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ON THE TRUE AND SPURIOUS EIGENVALUES BY USING THE REAL OR THE IMAGINARY-PART OF THE METHOD OF FUNDAMENTAL SOLUTIONS

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In this paper, the method of fundamental solutions (MFS) of real-part or imaginary-part kernels is employed to solve two-dimensional eigenproblems. The occurring mechanism of spurious eigenvalues for circular and elliptical membranes is examined. It is found that the spurious eigensolution using the MFS depends on the location of the fictitious boundary where the sources are distributed. By employing the singular value decomposition

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technique, the common left unitary vectors of the true eigenvalue for the single- and double-layer potential approaches are found while the common right unitary vectors of the spurious eigenvalue are obtained. Dirichlet and Neumann eigenproblems are both considered. True eigenvalues are dependent on the boundary condition while spurious eigenvalues are different in the different approach, single-layer or double-layer potential MFS. Two examples of circular and elliptical membranes are numerically demonstrated to see the validity of the present method and the results are compared well with the theoretical prediction.

Keywords: Method of fundamental solutions; singular value decomposition; spurious eigenvalue; right unitary; left unitary.

1. Introduction

Over past decades, researchers have paid attention to the meshless method without employing the concept of elements. The initial idea of the meshless method dates back to the smooth particle hydrodynamics (SPH) method for modeling astrophysical phenomena [Gingold and Maraghan (1977)]. Several meshless methods have also been reported in the literature, for example, the domain-based methods including the element-free Galerkin method [Belystcho *et al.* (1994)], the reproducing kernel method [Liu *et al.* (1995)], and the boundary-based methods including the boundary node method [Mukherjee and Mukherjee (1997)], the meshless local Petrov–Galerkin approach [Atluri and Zhu (1998)], the local boundary integral equation method [Sladek *et al.* (2000)], the RBF approach [Chen *et al.* (1998a); Golberg *et al.* (2000); Zhang *et al.* (2000)] and the method of fundamental solutions. The method of fundamental solutions (MFS) is a numerical approach by using the superposition concept. It is well known that the MFS can deal with engineering problems when a fundamental solution is known. The basic idea is to approximate the solution by forming a linear combination of fundamental solutions with sources distributing outside the problem domain. The coefficients of the linear combination are determined by satisfying boundary conditions. This method was attributed to Kupradze (1964). The MFS has been applied to potential [Fairweather and Karageorghis (1998)], eigenvalue [Fan *et al.* (2009)], diffusion [Chen *et al.* (1998b)], biharmonic [Poullikkas *et al.* (1998)] and elasticity problems [Kupradze (1964)]. A study for the fictitious frequency in the MFS was done by Chen [2006]. A nondimensional dynamic influence function (NDIF) method which is a special case of the MFS of imaginary-part kernel was applied to solve the eigenproblem of an elliptical membrane [Chen *et al.* (2012a)]. An extension to plate eigenproblems by using the MFS can be found in [Chen *et al.* (2005)].

Eigenanalysis is very important for vibration and acoustics, because it can provide some fundamental information. Since analytical solutions are sometimes not available, numerical methods are needed. Nearly 40 years ago, Tai and Shaw [1974] first employed the complex-valued BEM to solve membrane vibration problem. De Mey [1976; 1977] proposed a simplified approach by using only the real-part or imaginary-part kernel to solve the membrane problem. Hutchinson and Wong [1979] employed only the real-part kernel to solve plate vibrations. Although the complex-valued computation is avoided, they faced the occurrence of spurious eigenequations. The spurious eigensolutions for simply-connected problems are due to the loss of constraint by only using real-part or the imaginary-part kernel.

In the recent years, the singular value decomposition (SVD) technique has been applied to solve problems of continuum mechanics [Chen *et al.* (2002a)] and fictitious-frequency problems [Chen *et al.* (2006)]. Two ideas, updating term and updating document [Chen *et al.* (2002b)], were successfully applied to extract the true and spurious solutions, respectively. Based on these successful experiences, the SVD updating technique is employed to study the true and spurious eigenvalues in a unified manner for the MFS of real- and imaginary-part kernels.

In this paper, the MFS for solving the eigenfrequencies of circular and elliptical membranes are proposed. Both the MFS of real- and imaginary-part kernels are used for the Dirichlet as well as the Neumann problems. The appearance of the spurious eigensolution of the membrane and its treatment in the MFS are studied numerically. Two examples of circular and elliptical membranes are demonstrated. Besides, left and right unitary vectors are shown in bar charts. Common left and right unitary vectors in the four influence matrices of the MFS for the true and spurious eigenvalues are also examined.

2. Formulation of the Eigenproblem Using the Method of Fundamental Solutions

The governing equation for membrane vibration is the Helmholtz equation as follows:

$$(\nabla^2 + k^2)u(x) = 0, \quad x \in D, \tag{1}$$

where ∇^2 is the Laplacian operator, D is the domain of interest and k is the wave number. The fundamental solution $U(s,x)$ which satisfies

$$(\nabla^2 + k^2)U(s,x) = -\delta(x-s), \tag{2}$$

where $\delta(x-s)$ is the Dirac delta function. According to the dual formulation [Chen and Chen (1998)], we have the four kernels

$$U(s,x) = \frac{iJ_0(kr) - Y_0(kr)}{4}, \tag{3}$$

$$T(s,x) = \frac{\partial U(s,x)}{\partial n_s} = -\frac{k}{4}\left(\frac{iJ_1(kr) - Y_1(kr)}{r} y_i n_i\right), \tag{4}$$

$$L(s,x) = \frac{\partial U(s,x)}{\partial n_x} = \frac{k}{4}\left(\frac{iJ_1(kr) - Y_1(kr)}{r} y_i \bar{n}_i\right), \tag{5}$$

$$M(s,x) = \frac{\partial^2 U(s,x)}{\partial n_x \partial n_s} = \frac{k}{4}\left(\frac{k(-iJ_2(kr) + Y_2(kr))}{r^2} y_i y_j n_i \bar{n}_j + \frac{iJ_1(kr) - Y_1(kr)}{r} n_i \bar{n}_i\right), \tag{6}$$

where $r \equiv |s-x|$ is the distance between the source and collocation points; n_i is the ith component of the unit outnormal vector at s; $\bar{n}_i$ is the ith component of the unit outnormal vector at x, J_m and Y_m denote the first kind and second kind of the mth-order Bessel function of the first and second, respectively, and $y_i \equiv s_i - x_i$, $i = 1, 2, \ldots$ are the differences of the ith components of s and x, respectively. Based on the indirect method using the dual formulation, we can represent the field solution by

- Single-layer potential approach

$$u(x_i) = \sum_j U(s_j, x_i) p_j, \tag{7}$$

$$t(x_i) = \sum_j L(s_j, x_i) p_j. \tag{8}$$

- Double-layer potential approach

$$u(x_i) = \sum_j T(s_j, x_i) q_j, \tag{9}$$

$$t(x_i) = \sum_j M(s_j, x_i) q_j. \tag{10}$$

The matrix forms of Eqs. (7)–(10) are

- Single-layer potential approach

$$\{u_i\} = [U_{ij}]\{p_j\}, \tag{11}$$

$$\{t_i\} = [L_{ij}]\{p_j\}. \tag{12}$$

- Double-layer potential approach

$$\{u_i\} = [T_{ij}]\{q_j\}, \tag{13}$$

$$\{t_i\} = [M_{ij}]\{q_j\}, \tag{14}$$

where $\{p_j\}$ and $\{q_j\}$ are the generalized unknowns by using the single- and double-layer potential approaches, respectiely.

For simplicity, the boundary condition of Dirichlet type ($\bar{u} = 0$) or Neumann type ($\bar{t} = 0$) are considered. We distributed N collocation points at each real boundary and N source points at each fictitious boundary. By matching the boundary condition, the linear algebraic equations can be obtained by using the single- and double-layer potential approaches in Eqs. (7)–(10) as shown below:

For the Dirichlet problem

$$[U_{ij}]\{p_j\} = \{0\}, \tag{15}$$

$$[T_{ij}]\{q_j\} = \{0\}. \tag{16}$$

For the Neumann problem

$$[L_{ij}]\{p_j\} = \{0\}, \tag{17}$$

$$[M_{ij}]\{q_j\} = \{0\}. \tag{18}$$

The determinant of the matrices $[U_{ij}]$, $[L_{ij}]$, $[T_{ij}]$ and $[M_{ij}]$ must be zero to obtain the nontrivial eigensolution, i.e.,

$$\det[U_{ij}] = 0, \tag{19}$$

$$\det[L_{ij}] = 0, \tag{20}$$

$$\det[T_{ij}] = 0, \tag{21}$$

$$\det[M_{ij}] = 0. \tag{22}$$

By plotting the minimum singular value versus the wave number, the curve drops at the positions of eigenvalue [Chen *et al.* (2012a)]. In the real implementation, either the real part or the imaginary part of the fundamental solution in Eq. (3) can be adopted independently to avoid computational cost.

3. Treatments of Spurious Eigenvalues

The SVD technique is an important tool in the linear algebra. A matrix G with dimension M by N can be decomposed into a product of an orthogonal matrix Φ (M by M), a diagonal matrix $\sum$ (M by N) with positive or zero elements, and an orthogonal matrix Ψ (N by N), as shown below:

$$[G]_{M\times N} = [\Phi]_{M\times M}\left[\sum\right]_{M\times N}[\Psi]^T_{N\times N}, \tag{23}$$

where the superscript "T" is the transpose operator, Φ and Ψ are both orthogonal in the sense that their column vectors are orthogonal,

$$\phi_i \cdot \phi_j = \delta_{ij}, \tag{24}$$

$$\psi_i \cdot \psi_j = \delta_{ij}, \tag{25}$$

where $\Phi^T\Phi = I$, $\Psi^T\Psi = I$ and δ_{ij} is the Dirac delta symbol.

4. Common Left and Right Unitary Vectors for True and Spurious Eigenvalues

4.1. *True eigenvalue*

For the Dirichlet boundary condition, we have the specified displacement on the boundary. Similarly, the Neumann boundary condition gives the slope information on the boundary. Due to the existence of solution $\{p\}$ for the nonhomogeneous Dirichlet problem ($u = \bar{u}$ on the boundary), we have two alternative formulations

$$[U(k_{\mathrm{T}})]\{p\} = [T(k_{\mathrm{T}})]\{q\} = \{\bar{u}\}, \tag{26}$$

where k_{T} is the true eigenvalue. It must be noted that the true eigenvalue k_{T} makes $[U]$ and $[T]$ both singular. According to a necessary and sufficient solvability condition [Stakgold (1998)], the existence of solution for the nonhomogeneous Dirichlet problem implies that there exists $\{\alpha\}$ such that

$$\{\bar{u}\}^T\{\alpha\} = 0, \tag{27}$$

where $\{\alpha\}$ satisfies

$$\{p\}^T[U(k_{\mathrm{T}})]^T\,\{\alpha\} = 0. \tag{28}$$

Substituting Eq. (26) into Eq. (27), we have

$$\{q\}^T[T(k_{\mathrm{T}})]^T\{\alpha\} = 0. \tag{29}$$

Since $\{p\}$ and $\{q\}$ can be arbitrary for any specified $\{\bar{u}\}$, Eqs. (28) and (29) imply

$$[U(k_{\mathrm{T}})]^T\,\{\alpha\} = \{0\}, \tag{30}$$

$$[T(k_{\mathrm{T}})]^T\,\{\alpha\} = \{0\}. \tag{31}$$

Combining Eq. (30) with Eq. (31), we have

$$\begin{bmatrix} [U(k_{\mathrm{T}})]^T \\ [T(k_{\mathrm{T}})]^T \end{bmatrix} \{\alpha\} = \{0\}, \tag{32}$$

or

$$\{\alpha\}^T[U(k_{\mathrm{T}}) \quad T(k_{\mathrm{T}})] = \{0\}^T. \tag{33}$$

Equation (33) indicates that there is a common left unitary vector $\{\alpha\}$ in both $[U]$ and $[T]$ matrices in the single- and double-layer potential approaches, respectively, from the concept of the SVD updating document. The aforementioned derivation can be similarly extended to the Neumann eigenproblem.

4.2. *Spurious eigenvalue*

Since spurious eigenvalues are imbedded in the formulation, a spurious eigenvalue exists if the single-layer potential approach is employed to solve either the Dirichlet or the Neumann problem. By assuming a solution for the Helmholtz boundary value problem, we can have the corresponding nonhomogeneous B.Cs ($u = \bar{u}$ or $t = \bar{t}$ on the boundary). Single-layer potential approach yields

$$[U(k_{\mathrm{S}})]\{p\} = \{\bar{u}\}, \tag{34}$$

$$[L(k_{\mathrm{S}})]\{p\} = \{\bar{t}\}, \tag{35}$$

where k_{S} denotes the spurious eigenvalue. Since the spurious eigenvalue k_{S} makes $[U]$ and $[L]$ both singular, we have

$$[U(k_{\mathrm{S}})]\{p\} = \{0\}, \tag{36}$$

$$[L(k_S)]\{p\} = \{0\}. \tag{37}$$

By combining Eq. (36) with Eq. (37), we have

$$\begin{bmatrix} [U(k_S)] \\ [L(k_S)] \end{bmatrix} \{p\} = \{0\}. \tag{38}$$

Equation (38) indicates that there is a common right unitary vector $\{p\}$ in both $[U]$ and $[L]$ matrices from the concept of the SVD updating term. Similarly, the aforementioned derivation can be similarly extended to the double-layer potential approach.

5. Numerical Results and Discussion

5.1. *Case 1: A circular membrane*

5.1.1. *Common left unitary vectors by using the MFS of real-part kernel (true eigenvalue)*

Since the real-part of the fundamental solution is singular if the source is distributed on the real boundary, an expanded idea by putting sources outside the domain (i.e., fictitious boundary) is used. In our work, the source points were located on the fictitious boundary where R is equal to *1.2a.* Based on our experience, the best range of the fictitious boundary is between *1.1a* and *1.4a.* Therefore, we choose *1.2a* to be an illustrative example. It is shown that the location of the spurious eigenvalue dependents on the fictitious boundary. The optimum location of the source points were mentioned by Alves [2009a], Alves and Antunes [2009b]. The distribution of source is shown in Fig. 1. Since the circular geometry has symmetry property, it is found that the left and right unitary vectors happen to be the same. For the Dirichlet or Neumann eigenproblem, the common left unitary vectors are numerically obtained in Fig. 1 as predicted in Eq. (33). Also, the true eigenvalue matches well with zeros of the Bessel function of $J_n(k_T)$ and its derivative $J_n'(k_T)$ for the Dirichlet and the Neumann problems, respectively.

5.1.2. *Common right unitary vectors by using the MFS of real-part kernel (spurious eigenvalue)*

The distribution of sources is shown in Fig. 2 on the expanded boundary. Regarding the single-layer or double-layer potential approach, the common left unitary vectors are numerically obtained in Fig. 2 as predicted in Eq. (38). The spurious eigenvalue matches well with zeros of the Bessel function $Y_n(1.2k_S)$ and its derivative $Y_n'(1.2k_S)$ for the single- and double-layer potential approaches, respectively.

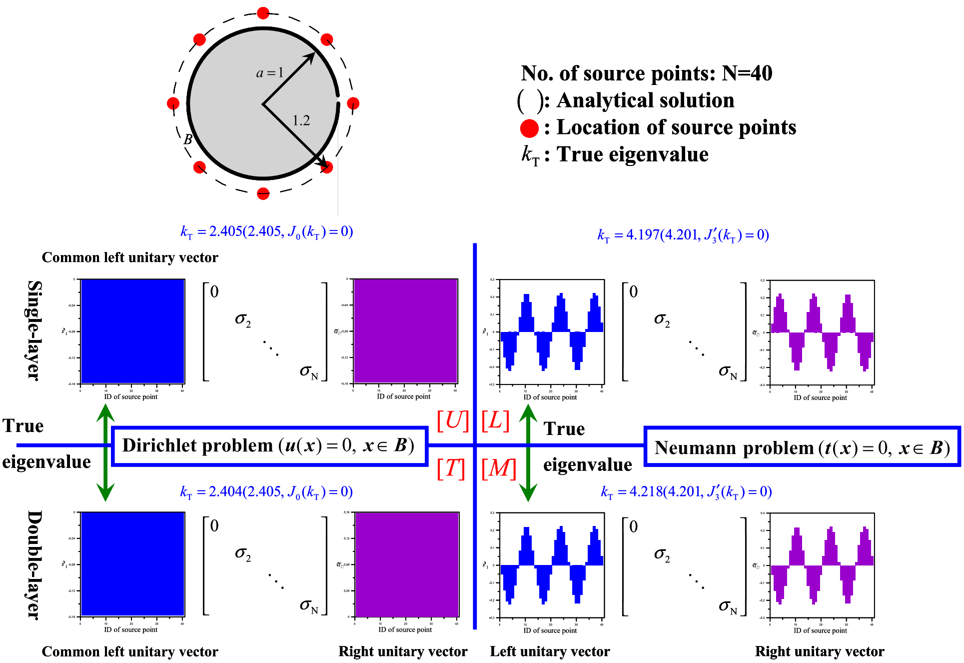

Fig. 1. (Color online) Common left unitary vectors of single- and double-layer real-part MFS for the Dirichlet and the Neumann problems (true eigenvalue).

5.1.3. *Common left unitary vectors by using the MFS of imaginary-part kernel (true eigenvalue)*

Since the imaginary-part of the fundamental solution is nonsingular even if the source point and the collocation point coincides together. Therefore, we can distribute the source points on the real boundary as shown in Fig. 3. Regarding the Dirichlet or Neumann eigenproblem, the common left unitary vectors are numerically obtained in Fig. 3 as predicted in Eq. (33). The true eigenvalue matches well with zeros of Bessel function $J_n(k_{\mathrm{T}})$ and its derivative $J_n'(k_{\mathrm{T}})$ for the Dirichlet and the Neumann problems, respectively.

5.1.4. *Common right unitary vectors by using the MFS of imaginary-part kernel (spurious eigenvalue)*

The distribution of source is also shown in Fig. 4 on the real boundary. Regarding the single-layer or double-layer potential approach, the common right unitary vectors are numerically obtained in Fig. 4 as predicted in Eq. (38). The spurious eigenvalue matches well with zeros of Bessel function $J_n(k_{\mathrm{S}})$ and its derivative $J_n'(k_{\mathrm{S}})$ for single- and double-layer potential approaches, respectively.

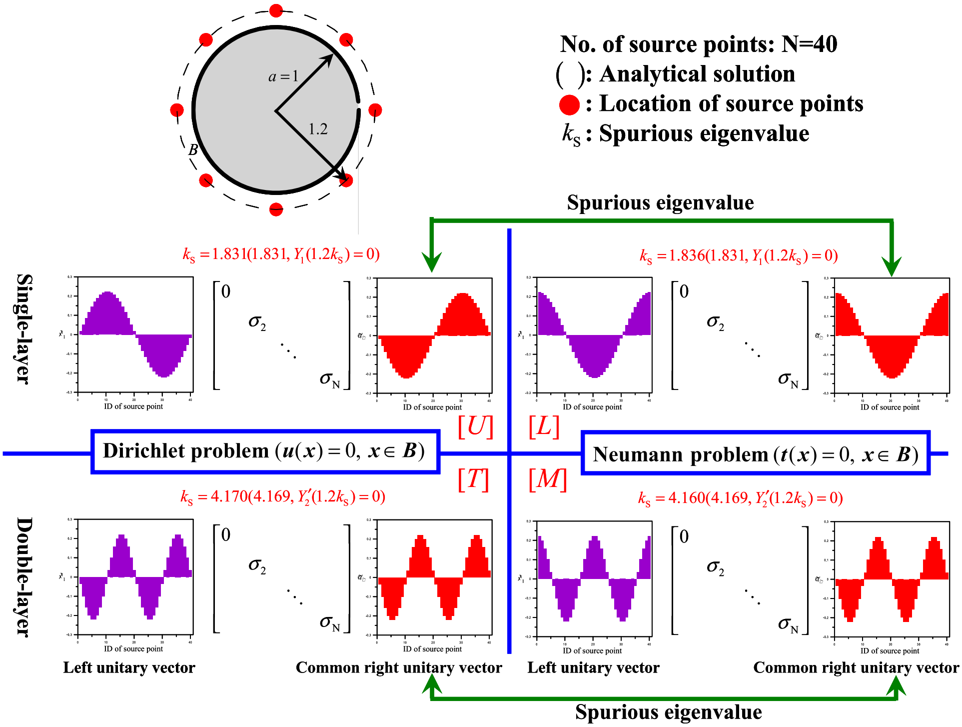

Fig. 2. (Color online) Common right unitary vectors of single- and double-layer real-part MFS for the Dirichlet and the Neumann problems (spurious eigenvalue).

5.2. *Case 2: An elliptical membrane*

5.2.1. *Common left unitary vectors by using the MFS of real-part kernel (true eigenvalue)*

As previous mentioned, the MFS of real-part kernel is singular if the source is distributed on the real boundary. The fictitious boundary of source distribution is put outside the physical domain at $\xi_S = 1.2\xi_1$, where ξ_1 is defined by

$$\xi_1 = \tanh^{-1}\left(\frac{b_1}{a_1}\right), \tag{39}$$

in which a_1 and b_1 are the half lengths of major and minor axes of the ellipse.

The distribution of source is also shown in Fig. 5. Regarding the Dirichlet or Neumann eigenproblem, the common left unitary vectors are numerically obtained in Fig. 5 as predicted in Eq. (33). The true eigenvalue matches well with zeros of the modified Mathieu function $Je_n(q, \xi_1)$ [Chen *et al.* (2012b)] and its derivative $Je_n'(q, \xi_1)$ [Chen *et al.* (2012b)] for the Dirichlet and the Neumann problems,

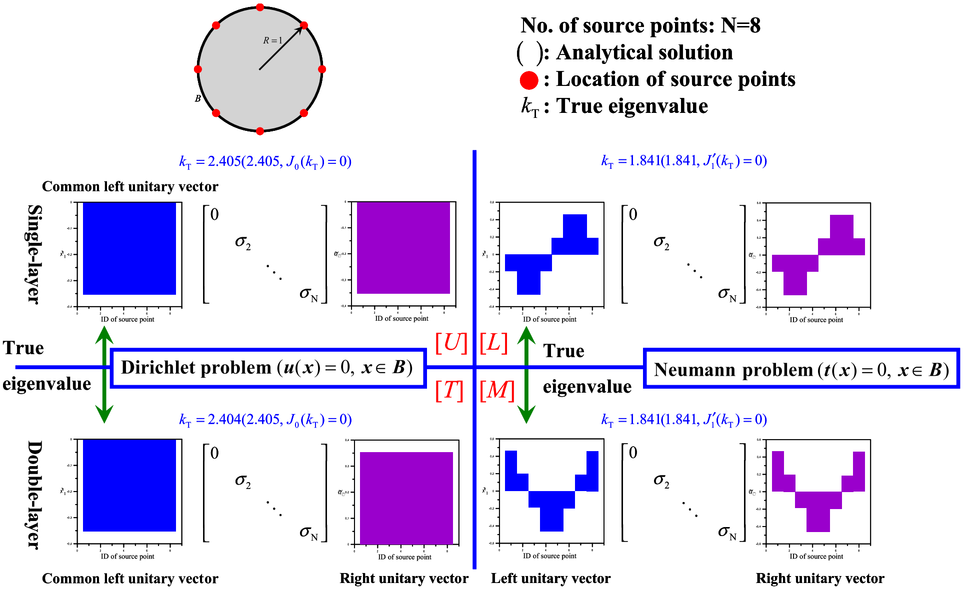

Fig. 3. (Color online) Common left unitary vectors of single- and double-layer imaginary-part MFS for the Dirichlet and the Neumann problems (true eigenvalue).

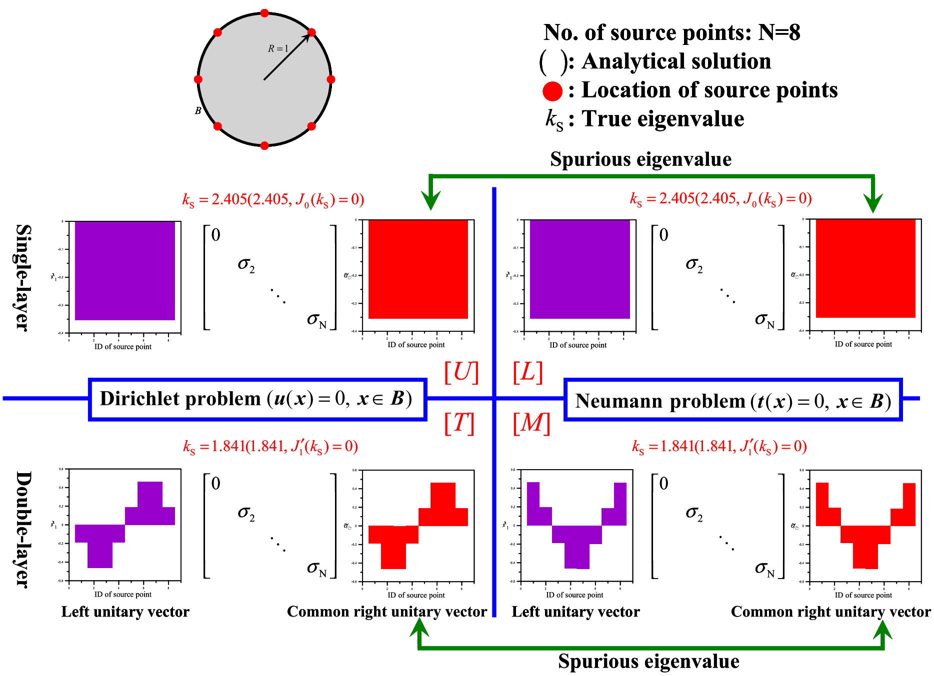

Fig. 4. (Color online) Common right unitary vectors of single- and double-layer imaginary-part MFS for the Dirichlet and the Neumann problems (spurious eigenvalue).

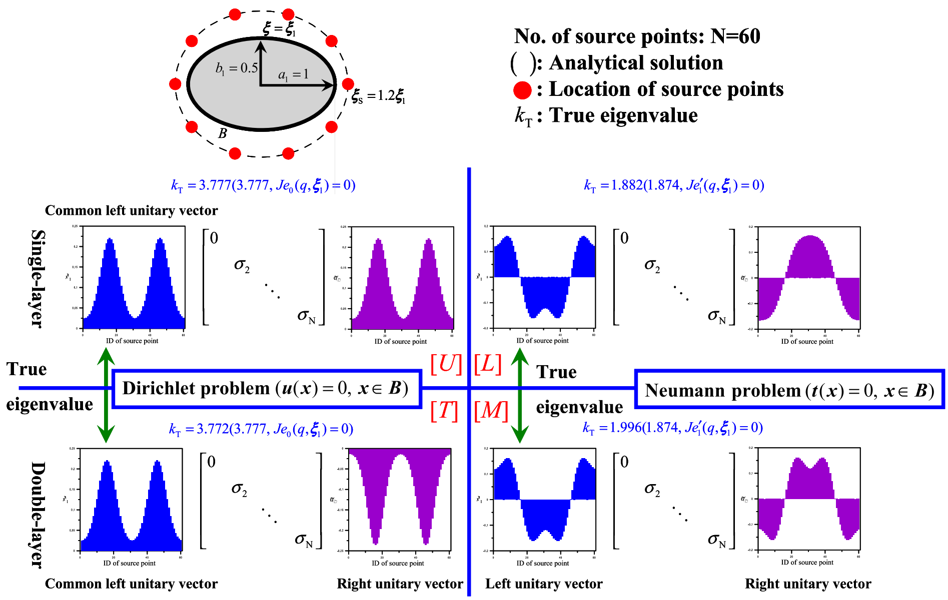

Fig. 5. (Color online) Common left unitary vectors of single- and double-layer real-part MFS for the Dirichlet and the Neumann problems (true eigenvalue).

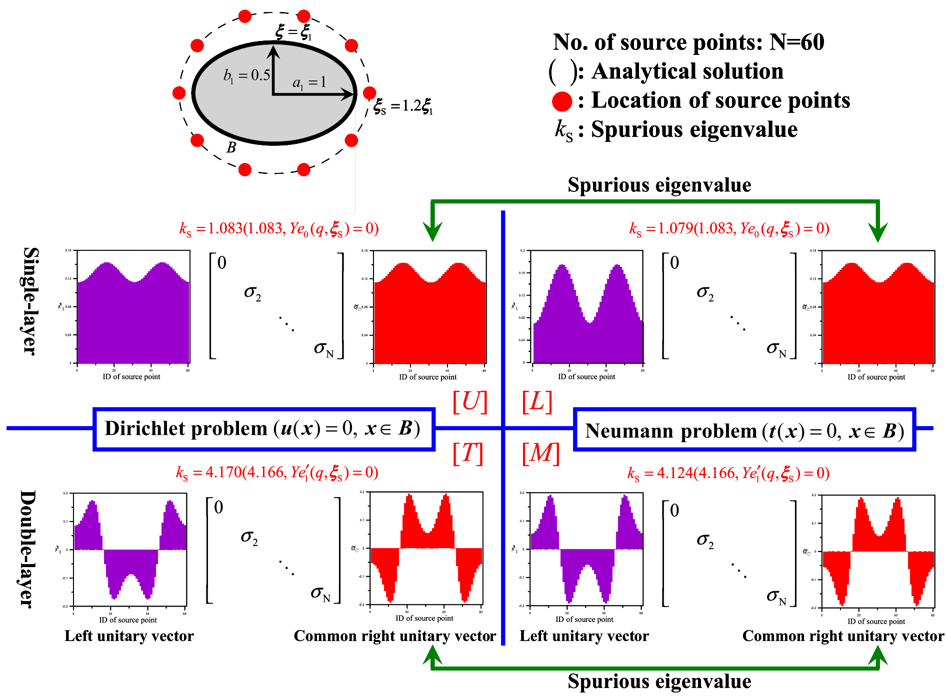

Fig. 6. (Color online) Common right unitary vectors of single- and double-layer real-part MFS for the Dirichlet and the Neumann problems (spurious eigenvalue).

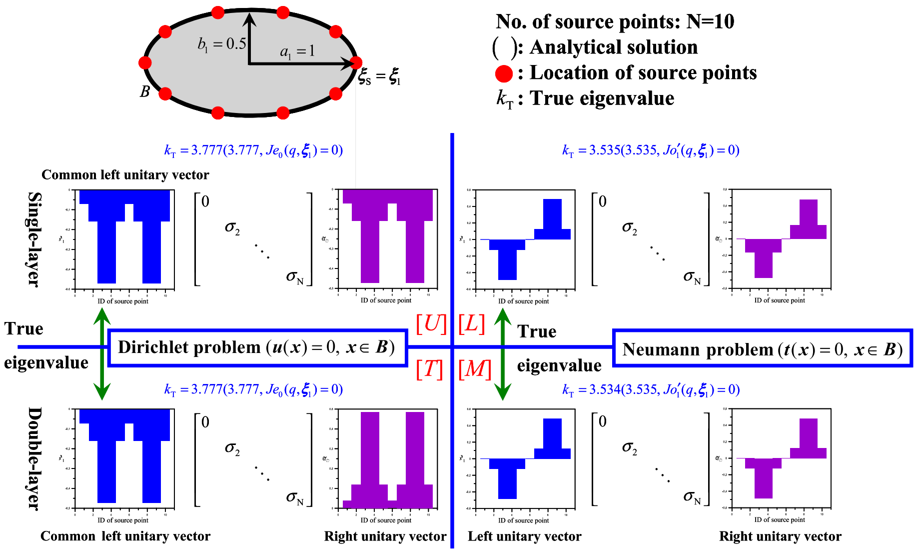

Fig. 7. (Color online) Common left unitary vectors of single- and double-layer imaginary-part MFS for the Dirichlet and the Neumann problems (true eigenvalue).

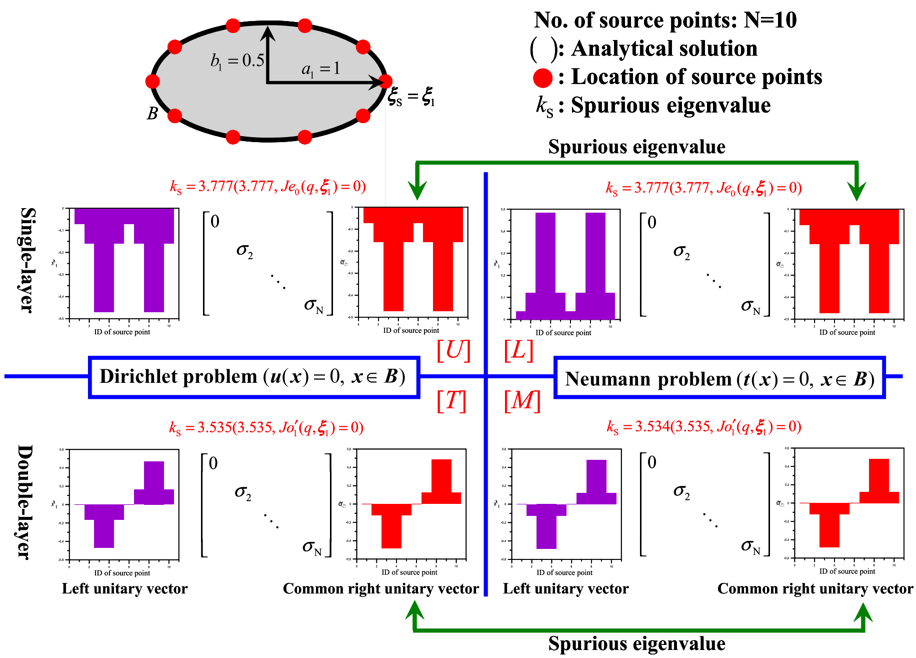

Fig. 8. (Color online) Common right unitary vectors of single- and double-layer imaginary-part MFS for the Dirichlet and the Neumann problems (spurious eigenvalue).

Table 1. The analytical true and spurious eigenequations for a circular membrane.

		Real-part MFS		Imaginary-part MFS	
Boundary condition	Eigenequation	Single-layer approach	Double-layer approach	Single-layer approach	Double-layer approach
Dirichlet	True eigenequation	$J_n(k_T) = 0$	$J_n(k_T) = 0$	$J_n(k_T) = 0$	$J_n(k_T) = 0$
	Spurious eigenequation	$Y_n(1.2k_S) = 0$	$Y'_n(1.2k_S) = 0$	$J_n(k_S) = 0$	$J'_n(k_S) = 0$
Neumann	True eigenequation	$J'_n(k_T) = 0$	$J'_n(k_T) = 0$	$J'_n(k_T) = 0$	$J'_n(k_T) = 0$
	Spurious eigenequation	$Y_n(1.2k_S) = 0$	$Y'_n(1.2k_S) = 0$	$J_n(k_S) = 0$	$J'_n(k_S) = 0$

Table 2. The analytical true and spurious eigenequations for an elliptical membrane.

		Real-part MFS		Imaginary-part MFS	
Boundary condition	Eigenequation	Single-layer approach	Double-layer approach	Single-layer approach	Double-layer approach
Dirichlet	True eigenequation	$Je_n(q,\xi_1) = 0$ $Jo_n(q,\xi_1) = 0$	$Je_n(q,\xi_1) = 0$ $Jo_n(q,\xi_1) = 0$	$Je_n(q,\xi_1) = 0$ $Jo_n(q,\xi_1) = 0$	$Je_n(q,\xi_1) = 0$ $Jo_n(q,\xi_1) = 0$
	Spurious eigenequation	$Ye_n(q,\xi_S) = 0$ $Yo_n(q,\xi_S) = 0$	$Ye'_n(q,\xi_S) = 0$ $Yo'_n(q,\xi_S) = 0$	$Je_n(q,\xi_1) = 0$ $Jo_n(q,\xi_1) = 0$	$Je'_n(q,\xi_1) = 0$ $Jo'_n(q,\xi_1) = 0$
Neumann	True eigenequation	$Je'_n(q,\xi_1) = 0$ $Jo'_n(q,\xi_1) = 0$	$Je'_n(q,\xi_1) = 0$ $Jo'_n(q,\xi_1) = 0$	$Je'_n(q,\xi_1) = 0$ $Jo'_n(q,\xi_1) = 0$	$Je'_n(q,\xi_1) = 0$ $Jo'_n(q,\xi_1) = 0$
	Spurious eigenequation	$Ye_n(q,\xi_S) = 0$ $Yo_n(q,\xi_S) = 0$	$Ye'_n(q,\xi_S) = 0$ $Yo'_n(q,\xi_S) = 0$	$Je_n(q,\xi_1) = 0$ $Jo_n(q,\xi_1) = 0$	$Je'_n(q,\xi_1) = 0$ $Jo'_n(q,\xi_1) = 0$

Note: $\xi_S = 1.2\xi_1$.

respectively, where the parameter q is defined by

$$q = \left(\frac{ck}{2}\right)^2, \tag{40}$$

in which c is the half distance between the two foci.

5.2.2. *Common right unitary vectors by using the MFS of real-part kernel (spurious eigenvalue)*

The distribution of source is also shown in Fig. 6 on the extracted boundary. Regarding the single-layer or double-layer potential approach, the common right unitary vectors are numerically obtained in Fig. 6 as predicted in Eq. (38). The spurious eigenvalue matches well with zeros of modified Mathieu function $Ye_n(q,\xi_S)$ and its derivative $Ye'_n(q,\xi_S)$ for the single- and double-layer potential approaches, respectively.

5.2.3. *Common left unitary vectors by using the MFS of imaginary-part kernel (true eigenvalue)*

Since the MFS of imaginary-part kernel is nonsingular even if the source point and the collocation point coincides together. Therefore, we distribute the source points on the real boundary as shown in Fig. 7. Regarding the Dirichlet or Neumann eigenproblem, the common right unitary vectors are numerically obtained in Fig. 7 as predicted in Eq. (33). The true eigenvalue matches well with zeros of modified Mathieu function $Je_n(q, \xi_1)$ [Chen *et al.* (2012b)] and its derivative $Je'_n(q, \xi_1)$ [Chen *et al.* (2012b)] for the Dirichlet and the Neumann problems, respectively.

5.2.4. *Common right unitary vectors by using the MFS of imaginary-part kernel (spurious eigenvalue)*

The distribution of source is shown in Fig. 8 on the real boundary. Regarding the Dirichlet or Neumann eigenproblem, the common right unitary vectors are numerically obtained in Fig. 8 as predicted in Eq. (38). The spurious eigenvalue matches well with zeros of the modified Mathieu function $Je_n(q, \xi_1)$ and its derivative $Je'_n(q, \xi_1)$ for the single- and double-layer potential approaches, respectively, by using the MFS of real- and imaginary-part kernels.

To summarize the results, the true and spurious eigenequations are listed in Tables 1 and 2 for circular and elliptical membranes, respectively.

6. Conclusions

We have shown that the spurious eigenvalues for the eigenproblems occur when both the MFS of real- and imaginary-part kernels were used. The positions of spurious eigenvalues for the eigenproblem depend on the location of fictitious boundary for the real-part kernel method where the sources are distributed. For the imaginary-part kernel method, source points can be directly collocated on the real boundary. By using the SVD technique, the common left unitary vectors are found in the case of true eigenvalue. For the spurious eigenvalue, the common right unitary vector is obtained. In this study, we have employed the SVD updating techniques to see the common left and right unitary vectors.

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