique.

For the numerical implementation, the specified displacement $\underset{\sim}{u}$ is the same as Eq. (42) and the regularized force $\underset{\sim}{p}_r$ can be obtained as shown below:

$$\underset{\sim}{p}_r^T = \{1 \quad 0 \quad 0 \quad 0 \quad -1 \quad 0\}. \tag{47}$$

and constants d_1, d_2 and d_3 are all zeros. The spurious force mode and numerical result are shown in Figs. 8 and 9, respectively.

However, the specified force vector not satisfying the force equilibrium is shown below:

$$\underset{\sim}{p}^T = \{1 \quad 0 \quad 0 \quad 0 \quad 1 \quad 0\}. \tag{48}$$

We obtain

$$\underset{\sim}{u}_r^T = \left\{\tfrac{1}{9} \quad \tfrac{-1}{3\sqrt{3}} \quad \tfrac{-2}{9} \quad 0 \quad \tfrac{1}{9} \quad \tfrac{1}{3\sqrt{3}}\right\}, \tag{49}$$

and the constants $c_1 = \frac{-1}{2\sqrt{3}}$, $c_2 = \frac{-5}{6}$ and $c_3 = 0$. The numerical result is shown in Fig. 10. In this case, the constants c_1 and c_2 are not zeros. It is interesting to find that the specified force vector in Eq. (48) does not satisfy the moment equilibrium and the horizontal force equilibrium. Mathematically speaking, c_i is equal to the inner product

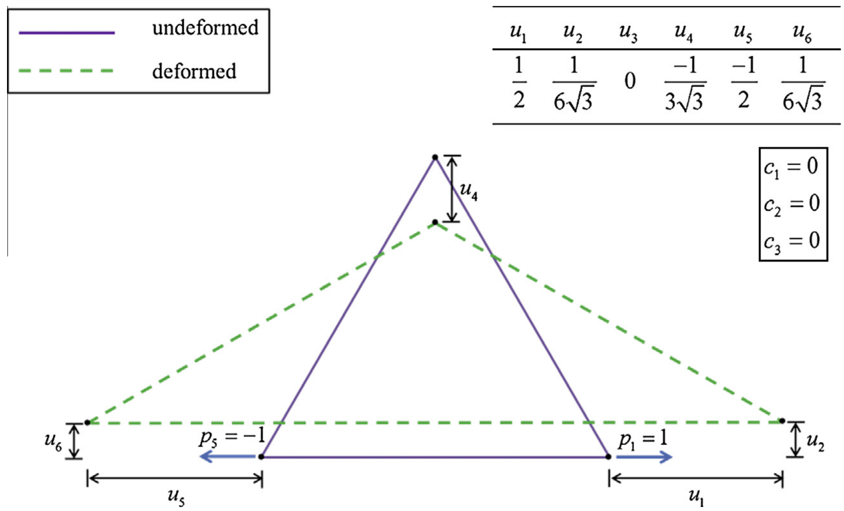


Fig. 7. The regularized displacement $\underset{\sim}{u}_r$ for a truss subject to equilibrium loading by using the self-regularized approach.

of $\underset{\sim}{\phi}_i$ and $\underset{\sim}{b}$. This matches nonzero c_1 (moment unequilibrium) and c_2 (horizontal force unequilibrium).

Example 3: A linear plane beam element

For a 2-node and 4-dof Bernoulli–Euler prismatic plane beam element as shown in Fig. 11, the free–free stiffness matrix **K** is

$$\mathbf{K} = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}, \tag{50}$$

where $\frac{EI}{L^3}$ is the flexural rigidity.

For simplicity, we set $EI = 1$ and $L = 1$. After employing the SVD, we obtain

$$\Sigma = \begin{bmatrix} 0 & & & \\ & 0 & & \\ & & 2 & \\ & & & 30 \end{bmatrix}, \tag{51}$$

$$\Phi = \Psi = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{10}} & 0 & \frac{2}{\sqrt{10}} \\ 0 & \frac{2}{\sqrt{10}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{10}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{10}} & 0 & -\frac{2}{\sqrt{10}} \\ 0 & \frac{2}{\sqrt{10}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{10}} \end{bmatrix}. \tag{52}$$

Since the stiffness matrix is a symmetric matrix, the right and left singular vectors are identical.

According to Eq. (12), the linear algebraic system $\mathbf{K}\underset{\sim}{u} = \underset{\sim}{p}$ can be bordered to

$$\begin{bmatrix} \mathbf{K} & \underset{\sim}{\phi}_1 & \underset{\sim}{\phi}_2 \\ \underset{\sim}{\psi}_1^T & 0 & 0 \\ \underset{\sim}{\psi}_2^T & 0 & 0 \end{bmatrix}\begin{Bmatrix} \underset{\sim}{u}_r \\ c_1 \\ c_2 \end{Bmatrix} = \begin{Bmatrix} \underset{\sim}{p} \\ 0 \\ 0 \end{Bmatrix}, \tag{53}$$

The flexibility matrix **F** can be obtained by inversing the regularized matrix as shown below

$$\mathbf{F} = \frac{1}{75}\begin{bmatrix} 1 & \frac{1}{2} & -1 & \frac{1}{2} \\ \frac{1}{2} & 19 & -\frac{1}{2} & \frac{37}{2} \\ -1 & -\frac{1}{2} & 1 & -\frac{1}{2} \\ \frac{1}{2} & \frac{37}{2} & -\frac{1}{2} & 19 \end{bmatrix}. \tag{54}$$

Since the singular values σ_1 and σ_2 are both zeros, the rigid body mode can be found in the right singular vectors $\underset{\sim}{\psi}_1$ and $\underset{\sim}{\psi}_2$, respectively. The rigid body modes are shown below

$$\underset{\sim}{\psi}_1^T = \frac{1}{\sqrt{2}}\{1 \quad 0 \quad 1 \quad 0\}, \tag{55}$$

$$\underset{\sim}{\psi}_2^T = \frac{1}{\sqrt{10}}\{-1 \quad 2 \quad 1 \quad 2\} \tag{56}$$

For the numerical implementation, the specified generalized force vector is pure bending of moment as given below

$$\underset{\sim}{p}^T = \{0 \quad 1 \quad 0 \quad -1\}. \tag{57}$$

Therefore, we obtain

$$\underset{\sim}{u}_r^T = \{0 \quad \tfrac{1}{2} \quad 0 \quad -\tfrac{1}{2}\} \tag{58}$$

and the constants c_1 and c_2 are both zeros. The pure bending mode $\underset{\sim}{u}_r$ for the beam element is shown in Fig. 12.

Following the result of the above section, the free–free flexibility matrix **F** can be obtained from the free–free stiffness matrix **K**, even though the stiffness matrix **K** is singular. According to Eqs. (12) and (54), the linear algebraic system $\mathbf{F}\underset{\sim}{p} = \underset{\sim}{u}$ can be similarly rewritten as

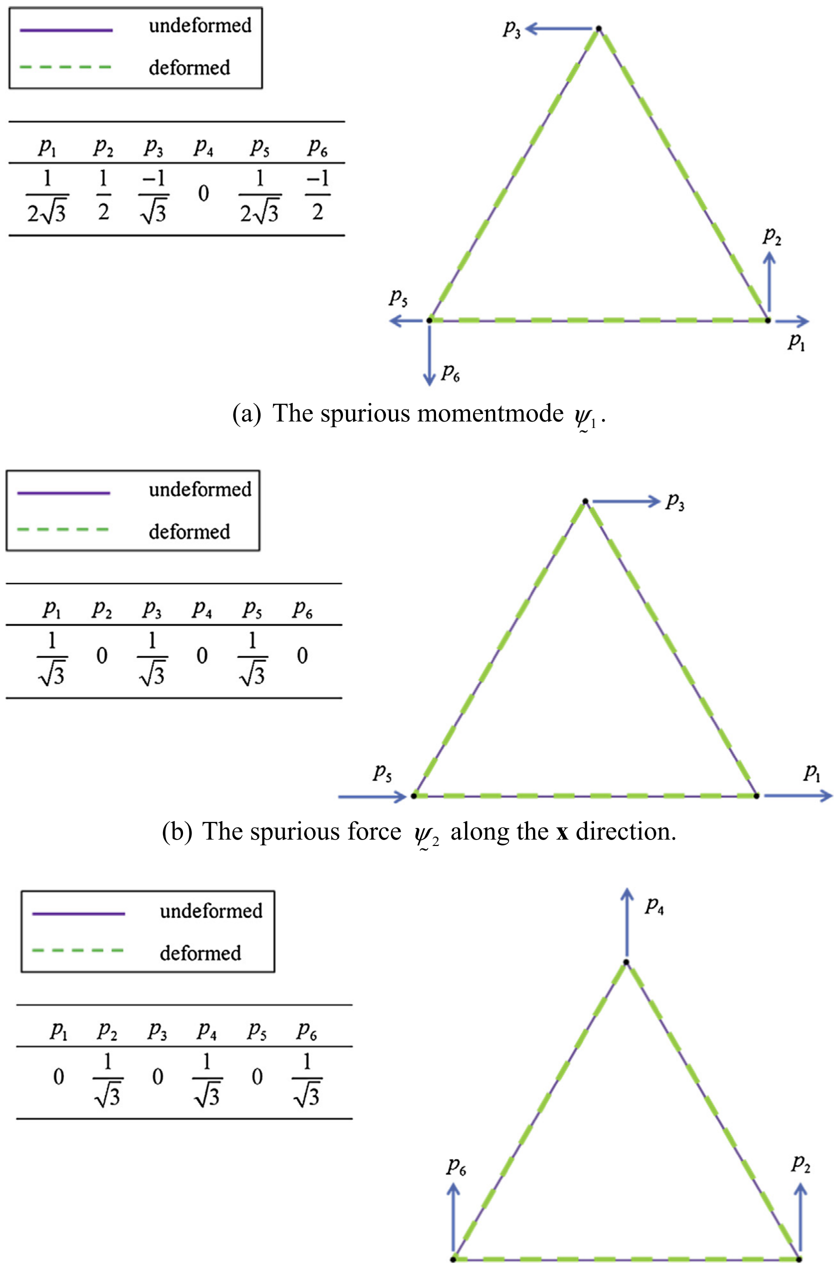
$$\begin{bmatrix} \mathbf{F} & \underset{\sim}{\phi}_1 & \underset{\sim}{\phi}_2 \\ \underset{\sim}{\psi}_1^T & 0 & 0 \\ \underset{\sim}{\psi}_2^T & 0 & 0 \end{bmatrix}\begin{Bmatrix} \underset{\sim}{p}_r \\ d_1 \\ d_2 \end{Bmatrix} = \begin{Bmatrix} \underset{\sim}{u} \\ 0 \\ 0 \end{Bmatrix}. \tag{59}$$

According to Eq. (59), we obtain the stiffness matrix **K** to be equal to that of Eq. (50) and the right singular vectors corresponding to zero singular values (spurious force mode) are shown below:

$$\underset{\sim}{\psi}_1^T = \frac{1}{\sqrt{2}}\{1 \quad 0 \quad 1 \quad 0\}, \tag{60}$$

$$\underset{\sim}{\psi}_2^T = \frac{1}{\sqrt{10}}\{-1 \quad 2 \quad 1 \quad 2\}. \tag{61}$$

For the numerical study, the specified displacement condition $\underset{\sim}{u}$ is the same as Eq. (58) and the regularized force $\underset{\sim}{p}_r$ can be obtained as shown below



p_1	p_2	p_3	p_4	p_5	p_6
$\frac{1}{2\sqrt{3}}$	$\frac{1}{2}$	$\frac{-1}{\sqrt{3}}$	0	$\frac{1}{2\sqrt{3}}$	$\frac{-1}{2}$

(a) The spurious momentmode $\underset{\sim}{\psi}_1$.

p_1	p_2	p_3	p_4	p_5	p_6
$\frac{1}{\sqrt{3}}$	0	$\frac{1}{\sqrt{3}}$	0	$\frac{1}{\sqrt{3}}$	0

(b) The spurious force $\underset{\sim}{\psi}_2$ along the **x** direction.

p_1	p_2	p_3	p_4	p_5	p_6
0	$\frac{1}{\sqrt{3}}$	0	$\frac{1}{\sqrt{3}}$	0	$\frac{1}{\sqrt{3}}$

(c) The spurious force mode $\underset{\sim}{\psi}_3$ along the **y** direction.

Fig. 8. The spurious force modes for a truss by using the SVD technique.

$$\underset{\sim}{p}_r^T = \{0 \quad 1 \quad 0 \quad -1\}. \tag{62}$$

Constants of d_1 and d_2 are both zeros. The numerical result is shown in Fig. 12.

Example 4: A unsymmetric bar stiffness

The last example is a geometrically nonlinear bar with 2-node element moving in the xy plane subjected to a lateral pressure q as shown in Fig. 13 and the free–free stiffness matrix **K** is

$$\mathbf{K} = \frac{EA}{L}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \frac{P}{L}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} + \frac{q}{2}\begin{bmatrix} 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{bmatrix}, \tag{63}$$

where P is a axial prestress force. The free–free stiffness matrix **K** is singular and asymmetric. By setting $EA = 1$, $P = 1$, $q = 2$, $L = 1$ for simplicity and by employing the SVD, we obtain

$$\Phi = \begin{bmatrix} \frac{1}{\sqrt{26}} & \frac{-9}{2\sqrt{65}} & \frac{1}{2} & \frac{2}{\sqrt{10}} \\ \frac{4}{\sqrt{26}} & \frac{3}{2\sqrt{65}} & \frac{-1}{2} & \frac{1}{\sqrt{10}} \\ \frac{3}{\sqrt{26}} & \frac{-1}{2\sqrt{65}} & \frac{1}{2} & \frac{-2}{\sqrt{10}} \\ 0 & \frac{\sqrt{13}}{2\sqrt{5}} & \frac{1}{2} & \frac{1}{\sqrt{10}} \end{bmatrix}, \tag{64}$$

$$\Sigma = \begin{bmatrix} 0 & & & \\ & 0 & & \\ & & 2\sqrt{2} & \\ & & & 2\sqrt{5} \end{bmatrix}, \tag{65}$$

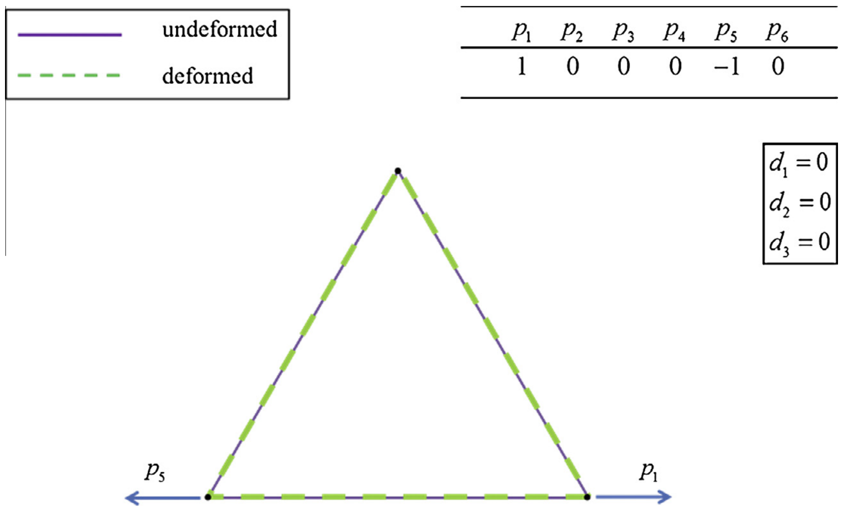


Fig. 9. The regularized force $\underset{\sim}{p}_r$ for a truss by using the self-regularized approach.

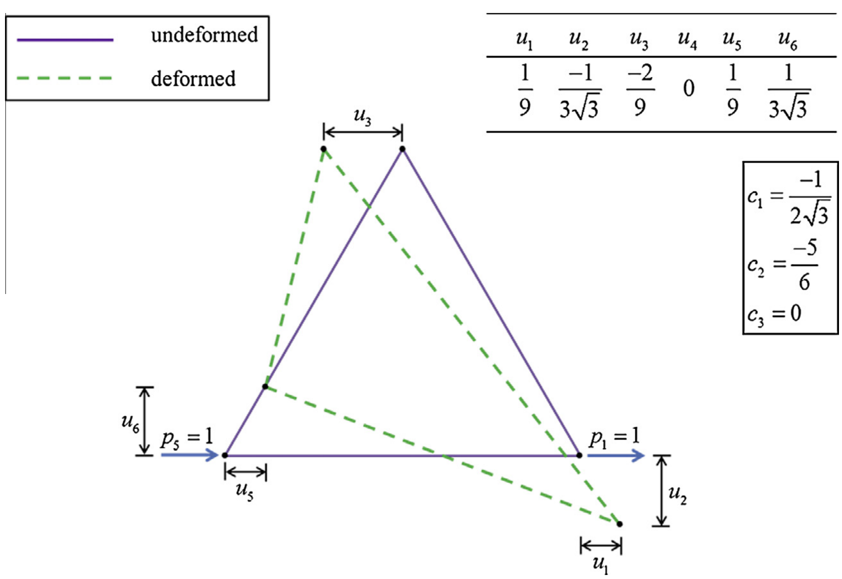


Fig. 10. The numerical result $\underset{\sim}{u}_r$ of a truss under unbalanced force system.

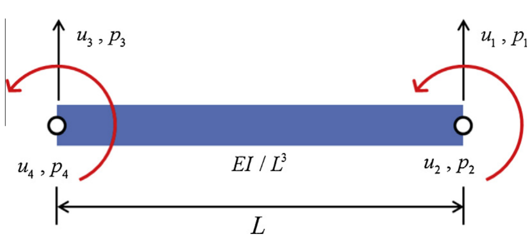


Fig. 11. The beam element for the example 3.

$$\Psi = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}. \qquad (66)$$

According to Eq. (12), the linear algebraic system $\mathbf{K}\underset{\sim}{u} = \underset{\sim}{p}$ can be bordered as

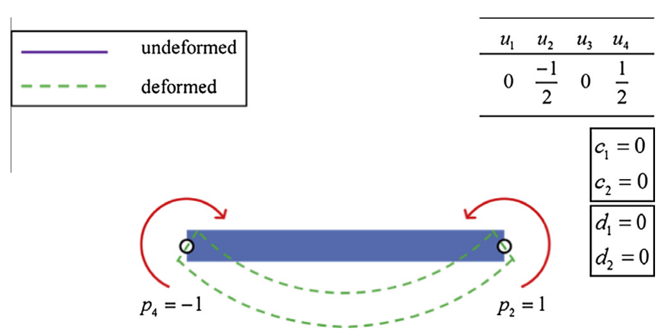


Fig. 12. The regularized displacement $\underset{\sim}{u}_r$ and the regularized force $\underset{\sim}{p}_r$ for a beam subject to pure bending by using the self-regularized approach.

$$\begin{bmatrix} \mathbf{K} & \underset{\sim}{\phi}_1 & \underset{\sim}{\phi}_2 \\ \underset{\sim}{\psi}_1^T & 0 & 0 \\ \underset{\sim}{\psi}_2^T & 0 & 0 \end{bmatrix} \begin{Bmatrix} \underset{\sim}{u}_r \\ c_1 \\ c_2 \end{Bmatrix} = \begin{Bmatrix} \underset{\sim}{p} \\ 0 \\ 0 \end{Bmatrix}, \qquad (67)$$

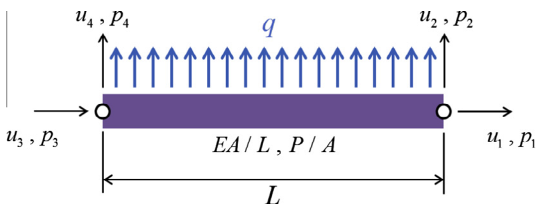


Fig. 13. The rod element with unsymmetric stiffness for the example 4.

where $\underset{\sim}{\phi}_1, \underset{\sim}{\phi}_2$ and $\underset{\sim}{\psi}_1, \underset{\sim}{\psi}_2$ are the left singular vectors and the right singular vectors according to zero singular values of the stiffness matrix $\mathbf{K}$, respectively. The free–free flexibility matrix $\mathbf{F}$ can be determined by inversing the bordered matrix as shown below

$$\mathbf{F} = \frac{1}{4}\begin{bmatrix} \frac{2}{5} & \frac{1}{5} & \frac{-2}{5} & \frac{1}{5} \\ \frac{-1}{2} & \frac{1}{2} & \frac{-1}{2} & \frac{-1}{2} \\ \frac{-2}{5} & \frac{-1}{5} & \frac{2}{5} & \frac{-1}{5} \\ \frac{1}{2} & \frac{-1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}. \tag{68}$$

Since singular values σ_1 and σ_2 are both zeros, the rigid body mode can be found in the right singular vectors $\underset{\sim}{\psi}_1$ and $\underset{\sim}{\psi}_2$, respectively. The rigid body modes are shown below:

$$\underset{\sim}{\psi}_1^T = \frac{1}{\sqrt{2}}\{1 \quad 0 \quad 1 \quad 0\}, \tag{69}$$

$$\underset{\sim}{\psi}_2^T = \frac{1}{\sqrt{2}}\{0 \quad 1 \quad 0 \quad 1\}. \tag{70}$$

Following the result of the above section, the free–free flexibility matrix $\mathbf{F}$ can be obtained from the free–free stiffness matrix $\mathbf{K}$, even though the stiffness matrix $\mathbf{K}$ is singular. Similarly, according to Eqs. (12) and (68), the linear algebraic system $\mathbf{F}\underset{\sim}{p} = \underset{\sim}{u}$ can be rewritten as

$$\begin{bmatrix} \mathbf{F} & \underset{\sim}{\phi}_1 & \underset{\sim}{\phi}_2 \\ \underset{\sim}{\psi}_1^T & 0 & 0 \\ \underset{\sim}{\psi}_2^T & 0 & 0 \end{bmatrix} \begin{Bmatrix} \underset{\sim}{p}_r \\ d_1 \\ d_2 \end{Bmatrix} = \begin{Bmatrix} \underset{\sim}{u} \\ 0 \\ 0 \end{Bmatrix}, \tag{71}$$

where $\underset{\sim}{\phi}_1, \underset{\sim}{\phi}_2$ and $\underset{\sim}{\psi}_1, \underset{\sim}{\psi}_2$ are the left singular vectors and the right singular vectors according to zero singular values of the flexibility matrix $\mathbf{F}$), respectively.

According to Eq. (71), we obtain the stiffness matrix $\mathbf{K}$ to be equal to that of Eq. (63) and the right singular vectors corresponding to zero singular values (spurious force mode) are shown below

$$\underset{\sim}{\psi}_1^T = \frac{1}{2\sqrt{35}}\{3 \quad 9 \quad 7 \quad -1\}, \tag{72}$$

$$\underset{\sim}{\psi}_2^T = \frac{1}{\sqrt{14}}\{-2 \quad 1 \quad 0 \quad 3\}. \tag{73}$$

and corresponding sketches are shown in Fig. 14.

By using the self-regularized approach, both satisfy the Moore–Penrose pseudo inverse as shown below:

$$\mathbf{KFK} = \mathbf{K}, \tag{74}$$

and

$$\mathbf{FKF} = \mathbf{F}. \tag{75}$$

No matter the derivation of the free–free flexibility matrix $\mathbf{F}$ from the free–free stiffness matrix $\mathbf{K}$ or the derivation of the free–free stiffness matrix $\mathbf{K}$ from the free–free flexibility matrix $\mathbf{F}$, our approach can be used.

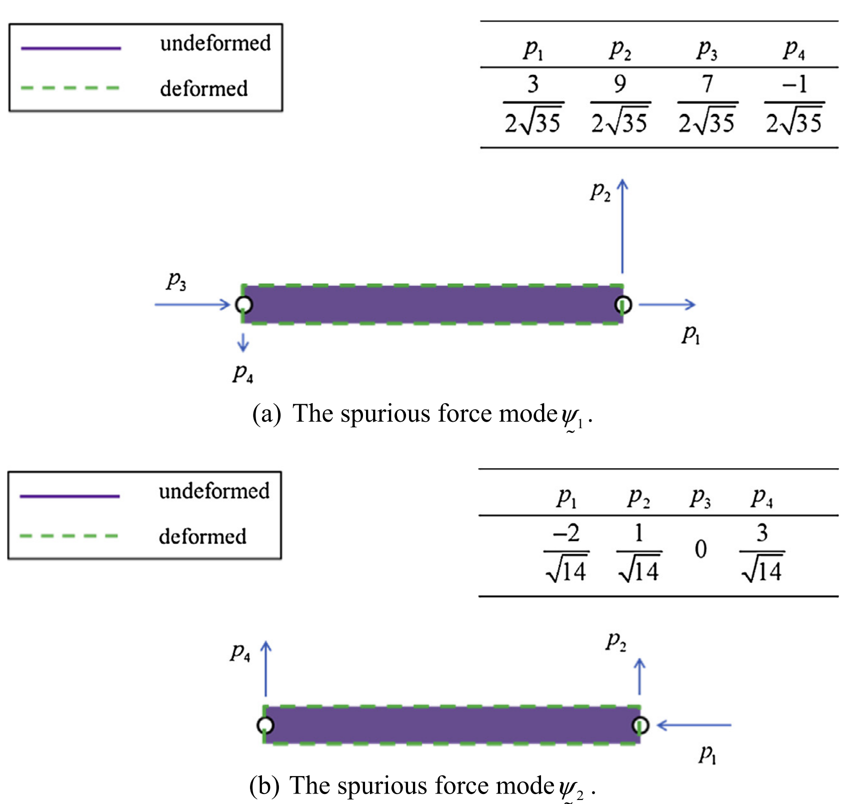


Fig. 14. The spurious force modes for a rod with unsymmetric stiffness by using the SVD technique.

4. Conclusions

In this paper, we remove the indeterminacy in the rigid body mode or spurious mode for the free–free stiffness and flexibility matrices by enforcing the solution to satisfy the constraint of orthogonality, respectively. Then, we add the corresponding null space since the range of the mapping is deficient. Finally, a singular stiffness or flexibility matrix can be bordered to a nonsingular matrix. By inversing the nonsingular matrix, we can obtain the generalized inverse matrix from the submatrix of the ordinary inversion with respect to the nonsingular matrix. Not only can we derive the flexibility matrix from the singular stiffness matrix but also we can derive the stiffness matrix from the singular flexibility matrix. Both stiffness and flexibility matrices satisfy the well-known Moore–Penrose equations. Four examples were demonstrated to see the validity of the present formulation. The physically unrealizable loading and incompatibility of the displacement constraint can be both examined by detecting from the nonzero parameter c in the regularized formulation.

Acknowledgements

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